String Orbifolds and Quotient Stacks

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In this note we observe that, contrary to the usual lore, string orbifolds do *not* describe strings on quotient spaces, but rather seem to describe strings on objects called quotient stacks, a result that follows from simply unraveling definitions, and is further justified by a number of results. Quotient stacks are very closely related to quotient spaces; for example, when the orbifold group acts freely, the quotient space and the quotient stack are homeomorphic. We explain how sigma models on quotient stacks naturally have twisted sectors, and why a sigma model on a quotient stack would be a nonsingular CFT even when the associated quotient space is singular. We also show how to understand twist fields in this language, and outline the derivation of the orbifold Euler characteristic purely in terms of stacks. We also outline why there is a sense in which one naturally finds $B \neq 0$ on exceptional divisors of resolutions. These insights are not limited to merely understanding existing string orbifolds: we also point out how this technology enables us to understand orbifolds in M-theory, as well as how this means that string orbifolds provide the first example of an entirely new class of string compactifications. As quotient stacks are not a staple of the physics literature, we include a lengthy tutorial on quotient stacks, describing how one can perform differential geometry on stacks.

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Figure 1: Contribution to the (g, h) twisted sector of a string orbifold on T^2

1 Introduction

One often hears that string orbifolds [1, 2, 3] define strings propagating on quotient spaces (with some 'stringy' effects at singularities). Of course, a string orbifold is not described in terms of maps into a quotient space, but rather is set up in terms of group actions on covering spaces.

For example, recall that the partition function for a string orbifold (at 1-loop, say) is of the form

$$Z = \frac{1}{|G|} \sum_{g,h,gh=hg} Z_{g,h}$$

where each $Z_{g,h}$ is the partition function of a sigma model of maps from a square into the covering space X, such that the images of the sides of the square are identified under the action of the orbifold group Γ, as illustrated in figure 1. In particular, the partition function of the original, "unorbifolded" theory is just $Z_{1,1}$. (From a Hamiltonian perspective, summing over twisted sectors is equivalent to inserting projection operators that only allow Γ-invariant states to propagate in loops.) Clearly this is not a sigma model on the quotient space X/Γ, if for no other reason than the fact that one does not sum over maps into X/Γ, but rather is a sigma model on X in which the Γ-action on X has been gauged.

It seems reasonable that a theory defined in this fashion determines some physical theory living on the quotient space. Let us take a moment to study this matter more closely.

Traditionally, contributions to twisted-sector partition functions are described as maps from polygons into X, such that inside X, the sides of the polygon are identified by the group action. One can set up a category of continuous maps of this form, and we shall check in section 5 that this category is strictly not the same as the category of continuous maps into the quotient space X/Γ when Γ does not act freely. (The problem is that lying over any given

map into X/Γ, there can be several inequivalent maps of the form illustrated in figure 1.) The category of continuous maps into a space completely determines that space (see appendix A for more details), so if we can think of a string orbifold as describing strings on *some* space, then that space cannot be the same as X/Γ, but rather appears to be something containing more information at singularities than X/Γ.

Another way of thinking about the partition function Z of a string orbifold is that we have gauged the action of the orbifold group Γ on the covering space. However, the projection $X \to X/\Gamma$ is only a principal Γ-bundle when Γ acts freely on X. Thus, again, if we try to understand physical properties of string orbifolds in terms of quotient spaces, we seem to have a minor difficulty.

Another difficulty lies in interpreting twist fields. Many physicists have believed that twist fields could be understood in terms of some cohomology theory of the quotient space[1] At the moment, there does not seem to exist any completely satisfactory calculation that reproduces twist fields directly in terms of some cohomology of the quotient space. Now, although many physicists have this perspective, there does seem to be a bit of confusion regarding twist fields in the community, and we shall take a moment to speak to two classes of misconceptions that have arisen in some places:

1. Equivariant cohomology: Some physicists have claimed that the twist fields can be reproduced using equivariant cohomology. A simple counterexample will convince the reader that this is incorrect. Consider the $\mathbf{Z}_2$ action on $\mathbf{C}^2$ that leaves the origin invariant. Now,

   $$H^*_{\mathbf{Z}_2}\left(\mathbf{C}^2, \mathbf{Z}\right) = H^*_{\mathbf{Z}_2}(\text{point}, \mathbf{Z}) = H^*(B\mathbf{Z}_2, \mathbf{Z}) = H^*(\mathbf{RP}^\infty, \mathbf{Z}) = \{\mathbf{Z}, 0, \mathbf{Z}_2, 0, \mathbf{Z}_2, \cdots\}$$

   On the other hand,

   $$H^*\left(\widetilde{\mathbf{C}^2/\mathbf{Z}_2}, \mathbf{Z}\right) = H^*\left(\mathbf{P}^1, \mathbf{Z}\right) = \{\mathbf{Z}, 0, \mathbf{Z}, 0, 0, 0, \cdots\}$$

   Clearly, equivariant cohomology does not compute twist fields, as not even its freely-generated part is appropriate.

2. Resolutions. Because string orbifolds behave as though they described strings on smooth spaces, and because their massless spectrum contains twist fields, which often couple to exceptional divisors of resolutions, people sometimes speak loosely of string orbifolds describing strings on resolutions of quotient spaces. If that were literally true, then twist fields would be naturally understood in terms of properties of resolutions. However, not all singularities that one encounters can be resolved; for example, $\mathbf{C}^4/\mathbf{Z}_2$ admits no Calabi-Yau resolution, yet a $\mathbf{C}^4/\mathbf{Z}_2$ orbifold possesses a twist field.

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[1]As opposed to merely a description in terms of group actions on covers, as in, for example, [7].

We shall propose in this paper that string orbifolds are literally strings compactified on spaces known as quotient *stacks* (denoted $[X/\Gamma]$), which are closely related to quotient spaces X/Γ. Quotient stacks differ from quotient spaces by possessing extra structure at singularities, so in effect, we are arguing that the "stringy" structure believed to exist at singularities in string orbifolds, actually has a geometric interpretation. This picture of string orbifolds sheds new light on their physical properties.

At first blush, this may sound radical, but actually it is quite conservative. In particular, to a mathematician familiar with stacks, this would be very natural. Among other things, quotient stacks are one way of describing group actions on covers – the term "quotient stack" could be used synonymously with "orbifold" – and moreover, as stacks, they have the structure of a "generalized space," so that one can, for example, perform differential geometry on a (quotient) stack. (One way to think of stacks is as slight generalizations of spaces.) In a nutshell, we are proposing that the extra structure of a generalized space has physical relevance.

Quotient stacks have the following basic properties:

1. Maps from the string worldsheet into $[X/\Gamma]$ are described by maps from twisted sectors into the cover X. Put another way, a string orbifold is literally a weighted sum over maps into $[X/\Gamma]$, a smoking gun for an interpretation as a sigma model on $[X/\Gamma]$. This is, in the author's mind, the single most important point – by merely unraveling definitions, one finds the signature of an interpretation of string orbifolds in terms of a sigma model on a quotient stack $[X/\Gamma]$. We also propose a classical action for a sigma model on a stack that reproduces string orbifolds (down to $|G|^{-1}$ factors in partition functions) for the special case of global quotient stack targets.

2. If Γ acts freely, so that X/Γ is nonsingular, then $[X/\Gamma] \cong X/\Gamma$.

3. Regardless of whether or not Γ acts freely, there is a natural map $X \to [X/\Gamma]$ which describes X as a principal Γ-bundle over $[X/\Gamma]$.

4. The quotient stack $[X/\Gamma]$ is smooth if X is smooth and Γ acts by diffeomorphisms, regardless of whether or not Γ acts freely. Thus, the CFT of a sigma model on $[X/\Gamma]$ is naturally well-behaved, regardless of whether X/Γ is singular. In the past, people have claimed that string orbifolds are nonsingular because "string theory smooths out singularities;" we instead are claiming that string orbifolds are nonsingular because they describe sigma models into smooth spaces – nonsingularity has nothing to do with string theory. The old lore about "strings smoothing out singularities" is just a result of misinterpreting the underlying mathematical structure. Later we shall argue that, moreover, quotient stacks are smooth in precisely the sense relevant for physics.

In addition,

5. We can now naturally interpret twist fields, in terms of cohomology of a stack associated to a quotient stack rather than a quotient space. More precisely, we shall argue later that the low-energy spectrum of a sigma model on $[X/\Gamma]$ is determined not by a cohomology of $[X/\Gamma]$ (a subtlety of generalized spaces), but rather by a cohomology of the associated inertia group stack $I_{[X/\Gamma]}$. This stack has the pleasant property that [9]

$$I_{[X/\Gamma]} \cong \coprod_{[g]} [X^g/C(g)]$$

which is the precise form of the Hirzebruch-Höfer [10] expression for the orbifold Euler characteristic

$$\chi_{orb}(X, \Gamma) = \sum_{[g]} \chi(X^g/C(g))$$

For physicists who have been expecting a description of twist fields in terms of a cohomology of a quotient space, this is rather new. However, this result is not new to some mathematicians, who may recognize it as an overcomplicated way of describing twist fields in terms of group actions on covers. In essence, we are arguing that a description of twist fields in terms of group actions on covers is the best one can expect.

6. The lore that string orbifolds have "nonzero B fields at quotient singularities" also has a natural understanding in this framework, which we shall explore in detail. Very briefly, if we think of a quotient stack as looking like a quotient space with some sort of "extra structure" at the singularities, then that extra structure is precisely a gerbe over the singularities, precisely right to duplicate the existing lore.

7. The role of equivariance in string orbifolds is now clarified. We shall show that a bundle (or sheaf) on the quotient stack $[X/\Gamma]$ is the same thing as a Γ-equivariant bundle (or sheaf) on X. Thus, the standard intuition that one defines an orbifold group action by specifying an equivariant structure, is clarified.

So far we have merely put a new perspective on existing physics. However, there are some important new directions we can use this technology to pursue:

1. This now gives us an entirely new family of spaces to compactify on. Quotient stacks are examples of "generalized" spaces, and cannot be understood within the usual notions of point-set topology. Since we now have an example of strings on generalized spaces (provided by string orbifolds), we see that strings can indeed consistently propagate on these generalized spaces, and so these can be used for compactification in general.

2. We can now understand orbifolds in M-theory (in the sense of, quantum theory underlying eleven-dimensional supergravity). Previously, one might have argued that understanding orbifolds in M-theory was unclear, as twisted sectors seem to be purely

stringy. However, now that we properly understand the underlying mathematics, we see that twisted sectors are not in any way stringy, and can now make sense of the notion of an orbifold in M-theory, as M-theory compactified on a quotient stack $[X/\Gamma]$.

3. As a further minor note, this appears to give an alternative way of understanding the computation of B fields at quotient singularities.

Unfortunately, the picture is not completely set. One of the immediate consequences of a description of string orbifolds as sigma models on quotient stacks, is that deformation theory of the string orbifold CFT must be understood in terms of deformation theory of the quotient stack, as opposed to the quotient space. We shall give indirect evidence that this may coincide with the usual picture, in terms of a quotient space "with B fields," but much work remains to be done to check whether this is actually consistent.

Phrased another way, although we describe sigma models on stacks classically, we have not checked whether there are any obstructions to quantization, which would be due to global effects (see for example [43] for a discussion of this matter in the context of sigma models on spaces). By performing consistency checks (such as checks of the deformation theory, as above), one can hope to gain some insight into whether this picture is consistent. In any event, much work remains to be done.

We begin in section 2 with an overview of groupoids and stacks, the technology that will be central to this paper. In section 3 we then give a lengthy pedagogical tutorial on the basics of quotient stacks. In section 4 we consider sigma models on quotient stacks, and argue that immediately from the definitions, a sum over maps into a quotient stack duplicates both the twisted sector sum and the functional integrals within each twisted sector, a smoking gun for an interpretation of string orbifolds in terms of sigma models on stacks. In section 5 we examine quotient stacks when the orbifold group action is free, and argue that in this special case, $[X/\Gamma] \cong X/\Gamma$. In section 6 we point out that X is always the total space of a principal Γ-bundle over $[X/\Gamma]$, regardless of whether or not Γ acts freely, a fact we will use often. In section 7 we examine the lore on equivariance in string orbifolds, and show how a sheaf on $[X/\Gamma]$ is equivalent to a Γ-equivariant sheaf on X. In section 8 we show that quotient stacks are always smooth (so, as we argue later, a sigma model on a quotient stack should be well-behaved, as indeed string orbifold CFT's are). In section 9 we complete the picture begun in section 4 by writing down a proposal for a classical action for a sigma model on a stack, which generalizes both ordinary sigma models and string orbifolds. In section 10 we briefly outline what conditions would naively be necessary for a sigma model on a stack to yield a well-behaved CFT, and argue that smoothness of the target stack are the conditions needed – so, in other words, the notion of 'smoothness' for stacks has physical relevance in this context. In section 11 we examine the low-energy spectrum of strings propagating on a quotient stack, and argue that it is computed by the cohomology of an auxiliary stack $I_{[X/\Gamma]}$, a result that, although known to some mathematicians, has a very different form from what many physicists have expected. In section 12 we talk about

non-effective orbifold group actions. In section 13 we describe how the old lore about B fields and string orbifolds seems to fit into this picture. In section 14 we talk about what our results implies about understanding orbifolds in M theory. In section 15 we very briefly speak about the possibility of getting new string compactifications from stacks. In section 16, on a slightly different note, we discuss some mathematical lore concerning the relationship between stacks and noncommutative geometry. Finally we conclude in section 17 with a list of followup projects that we encourage readers to pursue. We have also included several appendices on more technical aspects of quotient stacks.

Before we proceed with the rest of this paper, a few comments are in order. First, we should comment on our quotient stacks. Historically, stacks have been used primarily by algebraic geometers, so many discussions of stacks refer to schemes. However, stacks have nothing to do *per se* with schemes, and can be described purely topologically (see [8, 11, 12, 13, 14, 15] for examples). Now, although stacks in general are not specific to algebraic geometry, all previous discussions of quotient stacks we have seen are written in terms of schemes. To make them more useful for physics (and accessible for physicists), we have written up a description of quotient stacks from purely topological and differential-geometric perspectives. The details and analysis of such "topological quotient stacks" is virtually identical to that of traditionally-defined quotient stacks. In particular, in this paper we really only use very basic properties of quotient stacks, and these properties all go over without modification to our "topological quotient stacks" (essentially because the underlying analysis is identical between the cases). Put another way, because our "topological quotient stacks" are in principle very slightly different from traditional quotient stacks, one has to check that everything works – but after checking, one finds that everything basic works just as usual, and essentially for the same reasons. Because these "topological quotient stacks" are virtually identical to traditional quotient stacks, we have opted to refer to them as quotient stacks; dignifying them with a new name seems excessive. Furthermore, although our description of and perspective on quotient stacks is very slightly novel, the results on quotient stacks that we use are very well-entrenched in the existing mathematics literature.

It should also be said that there exists a group of mathematicians who already use quotient stacks to describe string orbifolds. Indeed, as described above, quotient stacks are an overcomplicated way to describe group actions on covering spaces, and also possess extra structure that gives them an interpretation as a sort of 'generalized space,' so that, for example, one can make sense of differential geometry on a quotient stack. However, as far as the author has been able to determine, the mathematicians in question have not done any of the work required to justify making the claim that a string orbifold CFT coincides with the CFT for a string compactified on a quotient stack, or even to justify the claim that the notion of string compactification on a stack makes sense. They do not seem to have attempted to study sigma models on stacks, and do not even realize why this is relevant. They also do not seem to be aware of even the most elementary physical implications of such a claim, such as the fact that, to be consistent, any deformations of a string orbifold CFT would have to be interpreted in terms of deformations of the quotient stack, rather than the

quotient space. Thus, mathematicians reading this paper should interpret our work as the beginnings of a program to fill in the logical steps that they seem to have omitted. In fact, this paper was written for a physics audience; mathematicians are encouraged to instead read lecture notes [6] we shall publish shortly.

Finally, in our discussion of quotient stacks as generalized topological spaces, we assume that Γ is discrete[2] and acts by homeomorphisms. When we discuss quotient stacks as manifolds, we assume that X is a smooth manifold and that Γ acts by diffeomorphisms.

2   Generalities on groupoids and stacks

We shall begin with a brief overview of groupoids and stacks, which we shall be manipulating for the rest of this paper. The original reference for the material present is [16, section 4]. For algebraic stacks in general, and quotient stacks in particular, [17] is an excellent resource, and we also recommend [18, appendix A].

For alternative perspectives on stacks, less useful in the present context but more useful for other things, see for example [8, 11, 12, 13, 14, 15].

2.1   Groupoid basics

Let $\mathcal{C}$ be any category. Let $\mathcal{F}$ be another category and $p : \mathcal{F} \to \mathcal{C}$ a functor. We say that $\mathcal{F}$ is a groupoid over $\mathcal{C}$ [16, section 4] if it obeys the following two axioms:

1. If $\rho : U \to V$ is a morphism in $\mathcal{C}$ and $\eta \in \mathrm{Ob}\,\mathcal{F}$ with $p(\eta) = V$, then there exists an object $\xi \in \mathrm{Ob}\,\mathcal{F}$ and a morphism $f : \xi \to \eta$ in $\mathcal{F}$ such that $p(\xi) = U$ and $p(f) = \rho$.

2. If $\phi : \xi \to \zeta$ and $\psi : \eta \to \zeta$ are morphisms in $\mathcal{F}$ and $h : p(\xi) \to p(\eta)$ is a morphism in $\mathcal{C}$ such that $p(\psi) \circ h = p(\phi)$, then there exists a unique morphism $\chi : \xi \to \eta$ such that $\psi \circ \chi = \phi$ and $p(\chi) = h$.

It is straightforward to check that the object ξ determined in the first axiom is unique up to unique isomorphism, from the second axiom. Since ξ is determined uniquely (up to unique isomorphism), ξ is commonly denoted $\rho^*\eta$ (remembering that it comes with a morphism $\rho^*\eta \to \eta$).

It is also straightforward to check that if α is a morphism in $\mathcal{F}$ such that $p(\alpha) = \mathrm{Id}$, then α is invertible.

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[2]This condition can be dropped in general, but it simplifies certain technical aspects, and is sufficient for our purposes.

Any object of $\mathcal{C}$ naturally defines a groupoid over $\mathcal{C}$. Specifically, let $X \in \mathrm{Ob}\,\mathcal{C}$, and define $\mathcal{F} = \mathrm{Hom}_{\mathcal{C}}(-, X)$, i.e., the category whose

1. Objects are morphisms $Y \to X$ in $\mathcal{C}$, for any other object $Y \in \mathrm{Ob}\,\mathcal{C}$, and

2. Morphisms $(Y \to X) \longrightarrow (Z \to X)$ are morphisms $Y \to Z$ such that the diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
$$

commutes.

Later in this paper, we shall often use the notation $(X)_{map}$ in place of $\mathrm{Hom}(-, X)$, in an attempt to give readers inexperienced in these matters a better grip. We can define a functor $p : \mathrm{Hom}(-, X) \to \mathcal{C}$, that maps an object $(Y \to X)$ to the object $Y \in \mathrm{Ob}\,\mathcal{C}$, and acts in the obvious trivial way upon morphisms. It is straightforward to check that with these definitions, $\mathrm{Hom}(-, X)$ is a groupoid over $\mathcal{C}$.

A morphism of groupoids $\mathcal{F} \to \mathcal{G}$ over $\mathcal{C}$ is a functor between the categories such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \mathcal{G} \\
& \searrow \quad \swarrow & \\
& \mathcal{C} &
\end{array}
$$

Note that since we have defined groupoids associated to objects $X \in \mathrm{Ob}\,\mathcal{C}$, we can now talk about maps $X \to \mathcal{F}$, i.e., maps from objects of $\mathcal{C}$ into groupoids $\mathcal{F}$ over $\mathcal{C}$. Such a map is simply a functor

$$\mathrm{Hom}_{\mathcal{C}}(-, X) \longrightarrow \mathcal{F}$$

compatible with the projections to $\mathcal{C}$.

We can now talk about representable groupoids. We say that a groupoid $\mathcal{F}$ is representable, or represented by $X \in \mathrm{Ob}\,\mathcal{C}$, if there exists an object $X \in \mathrm{Ob}\,\mathcal{C}$ and an equivalence of categories $F : \mathcal{F} \longrightarrow \mathrm{Hom}(-, X)$ (compatible with the projection maps to $\mathcal{C}$).

Now that we have described groupoids for general categories $\mathcal{C}$, what categories does one actually use in practice? Sometimes, especially if one wants to think of a groupoid as being a "presheaf of categories" on a space, one takes $\mathcal{C}$ to be the category of open sets on that space (see for example [8, 11, 12, 13].)

However, when describing quotient stacks and other algebraic stacks, one typically wants $\mathcal{C}$ to be Sch/S (the category of schemes relative to a fixed scheme), whose topological equivalent is Top, the category of topological spaces.

Now, in this paper we want to describe stacks from a topological and differential-geometric perspective, so we shall refer to groupoids over Top, instead of Sch/S. Also, we shall be specifically concerned with quotient stacks, which can also be understood as groupoids over quotient spaces. This fact is often obscure in usual discussions of quotient stacks (see [18] for a recent discussion), but as it adds a great deal of geometric insight, we shall emphasize this connection in this paper.

2.2   Stack basics

A stack is a special kind of groupoid, one that satisfies certain gluing axioms (just as a sheaf is a special kind of presheaf, satisfying certain gluing axioms). The relation to sheaves is more than just an analogy – stacks are often thought of as, essentially, sheaves of categories. One demands that objects and morphisms satisfy certain gluing axioms. Of course, to make sense of those gluing axioms, one must define a notion of covering, which requires putting what is known as a Grothendieck topology on the categories in question.

