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FUNCTION EXPANSION METHODS FOR SOLVING AUTONOMOUS NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we propose some algorithms for analytical solution construction to nonlinear polynomial partial differential equations with constant function coefficients. These schemes are based on one-(single), two- (double) or three- (triple) function expansion methods. Most of the existing expansion function methods are well recovered from the mentioned schemes. The effectiveness of these methods has been tested on some nonlinear partial differential equations (NLPDEs) describing important phenomena in physics.

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1 Introduction

A variety of methods, e.g. the exponential function [12], the hyperbolic tangent function [28], the Jacobi elliptic function expansion [26], the sine-cosine function [4], the F-expansion [24], the projective Riccati equation expansion [1], the $\frac{G'}{G}$-expansion [5], the $(\frac{G'}{G},\frac{1}{G})$-expansion [25] methods and their numerous extensions [2, 3, 6, 9, 11, 10, 21, 27, 30, 31, 33, 34, 35, 36], etc. to cite the main methods, are used to obtain analytical solutions for nonlinear differential equations. The common issue to all these methods is the rational form of the final solution. The rationality here is with respect to either trigonometric, hyperbolic, elliptic, exponential functions, or other given functions. Furthermore, the scope of these methods is restricted to polynomial differential equations. We describe in this paper the general principle from which all the above mentioned methods derive for the integration of autonomous nonlinear partial differential equations (NLPDEs). For the notations used in this work, see [14] - [18].

2 Function expansion methods: general procedure

Consider the constant coefficient partial differential equations

$$F\left(u^{(s)}(x)\right)=0, \tag{2.1}$$

where the nonzero positive integer s is the order of the equation and $x=\left(x^1,\cdots,x^n\right)$ are independent variables. The dependent variable $u=u(x)$ is a scalar valued function. If

$$F\left(u^{(s)}(x)\right)=P\left(u_{(k_1)}[h_1](x),u_{(k_2)}[h_2](x),\cdots,u_{(k_r)}[h_r](x)\right), \tag{2.2}$$

where $r,k_i,h_i\in\mathbb{N}$ with $\max\{k_i,\,i=1,\cdots,r\}=s$, $h_i\in\{1,2,\cdots,p_{k_i}\}$ and P is a polynomial whose the indeterminates are the r functions $u_{(k_i)}[h_i](x)$, the equation (2.1) becomes

$$P\left(u_{(k_1)}[h_1](x),u_{(k_2)}[h_2](x),\cdots,u_{(k_r)}[h_r](x)\right)=0 \tag{2.3}$$

which is called a polynomial autonomous partial differential equation.

The change of variables $u(x)=v(\xi)$ with $\xi=\alpha_1x^1+\alpha_2x^2+\cdots+\alpha_nx^n$, where $\alpha_i\in\mathbb{R}$ transforms (2.3) into an ordinary differential equation

$$Q\left(v_{(k_1)}[1](\xi),v_{(k_2)}[1](\xi),\cdots,u_{(k_r)}[1](\xi)\right)=0, \tag{2.4}$$

where Q is also a polynomial function whose the indeterminates are $v_{(k_i)}[1](\xi)$, $i=1,\cdots,r$.
If the polynomial Q has only one monomial, then the equation (2.4) is simply solved by successive integrations. Thus, without loss of generality, one can assume that the polynomial Q is a sum of at least two monomials. We propose in this paper to search for a solution of the equation (2.4) by using single, double or triple function expansion methods.

2.1 Single function expansion method

It consists to seek the function $v=v(\xi)$, solution of the equation (2.4), into the form

$$v=A(F)+F_{(1)}[1]B(F), \tag{2.5}$$

where

$$A(F)=\sum_{i=-m}^{m} a_i F^i, \quad B(F)=\sum_{i=-\hat{m}}^{\hat{m}} b_i F^i$$

and the function $F=F(\xi)$ is a solution to an auxiliary differential equation which is in one of the forms

$$F_{(1)}[1]=R \quad \text{or} \quad \left(F_{(1)}[1]\right)^2=R, \tag{2.6}$$

R being a rational function in F and $m,\hat{m}\in\mathbb{N}$, $a_i,b_i\in\mathbb{R}$. The higher order derivatives of F can be written into the form

$$F_{(k)}[1]=R^{0,k}+F_{(1)}[1]R^{1,k},$$

where the positive integer $k\geq 2$ and $R^{i,k}$, $i=0,1$, are also rational functions in F.