In this paper, although we will be manipulating groupoids a very great deal, nothing we shall be doing will require us to work with stacks specifically. Put another way, although groupoids are very important in this paper, stacks *per se* have no role at all. Although we will be describing quotient "stacks," we will only be manipulating them at the level of the underlying groupoid.

As stacks *per se* (as opposed to groupoids) play such a marginal role in this paper, we shall not discuss them further.

For more information on stacks specifically, see for example [8, 11, 17, 18, 16, 12, 13].

3   Quotient stacks

In this section we shall give a basic description of the quotient stack $[X/\Gamma]$. Our presentation is ordered pedagogically. We begin with some generalities on generalized spaces, then discuss the points of a quotient stack, and then outline how one could understand the open sets of a quotient stack. Then, finally, we come to a thorough technically useful description, in terms of the continuous maps into a quotient stack.

In passing, for the benefit of the expert on stacks, we shall say a few things about

our presentation. First, since we are interested in describing stacks from a topological and differential-geometric perspective, our groupoids will be defined over Top, the category of topological spaces, rather than Sch/S, the category of schemes relative to some fixed scheme S. Second, we are primarily interested in quotient stacks in this paper, which can be understood as groupoids over quotient spaces as well as Top (see [18] for a recent discussion of this matter). In order to help the reader gain some geometric insight into quotient stacks, we shall primarily refer to them in terms of groupoids over quotient spaces. In particular, we shall talk about the category of continuous maps $[X/\Gamma]_{map}$ into a quotient stack, as a groupoid over the category of continuous maps $(X/\Gamma)_{map}$ into the quotient space. For pedagogical reasons we shall also set up the category of points of $[X/\Gamma]$, as a groupoid over the discrete[3] category of points of X/Γ.

The general intuition for quotient stacks is that a quotient stack $[X/\Gamma]$ looks essentially like a quotient space X/Γ, but with extra structure at the singularities. That extra structure makes quotient stacks far better behaved than quotient spaces. The technical description of quotient stacks rapidly obscures this intuition; the reader may want to keep this picture in mind.

3.1   Generalized spaces

Quotient stacks are examples of "generalized spaces," which cannot be understood within the usual notions of point-set topology. In a nutshell, in generalized spaces, all occurrences of "set" in point-set topology should be replaced by "category." For example, instead of having a *set* of points, a generalized space has a category of points. In other words, distinct points can be isomorphic, and a single point can have nontrivial automorphisms.

Ordinary spaces can also be thought of as generalized spaces. For example, since sets are discrete categories, we can think of the set of points of an ordinary space as a (discrete) category of points.

Generalized spaces are perhaps best defined in terms of the category of maps into them. After all, certainly ordinary topological spaces are completely determined by the categories of maps into them (see appendix A), so this is a very natural approach. One can then recover, for example, the category of points as the category of maps from a fixed point into the space. This is somewhat analogous to noncommutative geometry, where spaces are defined by the rings of functions on them. Instead of altering the product structure of the ring to describe new spaces, we alter the category structure of maps. This may sound somewhat cumbersome, but it is actually an ideal setup for string sigma models.

In passing, we should also mention that the idea of defining spaces in terms of the maps

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[3]A discrete category is the same thing as a set – it is a category in which the only morphisms are the identity morphisms.

into them is a commonly-used setup in algebraic geometry (see discussions of "Grothendieck's functor of points" in, for example, [19, section II.6] or [20, section VI]). For example, Hilbert schemes are defined by the maps into them; the defining statement [21, chapter 1.1] is that a map from a scheme U into a Hilbert scheme of ideals over X (i.e., an object of the category) is a closed subscheme of $U \times X$, flat over U, of fixed Hilbert polynomial.

We shall introduce quotient stacks in two successive stages. First, we shall discuss the category of points of a quotient stack. The category of points does not completely characterize quotient stacks in a useful form, however. Completely characterizing quotient stacks is the business of the second stage, where we discuss the continuous maps into a quotient stack.

For an introduction to generalized spaces, see for example [8, 22, 23, 24, 25, 26].

3.2   Points

3.2.1   Definition and intuition

The category $[X/\Gamma]_{pt}$ of points of $[X/\Gamma]$ has the following objects and morphisms:

1. Objects are Γ-equivariant maps $\Gamma \to X$, i.e., maps $f : \Gamma \to X$ such that $f(gh) = gf(h)$ for $g, h \in \Gamma$.

2. Morphisms $f_1 \to f_2$ are Γ-equivariant bijections $\lambda : \Gamma \to \Gamma$ such that $f_2 \circ \lambda = f_1$

Loosely speaking, the category $[X/\Gamma]_{pt}$ of points of $[X/\Gamma]$ consists of the orbits of points of X under Γ.

Note that there is a natural projection functor $[X/\Gamma]_{pt} \to (X/\Gamma)_{pt}$ (where $(X/\Gamma)_{pt}$ denotes the discrete category (i.e., set) of points of X/Γ): any object $(f : \Gamma \to X)$ maps to the image of $f(e)$ in X/Γ, and any morphism maps to the identity morphism. It is straightforward to check that, with this projection functor, $[X/\Gamma]_{pt}$ is a groupoid over $(X/\Gamma)_{pt}$.

Let us pause to study these definitions. For any one orbit of a point on X, the various elements of the corresponding isomorphism class of points on $[X/\Gamma]$ consists of the various ways to map points of Γ into the points of the orbit. Thus, isomorphism classes of points of $[X/\Gamma]$ are in one-to-one correspondence with points of X/Γ. In fact, when Γ acts freely, we shall see later that $[X/\Gamma]$ and X/Γ are homeomorphic[4], so away from fixed points of Γ, $[X/\Gamma]$ and X/Γ look the same.

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[4]For example, it is straightforward to check that if Γ acts freely, then the category of points of $[X/\Gamma]$ is equivalent to the discrete category of points of X/Γ.

What happens at fixed points of Γ? To gain a little insight, first consider the case that $\Gamma = \mathbf{Z}_2$. At a fixed point, there will be a single corresponding point of $[X/\Gamma]$, but this point will have a nontrivial automorphism in the category of points.

More generally, it is straightforward to check that if f_1, f_2 are two objects in the category of points, then the set of morphisms $f_1 \to f_2$ will be

1. empty if f_1 and f_2 do not have the same image on X, or

2. nonempty if they do have the same image on X, and containing as many elements as the number of elements of the isotropy group of a point in the orbit

As mentioned earlier, the intuitive picture of quotient stacks is that $[X/\Gamma]$ looks like X/Γ except at singularities, where the quotient stack has extra structure. Here, we have seen this explicitly, at the level of points of the stack.

3.2.2 The canonical functor $(X)_{pt} \to [X/\Gamma]_{pt}$

In general, the projection map $X \to X/\Gamma$ only defines X as a principal Γ-bundle over X/Γ if Γ acts freely. However, there is a natural map $X \to [X/\Gamma]$.

We can understand this map at the level of points as a functor $\pi : (X)_{pt} \to [X/\Gamma]_{pt}$, defined as follows. For any point $x \in X$ (i.e., any object of the discrete category $(X)_{pt}$), define $f_x : \Gamma \to X$ by, $f_x(g) = g \cdot x$. (The only morphisms in $(X)_{pt}$ are the identity morphisms, which are mapped to identity morphisms.) We shall show later that this map always defines X as the total space of a principal Γ-bundle over $[X/\Gamma]$, regardless of whether or not Γ acts freely.

Before continuing, we shall make a technical observation concerning this functor, that will be useful later. In principle, one could define other functors $\pi_g : X \to [X/\Gamma]$, one for each $g \in \Gamma$, as the composition

$$X \xrightarrow{g} X \xrightarrow{\pi} [X/\Gamma]$$

However, these maps are all equivalent, in the sense that the corresponding functors are always isomorphic. (Indeed, the reader might have expected this, since the analogous maps into the quotient space X/Γ are all equivalent.) Specifically, at the level of points, we can explicitly define a natural transformation $\omega : \pi \Rightarrow \pi_g$ as follows: for each $x \in X$, define $\omega(x) : \Gamma \to \Gamma$ to be the Γ-equivariant map defined by $\omega(x)(e) = g$. It is straightforward to check that this is an invertible natural transformation, and so, at least at the level of points, $\pi \cong \pi_g$. We shall see this is also the case in general later.

3.3   Continuous maps

As mentioned earlier, quotient stacks $[X/\Gamma]$ are defined by the category $[X/\Gamma]_{map}$ of maps into them. In this section we shall describe that category in detail.

Much of this section is rather technical. The basic idea is motivated in the next subsection; a reader working through this material for the first time might want to read only the next subsection and leave the rest for later study.

3.3.1   Intuition

In this subsection we shall merely motivate the definition of a continuous map into a quotient stack $[X/\Gamma]$, rather than try to give an intrinsic definition of continuity for generalized spaces.

Recall points of $[X/\Gamma]$ are Γ-equivariant maps $\Gamma \to X$. What should a "continuous" map $f : Y \to [X/\Gamma]$ look like?

Well, for each point $y \in Y$, one would have a copy of Γ, together with a Γ-equivariant map $f_y : \Gamma \to X$. So, naively, it looks as though continuous maps $f : Y \to [X/\Gamma]$ should be described by Γ-equivariant maps $Y \times \Gamma \to X$.

However, this description of the possible maps is slightly naive. Ordinarily, when we think of a function $f : Y_1 \to Y_2$ between two (ordinary) spaces Y_1, Y_2, the function should assign a unique point of Y_2 to each point of Y_1. That is what it means for a function to be well-defined, after all. However, for generalized spaces, matters are somewhat more interesting. For a function $f : Y \to [X/\Gamma]$ to be well-defined, we do not need to assign a unique point of $[X/\Gamma]$ to each point of Y, but rather only a unique equivalence class of points to each point of Y – it does not matter if f does not assign a well-defined point to each point of Y, so long as the possible points of $[X/\Gamma]$ assigned to any one point of Y are isomorphic.

Thus, in general, a continuous map $Y \to [X/\Gamma]$ is defined by a principal Γ-bundle $E \to Y$, together with a Γ-equivariant map $f : E \to X$.

3.3.2   Technical definition

Now that we have developed some intuition for continuous maps into $[X/\Gamma]$, we shall give the complete technical definition of the category of continuous maps into $[X/\Gamma]$. First, however, we shall define the category of continuous maps $(Y)_{map}$ into a topological space Y, both for purposes of comparison, and for later use.

The remainder of this section is rather technical; the reader may wish to skip it upon a

first reading.

   Given any topological space Y, we can define a category $(Y)_{map}$ (also denoted $\mathrm{Hom}(-, Y)$) consisting of all the continuous maps into Y:

1. Objects are continuous maps $f : W \to Y$, where W is any topological space

2. Morphisms $(W_1 \xrightarrow{f_1} Y) \longrightarrow (W_2 \xrightarrow{f_2} Y)$ are maps $\lambda : W_1 \to W_2$ such that the following diagram commutes:

$$
\begin{array}{ccc}
W_1 & \xrightarrow{\ \ \lambda\ \ } & W_2 \\
& {}_{f_1}\searrow \quad \swarrow{}_{f_2} & \\
& Y &
\end{array}
$$

It is straightforward to check that, for any topological space Y, the category $(Y)_{map}$ is a groupoid over the category *Top* of topological spaces, whose objects are topological spaces and morphisms are continuous maps. (The projection functor is defined by sending an object $(W \to Y)$ of $(Y)_{map}$ to the object $W \in \mathrm{Ob}\ Top$, and in the obvious fashion on morphisms.)

   Now, we shall finally define the category of continuous maps into a quotient stack. Define the category $[X/\Gamma]_{map}$ of continuous maps into the quotient stack $[X/\Gamma]$ as follows:

1. Objects are pairs $(E \longrightarrow Y, E \xrightarrow{f} X)$, where

   Y is any topological space,
   $E \to Y$ is a principal Γ-bundle, and
   $f : E \to X$ is a continuous Γ-equivariant map.

2. Morphisms

$$
\left(E_1 \longrightarrow Y_1,\ E_1 \xrightarrow{f_1} X \right) \longrightarrow \left(E_2 \longrightarrow Y_2,\ E_2 \xrightarrow{f_2} X \right)
$$

   are pairs (ρ, λ), where $\rho : Y_1 \to Y_2$ is a continuous map, and $\lambda : E_1 \to E_2$ is a bundle morphism, meaning

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \lambda\ } & E_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\ \rho\ } & Y_2
\end{array}
$$

   commutes, and is constrained to make the diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \ \lambda\ \ } & E_2 \\
& {}_{f_1}\searrow \quad \swarrow{}_{f_2} & \\
& X &
\end{array}
$$

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commute.

It is straightforward to check that $[X/\Gamma]_{map}$ is a groupoid over the category *Top* of topological spaces. (The projection functor sends pairs $(E \to Y, E \to X)$ to $Y \in \text{Ob } Top$, and sends a morphism (ρ, λ) to ρ.) Now, we shall argue later that $[X/\Gamma]_{map}$ is also a groupoid over the category $(X/\Gamma)_{map}$ of continuous maps into the quotient space X/Γ, and it is this second groupoid structure that we shall emphasize in this paper.

In passing, note that we can now recover the points of $[X/\Gamma]$ that we described earlier, simply as the maps from a fixed point into $[X/\Gamma]$. More generally, if one knows the category of maps into some space, then the points of that space can be identified with maps from a fixed point into the space.

3.3.3 Philosophy

If the reader pauses to think carefully about what we have described so far, they may be somewhat confused about our description of continuous maps into a quotient stack. Given two spaces X and Y, a continuous map $f : X \to Y$ should be equivalent to a functor between the categories $(X)_{map} \to (Y)_{map}$. (After all, given any $h : Z \to X$, one can compose with f to form $f \circ h : Z \to Y$. One could recover f itself as the image of the identity map $X \to X$.) Thus, a continuous map from any space Y into the quotient stack $[X/\Gamma]$ should be equivalent to a functor $(Y)_{map} \to [X/\Gamma]_{map}$.

On the other hand, these functors should somehow be defined by objects of $[X/\Gamma]_{map}$ – after all, we defined $[X/\Gamma]_{map}$ to be the category of maps into $[X/\Gamma]$.

In fact, this is precisely the case, thanks to the Yoneda lemma (see for example [8, 17, 18]), which says that a functor $(Y)_{map} \to \mathcal{F}$, for any groupoid $\mathcal{F}$, is determined (up to equivalence) by the image of the identity map $\text{Id} : Y \to Y$.

In the present case, this means the following. Let

$$\left(E \longrightarrow Y, \ E \xrightarrow{f} X \right)$$

be an object of $[X/\Gamma]_{map}$. Given this object, define a functor $(Y)_{map} \to [X/\Gamma]_{map}$ as follows:

1. Objects: Let $(h : Z \to Y) \in \text{Ob } (Y)_{map}$. The image of this object under the functor we are defining is the pair

$$\left(h^*E \longrightarrow Z, \ h^*E \xrightarrow{canonical} E \xrightarrow{f} X \right)$$

2. Morphisms: Let

$$\lambda : (\, h_1 : Z_1 \longrightarrow Y \,) \longrightarrow (\, h_2 : Z_2 \longrightarrow Y \,)$$

be a morphism in $(Y)_{map}$, meaning that $\lambda : Z_1 \to Z_2$ is a map such that the diagram

$$
\begin{array}{ccc}
Z_1 & \xrightarrow{\;\lambda\;} & Z_2 \\
& h_1 \searrow \quad \swarrow h_2 & \\
& Y &
\end{array}
$$

commutes. The map λ induces a canonical map

$$\lambda^{\#} : h_1^* E \; (= \; \lambda^* h_2^* E \,) \longrightarrow h_2^* E$$

so the image of the morphism λ under the functor $(Y)_{map} \to [X/\Gamma]_{map}$ is the morphism $(\lambda, \lambda^{\#})$.

Conversely, given a functor $(Y)_{map} \to [X/\Gamma]_{map}$, the object of $[X/\Gamma]_{map}$ corresponding to this functor is the image of the object $(\mathrm{Id} : Y \to Y) \in \mathrm{Ob}\, (Y)_{map}$.

For example, there is a canonical map $X \to [X/\Gamma]$ defined by the object

$$\left(X \times \Gamma \xrightarrow{\;p_1\;} X, \; X \times \Gamma \xrightarrow{\;eval\;} X \right)$$

of $[X/\Gamma]_{map}$. This map plays the same role for quotient stacks that the canonical projection $X \to X/\Gamma$ plays for quotient spaces. In subsection 3.3.5 we shall discuss this example in more detail.

3.3.4 $[X/\Gamma]_{map}$ is a groupoid over $(X/\Gamma)_{map}$

It is straightforward to check that the category $[X/\Gamma]_{map}$ of maps into the quotient stack $[X/\Gamma]$ is a groupoid over the category $(X/\Gamma)_{map}$ of maps into the quotient space X/Γ.

Define a projection functor $p : [X/\Gamma]_{map} \to (X/\Gamma)_{map}$ as follows:

1. Objects: Let $(E \xrightarrow{\;\pi\;} Y, E \xrightarrow{\;f\;} X)$ be an object of $[X/\Gamma]_{map}$. Given this object, define a map $g : Y \to X/\Gamma$ as follows: for any $y \in Y$, let $e \in \pi^{-1}(y)$, and define $g(y) = (\pi_0 \circ f)(e)$, where $\pi_0 : X \to X/\Gamma$ is the canonical map. It is straightforward to check that this map is well-defined and continuous.[5]

   So, define

$$p\left(\left(E \xrightarrow{\;\pi\;} Y, \; E \xrightarrow{\;f\;} X \right) \right) = \left(Y \xrightarrow{\;g\;} X/\Gamma \right)$$

---

[5]To show that this map is continuous, we have used the assumption that Γ acts by homeomorphisms.

2. Morphisms: Let

$$(\rho, \lambda) : \left(E_1 \xrightarrow{\pi_1} Y_1, \ E_1 \xrightarrow{f_1} X \right) \longrightarrow \left(E_2 \xrightarrow{\pi_2} Y_2, \ E_2 \xrightarrow{f_2} X \right)$$

be a morphism. Define $p\left((\rho, \lambda) \right) = \rho$.

It is straightforward to check that this definition makes p a well-defined functor

$$[X/\Gamma]_{map} \to (X/\Gamma)_{map}$$

Furthermore, it is also straightforward to check that $[X/\Gamma]_{map}$ is a groupoid over $(X/\Gamma)_{map}$.

3.3.5 The canonical functor $(X)_{map} \to [X/\Gamma]_{map}$

At the level of points, recall we discussed a map $X \to [X/\Gamma]$, given by mapping any point $x \in X$ to the function $f_x : \Gamma \to X$ defined by $f_x(g) = g \cdot x$.

We can now define this map more thoroughly.

At the level of maps, the functor $\pi : (X)_{map} \to [X/\Gamma]_{map}$ is defined by the pair

$$\left(X \times \Gamma \xrightarrow{p_1} X, \ X \times \Gamma \xrightarrow{eval} X \right)$$

In other words, at the level of maps, this functor sends the object $(Y \xrightarrow{f} X)$ in $(X)_{map}$ to the object

$$\left(f^*(X \times \Gamma \to X), \ f^*(X \times \Gamma) \longrightarrow X \times \Gamma \xrightarrow{eval} X \right)$$

Now, for later use, note we could also define other functors $\pi_g : (X)_{map} \to [X/\Gamma]_{map}$, for any $g \in \Gamma$, by the composition

$$(X)_{map} \xrightarrow{g} (X)_{map} \xrightarrow{\pi} [X/\Gamma]_{map}$$

However, these functors are all equivalent. Specifically, define an invertible natural transformation $\omega : \pi_g \Rightarrow \pi$ as follows. For any object $(Y \xrightarrow{f} X)$, define $\omega\left((Y \xrightarrow{f} X) \right)$ to be the pair $(\mathrm{id}_Y, 1 \times g)$, where id_Y is the identity map $Y \to Y$, and $1 \times g : Y \times \Gamma \to Y \times \Gamma$ acts as, $(1 \times g)(y, h) = (y, hg)$. It is straightforward to check that this is a well-defined and invertible morphism in $[X/\Gamma]_{map}$, and moreover that such ω define an invertible natural transformation $\pi_g \Rightarrow \pi$. Thus, the functors π_g and π are isomorphic.

3.3.6 General properties of continuous maps

Continuous maps between ordinary topological spaces are often classified by certain extra properties they may possess. For example, they may be open, closed, surjective, injective, local homeomorphisms, and so forth.

Closely analogous notions exist for certain continuous maps into stacks. In order to understand how to phrase notions such as "open" and "surjective" in language that makes sense for stacks, we need to reexamine these notions for ordinary topological spaces. Now, although properties such as open and surjective do not, as ordinarily formulated, make sense for stacks, the notion of a fiber product does make sense for stacks. With this in mind, let us rewrite such notions in terms of fiber products, so as to be potentially useful for stacks.

Define a property "P" of continuous maps between (ordinary) topological spaces to be local if it is true that, a map $f : Y \to X$ has property "P" if and only if for all continuous maps $g : Z \to X$, the first projection map $Z \times_X Y \xrightarrow{p_1} Z$ has property "P" also.

It is straightforward to check that the properties open, surjective, injective, and local homeomorphism are all examples of local properties. In other words, for example, a continuous map $f : Y \to X$ between ordinary topological spaces is open if and only if for all continuous maps $g : Z \to X$, the projection $Z \times_X Y \xrightarrow{p_1} Z$ is also open.

In order to be able to apply this to stacks, we still need to relate fiber products of stacks to ordinary spaces. Now, many morphisms $f : Y \to \mathcal{F}$ into a stack $\mathcal{F}$ have the useful property that for all continuous maps $g : Z \to \mathcal{F}$, the fiber product $Z \times_\mathcal{F} Y$ is an ordinary space, not another stack. We call a morphism $f : Y \to \mathcal{F}$ representable if it has this property. Intuitively, for a morphism to be representable means that the fibers of the morphism are ordinary spaces, not stacks. (For a useful theorem on representable morphisms, see appendix C.)

At least for the representable morphisms, we can now formulate for stacks the local properties of morphisms. Specifically, if "P" is a local property of morphisms between ordinary spaces, then we say that a representable map $f : Y \to \mathcal{F}$ has property "P" precisely when for all continuous maps $g : Z \to \mathcal{F}$, the projection map $Z \times_\mathcal{F} Y \xrightarrow{p_1} Z$ has property "P." Since, by assumption, $Z \times_\mathcal{F} Y$ is an ordinary space, it is sensible to talk about properties of the continuous map $Z \times_\mathcal{F} Y \to Z$ in the ordinary sense.

In this paper, essentially the only map into a stack for which we shall be concerned with local properties is the canonical map $X \to [X/\Gamma]$, and we shall show in section 6 that this map is representable. In fact, we shall show that if $Y \to [X/\Gamma]$ is defined by the pair

$$\left(E \xrightarrow{\pi} Y, \ E \xrightarrow{f} X \right)$$

then $Y \times_{[X/\Gamma]} X \cong E$, and the projection map $Y \times_{[X/\Gamma]} X \xrightarrow{p_1} Y$ is isomorphic to the
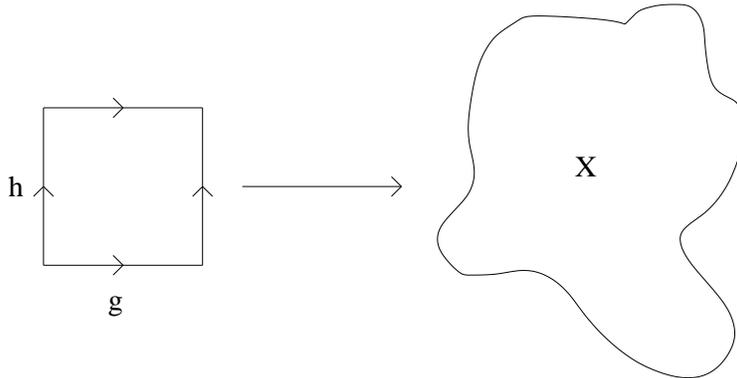
Figure 2: Contribution to the (g, h) twisted sector of a string orbifold on T^2

projection $E \xrightarrow{\pi} Y$.

Let us put our description of properties of morphisms into practice. For example, we can now ask whether the canonical (and representable) map $X \to [X/\Gamma]$ is open. Let $Y \to [X/\Gamma]$ be any continuous map, defined by some pair

$$\left(E \xrightarrow{\pi} Y, \ E \xrightarrow{f} X \right)$$

As described above, the projection $Y \times_{[X/\Gamma]} X \longrightarrow Y$ is isomorphic to the projection $\pi :$ $E \to Y$. The projection map π is always open, for all principal Γ-bundles E over any Y, hence the canonical map $X \to [X/\Gamma]$ is open.

Similarly, if Γ is discrete, then the projection map $\pi : E \to Y$ is always a local homeomorphism, hence if Γ is discrete, the canonical map $X \to [X/\Gamma]$ is a local homeomorphism.

4   Sigma models on $[X/\Gamma]$ – first pass

What would a sigma model on a quotient stack look like? We shall argue that, after unraveling definitions, a string orbifold is precisely a sigma model on a quotient stack.

How does this work? A sigma model on $[X/\Gamma]$ is a weighted sum over maps into $[X/\Gamma]$. Now, we have just discussed continuous maps $Y \to [X/\Gamma]$ – these are the same as pairs

$$\left(E \xrightarrow{\pi} Y, \ E \xrightarrow{f} X \right)$$

where E is a principal Γ-bundle over Y, and f is Γ-equivariant.

Now, it is straightforward to check that this data is the same thing as a map from a twisted sector into X, as illustrated in figure 2. Over any Riemann surface, restrict to a

maximally large contractible open subset (the interior of the square in figure 2). Over that open subset, E is trivializable, and so admits a section. That section is precisely a twisted sector – the choice of boundary conditions dictates the extent to which the section fails to extend globally, and also dictates the equivalence class[6] of the bundle E. Now, a map from that local section into X is the same thing as a Γ-equivariant map from the entire bundle E into X. (Although the map $E \to X$ may naively appear to contain more information, Γ-equivariance dictates that the action on a choice of a local section of the form above completely determines the map.)

In other words, the maps one sums over in twisted sectors, as illustrated in figure 2, are the same thing as pairs

$$\left(E \xrightarrow{\pi} Y, \ E \xrightarrow{f} X \right)$$

and hence the same thing as maps into $[X/\Gamma]$.

So far so good, but why cannot we understand these maps as maps into the quotient space? Given a pair $(E \to Y, E \to X)$, we can certainly define a map $Y \to X/\Gamma$, as we shall study in more detail later. However, the category of continuous maps of the form summed over in twisted sectors (the category of continuous maps into $[X/\Gamma]$) is only equivalent to the category of continuous maps into the quotient space X/Γ in the special case that Γ acts freely. If Γ does not act freely, then these categories are *not* equivalent (essentially because there are more twisted-sector-type maps than corresponding maps into X/Γ). Since a topological space is completely determined by the maps into it (see, for example, appendix A), we see a string orbifold can *not* be understood as a sigma model on a quotient space X/Γ in general. We shall study this further in the next section.

So far, we have argued that after unraveling definitions, the maps one sums over in a string orbifold are naturally understood as maps into the quotient stack $[X/\Gamma]$, and in general not as maps into the quotient space X/Γ. Put another way, twisted sectors are not naturally understood in terms of quotient spaces, but are naturally understood in terms of quotient stacks. Put another way still, we have argued that, after unraveling definitions, a string orbifold is a sum over maps into the quotient stack $[X/\Gamma]$, a 'smoking gun' for an interpretation as a sigma model.

---

[6] As a check, it is straightforward to compute that the classification of twisted sectors agrees with the classification of principal Γ-bundles on the worldsheet (given by $H^1(\Sigma, C^\infty(\Gamma))$, for Σ the worldsheet). For example, for $\Sigma = T^2$, we can calculate

$$
\begin{aligned}
H^1\left(T^2, C^\infty(\Gamma)\right) &= H^1\left(T^2, \Gamma\right) \text{ for } \Gamma \text{ discrete} \\
&= \mathrm{Hom}\left(H_1(T^2), \Gamma\right) \\
&= \mathrm{Hom}\left(\mathbf{Z}^2, \Gamma\right) \\
&= \{(g, h) \mid g, h \in \Gamma, gh = hg\}
\end{aligned}
$$

We would like to emphasize that our analysis really does proceed by unraveling definitions. In particular, we are not making some wild guess and trying to justify it by checking a handful of examples.

Now, in order to positively identify string orbifolds with sigma models on quotient stacks, there is at least one more thing we need to do. Namely, we need to identify the action weighting the contributions to the string orbifold path integral with the action weighting the contributions to the sigma model partition function. In section 9 we shall describe a proposal for a classical action for a sigma model on a stack, which generalizes both ordinary sigma models and string orbifolds. Of course, beyond classical actions one also needs to check whether such a sigma model can be consistently quantized, which involves thinking about global properties of the target; this we shall not do in this paper. We are really just trying to set up the basics of a program, and such considerations are beyond what we hope to accomplish in this particular paper.

Given that a string orbifold is a sigma model on a quotient stack $[X/\Gamma]$, what can we now understand that was unclear before? Many things! For example, if X is smooth, then $[X/\Gamma]$ is smooth, regardless of whether Γ acts freely; hence, a string orbifold should always be well-behaved, regardless of whether the quotient space X/Γ is singular, and indeed this is a well-known property. The story that the B field is somehow nonzero (in simple examples) in string orbifolds can also be naturally understood. Twist fields and orbifold Euler characteristics also have a natural explanation.

In the rest of this paper, we shall describe why quotient stacks have these nice properties. We begin in the next section with an analysis of how $[X/\Gamma]$ is related to X/Γ in the special case that Γ acts freely.

5   The case that Γ acts freely

One can check that when Γ acts freely, $[X/\Gamma]$ and X/Γ are homeomorphic. As a result of the computations below, one often thinks of the quotient stack $[X/\Gamma]$ heuristically as roughly being the quotient space X/Γ with some sort of extra structure at the singularities.

What does it mean for $[X/\Gamma]$ to be homeomorphic to X/Γ? Simply that there is an equivalence of categories, compatible with the projection map, that respects coverings defining the Grothendieck topology on $[X/\Gamma]$. For the most part we shall continue to ignore questions regarding Grothendieck topologies on $[X/\Gamma]$, as our manipulations really only involve the underlying groupoid structure.

We shall check this at three levels: in terms of points, in terms of our ansatz for open sets, and in terms of the continuous maps. Of course, the only case that matters is the continuous maps case; the other two cases are presented for pedagogical purposes.

5.1 Points

In this section we shall show that the functor $p : [X/\Gamma]_{pt} \to (X/\Gamma)_{pt}$ defines an equivalence of categories in the case that Γ acts freely on X. To do this, we shall construct a functor $q : (X/\Gamma)_{pt} \to [X/\Gamma]_{pt}$, and show that when Γ acts freely, then $p \circ q \cong \mathrm{Id}$ and $q \circ p \cong \mathrm{Id}$.

Define a functor $q : (X/\Gamma)_{pt} \to [X/\Gamma]_{pt}$ as follows. For any $y \in \mathrm{Ob}\ (X/\Gamma)_{pt}$, let $y(x) \in \pi_0^{-1}(x) \subset X$, and define a Γ-equivariant map $f : \Gamma \to X$ by, $f(1) = y(x)$. Then, define q to map y to $(f : \Gamma \to X)$. (As the only morphisms in $(X/\Gamma)_{pt}$ are the identity morphisms, defining the action of q on them is trivial.)

It should be clear that $p \circ q = \mathrm{Id}$. In the case that Γ acts freely on X, we can also show that $q \circ p \cong \mathrm{Id}$. Define a natural transformation $q \circ p \Rightarrow \mathrm{Id}$ as follows. For any object $(f : \Gamma \to X) \in \mathrm{Ob}\ [X/\Gamma]_{pt}$, let $\eta : \Gamma \to \Gamma$ be the unique Γ-equivariant map making the diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ \ \eta\ \ } & \Gamma \\
& {\scriptstyle (q \circ p)(f)} \searrow \quad \swarrow {\scriptstyle f} & \\
& X &
\end{array}
$$

commute. (Uniqueness follows from the fact that Γ acts freely on X.) This is clearly a morphism in $[X/\Gamma]_{pt}$, and it is also straightforward to check that η is a natural transformation.

Thus, there exists a functor $q : (X/\Gamma)_{pt} \to [X/\Gamma]_{pt}$ such that $p \circ q = \mathrm{Id}$ and $q \circ p \cong \mathrm{Id}$, so p defines an equivalence of categories.

Next we shall give a more thorough (and more technical) arguments that when Γ acts freely, $[X/\Gamma] \cong X/\Gamma$.

5.2 Continuous maps

In this subsection we shall check that when Γ acts freely, $[X/\Gamma]_{map} \cong (X/\Gamma)_{map}$, i.e., p is an equivalence of categories. In general terms, the idea is that if $g : Y \to X/\Gamma$ is any continuous map and Γ acts freely, then we can construct a bundle over Y as $g^*(X \to X/\Gamma)$, and this construction provides an inverse to the projection $[X/\Gamma]_{map} \to (X/\Gamma)_{map}$.

The remainder of this section is rather technical; the reader may wish to skip it on a first reading.

To prove the claim, we need to find a functor $q : (X/\Gamma)_{map} \to [X/\Gamma]_{map}$ such that $p \circ q \cong \mathrm{Id}$ and $q \circ p \cong \mathrm{Id}$.

Define a functor $q : (X/\Gamma)_{map} \to [X/\Gamma]_{map}$ as follows:

1. Objects: Let $(Y \xrightarrow{g} X/\Gamma)$ be an object of $(X/\Gamma)_{map}$, i.e., let $g : Y \to X/\Gamma$ be a continuous map.

   Define a principal Γ-bundle E over Y by, $E = g^*(X \to X/\Gamma)$. (Since Γ acts freely, X is itself a principal Γ-bundle over X/Γ.)

   Since we defined E in terms of a pullback by g, there is a canonical bundle map $f : E \to X$ that makes the following diagram commute:

   $$\begin{array}{ccc} E & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & X/\Gamma \end{array}$$

   So, we can now consistently define

   $$q : \left(Y \xrightarrow{g} X/\Gamma \right) \mapsto \left(E \longrightarrow Y, \; E \xrightarrow{f} X \right)$$

2. Morphisms: Let

   $$\rho : \left(Y_1 \xrightarrow{g_1} X/\Gamma \right) \longrightarrow \left(Y_2 \xrightarrow{g_2} X/\Gamma \right)$$

   be a morphism in $(X/\Gamma)_{map}$, i.e., the diagram

   $$\begin{array}{ccc} Y_1 & \xrightarrow{\quad\rho\quad} & Y_2 \\ & {\scriptstyle g_1} \searrow \quad \swarrow {\scriptstyle g_2} & \\ & X/\Gamma & \end{array}$$

   commutes. Define $E_1 = g_1^*(X \to X/\Gamma)$ and $E_2 = g_2^*(X \to X/\Gamma)$. Since $g_1 = g_2 \circ \rho$, there is a canonical map $\lambda : E_1(= \rho^* E_2) \to E_2$ such that the diagram

   $$\begin{array}{ccc} E_1 & \xrightarrow{\lambda} & E_2 \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow{\rho} & Y_2 \end{array}$$

   commutes. Also, defining $f_i : E_i \to X$ to be the canonical maps, we also have that

   $$\begin{array}{ccc} E_1 & \xrightarrow{\quad\lambda\quad} & E_2 \\ & {\scriptstyle f_1} \searrow \quad \swarrow {\scriptstyle f_2} & \\ & X & \end{array}$$

With this definition, $q : (X/\Gamma)_{map} \to [X/\Gamma]_{map}$ is a well-defined functor.

Finally, we must show that p and q are inverses. It is immediately clear that $p \circ q = \mathrm{Id}_{(X/\Gamma)_{map}}$. Define a natural transformation $\psi : \mathrm{Id}_{[X/\Gamma]_{map}} \Rightarrow q \circ p$ as follows.

Let $\left(E \xrightarrow{\pi} Y,\ E \xrightarrow{f} X \right)$ be an object of $[X/\Gamma]_{map}$, and let $\left(Y \xrightarrow{g} X/\Gamma \right)$ be its image under p. Define $E' = g^*(X \to X/\Gamma)$, and $f' : E' \to X$ the canonical map.

Define $\psi : E \to E'$ by,
$$\psi(e) \ = \ (\,\pi(e),\ f(e)\,)$$

It is straightforward to check that this definition makes $\psi : E \to E'$ a well-defined bundle isomorphism, such that

$$
\begin{array}{ccc}
E & \xrightarrow{\ \ \psi\ \ } & E' \\
 & {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle f'} & \\
 & X &
\end{array}
$$

commutes. In other words, ψ defines a morphism in $[X/\Gamma]_{map}$. Furthermore, one can check that such ψ's define an invertible natural transformation $\mathrm{Id}_{[X/\Gamma]_{map}} \Rightarrow q \circ p$.

Thus, p and q are inverses, and so, for freely-acting Γ,

$$[X/\Gamma]_{map} \ \cong \ (X/\Gamma)_{map}$$

In passing, when Γ does not act freely, it is easy to check that $[X/\Gamma]_{map}$ and $(X/\Gamma)_{map}$ are not equivalent as categories, because $[X/\Gamma]_{map}$ has extra non-isomorphic objects that all project down to the same object of $(X/\Gamma)_{map}$. For example, suppose X has a point that is fixed under all of Γ, for simplicity. Then for any principal Γ-bundle E over a space Y, there is a Γ-equivariant map $f : E \to X$ that sends all of E to that fixed point. Each of these objects projects down to the same object of $(X/\Gamma)_{map}$, namely the map that sends all of Y to the image of this fixed point on X/Γ; however, for general Y, there can be objects with non-isomorphic bundles E. Thus, in this case, $[X/\Gamma]_{map}$ contains non-isomorphic objects that project down to the same object of $(X/\Gamma)_{map}$, and so these categories cannot be the same. Put another way, when Γ does not act freely, the quotient stack $[X/\Gamma]$ and the quotient space X/Γ can not be homeomorphic.

6 $X \to [X/\Gamma]$ is a principal Γ-bundle

6.1 Generalities

Earlier in this paper we claimed that the natural map $X \to [X/\Gamma]$ described X as the total space of a principal Γ-bundle over $[X/\Gamma]$, regardless of whether or not Γ acted freely. In this section we shall explain this comment.

In order to have a precise understanding of this claim, we must explain how to make sense out of the notion of bundle when the base space is a generalized space, and the projection map into the base does not carry open sets to objects in the groupoid (i.e., open sets in the generalized space).

In particular, we need to find a way of defining bundles that involves no more than fibered products. Given such a definition for bundles over ordinary spaces, we can then define bundles over generalized spaces. As a first step, note that if $\pi : E \to X$ is a fiber bundle over X, an ordinary topological space, then for all open $U \subseteq X$, $U \times_X E = \pi^{-1}(U)$, so we can say that $\pi : E \to X$ is a fiber bundle if, for all open $U \subseteq X$, $U \times_X E \to U$ is a fiber bundle over U.

It is straightforward to check that this definition is equivalent to the following: $\pi : E \to X$ is a fiber bundle if, for all spaces Y and continuous maps $f : Y \to X$, $Y \times_X E \to Y$ is a fiber bundle over Y, with fixed (Y-independent) fiber. This new definition has the useful property that we can make sense out of it when the base space is a generalized space.

So, we say that $X \to [X/\Gamma]$ is a principal Γ-bundle if, for all $Y \to [X/\Gamma]$, $Y \times_{[X/\Gamma]} X \to Y$ is a principal Γ-bundle, where the map $X \to [X/\Gamma]$ is the canonical map discussed earlier. Although this definition may look somewhat unusual, it agrees with the usual notion of bundle in the special case that the base space is an ordinary space. We shall first show how this works at the level of points, where one can gain some insight with only a little work, and then show how this is proven in general.

6.2 Points

Let Y be any topological space, and $\psi : Y \to [X/\Gamma]$ a map. It is straightforward to check that, at the level of points, $Y \times_{[X/\Gamma]} X \to Y$ is a principal Γ-bundle. After all, a point in $Y \times_{[X/\Gamma]} X$ is a triple (y, x, λ), $y \in Y$, $x \in X$, and $\lambda : \psi(y) \to h_X(x)$ is an isomorphism in the category of points of $[X/\Gamma]$ ($h_X : X \to [X/\Gamma]$ is canonical). If Γ acts freely on the orbit of x, then for each $y \in Y$, there are $|\Gamma|$ points in X and for each, a unique λ. Thus, one has a principal Γ-bundle. If Γ does not act freely, then there are fewer points $x \in X$, but their images in $[X/\Gamma]$ have extra automorphisms, hence there are extra maps λ, so that one still has a principal Γ-bundle.

6.3 Continuous maps

Now that we have seen how $Y \times_{[X/\Gamma]} X$ is a bundle at the level of points, let us examine the matter rigorously by considering the category of maps into $Y \times_{[X/\Gamma]} X$.

Let Y be any topological space, and $Y \to [X/\Gamma]$ a continuous map. As remarked earlier, specifying such a continuous map is equivalent to specifying an object of $[X/\Gamma]_{map}$, i.e., a pair

$$\left(E \xrightarrow{\pi} Y, \; E \xrightarrow{f} X \right)$$

A lengthy definition chase reveals that $Y \times_{[X/\Gamma]} X \cong E$, and the projection maps $Y \times_{[X/\Gamma]} X \to Y, X$ are equivalent to the maps $E \to Y, X$. Technically this is shown by proving that the categories

$$\left(Y \times_{[X/\Gamma]} X \right)_{map} \cong (E)_{map}$$

are equivalent, and furthermore the functors

$$\left(Y \times_{[X/\Gamma]} X \xrightarrow{p_1} Y \right) \; \cong \; \left(E \xrightarrow{\pi} Y \right)$$

$$\left(Y \times_{[X/\Gamma]} X \xrightarrow{p_2} X \right) \; \cong \; \left(E \xrightarrow{f} X \right)$$

are also equivalent. Because of their technical nature and length, we have banished the formal proofs of these assertions to an appendix. However, it is now clear that, as claimed, $Y \times_{[X/\Gamma]} X$ is indeed a principal Γ-bundle over Y, and in fact is isomorphic to the bundle E defining the map $Y \to [X/\Gamma]$.

Thus, $X \to [X/\Gamma]$ is a principal Γ-bundle.

7 The lore on equivariance

In studying string orbifolds mathematically, one often uses the notion of equivariance. For example, one often speaks of defining an orbifold group action on a bundle by putting a Γ-equivariant structure on it. Although this is a very natural thing to do, a very heuristically reasonable thing to do, and there are no other reasonable alternatives, one might still ask if there is a deeper reason why this must work. Put another way, why must one use honest equivariant structures, and not, say, projectivized[7] equivariant structures?

Now that we have a deeper understanding of string orbifolds (i.e., as sigma models on quotient stacks), we can now shed some light on this matter. In a nutshell, a Γ-equivariant

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[7]Projectivized equivariant structures appear in [27] in describing discrete torsion for D-branes; however, it was argued in [28] that this projectivized equivariant structure on the D-brane bundle should really be understood as an honest equivariant structure on a twisted "bundle."

or -invariant object on X is the same thing as a corresponding object on the quotient stack $[X/\Gamma]$. (This is one of the famous properties of quotient stacks.) For example, a sheaf on $[X/\Gamma]$ is equivalent to a Γ-equivariant sheaf on X. So, by putting Γ-equivariant structures on bundles, sheaves, *et cetera* on X, we are really specifying bundles, sheaves, *et cetera* on $[X/\Gamma]$. Put another way, the fundamental thing to do is to construct a bundle, sheaf, whatever on $[X/\Gamma]$, and then the fact that this corresponds to a bundle, sheaf, whatever on X with an honest equivariant structure is dictated by the geometry of $[X/\Gamma]$.