One can adopt the following iterative process in practice for the determination of different parameters of v.

Estimation of the integers $m,\hat{m}$.

Let $M_1,M_2,\cdots,M_\nu$ be the monomials of Q such that M_1 contains the highest order derivative of the function $v(\xi)$ and let M_2 be a nonlinear or linear monomial. M_2 is linear only in the case where all remaining monomials are linear. Substitute in (2.4) the expression (2.5) of $v(\xi)$ along with (2.6) and write the result in the form

$$Q=\frac{1}{T}\sum_{i=1}^{\nu}\left(K_i+F_{(1)}[1]L_i\right), \tag{2.7}$$

where K_i,L_i,T are polynomials in F whose degrees depend linearly on $m,\hat{m}$ such that

$$M_i=\frac{1}{T}\left(K_i+F_{(1)}[1]L_i\right). \tag{2.8}$$

Solve the system of linear algebraic equations obtained by balancing the degree of K_1 with that of K_2 and the degree of L_1 with that of L_2 in F to determine the values of integers $m,\hat{m}$.

Estimation of the constants a_i,b_i,α_i.

Introduce into (2.7) the obtained values of $m,\hat{m}$. Write the result in the form

$$Q=\frac{1}{T}\left(K+F_{(1)}[1]L\right), \tag{2.9}$$

where $K=\sum_{i=1}^{\nu}K_i$ and $L=\sum_{i=1}^{\nu}L_i$. Set to zero all coefficients of distinct monomials in K and L. This gives a system of algebraic equations whose the unknowns are the constants a_i,b_i and α_i.

Remark 2.1. We have the tanh-expansion method if we take $F(\xi)=\tanh(\xi)$, the tan-expansion method if $F(\xi)=\tan(\xi)$, the exp-expansion method if $F(\xi)=\exp(\xi)$, the $\frac{G'}{G}$-expansion method if $F(\xi)=\frac{G'(\xi)}{G(\xi)}$, where $G''(\xi)=\alpha G'(\xi)+\beta G(\xi)$ with $\alpha,\beta\in\mathbb{R}$. It is noticeable that in all these cases $F'(\xi)$ is a polynomial function in $F(\xi)$ and hence a rational function as required.

2.2 Double function expansion method

It aims at finding the function $v = v(\xi)$, solution of the equation (2.4), into the form

$$v = A(F,G) + F_{(1)}[1]B(F,G) + G_{(1)}[1]C(F,G) + F_{(1)}[1]G_{(1)}[1]D(F,G), \tag{2.10}$$

where

$$A(F,G) = \sum_{i=-m_1}^{m_1}\sum_{j=-m_2}^{m_2} a_{i,j}F^iG^j, \quad B(F,G) = \sum_{i=-\hat{m}_1}^{\hat{m}_1}\sum_{j=-\hat{m}_2}^{\hat{m}_2} b_{i,j}F^iG^j,$$

$$C(F,G) = \sum_{i=-\check{m}_1}^{\check{m}_1}\sum_{j=-\check{m}_2}^{\check{m}_2} c_{i,j}F^iG^j, \quad D(F,G) = \sum_{i=-\bar{m}_1}^{\bar{m}_1}\sum_{j=-\bar{m}_2}^{\bar{m}_2} d_{i,j}F^iG^j$$

and the functions $F = F(\xi)$ and $G = G(\xi)$ are solutions of an auxiliary system of differential equations of the form

$$F_{(1)}[1] = R_1 \quad \text{or} \quad \left(F_{(1)}[1]\right)^2 = R_1, \tag{2.11}$$

$$G_{(1)}[1] = R_2 \quad \text{or} \quad \left(G_{(1)}[1]\right)^2 = R_2, \tag{2.12}$$

R_k, $k = 1,2$ being rational functions in F, G and $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2 \in \mathbb{N}$, $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} \in \mathbb{R}$. The higher order derivatives of F and G can be written into the form

$$\begin{aligned} F_{(k)}[1] &= R_1^{0,k} + F_{(1)}[1]R_1^{1,k} + G_{(1)}[1]R_1^{2,k} + F_{(1)}[1]G_{(1)}[1]R_1^{3,k}, \\ G_{(k)}[1] &= R_2^{0,k} + F_{(1)}[1]R_2^{1,k} + G_{(1)}[1]R_2^{2,k} + F_{(1)}[1]G_{(1)}[1]R_2^{3,k}, \end{aligned}$$

where the positive integer $k \geq 2$ and $R_j^{i,k}$, $i = 0,1,2,3,\ j = 1,2$, are also rational functions in F, G.