In this section, we will show specifically that a sheaf on the quotient stack $[X/\Gamma]$ is the same thing, after unraveling definitions, as a Γ-equivariant sheaf on X, another standard fact about quotient stacks (see for example [17, 18]). From this fact, one can quickly deduce that many other objects on $[X/\Gamma]$ are Γ-equivariant or -invariant objects on X – for example, bundles are just special kinds of sheaves, spinors and differential forms can be understood as sections of sheaves, and so forth. However, technically we shall only explicitly consider the general construction for sheaves.

Since understanding how to define a sheaf on a stack is a rather technical matter, we begin our discussion with a much simpler prototype: functions on stacks. Just as a sheaf on the quotient stack $[X/\Gamma]$ is a Γ-equivariant sheaf on X, it is also true that a function on the quotient stack $[X/\Gamma]$ is the same as a Γ-invariant function on X. Also, the analysis for functions is not only a prototype of the analysis for sheaves, but it is technically much simpler, so for pedagogical purposes, we begin with functions. After having described functions on stacks, we shall motivate the relevant definition of a sheaf on a stack, then we shall show the equivalence between sheaves on $[X/\Gamma]$ and Γ-equivariant sheaves on X.

As this section is rather technical, the reader may wish to skip this section on a first reading.

7.1 Functions on stacks

The ordinary definition of a function on a space does not quite make sense for generalized spaces, a problem similar to that we encountered when trying to understand in what sense $X \to [X/\Gamma]$ could be a principal Γ-bundle. In order to make sense of functions in this context, we need a somewhat different definition, that makes sense for stacks. The answer is to define functions in terms of their pullbacks – instead of trying to define a function f directly on the stack, we instead speak of functions "$\phi^* f$" for each map $\phi : Y \to \mathcal{F}$ into a stack $\mathcal{F}$.

Technically we define functions on stacks as follows. For any stack $\mathcal{F}$, a function $f : \mathcal{F} \to \mathbf{R}$ is defined by associating, to each map $Y \to \mathcal{F}$, a function $f_Y : Y \to \mathbf{R}$, such that for any

commuting diagram

$$Y_1 \xrightarrow{\ g\ } Y_2 \qquad\qquad (1)$$
$$Y_1 \searrow \quad \swarrow Y_2$$
$$\mathcal{F}$$

one has $f_2 \circ g = f_1$.

Note that if $\mathcal{F}$ is itself an honest space, then this definition uniquely specifies a function on $\mathcal{F}$ in the usual sense.

In the special case that $\mathcal{F} = [X/\Gamma]$, it is straightforward to check that a function on $[X/\Gamma]$ is the same thing as a Γ-invariant function on X.

First, to show that a function on $[X/\Gamma]$ defines a Γ-invariant function on X, note that by virtue of the definition, one immediately has a function f_X on X. Moreover, because the diagram

$$X \xrightarrow{\ g\ } X$$
$$[X/\Gamma]$$

commutes for any $g \in \Gamma$, we see that $f_X = f_X \circ g$, hence f_X is Γ-invariant.

Next, let f_X be a Γ-invariant function on X, and let $Y \to [X/\Gamma]$ be any map. We have the commuting diagram

$$\begin{array}{ccc} Y \times_{[X/\Gamma]} X & \xrightarrow{\ p_2\ } & X \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ Y & \longrightarrow & [X/G] \end{array}$$

where p_2 is Γ-equivariant. Now, $f_X \circ p_2$ is a function on $\mathbf{R}$, and because p_2 is Γ-equivariant, we see that $(f_X \circ p_2)(g \cdot e) = (f_X \circ p_2)(e)$ for all $g \in \Gamma$, $e \in Y \times_{[X/\Gamma]} X$. Hence, $f_X \circ p_2$ descends to a function on Y, and it is also straightforward to check that diagram (1) is satisfied.

7.2   Sheaves on stacks

The ordinary definition of a sheaf on a space does not quite make sense for generalized spaces. We had a similar problem in understanding in what sense $X \to [X/\Gamma]$ could be a principal Γ-bundle. We shall proceed here in the same fashion that we did there – specifically, we shall find a way to define sheaves on ordinary spaces that also makes sense for generalized spaces.

We shall begin with the following observation [20, cor. I-14]. Let $\{U_\alpha\}$ be an open cover

of a space X. If $\mathcal{S}_\alpha$ is a sheaf on U_α for each α, and if

$$\varphi_{\alpha\beta} : \mathcal{S}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{S}_\beta|_{U_\alpha \cap U_\beta}$$

are isomorphisms such that

$$\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma$, then [20, cor. I-14] there exists a unique sheaf $\mathcal{S}$ on X such that $\mathcal{S}|_{U_\alpha} = \mathcal{S}_\alpha$.

That definition is not quite useful for generalized spaces, but we can step towards a more useful definition as follows. It is straightforward to check that the definition above is equivalent to the following data:

1. For all open $U \subseteq X$, and all maps $f_U : U \to X$, a sheaf $\mathcal{S}_U$ on U.

2. For all inclusion maps $\rho : U \hookrightarrow V$ such that the following diagram commutes

$$
\begin{array}{ccc}
U & \xrightarrow{\;\;\rho\;\;} & V \\
& \searrow^{f_U} \quad \swarrow^{f_V} & \\
& X &
\end{array}
$$

an isomorphism $\varphi_\rho : \mathcal{S}_U \xrightarrow{\sim} \rho^* \mathcal{S}_V$, such that for any triple

$$
\begin{array}{ccccc}
U & \xrightarrow{\;\rho_1\;} & V & \xrightarrow{\;\rho_2\;} & W \\
& \searrow_{f_U} & \downarrow^{f_V} & \swarrow_{f_W} & \\
& & X & &
\end{array}
$$

In particular, to show that the first definition implies the second, work with a cover containing U and V as elements, then the isomorphism $\varphi_{\alpha\beta} : \mathcal{S}_\alpha|_{U_\alpha \cap U_\beta} \to \mathcal{S}_\beta|_{U_\alpha \cap U_\beta}$, where $U_\alpha = U$ and $U_\beta = V$, is the same as an isomorphism $\varphi_\rho : \mathcal{S}_U \to \rho^* \mathcal{S}_V$, since $U \cap V = U$, and the triple compatibility condition follows similarly.

To show that the second implies the first, consider sheaves $\mathcal{S}_\alpha$, $\mathcal{S}_{\alpha\beta}$, and $\mathcal{S}_{\alpha\beta\gamma}$ defined on elements of a cover $\{U_\alpha\}$ and its intersections. Given an isomorphism

$$\alpha_{\alpha\beta}^\alpha : \mathcal{S}_{\alpha\beta} \xrightarrow{\sim} \mathcal{S}_\alpha|_{U_\alpha \cap U_\beta}$$

one can define an isomorphism $\varphi_{\alpha\beta} : \mathcal{S}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{S}_\beta|_{U_{\alpha\beta}}$ as the composition

$$\varphi_{\alpha\beta} \equiv \alpha_{\alpha\beta}^\beta \circ \left(\alpha_{\alpha\beta}^\alpha\right)^{-1} : \mathcal{S}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{S}_\beta|_{U_\alpha \cap U_\beta}$$

and it is straightforward to check that these $\varphi_{\alpha\beta}$ satisfy the requisite condition on triple intersections.

From the second definition, we have a third equivalent definition, which finally does make sense for generalized spaces. A sheaf $\mathcal{S}$ on X is defined by the following data:

1. For all spaces Y and continuous maps $f : Y \to X$, a sheaf $\mathcal{S}_f$ on Y.

2. For all commuting diagrams

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{\ \rho\ } & Y_2 \\
 & \!\!\!{}_{f_1}\searrow \quad \swarrow{}_{f_2}\!\!\! & \\
 & X & \\
\end{array}
$$

an isomorphism $\varphi_\rho : \mathcal{S}_{f_1} \xrightarrow{\ \sim\ } \rho^* \mathcal{S}_{f_2}$ which obeys the consistency condition that for all commuting triples

$$
\begin{array}{ccccc}
Y_1 & \xrightarrow{\ \rho_1\ } & Y_2 & \xrightarrow{\ \rho_2\ } & Y_3 \\
 & \!\!{}_{f_1}\searrow & \ \downarrow{}_{f_2} & \swarrow{}_{f_3} & \\
 & & X & & \\
\end{array}
$$

the isomorphisms obey

$$
\varphi_{\rho_2 \circ \rho_1} \;=\; \rho_1^* \varphi_{\rho_2} \circ \varphi_{\rho_1} : \mathcal{S}_{f_1} \xrightarrow{\ \sim\ } (\rho_2 \circ \rho_1)^* \mathcal{S}_{f_3}
$$

To see that the definition above implies the second definition is clear; just restrict the possible Y's to open subsets of X. Conversely, since the second definition gave rise to an ordinary sheaf $\mathcal{S}$ on X, we can define $\mathcal{S}_f \equiv f^* \mathcal{S}$, and it is straightforward to check that with this definition, $\varphi_\rho = \text{Id}$ for all ρ, which automatically satisfies the condition on triples.

So far all we have done is wander through a succession of equivalent definitions of a sheaf on an ordinary space. The reader may well wonder why we bothered. The answer is that we have, finally, generated a definition of sheaf which makes sense on generalized spaces as well as ordinary spaces.

As the reader may have guessed, we define a sheaf of sets on a groupoid $\mathcal{F}$ by the following data:

1. For all spaces Y and maps $f : Y \to \mathcal{F}$, a sheaf $\mathcal{S}_f$ on Y.

2. For each commutative diagram

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{\ \rho\ } & Y_2 \\
 & \!\!\!{}_{f_1}\searrow \quad \swarrow{}_{f_2}\!\!\! & \\
 & \mathcal{F} & \\
\end{array}
$$

an isomorphism $\varphi_\rho : \mathcal{S}_{f_1} \xrightarrow{\ \sim\ } \rho^* S_{f_2}$.

The isomorphisms φ are required to obey a consistency condition, namely for each commutative diagram

$$Y_1 \xrightarrow{\rho_1} Y_2 \xrightarrow{\rho_2} Y_3$$

the associated isomorphisms must obey

$$\varphi_{\rho_2 \circ \rho_1} = \rho_1^* \varphi_{\rho_2} \circ \varphi_{\rho_1} : \mathcal{S}_{f_1} \xrightarrow{\sim} (\rho_2 \circ \rho_1)^* \mathcal{S}_{f_3} = \rho_1^* (\rho_2^* \mathcal{S}_{f_3})$$

As should be clear from the preceding analysis, this definition agrees with the notion of "sheaf" for ordinary topological spaces. There is a technical point we should mention – the notion "φ_ρ" omits some information. For a diagram of functors to commute means merely that the two compositions must agree up to an invertible natural transformation, and in fact the isomorphisms φ_ρ depend upon that natural transformation in addition to ρ. However, it seems customary to omit mention of this dependence from the notation.

Now that we have defined sheaves on stacks $\mathcal{F}$, we shall show that a sheaf on the quotient stack $[X/\Gamma]$ is the same thing as a Γ-equivariant sheaf on X.

7.3 Sheaves on $[X/\Gamma]$ define Γ-equivariant sheaves on X

First, we shall show that a sheaf on $[X/\Gamma]$ defines a Γ-equivariant sheaf on X.

Earlier we described a natural map $X \to [X/\Gamma]$, that also describes X as a principal Γ-bundle over $[X/\Gamma]$. By composing that map with actions of $g \in \Gamma$, which map $X \xrightarrow{g} X$, we can form several maps $X \to [X/\Gamma]$, which we shall denote by π_g.

Now, from the definition of sheaf on a stack given in the previous subsection, for each map $\pi_g : X \to [X/\Gamma]$, one associates a sheaf $\mathcal{S}_g$ on X. Also, since $\pi_g = \pi_1 \circ g$, there are isomorphisms $\varphi_g : \mathcal{S}_g \xrightarrow{\sim} g^* \mathcal{S}_1$.

Now, a Γ-equivariant structure on $\mathcal{S}_1$ consists of isomorphisms $\mathcal{S}_1 \xrightarrow{\sim} g^* \mathcal{S}_1$, respecting the group law. Although we have some isomorphisms, they are between *a priori* distinct sheaves, and are not sufficient.

The remaining isomorphisms emerge from the fact that $\pi_1 = \pi_g$, in the sense that they define isomorphic functors, and so the diagram

$$\pi_1 \left(\begin{matrix} X \\ \\ [X/\Gamma] \end{matrix} \right) \pi_g$$

commutes. (This was discussed earlier when the canonical map $X \to [X/\Gamma]$ was originally introduced.) As a result, we have additional isomorphisms $\phi_g : \mathcal{S}_1 \to \mathcal{S}_g$.

Given these additional isomorphisms, we can now define an isomorphism $\alpha_g : \mathcal{S}_1 \xrightarrow{\sim} g^*\mathcal{S}_1$, for each $g \in \Gamma$. Specifically, define α_g to be the composition

$$\alpha_g : \ \mathcal{S}_1 \ \xrightarrow{\phi_g} \ \mathcal{S}_g \ \xrightarrow{\varphi_g} \ g^*\mathcal{S}_1$$

From the consistency condition for isomorphisms, it is straightforward to check that $\alpha_{gh} = h^*\alpha_g \circ \alpha_h$ for all $g, h \in \Gamma$, and so our isomorphisms obey the group law.

Thus, given a sheaf on $[X/\Gamma]$, we can construct a Γ-equivariant sheaf on X.

For an example, consider the principal Γ-bundle $X \to [X/\Gamma]$ described earlier. For any map $Y \to [X/\Gamma]$, there is a principal Γ-bundle over Y, namely $Y \times_{[X/\Gamma]} X$, so we can explicitly recover a description of the bundle $X \to [X/\Gamma]$ in terms of data over spaces Y. The corresponding Γ-equivariant sheaf on X itself is then $X \times_{[X/\Gamma]} X = X \times \Gamma$. So, the Γ-equivariant sheaf on X corresponding to the principal Γ-bundle $X \to [X/\Gamma]$ is the trivial bundle $X \times \Gamma$ on X.

Next, we shall show that a Γ-equivariant sheaf on X defines a sheaf on $[X/\Gamma]$.

7.4 Γ-equivariant sheaves on X define sheaves on $[X/\Gamma]$

Let $\mathcal{S}$ be a Γ-equivariant sheaf on X, and let $\alpha_g : \mathcal{S} \xrightarrow{\sim} g^*\mathcal{S}$ be the isomorphisms defining a Γ-equivariant structure on $\mathcal{S}$. We shall outline how to construct a sheaf on $[X/\Gamma]$. (The details are quite technical, so we have opted to only outline how this is done.) This means, for each map $\lambda : Y \to [X/\Gamma]$, we shall outline how to construct a sheaf $\mathcal{S}_\lambda$ on Y, as well as a set of consistent isomorphisms.

Let $\lambda : Y \to [X/\Gamma]$ be a continuous map, defined by some pair

$$\left(E \ \xrightarrow{\pi} \ Y, \ E \ \xrightarrow{f} \ X \right)$$

Since f and $\mathcal{S}$ is Γ-equivariant, $f^*\mathcal{S}$ is a Γ-equivariant sheaf on E, and so there is a sheaf $\mathcal{S}_\lambda$ such that $\pi^*\mathcal{S}_\lambda = f^*\mathcal{S}$. Take the sheaf on Y associated to λ to be $\mathcal{S}_\lambda$.

Now, for every commutative diagram

$$
\begin{array}{ccc}
Y_1 & \xrightarrow{\ \ \rho\ \ } & Y_2 \\
& \lambda_1 \searrow \quad \swarrow \lambda_2 & \\
& [X/\Gamma] &
\end{array}
\tag{2}
$$

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we need an isomorphism $\varphi_\rho : \mathcal{S}_{\lambda_1} \xrightarrow{\sim} \rho^*\mathcal{S}_{\lambda_2}$. Let λ_i be defined by a pair

$$\left(E_i \xrightarrow{\pi_i} Y_i,\ E_i \xrightarrow{f_i} X \right)$$

For diagram (2) to commute means the maps λ_1, λ_2 are isomorphic, i.e., there exists a morphism

$$(\rho, \gamma) : \left(E_1 \longrightarrow Y_1,\ E_1 \xrightarrow{f_1} X \right) \longrightarrow \left(E_2 \longrightarrow Y_2,\ E_2 \xrightarrow{f_2} X \right)$$

which is to say, $\gamma : E_1 \to E_2$ is a map of principal Γ-bundles making the diagrams

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\gamma} & E_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\rho} & Y_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
E_1 & \xrightarrow{\quad\gamma\quad} & E_2 \\
& {\scriptstyle f_1}\searrow \quad \swarrow {\scriptstyle f_2} & \\
& X &
\end{array}
$$

commute. Recall that $\pi_1^*\mathcal{S}_{\lambda_1} = f_1^*\mathcal{S}$ and $\pi_2^*\mathcal{S}_{\lambda_2} = f_2^*\mathcal{S}$. Now, since $f_1 = f_2 \circ \gamma$, $f_1^*\mathcal{S} \cong \gamma^* f_2^*\mathcal{S}$. It can be shown that this isomorphism descends to an isomorphism $\varphi_\rho : \mathcal{S}_{\lambda_1} \xrightarrow{\sim} \rho^*\mathcal{S}_{\lambda_2}$, and moreover these isomorphisms satisfy the requisite properties.

A simple example may help make this a bit more clear. Consider the principal Γ-bundle $X \to [X/\Gamma]$. We pointed out in the last section that this could be understood as the trivial Γ-equivariant bundle $X \times \Gamma$ on X. Now, let us recover bundles over Y, for each map $Y \to [X/\Gamma]$. Let such a map be defined by the object

$$\left(E \xrightarrow{\pi} Y,\ E \xrightarrow{f} X \right)$$

Use the fact that

$$
\begin{aligned}
f^*(X \times \Gamma) &= E \times \Gamma \\
&= \pi^* E
\end{aligned}
$$

and so we take the associated bundle on Y to be E. Indeed, earlier we began by describing $X \to [X/\Gamma]$ in terms of bundles $Y \times_{[X/\Gamma]} X \to Y$, and recall $Y \times_{[X/\Gamma]} X \cong E$. So, in this example, we have precisely inverted our previous construction, as one would have hoped.

8 $[X/\Gamma]$ is smooth

One of the most famous properties of quotient stacks is that, so long as X and Γ are smooth, and Γ acts by diffeomorphisms, the quotient stack $[X/\Gamma]$ is smooth, regardless of whether or not the quotient space X/Γ is smooth.

In this section, we shall review how this works. We shall show that, in the case X is a smooth manifold, Γ is discrete[8], and Γ acts on X by diffeomorphisms, the quotient stack $[X/\Gamma]$ naturally admits a smooth structure. We first show that $[X/\Gamma]$ is a topological manifold (that is, a space which admits coordinate charts, but those charts are not required to differ by diffeomorphisms on overlaps), and then show how it is furthermore a smooth manifold, not just a topological manifold.

Perhaps the fastest way to do this is to simply note that if we change the previous discussion in this paper by replacing all occurrences of "topological space" with "manifold" and, "continuous map" with "smooth map," then most of the discussion is unchanged. If Γ does not act freely, then of course the quotient space no longer makes sense in this context, but we could still define everything as groupoids over the category of manifolds rather than Top or X/Γ. However, the reader may find such an approach unsatisfactory, as it would not directly explain how to put a smooth structure on a topological quotient stack. Therefore, we shall work through direct constructions of smooth structures on topological quotient stacks in detail.

8.1 $[X/\Gamma]$ is a topological manifold

To begin, we shall first show that $[X/\Gamma]$ is a topological manifold. Specifically, this is a topological space that is locally homeomorphic to $\mathbf{R}^n$, though the coordinate charts are not required to be related by diffeomorphism on overlaps (though they are automatically homeomorphisms).

How should one define a notion of "topological manifold" for a stack? As usual, the correct procedure is to rewrite the definition for ordinary spaces in a manner that makes sense for stacks.

Here is one way to describe a topological manifold. Let N be a topological space. Then N can be given the structure of a topological manifold if and only if there exists another topological manifold M and a surjective local homeomorphism $f : M \to N$.

To check this statement, first note that given such a pair (M, f), it is straightforward to show that N is locally homeomorphic to $\mathbf{R}^n$. Let $\{V_\alpha\}$ be an open cover of M, such that for each α, the restriction of f to V_α is a homeomorphism, and without loss of generality assume furthermore that each V_α is contained within some coordinate neighborhood. Let $\psi_\alpha : V_\alpha \longrightarrow \psi_\alpha(V_\alpha) \subseteq \mathbf{R}^n$ be homeomorphisms onto open subsets of $\mathbf{R}^n$, defining M as a topological manifold. Then $\{f(V_\alpha)\}$ is an open cover of N, and

$$\psi_\alpha \circ (f|_{V_\alpha})^{-1} : \ f(V_\alpha) \ \longrightarrow \ \mathbf{R}^n$$

---

[8]The smoothness result commonly quoted does not require Γ to be discrete. However, this is all we require for physics purposes, and assuming that Γ is discrete simplifies certain technical aspects of the discussion.

is a homeomorphism, defining N as a topological manifold.

Conversely, if N is a topological manifold, let $\{U_\alpha\}$ be an open cover such that each U_α is contained within a coordinate patch. Then, define M to be the disjoint union of the U_α, and f to be the amalgamated inclusions.

So, if we want to describe a topological space N as a topological manifold, one way to proceed is to use a topological manifold M together with a surjective local homeomorphism $f : M \to N$.

Now, as the reader may have guessed, this particular definition generalizes immediately to stacks. We say a stack $\mathcal{F}$ is a topological (generalized) manifold provided that there exists a topological manifold M and a surjective local homeomorphism $f : M \to \mathcal{F}$ (assumed to be representable, of course). Such a pair (M, f) is known as a "topological atlas" for $\mathcal{F}$. (Recall from section 3.3.6 that we can make sense out of notions such as "surjective" and "local homeomorphism" for representable maps into stacks by saying that such a map is surjective or a local homeomorphism if and only if for all continuous maps $g : Y \to \mathcal{F}$, the projection map $Y \times_\mathcal{F} M \xrightarrow{p_1} Y$ is surjective or a local homeomorphism.)

Note that if $\mathcal{F}$ is a stack with topological atlas (M, f), then for any topological manifold Y and continuous map $Y \to \mathcal{F}$, the product $Y \times_\mathcal{F} M$ is a topological manifold, since $p_1 : Y \times_\mathcal{F} M \to Y$ is a local homeomorphism.

Now that we have defined what it means for a stack to be a topological manifold, and shown that this definition makes sense for ordinary spaces, let us check whether the quotient stack $[X/\Gamma]$ is a topological manifold.

For $[X/\Gamma]$ to be a topological manifold, we must give a topological manifold M together with a surjective local homeomorphism. For simplicity, assume that Γ is discrete. In that case, take $M = X$, which we assume is a topological manifold, and take f to be the canonical map $X \to [X/\Gamma]$. As observed earlier, the canonical map is always surjective, and in the special case that Γ is discrete, it is also a local homeomorphism. Thus, we have given a topological manifold M together with a surjective local homeomorphism $f : M \to [X/\Gamma]$, and so, so long as X is a topological manifold, $[X/\Gamma]$ is also a topological manifold. In other words, X together with the canonical map $X \to [X/\Gamma]$ form a topological atlas for $[X/\Gamma]$.

To check that this is not vacuous, let us try to do the same thing for the quotient space X/Γ, in the case that Γ is not freely acting. In this case, the canonical projection map $X \to X/\Gamma$ fails to be a local homeomorphism over the fixed points of Γ. In a nutshell, the difficulty is that $\mathbf{R}^n/\Gamma \not\simeq \mathbf{R}^n$. (For example, consider the special case that the only fixed point of Γ is at the origin of $\mathbf{R}^n$. Consider the space obtained by omitting that point, so Γ acts freely on what remains. Now, $\mathbf{R}^n - \{0\}$ is homotopic to S^{n_1}, and (for $n > 2$) is simply-connected. However, $(\mathbf{R}^n - \{0\})/\Gamma$ is homotopic to S^{n-1}/Γ, and is not simply-connected, but has $\pi_1 = \Gamma$.) Thus, if Γ does not act freely, then X/Γ is not a topological manifold.

8.2 $[X/\Gamma]$ is a smooth manifold

The difference between a smooth manifold and a topological manifold is that in a smooth manifold, the coordinate charts on overlapping open sets must be related by diffeomorphism. How can we set up that condition in the language above? As before, we shall first find a definition of smooth manifold that can be directly applied to stacks, and then check that the quotient stack $[X/\Gamma]$ can be given the relevant structure.

8.2.1 Rephrase smoothness for spaces

Let us first rephrase the usual definition of smooth manifold. Before we can do this, we need a short lemma.

The lemma is as follows: let M, N be topological spaces, $f : M \to N$ a surjective local homeomorphism, Y a smooth manifold, and $g : Y \to N$ any continuous map. Then $Y \times_N M$ is naturally a smooth manifold, with smooth structure induced by Y. To prove this, let $\{V_\alpha\}$ be a cover of M such that f restricted to any V_α is a homeomorphism. Then, $Y \times_N M$ has an open cover of the form

$$\left\{ g^{-1}(f(V_\alpha)) \times_N V_\alpha \cong g^{-1}(f(V_\alpha)) \right\}$$

Since g is continuous, and an open subset of a smooth manifold is itself a smooth manifold, we have an obvious smooth structure. (More concretely, if ψ is a coordinate chart mapping an open subset of $g^{-1}(f(V_\alpha))$ to an open subset of $\mathbf{R}^n$, define ϕ, a coordinate chart on an open subset of $g^{-1}(f(V_\alpha)) \times_N M$, by $\phi(a,b) = \psi(a)$. This is not only a well-defined chart, but also differs from other charts by diffeomorphisms on overlaps.)

Now we are ready to rephrase the usual definition of smooth manifold. Let N be a topological space. We claim that N can be naturally given the structure of a smooth manifold, if and only if there exists a smooth manifold M and a surjective local homeomorphism $f : M \to N$, such that both of the projection maps $M \times_N M \longrightarrow M, M$ are smooth. (Note that $M \times_N M$ is itself a smooth manifold by virtue of the lemma above, so it is indeed sensible to speak of smooth maps $M \times_N M \to M$.)

To show that this is true, first consider the case that N is a smooth manifold. Let $\{U_\alpha\}$ be an open cover of N, such that each U_α is contained within a coordinate chart. Let $\psi_\alpha : U_\alpha \to \mathbf{R}^n$ be those coordinate charts.

Define M to be the disjoint union of the U_α, and $f : M \to N$ to be the amalgamated inclusion. It is straightforward to check that smoothness of the projection maps

$$M \times_N M \left(= \coprod_{\alpha,\beta} U_\alpha \cap U_\beta \right) \xrightarrow{p_{1,2}} M, M$$

is equivalent to demanding that the coordinate charts be related by diffeomorphisms on overlaps, guaranteed by the fact that N is a smooth manifold.

Thus, one direction of the claim above is proven.

Conversely, let N be a topological space, with M and $f : M \to N$ as stated above. We shall construct a smooth structure on N.

To do this, let $\{V_\alpha\}$ be an open cover of M, such that each V_α is contained within a coordinate chart, and such that the restriction of f to each V_α is a homeomorphism onto its image. Let $\psi_\alpha : V_\alpha \to \mathbf{R}^n$ be the homeomorphisms defining the smooth structure on M. For notational convenience, define $f_\alpha = f|_{V_\alpha}$, and $U_\alpha = f_\alpha(V_\alpha)$.

Now, $\{U_\alpha\}$ is an open cover of N, and the maps $\psi_\alpha \circ f_\alpha^{-1} : U_\alpha \to \mathbf{R}^n$ define coordinate charts on N, as for topological manifolds. To show that N has the structure of a smooth manifold, not just a topological manifold, we need to show that these coordinate charts are related by diffeomorphism on overlaps, i.e., that the map between open subsets of $\mathbf{R}^n$ defined by the composition

$$\psi_\beta\left(V_\beta|_{f_\beta^{-1}(U_\alpha \cap U_\beta)}\right) \xrightarrow{\psi_\beta^{-1}} V_\beta|_{f_\beta^{-1}(U_\alpha \cap U_\beta)} \xrightarrow{f_\beta} U_\alpha \cap U_\beta \xrightarrow{f_\alpha^{-1}} V_\alpha|_{f_\alpha^{-1}(U_\alpha \cap U_\beta)} \xrightarrow{\psi_\alpha} \psi_\alpha\left(V_\alpha|_{f_\alpha^{-1}(U_\alpha \cap U_\beta)}\right) \tag{3}$$

is a diffeomorphism.

This overlap condition ultimately follows from the constraint that the projection map $M \times_N M \xrightarrow{p_2} M$ is smooth. The topological space $M \times_N M$ has an open cover of the form

$$\left\{ V_\alpha|_{f_\alpha^{-1}(U_\alpha \cap U_\beta)} \times_N V_\beta|_{f_\beta^{-1}(U_\alpha \cap U_\beta)} \right\}$$

These open sets naturally form coordinate charts on $M \times_N M$, with the maps

$$\phi_{\alpha\beta} : \ V_\alpha|_{f_\alpha^{-1}(U_\alpha \cap U_\beta)} \times_N V_\beta|_{f_\beta^{-1}(U_\alpha \cap U_\beta)} \ \longrightarrow \ \mathbf{R}^n$$

defined by $\psi_\beta : V_\beta \to \mathbf{R}^n$ by, $\phi_{\alpha\beta}(a, b) = \psi_\beta(b)$ for $(a, b) \in V_\alpha \times V_\beta$.

The projection map $p_1 : M \times_N M \to M$ acts by mapping $(a, b) \mapsto a$, so the constraint that the projection map be smooth simply means that the map between open sets of $\mathbf{R}^n$ defined by equation (3) must be smooth. Doing this for all α, β, we see that both the homeomorphism in equation (3) and its inverse must be smooth, hence it is a diffeomorphism, as required.

Thus, the coordinate charts on the topological manifold N are related by diffeomorphism on overlaps, and so N is a smooth manifold.

8.2.2 Smoothness for stacks

We are almost ready to rephrase the results above to apply to stacks. Before we do so, we need an analogue of the lemme that we introduced at the beginning of the last subsection.

Let Y be a smooth manifold, M a topological space, $f : M \to \mathcal{F}$ a surjective local homeomorphism, and $g : Y \to \mathcal{F}$ any continuous map. Then the smooth structure on Y induces a smooth structure on the topological manifold $Y \times_{\mathcal{F}} M$.

To prove this, let $p : Y \times_{\mathcal{F}} M \to Y$ denote the projection, and let $\{W_\alpha\}$ denote an open cover of $Y \times_{\mathcal{F}} M$ such that for all α, $p|_{W_\alpha}$ is a homeomorphism. Let $\psi_\alpha : p(W_\alpha) \to \mathbf{R}^n$ be charts on Y. We claim that $\psi_\alpha \circ p|_{W_\alpha} : W_\alpha \to \mathbf{R}^n$ are good charts on $Y \times_{\mathcal{F}} M$. To check this is very easy: after all, if $W_\alpha \cup W_\beta \neq \emptyset$, then

$$(\psi_\alpha \circ p|_{W_\alpha}) \circ \left(\psi_\beta \circ p|_{W_\beta}\right)^{-1} \;=\; \psi_\alpha \circ \psi_\beta^{-1}$$

we see immediately that our charts on $Y \times_{\mathcal{F}} M$ behave well on overlaps, hence $Y \times_{\mathcal{F}} M$ is a smooth manifold.

Now, given this lemma, we can now use the results of the previous subsection to give a definition of smoothness applicable to stacks. We say that a stack $\mathcal{F}$ is smooth if there exists a smooth manifold M together with a (representable) surjective local homeomorphism $f : M \to \mathcal{F}$, with the property that the projection maps $M \times_{\mathcal{F}} M \xrightarrow{p_{1,2}} M, M$ are smooth. Such a pair (M, f) is known as a "smooth atlas" for $\mathcal{F}$. (By virtue of the lemma, $M \times_{\mathcal{F}} M$ is itself a smooth manifold, so it makes sense to speak of a map $M \times_{\mathcal{F}} M \to M$ as being smooth.)

With this definition, it is straightforward to check that the quotient stack $[X/\Gamma]$ is smooth, so long as X is smooth, Γ is discrete, and Γ acts by diffeomorphisms. In this case, take $M = X$, and take f to be the canonical map $X \to [X/\Gamma]$. We showed in section 3.3.6 that the canonical map is a surjective local homeomorphism (when Γ is discrete). Also, from section 6 we know that $X \times_{[X/\Gamma]} X = X \times \Gamma$, and that the projection maps to X are, respectively, the obvious projection $X \times \Gamma \to X$, and the evaluation map $X \times \Gamma \to X$. The first is always smooth, and the second is smooth so long as Γ acts by diffeomorphisms. Thus, we have explicitly constructed a smooth structure on $[X/\Gamma]$.

Next, what does it mean for a continuous map into $[X/\Gamma]$ to be smooth? As before, we shall first examine what this means for ordinary manifolds, in a context that easily generalizes to stacks.

Let N be a topological space, M a smooth manifold, and $f : M \to N$ a surjective local homeomorphism as before defining a smooth structure on N. Let Y be another smooth manifold and $g : Y \to N$ a continuous map. Then, g is smooth if and only if the projection maps $Y \times_N M \longrightarrow Y, M$ are both smooth. (This is easy to check – smoothness of the second

projection map $Y \times_N M \longrightarrow M$ is equivalent to smoothness of the map $g : Y \to N$ between manifolds, with smooth structure defined on N by M.)

Thus, given a smooth manifold Y and a continuous map $Y \to \mathcal{F}$ into some stack with a smooth structure defined by a pair (M, f) as above, we say that the map g is smooth if both of the projection maps $Y \times_\mathcal{F} M \longrightarrow Y, M$ are smooth. (By virtue of the lemma given at the beginning of this subsection, the smooth structure on Y induces a smooth structure on $Y \times_\mathcal{F} M$, hence it makes sense to speak of maps from $Y \times_\mathcal{F} M$ as being smooth.)

For example, the surjective local homeomorphism $f : M \to \mathcal{F}$ that defines a smooth structure on $\mathcal{F}$, is itself smooth, by definition of smooth map. After all, by definition of smooth structure on $\mathcal{F}$, we assume that both of the projections $M \times_\mathcal{F} M \longrightarrow M, M$ are smooth.

8.3 Differential forms and metrics on stacks

We can define differential forms on stacks in a manner closely analogous to functions (as described in section 7.1). Let $\mathcal{F}$ be a stack with a smooth atlas (M, f). A differential n-form ω on $\mathcal{F}$ is an assignment of a differential n-form ω_Y to each smooth map $Y \to \mathcal{F}$ (for Y a smooth manifold), such that for any commuting diagram

$$Y_1 \xrightarrow{\ \ g\ \ } Y_2 \qquad\qquad (4)$$
$$\searrow \quad \swarrow$$
$$\mathcal{F}$$

one has $\omega_2 \circ g = \omega_1$.

As usual, note that if $\mathcal{F}$ is itself an honest space, then this definition uniquely specifies an n-form on $\mathcal{F}$ in the usual sense.

In the special case that $\mathcal{F} = [X/\Gamma]$, where X is smooth, and Γ is a discrete group acting by diffeomorphisms, it is straightforward to check that an n-form on $[X/\Gamma]$ is the same thing as a Γ-invariant n-form on X. As the argument is identical to that for functions on stacks, we shall omit the details.

At the risk of beating a dead horse, one can also define metrics on stacks. To each map $Y \to \mathcal{F}$, one assigns a symmetric two-tensor g_Y, obeying the same constraint on commutative diagrams as above for differential forms. If $\mathcal{F}$ is itself an honest space, this is equivalent to the usual notion of a metric, and if $\mathcal{F}$ is the quotient stack $[X/\Gamma]$, as above, then this description is equivalent to a Γ-invariant metric on X.

9   Sigma models on stacks, redux

In section 4 we pointed out that a sum over (equivalence classes of) maps into $[X/\Gamma]$ duplicated both the twisted sector sum and the functional integral in each twisted sector of a string orbifold, a smoking gun for an interpretation of a string orbifold as a sigma model on a stack. In this section, we shall complete this picture by describing the classical action for a sigma model on a stack (with an atlas), and show how this generalizes both ordinary sigma models and string orbifolds.

Let $\mathcal{F}$ be a stack with smooth atlas X, and let Σ be the worldsheet (more generally, worldspace) of the sigma model. For any map $\phi : \Sigma \to \mathcal{F}$, let $\Phi : \Sigma \times_{\mathcal{F}} X \to X$ denote the projection map (which encodes part of ϕ). ote that both X and $\Sigma \times_{\mathcal{F}} X$ are smooth spaces, so Φ is a map in the usual sense. Let ϕ^*G denote the pullback of the metric on $\mathcal{F}$ to Σ. (Recall that we defined metrics on stacks in terms of their pullbacks, so we are guaranteed to know ϕ^*G as an immediate consequence of the definition of G, and ϕ^*G is an honest symmetric two-tensor on Σ.) Then, the natural proposal for the basic bosonic terms in the classical action for a sigma model on $\mathcal{F}$ are given by

$$\int d^2\sigma \; (\phi^*G_{\mu\nu}) \, h^{\alpha\beta} \left(\frac{\partial\Phi^\mu}{\partial\sigma^\alpha}\right)\left(\frac{\partial\Phi^\nu}{\partial\sigma^\beta}\right) \tag{5}$$

where this is integrated over a lift[9] of Σ to $\Sigma \times_{\mathcal{F}} X$, and where $h^{\alpha\beta}$ denotes the worldsheet metric. (Working out analogues of the other terms in a sigma model is completely straightforward, given the form of the bosonic terms; we leave them as an exercise for the dedicated reader.) Then, the path integral sums over (equivalence classes[10] of) maps $\Sigma \to \mathcal{F}$.

First, let us compare to ordinary sigma models. Suppose $\mathcal{F}$ is an ordinary space, rather than a stack. Then we can[11] take $X = \mathcal{F}$. As a consequence, $\Sigma \times_{\mathcal{F}} X = \Sigma$, so the classical action is integrated over Σ, and the path integral is the usual sum over maps. Also, in this case, ϕ coincides with the 'projection' map Φ, so the classical action above completely agrees with the usual classical action for a sigma model on a space. In other words, if $\mathcal{F}$ is a space, then we recover the standard sigma model.

Now, suppose that $\mathcal{F}$ is a global quotient stack, $i.e.$, $\mathcal{F} = [X/\Gamma]$ where Γ is discrete and acts by diffeomorphisms. For simplicity, we shall also assume that X is smooth. Then, an atlas for this stack is simply X itself. A map $\phi : \Sigma \to \mathcal{F}$ is determined by two pieces of data, namely

- A principal Γ-bundle on Σ

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[9]Sensible essentially because the (projection) map $\Sigma \times_{\mathcal{F}} X \to \Sigma$ is a surjective local homeomorphism.

[10]After all, we wish to perform a sum, hence we must take equivalence classes.

[11]Other choices of atlases work similarly. For example, if we take X to be a disjoint union of elements of an open cover of the target space, then we recover an expression for the sigma model explicitly in terms of elements of a cover.

- A Γ-equivariant map from the total space of the bundle into X

(Less formally, a twisted sector and a twisted sector map, respectively.) The fibered product $\Sigma \times_{\mathcal{F}} X$ is given by the total space of the principal Γ bundle on Σ, and the projection map $\Sigma \times_{\mathcal{F}} X \to X$ is given by the Γ-equivariant map into X. In any event, it is now straightforward to check that the classical action for a sigma model on $\mathcal{F}$ described above coincides with the action appearing in string orbifolds (including branch cuts induced by the lift), and the path integral sum over maps $\phi : \Sigma \to \mathcal{F}$ duplicates both the twisted sector sum as well as the sum over maps within each twisted sector, as described earlier.

So far we have only recovered known results; let us now try something new. Suppose the target $\mathcal{F}$ is a gerbe. For simplicity, we shall assume that $\mathcal{F}$ is the canonical trivial G-gerbe on a space X. Such a gerbe is described by the quotient stack $[X/G]$, where the action of G on X is trivial. Using the notion of sigma model on a stack as above, one quickly finds that the path integral for this target space is the same as the path integral for a sigma model on X, up to an overall multiplicative factor (equal to the number of equivalence classes of principal G-bundles on Y). As overall factors are irrelevant in path integrals, the result appears to be that a string on the canonical trivial gerbe is the same as a string on the underlying space. More generally, it is natural to conjecture that strings on flat gerbes should be equivalent to strings on underlying spaces, but with flat B fields. Such a result which would nicely dovetail with the well-known fact that a coherent sheaf on a flat gerbe is equivalent to a 'twisted' sheaf on the underlying space, the same twisting that occurs in the presence of a B field. For physicists, this is an alternative to the description in terms of modules over Azumaya algebras that has recently been popularized [42].

So far we have only discussed classical actions for sigma models on stacks, but there is much more that must be done before one can verify that the notion of a sigma model on a stack is necessarily sensible. In effect, we have only considered local behavior, but in order to be sure this notion is sensible after quantization, one also needs to consider global phenomena. Such considerations were the source of much hand-wringing when nonlinear sigma models on ordinary spaces were first introduced (see for example [43]), and must be repeated for stacks.

10   Well-behavedness of sigma models on stacks

Assuming that our proposal for a sigma model on a stack makes sense, under what circumstances might one expect the corresponding CFT to be well-behaved?

In an ordinary sigma model on a space, the CFT will become badly behaved if the target space metric degenerates, making the kinetic terms in the sigma model action poorly-behaved. In other words, one usually expects an ordinary sigma model to be well-behaved

so long as the target space is smooth.

In the proposal of the last section, for a sigma model on stack $\mathcal{F}$ with atlas X, the kinetic terms in the action are formulated in terms of maps into X, hence one requirement for a well-behaved CFT (if the CFT actually exists) would be that X is smooth.

If our stack $\mathcal{F}$ were an honest space, we would now be done. However, for a general stack $\mathcal{F}$, our proposal was more complicated than just a sum over maps into the atlas – the action was formulated on a lift of the worldsheet Y to $Y \times_{\mathcal{F}} X$, which introduces branch cuts. We shall not attempt to analyze what additional constraints the presence of such branch cuts introduce into requirements for a well-behaved CFT, but notice that surely any such constraints would be constraints on the map $X \to \mathcal{F}$.

So far we have briefly (and loosely) argued that for the (hypothetical) CFT for a sigma model on a stack $\mathcal{F}$ to be well-behaved, one needs the atlas X to be smooth, in addition to some (undetermined) criteria on the map $X \to \mathcal{F}$. Recall from earlier that the same criteria are needed for the stack $\mathcal{F}$ to be smooth – one needs the atlas X to be smooth, and the map $X \to \mathcal{F}$ must satisfy additional criteria. Now, without knowing precisely which criteria the map $X \to \mathcal{F}$ must satisfy, we cannot claim that the criteria for a well-behaved CFT necessarily completely agree with the criteria for the target stack $\mathcal{F}$ to be smooth, but we are at least in the right ballpark, and some of the criteria (*i.e.*, X smooth) are indeed identical.