One can adopt the following iterative process in practice for the determination of different parameters of v.

Estimation of the integers $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$.

Let $M_1, M_2, \cdots, M_\nu$ be the monomials of Q such that M_1 contains the highest order derivative of the function $v(\xi)$ and let M_2 be a nonlinear or linear monomial. M_2 is linear only in the case where all remaining monomials are linear. Substitute in (2.4) the expression (2.10) of $v(\xi)$ along with (2.11)-(2.12) and write the result in the form

$$Q = \frac{1}{T}\sum_{i=1}^{\nu}\left(K_i + F_{(1)}[1]L_i + G_{(1)}[1]S_i + F_{(1)}[1]G_{(1)}[1]J_i\right), \tag{2.13}$$

where K_i, L_i, S_i, J_i, T are polynomials in F, G whose the degrees linearly depend on m_1, m_2, $\hat{m}_1$, $\hat{m}_2$, $\check{m}_1$, $\check{m}_2$, $\bar{m}_1$, $\bar{m}_2$ such that

$$M_i = \frac{1}{T}\left(K_i + F_{(1)}[1]L_i + G_{(1)}[1]S_i + F_{(1)}[1]G_{(1)}[1]J_i\right). \tag{2.14}$$

Solve the system of linear algebraic equations obtained by balancing the degree of K_1 with that of K_2, the degree of L_1 with that of L_2, the degree of S_1 with that of S_2 and the degree of J_1 with that of J_2 in F and G to determine the values of integers $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$.

Estimation of the constants $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, \alpha_i$.

Introduce into (2.13) the obtained values of $m_1, m_2, \hat{m}_1, \hat{m}_2, \check{m}_1, \check{m}_2, \bar{m}_1, \bar{m}_2$. Write the result in the form

$$Q = \frac{1}{T}\left(K + F_{(1)}[1]\,L + G_{(1)}[1]\,S + F_{(1)}[1]G_{(1)}[1]\,J\right), \tag{2.15}$$

where $K = \sum_{i=1}^{\nu} K_i, \quad L = \sum_{i=1}^{\nu} L_i, \quad S = \sum_{i=1}^{\nu} S_i$ and $J = \sum_{i=1}^{\nu} J_i$. Set to zero all coefficients of distinct monomials in K, L, S and J. This gives a system of algebraic equations whose the unknowns are the constants $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$ and α_i.

Remark 2.2. We have the $(\sinh, \cosh)$-expansion method if we take $F(\xi) = \sinh(\xi)$ and $G(\xi) = \cosh(\xi)$, the $(\sin, \cos)$-expansion method if $F(\xi) = \sin(\xi)$ and $G(\xi) = \cos(\xi)$. We obtain a more general formulation of the $\left(\frac{H'}{H}, \frac{1}{H}\right)$-expansion method by setting $F(\xi) = \frac{H'(\xi)}{H(\xi)}$ and $G(\xi) = \frac{1}{H(\xi)}$, where $H''(\xi) + \lambda H(\xi) = \mu$ with $\lambda, \mu \in \mathbb{R}$, and a more general formulation of the projective Riccati equation expansion method if we choose $F(\xi), G(\xi)$ such that $F'(\xi) = \alpha F(\xi)\,G(\xi)$ and $G'(\xi) = \mu + \alpha^2 G^2(\xi) - \beta F(\xi)$ with $\alpha, \beta, \mu \in \mathbb{R}^\star$. In all these cases, $F'(\xi)$ and $G'(\xi)$ are also polynomial functions in $F(\xi), G(\xi)$ and hence rational functions as required.

2.3 Triple function expansion method

Here, we need to express the function $v = v(\xi)$, solution of the equation (2.4), into the form

$$\begin{aligned} v &= A + B^{1,1}F_{(1)}[1] + B^{2,1}G_{(1)}[1] + B^{3,1}H_{(1)}[1] + C^{1,1}F_{(1)}[1]G_{(1)}[1] \\ &+ C^{2,1}F_{(1)}[1]H_{(1)}[1] + C^{3,1}G_{(1)}[1]H_{(1)}[1] + DF_{(1)}[1]G_{(1)}[1]H_{(1)}[1] \end{aligned} \tag{2.16}$$