In other words, conditions needed physically for a sigma model into $\mathcal{F}$ to be well-behaved, are at least in the same ballpark as the criteria for $\mathcal{F}$ to be smooth. At least very naively, it looks very likely that the notion of smoothness for stacks is the physically-relevant notion, *i.e.*, the conditions for a stack to be smooth appear to agree with the conditions for the corresponding CFT to be well-behaved.

11 Twist fields, associated stacks, and orbifold Euler characteristics

In this section, we shall explain how twist fields arise in this picture of string orbifolds as sigma models on certain stacks. We will be naturally led to a description of twist fields in terms of cohomology of stacks associated to quotient stacks, a description which is actually known to certain mathematicians, though its form will be somewhat different from what many physicists have expected.

To understand how twist fields arise in a sigma model on a quotient stack, we need to recall for a moment why massless states of a sigma model are ever described in terms of the cohomology of the target. Somewhat loosely, in general terms one expects the circumference
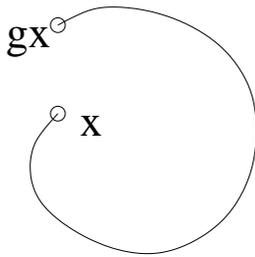
Figure 3:   A twist sector state; $g \in \Gamma$

of a closed string to be roughly proportional to the energy of the string, so massless modes should be described by strings that have shrunk to points. So, if one is only interested in massless modes, then it suffices to consider maps of points into the target space, and from the usual reasoning (see for example [35]), one recovers the cohomology of the target space.

Now, consider a string propagating[12] on a quotient stack. As emphasized earlier, a map from a closed string into the quotient stack $[X/\Gamma]$ is described by a principal Γ-bundle over S^1, together with a Γ-equivariant map from the total space of the bundle into X.

When the string shrinks to zero size, this data can be described more compactly. Naively, one would say that for a zero-size string, this data is the same as a Γ-equivariant map from a principal Γ-bundle over X, i.e., Γ itself, into X, which is the same as a point of $[X/\Gamma]$.

However, that is not quite the whole story – in particular, it ignores the possibility that the principal Γ-bundle on the shrinking S^1 is nontrivial. (Equivalently, it ignores the twisted sectors, illustrated in figure 3.) We need to include information about that bundle. Now, a principal bundle on S^1 can be described by a single transition function. As the S^1 shrinks, a map into $[X/\Gamma]$ is trying to look over each point of S^1 like a single map $\Gamma \to X$, together with a single transition function that should look increasingly like an automorphism of that single map $\Gamma \to X$. So, intuitively, in the limit that the S^1 shrinks to a point, a map will be described by a point of $[X/\Gamma]$, together with an automorphism of that point.

For the points of $[X/\Gamma]$ that project down to smooth points on X/Γ, the only automorphism is the identity, so there is no new information. However, over singularities of X/Γ, the points of $[X/\Gamma]$ have extra automorphisms, in addition to the identity. Thus, we get extra maps. Thus, the low-energy spectrum of a string orbifold is not precisely the cohomology of the quotient stack $[X/\Gamma]$, but rather some associated space which has extra points.

We can make this discussion somewhat more formal. For any stack $\mathcal{F}$ the inertia group

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[12]Here, we are taking a more Hamiltonian approach to matters than before. Previously we have considered path integrals for sigma models on quotient stacks, and so considered maps from the worldsheet into the quotient stack. By contrast, here we are interested in string states, not path integrals, so we consider maps from S^1 into the quotient stack.

stack[13] $I_{\mathcal{F}}$ of $\mathcal{F}$ [18, def'n 1.12] is defined by saying that a map $Y \to I_{\mathcal{F}}$ is given by a map $Y \to \mathcal{F}$ together with an automorphism of that map. (Phrased another way, the category $I_{\mathcal{F}}$ has as objects, pairs consisting of an object of $\mathcal{F}$ together with an automorphism of that object. Morphisms in the category are morphisms in $\mathcal{F}$ which are compatible with the automorphisms of the objects.) More to the point, from the discussion above we see that our shrunken S^1's are mapped into the inertia group stack $I_{[X/\Gamma]}$ associated to the quotient stack, not the quotient stack itself.

In principle, twist fields are the cohomology of this auxiliary space, the inertia group stack $I_{[X/\Gamma]}$. To put this matter in better perspective, the inertia group stack $I_{[X/\Gamma]}$ associated to a quotient stack has the form [9, p. 38]

$$I_{[X/\Gamma]} \cong \coprod_{[g]} [X^g/C(g)] \tag{6}$$

(where $[g]$ denotes the conjugacy class of $g \in \Gamma$ and $C(g)$ is the centralizer of g in Γ) which compares very closely to the usual expression for the stringy Euler characteristic of a string orbifold [10]:

$$\chi_{orb}(X, \Gamma) = \sum_{[g]} \chi\left(X^g/C(g)\right)$$

Expression (6) is proven in [9, p. 38], as well as in appendix D in a different fashion. Hopefully the obvious relationship between the form of the inertia group stack and orbifold Euler characteristics should help convince skeptical physicists.

Now, although this particular description of twist fields does not seem to be widely known to physicists, it is known to some mathematicians – by thinking about string orbifolds in terms of sigma models, we have recovered a description of twist fields. Also, although this form is quite obscure, it can be shown to be equivalent to a much simpler description in terms of group actions on covers.

Another point we should speak to is the fact that this particular description of twist fields is rather different from what many physicists have traditionally expected. In some corners of the physics community, it has been conjectured that twist fields should be understandable in terms of some cohomology of the quotient *space*, and not only is the expression above not a cohomology of the quotient space, it is not even a cohomology of the quotient stack! In fact, after considerable work, it can be shown that the description above is essentially equivalent to a description in terms of group actions on covers, something that many physicists would not ordinarily consider a 'final' description. So, one way of rephrasing our argument is that,

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[13]In the francophone literature, this is also known as the ramification stack. Also, another way to describe the inertia group stack $I_{\mathcal{F}}$ associated to any stack $\mathcal{F}$ is as

$$I_{\mathcal{F}} = \mathcal{F} \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F}$$

where both the maps $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ are the diagonal.

if a string orbifold truly is a sigma model on a quotient stack, then a description of twist fields in terms of group actions on covers is perhaps the best one can hope for – a description in terms of a cohomology of the quotient space need not exist.

Another point we should speak to is the naive contradiction that has arisen, due to the face that the massless spectrum of our proposed sigma model is not the cohomology of the target. We implicitly resolved this paradox earlier when we noted that the 'zero-momentum' part of the loop space need no longer agree with the original stack, a fact which invalidates the standard result. Phrased another way, although a string orbifold describes strings propagating on a quotient stack $[X/\Gamma]$, it appears that twist fields appear via cohomology calculations in the associated inertia group stack $I_{[X/\Gamma]}$. Ordinarily, massless modes would come from cohomology of the space the string propagates on, but stacks are a little more subtle. If the stack $\mathcal{F}$ can be represented by an ordinary space, then all points have only the trivial automorphism, and so the inertia group stack $I_{\mathcal{F}}$ and $\mathcal{F}$ are the same – so, this subtlety that the space strings propagate on is distinct from the space that the massless modes are associated to, can only crop up on stacks, not ordinary spaces.

As our discussion so far has been rather abstract, let us take a moment to consider some concrete examples. For example, consider the orbifold of $\mathbf{C}^2$ by $\mathbf{Z}_2$. Most points of $[\mathbf{C}^2/\mathbf{Z}_2]$ do not have automorphisms beyond the identity. However, the point of $[\mathbf{C}^2/\mathbf{Z}_2]$ lying over the singularity of $\mathbf{C}^2/\mathbf{Z}_2$ does have a nontrivial automorphism. After all, that point is the map $\mathbf{Z}_2 \to \mathbf{C}^2$ whose image is the fixed point, and one can compose this map with a nontrivial action of $\mathbf{Z}_2$ to recover the same map again. Thus, the cohomology of $I_{[X/\Gamma]}$ should look like the cohomology of $[X/\Gamma]$, together with an extra generator reflecting the cohomology of an extra point. This extra point is the object of $[\mathbf{C}^2/\mathbf{Z}_2]$ lying over the singularity of $\mathbf{C}^2/\mathbf{Z}_2$, together with a nontrivial automorphism. The reader should recognize that nontrivial automorphism as precisely the "twist" in the twist field appearing in the low-energy spectrum of the string orbifold of $\mathbf{C}^2$ by $\mathbf{Z}_2$. In the next section we shall check more formally that the inertia group stack $I_{[X/\Gamma]}$ correctly reproduces the general formula for the orbifold Euler characteristic, but for the moment, the reader should pause to consider that we have directly computed a twist field in which the "twist" is manifest.

More generally, it should be clear that quotient stacks $[\mathbf{C}^n/\mathbf{Z}_k]$ should have extra maps over the singularities, precisely duplicating the appropriate twist fields.

For most of this paper, we have argued that standard properties of string orbifolds (twisted sectors, well-behaved CFT, and B fields, for example) are really properties of stacks, and have nothing to do with string theory *per se*. However, in order to properly understand twist fields, we have had to use strings – we have used the fact that the extended objects are one-dimensional crucially in determining that the massless fields should be described by cohomology of $I_{[X/\Gamma]}$. For other, higher-dimensional, extended objects, the analogous analysis would give a different result. See section 14.2 for more details on higher-dimensional analogues of twist fields.

12 Non-effective group actions

Sometimes in string theory we consider orbifolds whose group only acts on the fields of the theory, and leaves the underlying space invariant. For example, one sometimes orbifolds by $(-)^F$, commonly in situations where one wishes to break supersymmetry at string scale. In such cases, one still has the structure of twisted sectors, even though the group acts trivially on the underlying space. Is this consistent with the interpretation of string orbifolds as sigma models on quotient stacks?

Happily, this is completely consistent with the interpretation of strings as sigma models on quotient stacks. For example, if Γ acts trivially on X, then $[X/\Gamma]$ and X are not homeomorphic. In other words, even when Γ acts trivially on X, in the sense that $g \cdot x = x$ for all $g \in \Gamma$ and points $x \in X$, a sigma model on the quotient stack $[X/\Gamma]$ still has the structure of twisted sectors and so forth.

On a slightly different note, non-effective group actions can also be used to construct other interesting stacks. For example, the trivial gerbe over a space (the formal structure corresponding to B fields, analogous to bundles for gauge fields) can be described as the quotient stack of a $U(1)$ action that acts trivially on the space. (Although we have been primarily interested in quotients by discrete groups in this paper, most of our discussion also holds for the case that the orbifold group is continuous.)

In particular, the quotient stack $[\mathrm{pt}/\Gamma]$, where pt denotes some fixed point with a trivial Γ-action, is known as the classifying stack for Γ and denoted $B\Gamma$. The trivial gerbe over a space X is simply $X \times B\Gamma$, and other gerbes are twisted versions of this (i.e., other gerbes look locally, but not necessarily globally, like $X \times B\Gamma$).

13 B fields

The standard lore concerning string orbifolds nowadays says that in string orbifolds, the B field associated with blowup modes is naturally nonzero. (Such a B field could be rotated to take on other values; but there is a specific natural value associated with orbifold points.) Technically, for $\mathbf{C}^2/\mathbf{Z}_2$, by resolving the singularity one is led to believe that for a string orbifold, $B = \frac{1}{2}$ [29], and for $\mathbf{C}^2/\mathbf{Z}_n$, there are weaker arguments[14] suggesting that $B = \frac{1}{n}$ [32]. Few other cases have been analyzed in detail, though it has been suggested that there exist other examples in which the B field is naturally zero in a string orbifold.

Witten has observed [33] that if the B field associated to a blowup mode is naturally nonzero, then one should believe that the corresponding conformal field theory is nonsingular,

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[14]For some rigorous results, see instead [30]; also, [31].

judging by results on theta angles in linear sigma models [34].

How does this tie into the present work? If string orbifolds are sigma models on quotient stacks, then how can we see the B field? For that matter, what does it mean to have a B field on a shrunken divisor? How can one talk about holonomy on a zero-size object?

Surprisingly enough, one can get answers to these questions with comparatively little work. One way of describing a quotient stack $[X/\Gamma]$ is as a quotient space X/Γ with some "extra structure" at the singularities. In particular, recall that when Γ acts freely, $[X/\Gamma] \cong X/\Gamma$, so the difference between the quotient stack and the quotient space arises at the fixed points. Also recall that, in the category of points $[X/\Gamma]_{pt}$ of the quotient stack $[X/\Gamma]$, objects lying over singularities of X/Γ have extra automorphisms – these extra automorphisms are precisely the "extra structure" referred to in comparing $[X/\Gamma]$ to X/Γ.

How does the B field fit into this picture? A very quick way to see a connection is as follows. The extra structure possessed by the quotient stack $[X/\Gamma]$ over singularities of the quotient space X/Γ is precisely a gerbe at a point. In the literature on stacks, this seems to be known as the "residual gerbe." (More generally, recall that gerbes are merely special types of stacks, so one should not be at all surprised to find gerbes lurking in the present context.) Moreover, for $[\mathbf{C}^2/\mathbf{Z}_n]$ singularities, for example, the gerbe corresponds to a B field with holonomy quantized by $1/n$. Phrased another way, if one resolves the quotient space, and pulls back the quotient stack to the resolution of the quotient space, then the restriction of the pullback to the exceptional divisor will be a gerbe describing a B field with holonomy lying in $\{0, 1/n, 2/n, 3/n, \cdots, (n-1)/n\}$.

Now, this analysis was very quick, and with a bit of effort, one can do a much better job, by considerations of D-brane probes, described as coherent sheaves. We will need two important facts in this analysis:

- First, a coherent sheaf on a quotient stack $[X/\Gamma]$ is precisely a Γ-equivariant coherent sheaf on X (which is not quite the same as a sheaf on the quotient space X/Γ). Recall that the Douglas-Moore construction [44] of D-branes on string orbifolds describes Γ-equivariant objects on the covering space, so in other words, the Douglas-Moore construction of branes on orbifolds precisely corresponds to coherent sheaves on quotient stacks.

- Second, a coherent sheaf on a flat gerbe is the same thing as a 'twisted' sheaf on the underlying space, *i.e.*, twisted in the sense of 'bundles' on D-branes with B fields. Put another way, sheaves on gerbes are an alternative to modules over Azumaya algebras as popularized in [42]. A thoroughly ham-handed attempt to describe this phenomenon is presented in [8].

Given that all stacks look locally like either orbifolds or gerbes, these two cases justify using coherent sheaves to describe D-branes on more general stacks.

Now, in order to see what quotients stacks have to do with B fields, let us consider a naive 'blowup' of the stack $[\mathbf{C}^2/\mathbf{Z}_2]$. In particular, the minimal resolution of the quotient singularity $\mathbf{C}^2/\mathbf{Z}_2$ is the same as the quotient $(\mathrm{Bl}_1\mathbf{C}^2)/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ action has been extended trivially over the exceptional divisor of the blouwp. Thus, the quotient stack $[(\mathrm{Bl}_1\mathbf{C}^2)/\mathbf{Z}_2]$ is a naive stacky analogue of the resolution of the quotient space $\mathbf{C}^2/\mathbf{Z}_2$, and among other things, is a stack over the resolution.

Finally, consider D-brane probes of this stack, viewed as coherent sheaves. Away from the exceptional divisor, this stack looks like the corresponding space, so a D-brane away from the exceptional divisor thinks it is propagating on the underlying space. A coherent sheaf over the exceptional divisor, on the other hand, is a sheaf on a gerbe, and so describes a D-brane in the presence of a B field. Thus, we see the advertised B field. In fact, we can read off even more – the gerbe over the exceptional divisor is a $\mathbf{Z}_2$-gerbe, so the corresponding B field holonomy must be either 0 or 1/2. This certainly puts us in the right ballpark to recover the results in [29, 30, 31, 32]. In order to find out precisely which of these values the B field on the exceptional divisor would take on, one would have to do a more careful study of the deformation theory of the quotient stack, which we shall not attempt here. However, the fact that one naturally gets a B field associated with singularities, with, up to an as-yet-undetermined discrete choice, the right value, is enough for the moment.

More generally, it seems likely that one may be able to understand B fields in standard Calabi-Yau compactifications through a stack argument. It seems not unreasonable that the business of "analytically continuing around singularities" may now be understood as equivalently compactifying on a stack describing the B field[15]; presumably, one would find that the stacks in question are only singular at special points, hence one could have a smooth "analytic continuation."

14   M theory

Most of this paper has been devoted to giving a solid interpretation of string orbifolds in terms of quotient stacks. However, although we are providing a great deal of insight into the deep underpinnings of string orbifolds, we have not yet told the reader any new physics.

We shall now point out some possible new physics originating from these insights.

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[15] An analogy may make this more clear. For a line bundle with connection on a given space, there are often two very different looking representatives of the corresponding equivalence class of bundles with connection: one representative in which the transition functions of the bundle are all identically 1, and the connection is a globally-defined, closed (but not exact) form; another in which the connection is identically zero, and the information about Wilson lines is encoded in trivializable-but-not-trivial transition functions.

14.1 Generalities

It has sometimes been stated that orbifolds are not well-understood in the context of M-theory[16]. After all, in the past string orbifold constructions have seemed to rely deeply on having strings, which of course is not appropriate for M-theory.

However, in light of the arguments given in this paper, we can finally shed light on M-theory orbifolds. In essence, we have argued in this paper that string orbifolds can be understood without strings, so we can now apply these methods to M-theory.

If we think of M-theory as a limit of IIA string theory, then it is natural to believe that the limit of a IIA string theory on the generalized space $[X/\Gamma]$ (i.e., a string orbifold) will be M-theory on $[X/\Gamma]$.

Since the generalized space $[X/\Gamma]$ is smooth, regardless of whether or not Γ acts freely, M-theory on this generalized space should be well-behaved.

Just as strings propagating on $[X/\Gamma]$ possess twisted sectors, a membrane propagating on $[X/\Gamma]$ should also have twisted sectors, in the obvious sense (see [36] for a more pictorial explanation).

14.2 Hypothetical membrane twist fields

Now that we understand twist fields geometrically in strings, we can now speculate about their analogues in membranes. Such things may play an important role. For example, in the Horava-Witten [38] description of heterotic $E_8 \times E_8$ theory from M-theory, they were forced to add E_8 gauge multiplets to the boundary by hand. Their appearance was enforced by various anomaly cancellations, but still, they had to add them manually. If we understood the Horava-Witten picture in terms of stacks, then just as string twist fields naturally emerge, so might the E_8 multiplets from considerations of the M-theory three-form potential.

In this subsection, we shall not attack the problem of understanding the occurrence of E_8 boundary multiplets in Horava-Witten theory, but rather we shall attack a much simpler problem. We shall merely examine the twist fields that crop up on membranes if we formally follow the same argument from which we produced string twist fields. We will not claim to understand the physics, but the mathematics is clear.

One might naively guess that, if one could make sense out of the effective action of a membrane orbifold, that it would have the same twist fields as strings, but that does

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[16]We are here using "M-theory" to refer to the hypothetical quantum theory underlying eleven dimensional supergravity, in accordance with its original usage in the literature, as opposed to some unifying master theory.

not seem to be correct. If one takes a Hamiltonian approach as above, and considers, for example, membranes whose spatial cross-section looks like T^2, then to take into account the possibility of a nontrivial principal Γ-bundle on the shrinking T^2, one must include not a single automorphism, but a pair of commuting automorphisms. Thus, if we naively follow the same logic as for strings, we see that membranes would have different numbers of twist fields, i.e., the number of twist fields seen by an extended object on an orbifold depends upon the dimension and topology of its spatial cross-section.

Unfortunately there do not seem to be any examples in which the number of twist fields has any correlation with blowups in some natural family, unlike the case of strings. For example, for a T^3 membrane on $[\mathbf{C}^2/\mathbf{Z}_2]$, one would expect either 2 or 3 twist fields (depending upon an ordering issue we have not been careful about). The closest one can come is a 4-real-parameter family of blowups, describing a blowup of the Calabi-Yau resolution of $\mathbf{C}^2/\mathbf{Z}_2$. (One real parameter is the Kähler modulus of the resolution; two real parameters are the location of the second blowup on the first; and the last is the Kähler modulus of the second blowup.) The resulting blown-up space is not Calabi-Yau, but perhaps for membranes this is not unnatural. If there were some natural way to eliminate some of those four real parameters, perhaps by some sort of overall scaling or modding out an action of $GL(2, \mathbf{C})$ on the first exceptional divisor, then we would get the desired number of moduli, but we cannot see any natural reason to do such a thing.

The fact that, unlike strings, there do not seem to be any examples in which the number of "membrane twist fields" correlates with the number of parameters in a family of resolutions or deformations of the quotient space might be an indication that our purely formal motivation for "membrane twist fields," given above, has no physical basis, and so should be ignored. Alternatively, it might be an indication that the story is more interesting for membranes than previously assumed.

15   New compactifications

Although the emphasis in this paper has been on string orbifolds, part of the point of this paper is to lay the groundwork for a new class of string compactifications: compactification on stacks. One way of thinking about stacks is as slight generalizations of spaces, on which one can perform differential geometry, hence it is natural to ask whether one can compactify a string on such objects. Indeed, understanding whether string compactification on stacks even makes sense is a prerequisite to being able to fully understand whether, as suggested here, string orbifolds really are the same as strings compactified on quotient stacks.

Towards this goal of understanding string compactification on stacks, we have not only described in detail how one makes sense of differential geometry on stacks, but also, for example, described a proposal for a classical action for a sigma model on a stack (section 9),

which generalizes both ordinary sigma models and string orbifolds. Of course, possessing a proposal for a classical action is not sufficient; for example, when sigma models on ordinary spaces were first being considered, it was noted that global effects can lead to obstructions to quantization [43].