where $A, B^{1,1}, B^{2,1}, B^{3,1}, C^{1,1}, C^{2,1}, C^{3,1}, D$ are functions of F, G, H given by

$$A = \sum_{i=-m_1}^{m_1}\sum_{j=-m_2}^{m_2}\sum_{k=-m_3}^{m_3} a_{i,j,k}F^iG^jH^k, \qquad B^{1,1} = \sum_{i=-\hat{m}_{1,1}}^{\hat{m}_{1,1}}\sum_{j=-\hat{m}_{1,2}}^{\hat{m}_{1,2}}\sum_{k=-\hat{m}_{1,3}}^{\hat{m}_{1,3}} b^{1,1}_{i,j,k}F^iG^jH^k,$$

$$B^{2,1} = \sum_{i=-\hat{m}_{2,1}}^{\hat{m}_{2,1}}\sum_{j=-\hat{m}_{2,2}}^{\hat{m}_{2,2}}\sum_{k=-\hat{m}_{2,3}}^{\hat{m}_{2,3}} b^{2,1}_{i,j,k}F^iG^jH^k, \qquad B^{3,1} = \sum_{i=-\hat{m}_{3,1}}^{\hat{m}_{3,1}}\sum_{j=-\hat{m}_{3,2}}^{\hat{m}_{3,2}}\sum_{k=-\hat{m}_{3,3}}^{\hat{m}_{3,3}} b^{3,1}_{i,j,k}F^iG^jH^k,$$

$$C^{1,1} = \sum_{i=-\check{m}_{1,1}}^{\check{m}_{1,1}}\sum_{j=-\check{m}_{1,2}}^{\check{m}_{1,2}}\sum_{k=-\check{m}_{1,3}}^{\check{m}_{1,3}} c^{1,1}_{i,j,k}F^iG^jH^k, \qquad C^{2,1} = \sum_{i=-\check{m}_{2,1}}^{\check{m}_{2,1}}\sum_{j=-\check{m}_{2,2}}^{\check{m}_{2,2}}\sum_{k=-\check{m}_{2,3}}^{\check{m}_{2,3}} c^{2,1}_{i,j,k}F^iG^jH^k,$$

$$C^{3,1} = \sum_{i=-\check{m}_{3,1}}^{\check{m}_{3,1}}\sum_{j=-\check{m}_{3,2}}^{\check{m}_{3,2}}\sum_{k=-\check{m}_{3,3}}^{\check{m}_{3,3}} c^{3,1}_{i,j,k}F^iG^jH^k, \qquad D = \sum_{i=-\bar{m}_1}^{\bar{m}_1}\sum_{j=-\bar{m}_2}^{\bar{m}_2}\sum_{k=-\bar{m}_3}^{\bar{m}_3} d_{i,j,k}F^iG^jH^k$$

and $m_i, \bar{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$ are positive integers, $a_{i,j,k}, b^{l,1}_{i,j,k}, c^{l,1}_{i,j,k}, d_{i,j,k} \in \mathbb{R}$; the functions $F = F(\xi)$, $G = G(\xi)$ and $H = H(\xi)$ are solutions of the following auxiliary system of differential equations:

$$F_{(1)}[1] = R_1 \quad \text{or} \quad \left(F_{(1)}[1]\right)^2 = R_1, \tag{2.17}$$

$$G_{(1)}[1] = R_2 \quad \text{or} \quad \left(G_{(1)}[1]\right)^2 = R_2, \tag{2.18}$$

$$H_{(1)}[1] = R_3 \quad \text{or} \quad \left(H_{(1)}[1]\right)^2 = R_3, \tag{2.19}$$

R_k, $k=1,2,3$ being rational functions in F, G, H. The higher order derivatives of F, G and H can be written into the form

$$\begin{aligned}
F_{(k)}[1] &= R_1^{0,k}+F_{(1)}[1]R_1^{1,k}+G_{(1)}[1]R_1^{2,k}+H_{(1)}[1]R_1^{3,k}+F_{(1)}[1]G_{(1)}[1]R_1^{4,k}\\
&+ F_{(1)}[1]H_{(1)}[1]R_1^{5,k}+G_{(1)}[1]H_{(1)}[1]R_1^{6,k}+F_{(1)}[1]G_{(1)}[1]H_{(1)}[1]R_1^{7,k},\\
G_{(k)}[1] &= R_2^{0,k}+F_{(1)}[1]R_2^{1,k}+G_{(1)}[1]R_2^{2,k}+H_{(1)}[1]R_2^{3,k}+F_{(1)}[1]G_{(1)}[1]R_2^{4,k}\\
&+ F_{(1)}[1]H_{(1)}[1]R_2^{5,k}+G_{(1)}[1]H_{(1)}[1]R_2^{6,k}+F_{(1)}[1]G_{(1)}[1]H_{(1)}[1]R_2^{7,k},\\
H_{(k)}[1] &= R_3^{0,k}+F_{(1)}[1]R_3^{1,k}+G_{(1)}[1]R_3^{2,k}+H_{(1)}[1]R_3^{3,k}+F_{(1)}[1]G_{(1)}[1]R_3^{4,k}\\
&+ F_{(1)}[1]H_{(1)}[1]R_3^{5,k}+G_{(1)}[1]H_{(1)}[1]R_3^{6,k}+F_{(1)}[1]G_{(1)}[1]H_{(1)}[1]R_3^{7,k},
\end{aligned}$$

where the positive integer $k\geq 2$ and $R_j^{i,k}$, $i=0,1,2,3,4,5,6,7$, $j=1,2,3$, are also rational functions in F, G, H. In practice, the following iterative process leads to the determination of different parameters of v.

Estimation of the integer $m_i, \overline{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$.

Let $M_1, M_2, \cdots, M_\nu$ be the monomials of Q such that M_1 contains the highest order derivative of the function $v(\xi)$ and let M_2 be a nonlinear or linear monomial. M_2 is linear only in the case where all remaining monomials are linear. Substitute in (2.4) the expression (2.16) of $v(\xi)$ along with (2.17)-(2.19) and write the result in the form

$$\begin{aligned}
Q &= \frac{1}{T}\sum_{i=1}^{\nu}\left(K_i+F_{(1)}[1]L_{1,i}+G_{(1)}[1]L_{2,i}+H_{(1)}[1]L_{3,i}+F_{(1)}[1]G_{(1)}[1]S_{1,i}\right)\\
&+ \frac{1}{T}\sum_{i=1}^{\nu}\left(F_{(1)}[1]H_{(1)}[1]S_{2,i}+G_{(1)}[1]H_{(1)}[1]S_{3,i}+F_{(1)}[1]G_{(1)}[1]H_{(1)}[1]J_i\right), \qquad (2.20)
\end{aligned}$$

where K_i, $L_{j,i}$, $S_{j,i}$, J_i, T are polynomials in F,G,H whose the degrees linearly depend on m_i, $\overline{m}_i$, $\check{m}_{i,j}$, $\hat{m}_{i,j}$ and

$$\begin{aligned}
M_i &= \frac{1}{T}\left(K_i+F_{(1)}[1]L_{1,i}+G_{(1)}[1]L_{2,i}+H_{(1)}[1]L_{3,i}+F_{(1)}[1]G_{(1)}[1]S_{1,i}\right)\\
&+ \frac{1}{T}\left(F_{(1)}[1]H_{(1)}[1]S_{2,i}+G_{(1)}[1]H_{(1)}[1]S_{3,i}+F_{(1)}[1]G_{(1)}[1]H_{(1)}[1]J_i\right). \qquad (2.21)
\end{aligned}$$

Solve the system of linear algebraic equations obtained by balancing the degree of K_1 with that of K_2, the degree of $L_{j,1}$ with that of $L_{j,2}$, the degree of $S_{j,1}$ with that of $S_{j,1}$ and the degree of J_1 with that of J_2 in F, G and H to determine the values of the integers $m_i, \overline{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$.

Estimation of the constants $a_{i,j,k}, b_{i,j,k}^{l,1}, c_{i,j,k}^{l,1}, d_{i,j,k}, \alpha_i$.

Introduce into (2.20) the obtained values of $m_i, \overline{m}_i, \check{m}_{i,j}, \hat{m}_{i,j}$. Write the result in the form

$$\begin{aligned}
Q &= \frac{1}{T}\left(K+F_{(1)}[1]L_1+G_{(1)}[1]L_2+H_{(1)}[1]L_3+F_{(1)}[1]G_{(1)}[1]S_1\right)\\
&+ \frac{1}{T}\left(F_{(1)}[1]H_{(1)}[1]S_2+G_{(1)}[1]H_{(1)}[1]S_3+F_{(1)}[1]G_{(1)}[1]H_{(1)}[1]J\right). \qquad (2.22)
\end{aligned}$$

where $K=\sum_{i=1}^{\nu}K_i, \quad L_j=\sum_{i=1}^{\nu}L_{j,i}, \quad S_j=\sum_{i=1}^{\nu}S_{j,i}$ and $J=\sum_{i=1}^{\nu}J_i$.
Set to zero all coefficients of distinct monomials in K, L_j, S_j and J. This gives a system of algebraic equations whose the unknowns are the constants $a_{i,j,k}, b_{i,j,k}^{l,1}, c_{i,j,k}^{l,1}, d_{i,j,k}$ and α_i.