One of the most important ways to check whether the notion of string compactification on stacks is sensible, is to examine some nontrivial examples. Hence, we are faced with a chicken-and-egg problem: it is difficult to check whether the notion of compactification on stacks is truly sensible without nontrivial examples, yet until one understands whether this is a sensible notion, it is difficult to tell whether one has any nontrivial examples. String orbifolds appear to offer the first nontrivial examples, but we may have been misled by formalism.

In any event, although in principle one can now consider string compactification on stacks, it is not clear how useful this notion actually can hope to be. Not only are stacks closely related to spaces, but in fact, all stacks look locally either like spaces, orbifolds, or gerbes, a fact which greatly cuts down on possible compactifications. Also, in order to be consistent with standard results on string orbifolds, it seems very likely that compactification on many stacks is identical to compactification on some underlying space with a flat B field turned on. At the end of the day, we can at present only imagine two general directions along these lines that need to be pursued:

- Local orbifolds. Local orbifolds can be described in terms of stacks, in addition to global orbifolds. By considering string compactification on stacks, one may be able to give local orbifolds a solid physical foundation.

- Deformation theory of string orbifolds. If string orbifolds really are the same as strings compactified on quotient stacks, then their deformation theory must be understood in terms of deformation theory of the underlying quotient stack, not a quotient space. Hopefully this coincides with a quotient space "with B fields," and we have given indirect evidence to support such a conclusion, but much work remains to check whether this is reasonable.

16   Noncommutative geometry

On a slightly different topic, the reader might be curious if there is any relationship between stacks and noncommutative geometry. After all, both are defined indirectly, by deforming some structure that characterizes spaces:

1. Noncommutative geometry is defined via rings of functions. The ring of functions on a space characterizes the space, so one formally gets "noncommutative spaces" by

deforming the ring structure.

2. We have defined stacks (generalized spaces) via the category of maps into the space. The category of maps into a space characterizes the space, so one can formally define new "generalized spaces" by deforming the category.

Now, stacks and noncommutative spaces are *not* the same. However, there is lore in the mathematics community that they are related, that they can both be used to attack the same sorts of problems.

One example of this is well-known, and was given a ham-handed treatment in [8]. There, we pointed out that the twisted "bundles" on D-brane worldvolumes (twisted because of a nontrivial B field) can be understood either in terms of noncommutative geometry (as modules on Azumaya algebras, at least for torsion H, as seems to now be well-known), or in terms of sheaves on stacks.

Other examples also exist. For example, one of the original motivating examples for noncommutative geometry in physics [39] consisted of noting that understanding T-duality in certain B field backgrounds seemed to involve understanding quotients of tori by irrational rotations, which was made more nearly sensible via noncommutative geometry. However, one could also describe such a quotient in terms of a quotient stack – the basic properties we have discussed in this paper continue to hold regardless of the details of the group action on the space (just so long as the group acts by diffeomorphisms).

We shall not pursue possible connections further here, but hope to return to this matter in future work.

17   Conclusions

In this paper we have argued that string orbifolds appear to literally be sigma models on quotient stacks. We argued this first by unraveling definitions to see that a sum over maps into a quotient stack duplicated both the twisted sector sum as well as the path integral sum in each twisted sector, after which we pointed out that a natural ansatz for a classical action for a sigma model on a stack generalized both classical actions for ordinary sigma models and string orbifold classical actions. If all we had accomplished was to use quotient stacks as a highly overcomplicated mechanism for describing group actions on covers, then there would hardly have been a point; instead, we go on to point out that this new understanding of the target space geometry sheds new light a number of physical features of string orbifolds. For example, the fact that string orbifold CFT's behave as though they were sigma models on smooth spaces is an immediate consequence of the fact that quotient stacks are smooth spaces. The old lore concerning B fields and quotient singularities has a natural

understanding here, as do twist fields and orbifold Euler characteristics.

However, we feel we have barely scratched the surface of what can be done with quotient stacks in particular and stacks in general. There are a number of unanswered questions and natural followups that beg to be pursued:

1. New compactifications. Quotient stacks are examples of generalized spaces, which have not been considered previously in the context of string compactifications. In principle, given that we now realize we have an example of strings on stacks (i.e., string orbifolds), one could now study compactification on other stacks. We have only considered quotient stacks (in this paper) and gerbes (in [8], where we gave a ham-handed treatment of the well-known fact that the twisted "bundles" on D-brane worldvolumes were naturally understood as bundles on stacks). What is needed is some new family of stacks, Calabi-Yau in some appropriate sense, that one could study string compactification on.

2. Deformation theory of stacks. We know of no mathematical work on deformation theory of stacks, which would be very useful if one wished to seriously pursue studies of string compactifications on stacks, for example, or if one wished to completely and rigorously understand B fields on exceptional divisors of resolutions. Therefore, we encourage mathematicians reading this paper to embark on such studies, or at least send us references to existing work.

3. Relation between twist fields and blow-up modes. In the special cases of orbifolds with a lot of supersymmetry, there is a famous connection between twist fields and blow-up modes. Unfortunately, although in principle quotient stacks should surely shed light on this, we only have results at present that apply to all quotient stacks and string orbifolds – results that apply to special cases require a degree of refinement not possessed by the coarse analysis methods used in this paper. Using stacks should surely shed light on this matter, and may even give yet another way of understanding the McKay correspondence, and so should be pursued vigorously by interested mathematicians.

4. Asymmetric orbifolds. We have considered only geometric orbifolds in this paper; we do not yet understand asymmetric orbifolds. These are presently under investigation.

5. B fields in standard Calabi-Yau compactifications. At the end of section 13 we conjectured that it might be possible to understand the old business about "analytically continuing around singularities" in terms of, equivalently compactifying on a stack encoding the B field. As we elaborated on this notion at some length there, we will not say more here, aside from pointing out that this would be interesting to follow up.

6. Horava-Witten E_8 multiplets. In the Horava-Witten picture of heterotic $E_8 \times E_8$ theory from M-theory [38], the E_8 gauge multiplets are added by hand. Ideally, one would like to understand them as emerging somewhat more naturally, as some analogue of twist

fields. Now that we have a better understanding of twist fields, perhaps this could be worked out. Since the M-theory three-form, modulo the gravitational correction, is currently believed to be the Chern-Simons form of some underlying E_8 gauge theory, it seems not unreasonable that if one were to study possible twist fields in the same spirit as our discussion of discrete torsion and twist fields, one might find E_8 gauge multiplets appearing naturally.

18   Acknowledgements

A   A space is determined by the maps into it

In this section, we shall describe how a topological space is completely determined by the continuous maps into it.

How does this work? Before giving a thorough category-theoretic argument, there is some basic intuition that can be gained.

First, if we know all of the continuous maps from all topological spaces into X, then the points are easy to pick off. The points are just the maps, point $\rightarrow X$.

Now, knowing the points of a topological space is not sufficient to determine that space; one must also know the open sets. This also is relatively straightforward. For example, in the special case that X is known in advance to be compact and Hausdorff, we can use the fact that [41, chapter 3.5] every closed subset of a compact Hausdorff space is precisely the image of a compact space under some continuous map (and, conversely, the image of any compact space under a continuous map is a closed subset). So, in the special case that X is known to be compact and Hausdorff, the closed subsets of X are precisely the images of compact spaces. Knowing all of the closed sets is, of course, equivalent to knowing all of the open sets, and so this gives us a topology on X.

A much more efficient way to proceed is to use category theory. Here is the mathematically precise statement: topological spaces X and Y are homeomorphic if and only if

there is a groupoid morphism[17] $(X)_{map} \to (Y)_{map}$ over *Top* which defines an equivalence of categories.

If X and Y are homeomorphic, then it is easy to construct such an equivalence of categories. Let $f : X \to Y$ be a homeomorphism, then define a functor $F : (X)_{map} \to (Y)_{map}$ as, the functor that maps

$$\left(X \xrightarrow{Id} X \right) \mapsto \left(X \xrightarrow{f} Y \right)$$

The fact that such a functor is determined by the image of $(\mathrm{Id} : X \to X)$ is a result of the Yoneda lemma [8, 17, 18]. In more detail, an object $(Z \to X)$ maps to $(Z \longrightarrow X \xrightarrow{f} Y)$, and morphisms are mapped trivially. To show that this defines an equivalence of categories, define a functor $G : (Y)_{map} \to (X)_{map}$ as the functor that maps

$$\left(Y \xrightarrow{Id} Y \right) \mapsto \left(Y \xrightarrow{f^{-1}} X \right)$$

It is straightforward to check that $F \circ G = \mathrm{Id}$ and $G \circ F = \mathrm{Id}$.

Conversely, suppose that $(X)_{map}$ and $(Y)_{map}$ are equivalent as categories.[18] Let $F : (X)_{map} \to (Y)_{map}$ be a functor defining the equivalence, itself defined by the object $(X \xrightarrow{f} Y)$ of $(Y)_{map}$, as above. Let $G : (Y)_{map} \to (X)_{map}$ be the corresponding functor, defined by the object $(Y \xrightarrow{g} X)$ of $(X)_{map}$. Let $\eta : F \circ G \Longrightarrow \mathrm{Id}$, $\gamma : G \circ F \Longrightarrow \mathrm{Id}$ invertible natural transformations.

Now, $F \circ G$ maps

$$\left(Y \xrightarrow{Id} Y \right) \mapsto \left(Y \xrightarrow{g} X \xrightarrow{f} Y \right)$$

and $G \circ F$ maps

$$\left(X \xrightarrow{Id} X \right) \mapsto \left(X \xrightarrow{f} Y \xrightarrow{g} X \right)$$

Since there is an invertible natural transformation $\eta : F \circ G \Rightarrow \mathrm{Id}$, associated to the object $(Y \xrightarrow{Id} Y)$ there is a morphism

$$\eta : \left(Y \xrightarrow{f \circ g} Y \right) \longrightarrow \left(Y \xrightarrow{Id} Y \right)$$

For η to be a morphism means that the diagram

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[17]Defined over the category *Top* of topological spaces, whose objects are topological spaces and morphisms are continuous maps. In other words, the map $(X)_{map} \to (Y)_{map}$ is required to preserve the projection to *Top*, so that a map $(Z \to X)$ (an object of $(X)_{map}$) is mapped to a map of the form $(Z \to Y)$.

[18]With the equivalence preserving the projection map to *Top*.

commutes. For the natural transformation $\eta : F \circ G \Rightarrow \mathrm{Id}$ to be invertible means that the map η is a homeomorphism, hence $f \circ g$ is a homeomorphism. Similarly, using the existence of an invertible natural transformation $\gamma : G \circ F \Rightarrow \mathrm{Id}$, one can show that $g \circ f$ is a homeomorphism.

So far we have argued that $f \circ g : Y \to Y$ and $g \circ f : X \to X$ are homeomorphisms. For $f \circ g$ to be a homeomorphism means that f is surjective and g is injective; similarly, for $g \circ f$ to be a homeomorphism means that g is surjective and f is injective. Thus, f and g are bijections. By assumption, they are also continuous. Also, since $f^{-1} = g \circ (f \circ g)^{-1}$, i.e., since f^{-1} can be written as the composition of continuous maps, f^{-1} is continuous. Thus, $f : X \to Y$ is a homeomorphism, and we are done.

B $\quad Y \times_{[X/\Gamma]} X$ is a principal Γ-bundle over Y

In the main text, we made several claims regarding the fiber product $Y \times_{[X/\Gamma]} X$; in this appendix, we shall work out the technical details.

Recall a map $Y \to [X/\Gamma]$ is defined by a pair

$$\left(E \xrightarrow{\ \pi\ } Y, \ E \xrightarrow{\ f\ } X \right)$$

We shall show that the fiber product $Y \times_{[X/\Gamma]} X$ is naturally identified with E. Specifically, we shall show that the categories

$$\left(Y \times_{[X/\Gamma]} X \right)_{map} \ \cong \ (E)_{map}$$

are equivalent, and that the functors

$$\left(Y \times_{[X/\Gamma]} X \xrightarrow{\ p_1\ } Y \right) \ \cong \ \left(E \xrightarrow{\ \pi\ } Y \right)$$

$$\left(Y \times_{[X/\Gamma]} X \xrightarrow{\ p_2\ } X \right) \ \cong \ \left(E \xrightarrow{\ f\ } X \right)$$

are equivalent, in each of the next subsections.

B.1 Technical setup

Before giving the technical definition of the fiber product $(Y \times_{[X/\Gamma]} X)_{map}$, we shall first recall the form of functors $(Y)_{map} \to [X/\Gamma]_{map}$.

Recall that, given the pair $(E \to Y, E \to X)$ defining a map $Y \to [X/\Gamma]$, the corresponding functor $h_Y : (Y)_{map} \to [X/\Gamma]_{map}$ is defined as follows:

1. Objects: This functor maps

$$\left(W \xrightarrow{\ g\ } Y \right)$$

   to the object

$$\left(g^*E \longrightarrow W,\ g^*E \longrightarrow E \xrightarrow{\ f\ } X \right)$$

   where the map $g^*E \to E$ is canonical.

2. Morphisms: Let

$$\lambda : \left(W_1 \xrightarrow{\ g_1\ } Y \right) \longrightarrow \left(W_2 \xrightarrow{\ g_2\ } Y \right)$$

   be a morphism, i.e., the diagram

$$
\begin{array}{ccc}
W_1 & \xrightarrow{\ \ \lambda\ \ } & W_2 \\
& \underset{g_1}{\searrow} \quad \underset{g_2}{\swarrow} & \\
& Y &
\end{array}
$$

   commutes. The functor maps λ to $(\lambda, \lambda^{\#})$, where $\lambda^{\#} : \lambda^* g_2^* E (= g_1^* E) \longrightarrow g_2^* E$ is the canonical map.

   The functor $h_X : (X)_{map} \to [X/\Gamma]_{map}$ is defined similarly. In this case, one takes the trivial bundle $X \times \Gamma$ in place of E, and the Γ-equivariant map is the evaluation map.

   Now that we have recalled the definitions of the functors h_Y and h_X, we shall review the technical definition of the fiber product $(Y \times_{[X/\Gamma]} X)_{map}$, folliwng for example [8] and references therein.

   The category $(Y \times_{[X/\Gamma]} X)_{map}$ consists of

1. Objects are triples

$$\left(\gamma_1 : W \longrightarrow Y,\ \gamma_2 : W \longrightarrow X, \alpha : h_Y(\gamma_1) \xrightarrow{\ \sim\ } h_X(\gamma_2) \right)$$

   Note that since α is a morphism in $[X/\Gamma]_{map}$, α is not only a bundle isomorphism making

$$
\begin{array}{ccc}
\gamma_1^* E & \xrightarrow{\quad \alpha \quad} & \gamma_2^*(X \times \Gamma) = W \times \Gamma \\
& \searrow \qquad \swarrow & \\
& W &
\end{array}
$$

   commute, but in addition the diagram

$$
\begin{array}{ccc}
\gamma_1^* E & \xrightarrow{\quad \alpha \quad} & W \times \Gamma \\
& \searrow \qquad \swarrow & \\
& X &
\end{array}
$$

   is also required to commute.

2. Morphisms

$$\left(\gamma_1 : W_1 \longrightarrow Y, \; \gamma_2 : W_1 \longrightarrow X, \; \alpha_1 : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2) \right)$$

$$\longrightarrow \left(\delta_1 : W_2 \longrightarrow Y, \; \delta_2 : W_2 \longrightarrow X, \; \alpha_2 : h_Y(\delta_1) \xrightarrow{\sim} h_X(\delta_2) \right)$$

are given by pairs (ρ_Y, ρ_X), where $\rho_Y, \rho_X : W_1 \to W_2$ are continuous maps such that the diagrams

$$
\begin{array}{ccc}
W_1 & \xrightarrow{\rho_Y} & W_2 \\
{\scriptstyle \gamma_1} \searrow & & \swarrow {\scriptstyle \delta_1} \\
& Y &
\end{array}
\qquad\qquad
\begin{array}{ccc}
W_1 & \xrightarrow{\rho_X} & W_2 \\
{\scriptstyle \gamma_2} \searrow & & \swarrow {\scriptstyle \delta_2} \\
& X &
\end{array}
$$

and

$$
\begin{array}{ccc}
\gamma_1^* E & \xrightarrow{\alpha_1} & \gamma_2^*(X \times \Gamma) = W_1 \times \Gamma \\
{\scriptstyle h_Y(\rho_Y)} \big\downarrow & & \big\downarrow {\scriptstyle h_X(\rho_X)} \\
\delta_1^* E & \xrightarrow{\alpha_2} & \delta_2^*(X \times \Gamma) = W_2 \times \Gamma
\end{array}
$$

commute. In passing, we should mention that it is straightforward to show that the constraint that $h_X(\rho_X) \circ \alpha_1 = \alpha_2 \circ h_Y(\rho_Y)$ implies that $\rho_X = \rho_Y$; however, we will typically still refer to a morphism in this category as a pair (ρ_Y, ρ_X).

The projection maps $p_{1,2} : Y \times_{[X/\Gamma]} X \longrightarrow Y, X$ induce projection functors $P_{1,2} : (Y \times_{[X/\Gamma]} X)_{map} \longrightarrow (Y)_{map}, (X)_{map}$; we shall discuss these in more detail later.

B.2 $Y \times_{[X/\Gamma]} X$ versus E

In this subsection we shall argue that the categories $(Y \times_{[X/\Gamma]} X)_{map}$ and $(E)_{map}$ are equivalent, by explicitly constructing a functor $F : (Y \times_{[X/\Gamma]} X)_{map} \to (E)_{map}$ and showing that it defines an equivalence of categories.

Define a functor $F : (Y \times_{[X/\Gamma]} X)_{map} \to (E)_{map}$ as follows:

1. Objects: Let $(\gamma_1 : W \to Y, \gamma_2 : W \to X, \alpha : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2))$ be an object of $(Y \times_{[X/\Gamma]} X)_{map}$. Let $s : W \to \gamma_2^*(X \times \Gamma)(= W \times \Gamma)$ be the identity section. The image under F is the map $W \to E$ defined by

$$W \xrightarrow{s} \gamma_2^*(X \times \Gamma) = W \times \Gamma \xrightarrow{\alpha^{-1}} \gamma_1^* E \xrightarrow{canonical} E$$

2. Morphisms: Let

$$(\rho_Y, \rho_X): \quad \left(\gamma_1 : W_1 \to Y, \gamma_2 : W_1 \to X, \alpha_1 : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2)\right)$$
$$\longrightarrow \left(\delta_1 : W_2 \to Y, \delta_2 : W_2 \to X, \alpha_2 : h_Y(\delta_1) \xrightarrow{\sim} h_X(\delta_2)\right)$$

be a morphism in $(Y \times_{[X/\Gamma]} X)_{map}$. The image of (ρ_Y, ρ_X) under F is defined to be ρ_X.

With this definition, F is a well-defined covariant functor $(Y \times_{[X/\Gamma]} X)_{map} \to (E)_{map}$.

To show that the functor $F : (Y \times_{[X/\Gamma]} X)_{map} \to (E)_{map}$ defines an equivalence of categories, we shall next construct a functor $G : (E)_{map} \to (Y \times_{[X/\Gamma]} X)_{map}$ and show that $F \circ G \cong \mathrm{Id}$ and $G \circ F \cong \mathrm{Id}$.

Define a functor $G : (E)_{map} \to (Y \times_{[X/\Gamma]} X)_{map}$ as follows:

1. Objects: Let $(m : W \to E)$ be an object of $(E)_{map}$, i.e., a continuous map into E. The image under G is defined to be the object

$$(\gamma_1 : W \longrightarrow Y, \ \gamma_2 : W \longrightarrow X, \ \alpha : \gamma_1^* E \longrightarrow W \times \Gamma)$$

where

$$\begin{aligned} \gamma_1 &\equiv \pi \circ m \\ \gamma_2 &\equiv f \circ m \\ \alpha^{-1}(w, g) &\equiv (w, g \cdot m(w)) \in \gamma_1^* E \subseteq W \times E \end{aligned}$$

2. Morphisms: Let $\lambda : (W_1 \xrightarrow{m_1} E) \longrightarrow (W_2 \xrightarrow{m_2} E)$ be a morphism, i.e., the diagram

commutes. Define the image of G to be the morphism (λ, λ) in $(Y \times_{[X/\Gamma]} X)_{map}$.

With this definition, G is a well-defined covariant functor $(E)_{map} \to (Y \times_{[X/\Gamma]} X)_{map}$.

Finally, it is straightforward to check that $F \circ G = \mathrm{Id}_{(E)_{map}}$ and $G \circ F = \mathrm{Id}_{(Y \times_{[X/\Gamma]} X)_{map}}$. (This is actually a stronger statement than needed; one only requires $F \circ G \cong \mathrm{Id}$, for example.)

Thus, the categories $(Y \times_{[X/\Gamma]} X)_{map}$ and $(E)_{map}$ are equivalent, from which we conclude that $Y \times_{[X/\Gamma]} X \cong E$.

B.3 $(Y \times_{[X/\Gamma]} X \to Y)$ versus $(E \to Y)$

Given that $Y \times_{[X/\Gamma]} X \cong E$, one would naturally suspect that the map $Y \times_{[X/\Gamma]} X \xrightarrow{p_1} Y$ is equivalent to the map $E \xrightarrow{\pi} Y$, and indeed this is the case.

We shall verify this by showing that the natural functor $P_1 : (Y \times_{[X/\Gamma]} X)_{map} \to (Y)_{map}$ is equivalent to the functor $\Pi : (E)_{map} \to (Y)_{map}$.

First, we shall take a moment to carefully define these functors.

By definition of fiber product, the functor $P_1 : (Y \times_{[X/\Gamma]} X)_{map} \to (Y)_{map}$ is defined as follows:

1. Objects: Let $(\gamma_1 : W \to Y, \gamma_2 : W \to X, \alpha : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2))$ be an object of $(Y \times_{[X/\Gamma]} X)_{map}$. The functor P_1 maps $(\gamma_1, \gamma_2, \alpha)$ to γ_1.

2. Morphisms: Let

$$(\rho_Y, \rho_X) : \ \Big(\gamma_1 : W_1 \longrightarrow Y, \ \gamma_2 : W_1 \longrightarrow X, \ \alpha_1 : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2) \Big)$$
$$\longrightarrow \Big(\delta_1 : W_2 \longrightarrow Y, \ \delta_2 : W_2 \longrightarrow X, \ \alpha_2 : h_Y(\delta_1) \xrightarrow{\sim} h_X(\delta_2) \Big)$$

be a morphism in $(Y \times_{[X/\Gamma]} X)_{map}$. The functor P_1 maps (ρ_Y, ρ_X) to ρ_Y.

The functor $\Pi : (E)_{map} \to (Y)_{map}$ is defined as follows:

1. Objects: Let $(\gamma : W \to E)$ be an object of $(E)_{map}$. The functor Π maps this object to $(\pi \circ \gamma : W \to Y)$.

2. Morphisms: Let $\lambda : (\gamma_1 : W_1 \to E) \longrightarrow (\gamma_2 : W_2 \to E)$ be a morphism, i.e., the diagram

$$W_1 \xrightarrow{\ \ \lambda\ \ } W_2$$
$$\gamma_1 \searrow \quad \swarrow \gamma_2$$
$$E$$

commutes. The functor Π maps λ to itself.

Now, in what sense can these two functors be isomorphic? After all, they map between distinct categories! The answer is that one uses the functors $F : (Y \times_{[X/\Gamma]} X)_{map} \to (E)_{map}$ and $G : (E)_{map} \to (Y \times_{[X/\Gamma]} X)_{map}$, defined in the previous subsection, to relate the categories in question.

More specifically, it is straightforward to check that

$$P_1 \circ G \;=\; \Pi : (E)_{map} \;\longrightarrow\; (Y)_{map}$$

and

$$\Pi \circ F \;=\; P_1 : (Y \times_{[X/\Gamma]} X)_{map} \;\longrightarrow\; (Y)_{map}$$

(In fact, this result is stronger than necessary. One would merely need to require that $P_1 \circ G \cong \Pi$ and $\Pi \circ F \cong P_1$.)

Thus, we conclude that, as anticipated, the map $Y \times_{[X/\Gamma]} X \xrightarrow{p_1} Y$ is indeed the same as the map $E \xrightarrow{\pi} Y$.

B.4 $(Y \times_{[X/\Gamma]} X \to X)$ versus $(E \to X)$

Considering that $Y \times_{[X/\Gamma]} X \cong E$ and the projection map $Y \times_{[X/\Gamma]} X \xrightarrow{p_1} Y$ is isomorphic to the map $E \xrightarrow{\pi} Y$, one would naturally suspect that the other projection map $Y \times_{[X/\Gamma]} X \xrightarrow{p_2} X$ is isomorphic to the Γ-equivariant map $E \xrightarrow{f} X$.

In this subsection, we shall show that this suspicion is correct. Specifically, we shall show that the natural functor $P_2 : (Y \times_{[X/\Gamma]} X)_{map} \to (X)_{map}$ is isomorphic to the functor $\tilde{f} : (E)_{map} \to (X)_{map}$.

By definition of fiber product, the functor $P_2 : (Y \times_{[X/\Gamma]} X)_{map} \to (X)_{map}$ is defined as follows:

1. Objects: Let $(\gamma_1 : W \to Y, \gamma_2 : W \to X, \alpha : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2))$ be an object of $(Y \times_{[X/\Gamma]} X)_{map}$ The functor P_2 maps $(\gamma_1, \gamma_2, \alpha)$ to γ_2.

2. Morphisms: Let

$$(\rho_Y, \rho_X) : \; \left(\gamma_1 : W_1 \longrightarrow Y, \; \gamma_2 : W_1 \longrightarrow X, \; \alpha_1 : h_Y(\gamma_1) \xrightarrow{\sim} h_X(\gamma_2) \right)$$
$$\longrightarrow \left(\delta_1 : W_2 \to Y, \; \delta_2 : W_2 \to X, \; \alpha_2 : h_Y(\delta_1) \xrightarrow{\sim} h_X(\delta_2) \right)$$

be a morphism in $(Y \times_{[X/\Gamma]} X)_{map}$. The functor P_2 maps (ρ_Y, ρ_X) to ρ_X.

Define the functor $\tilde{f} : (E)_{map} \to (X)_{map}$ as follows:

1. Objects: Let $(m : W \to E)$ be an object of $(E)_{map}$. The functor $\tilde{f}$ maps m to $(f \circ m : W \to X)$.

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2. Morphisms: Let $\lambda : (m_1 : W_1 \to E) \longrightarrow (m_2 : W_2 \to E)$ be a morphism, i.e., the diagram

$$
\begin{array}{ccc}
W_1 & \xrightarrow{\ \lambda\ } & W_2 \\
& {\scriptstyle m_1}\searrow \quad \swarrow {\scriptstyle m_2} & \\
& E &
\end{array}
$$

The functor $\tilde{f}$ maps λ to itself.

Just as in the last subsection, we run into an apparent problem understanding how these functors can be isomorphic, as they map between distinct categories. The answer to this dilemma is to use the functors $F : (Y \times_{[X/\Gamma]} X)_{map} \to (E)_{map}$ and $G : (E)_{map} \to (Y \times_{[X/\Gamma]} X)_{map}$, defined earlier, to relate the categories in question.

It is straightforward to check that

$$\tilde{f} \circ F \;=\; P_2 : (Y \times_{[X/\Gamma]} X)_{map} \;\longrightarrow\; (X)_{map}$$

and

$$P_2 \circ G \;=\; \tilde{f} : (E)_{map} \;\longrightarrow\; (X)_{map}$$

As before, this result is stronger than strictly necessary – one merely requires $\tilde{f} \circ F \cong P_2$ and $P_2 \circ G \cong \tilde{f}$.

Thus, we conclude that the map $Y \times_{[X/\Gamma]} X \xrightarrow{\ p_2\ } X$ is indeed the same as the map $E \xrightarrow{\ f\ } Y$.

C   A theorem on representable morphisms

In this section we shall present a standard theorem regarding representable morphisms.

Specifically, let $\mathcal{F}$ be a groupoid over some category $\mathcal{C}$. Then the following are equivalent [16, section 4]:

1. The diagonal map $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable.

2. For all objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$, and maps $f : X \to \mathcal{F}$, $f : Y \to \mathcal{F}$, the fiber product $X \times_{\mathcal{F}} Y$ is representable.

3. All morphisms $f : X \to \mathcal{F}$ are representable.

The fact that the second and third items are equivalent follows directly from the definition of representable morphism.

Let us show that the first statement implies the second. Suppose Δ is representable. Let $f : X \to \mathcal{F}$, $g : Y \to \mathcal{F}$ be any two morphisms into $\mathcal{F}$. Now, for Δ to be representable means that for all objects $Z \in \mathrm{Ob}\,\mathcal{C}$ and for all morphisms $h : Z \to \mathcal{F}$, the fiber product $Z \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F}$ is a representable stack. Since Δ is representable, and $f \times g : X \times Y \to \mathcal{F} \times \mathcal{F}$ is a morphism, we have that $(X \times Y) \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F}$ is representable. However, from unraveling definitions one can quickly show

$$(X \times Y) \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F} \cong X \times_{\mathcal{F}} Y$$

and so we have the second statement.

Now, we shall show that the second statement implies the first. Suppose for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$, and for all maps $f \times g : X \times Y \to \mathcal{F} \times \mathcal{F}$, the fiber product $X \times_{\mathcal{F}} Y$ is representable. From the fact that

$$X \times_{\mathcal{F}} Y \cong (X \times Y) \times_{\mathcal{F} \times \mathcal{F}} \mathcal{F}$$

it should be immediately clear that Δ is also representable.

D   Proof of form of the associated inertia group stack

In this section we shall prove that

$$I_{[X/\Gamma]} \cong \coprod_{[g]} [X^g / C(g)]$$

which we used to understand orbifold Euler characteristics. This result is also proven in [9, p. 38], but we shall use somewhat different methods to arrive at this result, and we felt it important for completeness to include our methods here.

Very roughly, the idea is as follows. An object of $(I_{[X/\Gamma]})_{map}$ is a triple

$$\left(E \xrightarrow{\pi} Y, \; E \xrightarrow{f} X, \; E \xrightarrow{\lambda} E \right)$$

where E is a principal Γ-bundle over Y, $f : E \to X$ is Γ-equivariant, and $\lambda : E \to E$ is a base-preserving automorphism compatible with f. Now, the grading by conjugacy classes of Γ comes from the fact that equivalence classes of automorphisms λ are classified by conjugacy classes. The fact that the quotient stacks in each grading are quotients of X^g follows from the fact that if λ is an automorphism determined by $[g]$, then the image of f lies in X^g (or a naturally homeomorphic set). The appearance of the centralizer $C(g)$ reflects the fact that

forcing morphisms in the category to be compatible with the automorphism λ removes all gauge transformations except those determined by $C(g) \subseteq G$.

First, we shall prove the statement at the level of points, and then we shall study continuous maps.

For future reference, we shall assume to have fixed a set of representatives g of the conjugacy classes of Γ.

D.1 Points

Define a functor

$$F : \coprod_{[g]} [X^g/C(g)]_{pt} \longrightarrow \left(I_{[X/\Gamma]}\right)_{pt}$$

as follows:

1. Objects: Let $(f : C(g) \to X^g) \in \mathrm{Ob}\,[X^g/C(g)]_{pt}$. The functor F maps this object to the object

$$(f : \Gamma \longrightarrow X,\ g : \Gamma \longrightarrow \Gamma)$$

where the Γ-equivariant map $f : \Gamma \to X$ is determined by extending $f : C(g) \to X$ to all of Γ (after all, a Γ-equivariant map from any subset of Γ is completely determined by the image of the identity), and the map $g : \Gamma \to \Gamma$ is given by, $h \mapsto g \cdot h$ for $x \in \Gamma$.

2. Morphisms: Let

$$\lambda : (f_1 : C(g) \longrightarrow X^g) \longrightarrow (f_2 : C(g) \longrightarrow X^g)$$

be a morphism in $[X^g/C(g)]_{pt}$. Since λ is completely determined by $\lambda(1)$, it extends to a map $\Gamma \to \Gamma$, which commutes both with the extensions of f_1 and f_2, as well as with g. Hence, the functor F maps λ to its extension.

We claim that F defines an equivalence of categories. In order to prove this, we shall construct a functor

$$G : \left(I_{[X/\Gamma]}\right)_{pt} \longrightarrow \coprod_{[g]} [X^g/C(g)]_{pt}$$

such that $F \circ G \cong \mathrm{Id}$ and $G \circ F \cong \mathrm{Id}$.

Define a functor

$$G : \left(I_{[X/\Gamma]}\right)_{pt} \longrightarrow \coprod_{[g]} [X^g/C(g)]_{pt}$$

as follows:

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1. Objects: Let the pair

$$(f : \Gamma \longrightarrow X, \overline{g} : \Gamma \longrightarrow \Gamma)$$

be an object of $(I_{[X/\Gamma]})_{pt}$, meaning that the diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ \overline{g}\ } & \Gamma \\
& {\scriptstyle f}\searrow \quad \swarrow {\scriptstyle f} & \\
& X &
\end{array}
$$

commutes. Clearly, the map $\overline{g}$ is determined by some element of Γ, which we shall also denote $\overline{g}$. Let g denote the fixed representative of the conjugacy class of Γ to which $\overline{g}$ belongs, and define $h \in \Gamma$ by, $\overline{g} = hgh^{-1}$. Then the following diagram commutes:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ g\ } & \Gamma \\
& {\scriptstyle f \circ h}\searrow \quad \swarrow {\scriptstyle f \circ h} & \\
& X &
\end{array}
$$

Commutivity of the diagram above clearly implies that im $f \circ h \subseteq X^g$. Since $f \circ h$ is completely determined by the image of the identity, we can restrict it to arbitrary subgroups; define f' to be the restriction of $f \circ h$ to $C(g) \subseteq \Gamma$. Then, G maps the object given to $(f' : C(g) \to X^g) \in$ Ob $[X^g/C(g)]_{pt}$.

2. Morphisms: Let

$$k : (f_1 : \Gamma \longrightarrow X, \overline{g}_1 : \Gamma \longrightarrow \Gamma) \longrightarrow (f_2 : \Gamma \longrightarrow X, \overline{g}_2 : \Gamma \longrightarrow \Gamma)$$

be a morphism in $(I_{[X/\Gamma]})_{pt}$, which means that the following two diagrams commute:

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ k\ } & \Gamma \\
{\scriptstyle f_1}\searrow & & \swarrow {\scriptstyle f_2} \\
& X &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Gamma & \xrightarrow{\ k\ } & \Gamma \\
{\scriptstyle \overline{g}_1}\downarrow & & \downarrow {\scriptstyle \overline{g}_2} \\
\Gamma & \xrightarrow{\ k\ } & \Gamma
\end{array}
$$

We shall use k to denote the morphism and the corresponding group element (the image of the identity) interchangeably. From commutivity of the right diagram, we see that $\overline{g}_1$ and $\overline{g}_2$ are in the same conjugacy class. Let g denote the fixed representative of that conjugacy class, and define $h_1, h_2 \in \Gamma$ by, $\overline{g}_1 = h_1 g h_1^{-1}$ and $\overline{g}_2 = h_2 g h_2^{-1}$. Define f_1' to be the restriction of $f_1 \circ h_1$ to $C(g)$, define f_2' to be the restriction of $f_2 \circ h_2$ to $C(g)$, and define $k' = h_2^{-1} k h_1$. Using the right diagram it is straightforward to check that $k' \in C(g)$, and hence defines a $C(g)$-equivariant map $C(g) \to C(g)$. Also, using the left diagram it is straightforward to check that k' is compatible with f_1' and f_2'. Hence, define G to map k to the morphism

$$k' : (f_1' : C(g) \longrightarrow X^g) \longrightarrow (f_2' : C(g) \longrightarrow X^g)$$

in $[X^g/C(g)]_{pt}$.

Now, it is straightforward to check that $G \circ F = \mathrm{Id}$ on the category $\coprod_{[g]} [X^g/C(g)]_{pt}$. We shall also show that $F \circ G \cong \mathrm{Id}$ on the category $(I_{[X/\Gamma]})_{pt}$, by constructing an invertible natural transformation $\eta : F \circ G \Rightarrow \mathrm{Id}$.

Construct the invertible natural transformation $\eta : F \circ G \Rightarrow \mathrm{Id}$ as follows. For any object

$$(f : \Gamma \longrightarrow X, \overline{g} : \Gamma \longrightarrow \Gamma)$$

in $(I_{[X/\Gamma]})_{pt}$, define $\eta \equiv h$, where $h \in \Gamma$ is such that $\overline{g} = hgh^{-1}$. It is straightforward to check that this is a well-defined morphism in $(I_{[X/\Gamma]})_{pt}$, and furthermore that η is an invertible natural transformation.

Hence, $G \circ F = \mathrm{Id}$ and $F \circ G \cong \mathrm{Id}$, so F is an invertible natural transformation, which means that

$$\left(I_{[X/\Gamma]} \right)_{pt} \cong \coprod_{[g]} [X^g/C(g)]_{pt}$$

In the next subsection we shall check the statement completely, by showing that the categories of continuous maps into each are equivalent.

D.2 Continuous maps

In this section we shall show that

$$\left(I_{[X/\Gamma]} \right)_{map} \cong \coprod_{[g]} [X^g/C(g)]_{map}$$

in the special case that Γ is abelian. (As we just showed that the desired statement is true at the level of points for all Γ, not just those which are abelian, and reference [9] contains an independent proof, we are highly confident of the result.)

Define a functor

$$F : \coprod_{[g]} [X^g/C(g)]_{map} \longrightarrow \left(I_{[X/\Gamma]} \right)_{map}$$

as follows:

1. Objects: Let $(E \to Y, f : E \to X^g)$ be an object of $[X^g/C(g)]_{map}$ for some conjugacy-class-representative g. Define F to map this object to the triple

$$\left(E \longrightarrow Y, E \xrightarrow{f} X^g \hookrightarrow X, E \xrightarrow{g} E \right)$$

where $g : E \to E$ is defined by sending any $e \in E$ to $g \cdot E$.

2. Morphisms: Let

$$(\rho, \psi) : (\, E_1 \longrightarrow Y_1, \; f_1 : E_1 \longrightarrow X^g\,) \longrightarrow (\, E_2 \longrightarrow Y_2, \; f_2 : E_2 \longrightarrow X^g\,)$$

be a morphism in $\coprod_{[g]}[X^g/C(g)]_{map}$, i.e., $\rho : Y_1 \to Y_2$ and $\psi : E_1 \to E_2$ are morphisms making appropriate diagrams commute. Define F to send (ρ, ψ) to (ρ, ψ).

We claim that F defines an equivalence of categories. To prove this, we shall construct a functor

$$G : \left(I_{[X/\Gamma]} \right)_{map} \longrightarrow \coprod_{[g]} [X^g/C(g)]_{map}$$

and check that $F \circ G \cong \mathrm{Id}$ and $G \circ F \cong \mathrm{Id}$.

Define a functor

$$G : \left(I_{[X/\Gamma]} \right)_{map} \longrightarrow \coprod_{[g]} [X^g/C(g)]_{map}$$

as follows:

1. Objects: Let

$$(\, E \longrightarrow Y, \; f : E \longrightarrow X, \; \overline{g} : E \longrightarrow E\,)$$

be an object of $(I_{[X/\Gamma]})_{map}$, which means that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\;\overline{g}\;} & E \\
& f \searrow \quad \swarrow f & \\
& X &
\end{array}
$$

commutes. Without loss of generality, assume that Y is connected. Then, the base-preserving map $\overline{g} : E \to E$ is determined by some $\overline{g} \in \Gamma$, as $e \in E \mapsto \overline{g} \cdot e$. Let g denote the fixed representative of the conjugacy class of $\overline{g} \in \Gamma$, and define h by, $\overline{g} = hgh^{-1}$. Define G to map the object above to the pair

$$(\, E \longrightarrow Y, \; f \circ h : E \longrightarrow X\,)$$

which define an object of $\coprod_{[g]}[X^g/C(g)]_{map}$.

2. Morphisms: Let

$$
\begin{aligned}
(\rho, \psi) : (\, E_1 \to Y_1, \; f_1 : E_1 \to X, \; \overline{g}_1 : E_1 \to E_1\,) \\
\longrightarrow (\, E_2 \to Y_2, \; f_2 : E_2 \to X, \; \overline{g}_2 : E_2 \to E_2\,)
\end{aligned}
$$

be a morphism in $(I_{[X/\Gamma]})_{map}$, i.e., $\rho : Y_1 \to Y_2$ and $\psi : E_1 \to E_2$ are morphisms making the diagrams

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\psi} & E_2 \\
\downarrow & & \downarrow \\
Y_1 & \xrightarrow{\rho} & Y_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
E_1 & \xrightarrow{\quad\psi\quad} & E_2 \\
& \searrow{\scriptstyle f_1} \quad {\scriptstyle f_2}\swarrow & \\
& X &
\end{array}
\qquad\qquad
\begin{array}{ccc}
E_1 & \xrightarrow{\psi} & E_2 \\
{\scriptstyle \overline{g}_1}\downarrow & & \downarrow{\scriptstyle \overline{g}_2} \\
E_1 & \xrightarrow{\psi} & E_2
\end{array}
$$

commute. Now, it is straightforward to check that the rightmost diagram implies that $\overline{g}_1$ and $\overline{g}_2$ are conjugate. Define h_1 and h_2 by, $\overline{g}_1 = h_1 g h_1^{-1}$, $\overline{g}_2 = h_2 g h_2^{-1}$ for some fixed conjugacy-class representative $g \in \Gamma$. Define $\psi' \equiv h_2^{-1} \circ \psi \circ h_1$. Define G to map the morphism (ρ, ψ) to (ρ, ψ').

Next, in order to show that F is an equivalence of categories, we must show that $F \circ G \cong \mathrm{Id}$ and $G \circ F \cong \mathrm{Id}$. Now, is it straightforward to check that $G \circ F = \mathrm{Id}$ on the category $\coprod_{[g]} [X^g/C(g)]_{map}$. We shall show that $F \circ G \cong \mathrm{Id}$ on the category $(I_{[X/\Gamma]})_{map}$ by constructing an invertible natural transformation $\eta : F \circ G \Rightarrow \mathrm{Id}$.

Construct the invertible natural transformation $\eta : F \circ G \Rightarrow \mathrm{Id}$ as follows. For any object

$$
\left(E \longrightarrow Y, \; E \xrightarrow{f} X, \; E \xrightarrow{\overline{g}} E \right)
$$

(where we have assumed, for simplicity, that Y is connected) define $\eta \equiv h$, where $\overline{g} = hgh^{-1}$, for some fixed conjugacy-class representative $g \in \Gamma$. It is straightforward to check that this makes η a well-defined morphism in $(I_{[X/\Gamma]})_{map}$, and in fact makes η an invertible natural transformation.

Hence, $G \circ F = \mathrm{Id}$ and $F \circ G \cong \mathrm{Id}$, so F is an invertible natural transformation, which means that

$$
\left(I_{[X/\Gamma]} \right)_{map} \cong \coprod_{[g]} [X^g/C(g)]_{map}
$$

as advertised.

Again, in this subsection we have assumed that Γ is abelian. However, our proof that the categories of points are equivalent worked for all Γ, not just Γ abelian, and reference [9] contains an independent proof, so we are confident of the result.

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