Remark 2.3. We have the (sn,cn,dn)-expansion method if we take $F(\xi)=sn(\xi)$, $G(\xi)=cn(\xi)$ and $H(\xi)=dn(\xi)$, where $sn(\xi)=sn(\xi,k)$, $cn(\xi)=cn(\xi,k)$ and $dn(\xi)=dn(\xi,k)$ with $0<k<1$ are the basis Jacobi elliptic functions [7]. The function sn, cn and dn are solutions of the first order ordinary differential equations

$$\left(w'(\xi)\right)^2=\left(1-w^2(\xi)\right)\left(1-k^2w^2(\xi)\right), \tag{2.23}$$

$$\left(w'(\xi)\right)^2=\left(1-w^2(\xi)\right)\left(k^2w^2(\xi)+1-k^2\right), \tag{2.24}$$

$$\left(w'(\xi)\right)^2=\left(1+w^2(\xi)\right)\left(w^2(\xi)+k^2-1\right), \tag{2.25}$$

respectively. When $k\to 0$ the Jacobi functions degenerate to the trigonometric functions, i.e.,

$$\lim_{k\to 0}sn(\xi,k)=\sin(\xi), \quad \lim_{k\to 0}cn(\xi,k)=\cos(\xi), \quad \lim_{k\to 0}dn(\xi,k)=1.$$

When $k\to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.,

$$\lim_{k\to 1}sn(\xi,k)=\tanh(\xi), \quad \lim_{k\to 1}cn(\xi,k)=\frac{1}{\cosh(\xi)}, \quad \lim_{k\to 1}dn(\xi,k)=\frac{1}{\cosh(\xi)}.$$

The function sn, cn and dn have the algebraic properties

$$sn^2(\xi,k)+cn^2(\xi,k)=1, \quad k^2sn^2(\xi,k)+dn^2(\xi,k)=1,$$

$$k^2cn^2(\xi,k)+1-k^2=dn^2(\xi,k), \quad cn^2(\xi,k)+\left(1-k^2\right)sn^2(\xi,k)=dn^2(\xi,k)$$

and the differential properties

$$\frac{d}{d\xi}sn(\xi,k)=cn(\xi,k)\,dn(\xi,k), \quad \frac{d}{d\xi}cn(\xi,k)=-sn(\xi,k)\,dn(\xi,k),$$

$$\frac{d}{d\xi}dn(\xi,k)=-k^2sn(\xi,k)\,cn(\xi,k).$$

3 Application to some relevant NLPDEs in physics

(A) Case 1: Burger-Fisher equation

Let us use the single function expansion method to analyze the Burger-Fisher equation [32]

$$u_{xx}+uu_x-u_t+u-u^2=0, \tag{3.1}$$

where $u=u(t,x)$. We first substitute into (3.1) the variables $u(t,x)=v(\xi)$, $\xi=\alpha t+\beta x$, with $\alpha,\beta\in\mathbb{R}$ to obtain

$$\beta^2v_{\xi\xi}+\beta vv_\xi-\alpha v_\xi+v-v^2=0. \tag{3.2}$$

We take the expansion (2.5) to look for the solution of the equation (3.2) with the assumption that the function $F = F(\xi)$ satisfies the auxiliary equation $F'(\xi) = 1 - F^2(\xi)$. Balancing the degree of the nonlinear term vv_ξ with that of the linear term $v_{\xi\xi}$ yields $m = \hat{m} = 1$. This allows to take the ansatze

$$v(\xi) = \sum_{i=-1}^{1} a_i F^i(\xi) + F'(\xi) \sum_{j=-1}^{1} b_j F^j(\xi), \tag{3.3}$$

where a_i, b_j are constants to determine. Here $F(\xi) = \tanh(\xi)$ and (3.3) becomes

$$v(\xi) = -b_1 \tanh^3(\xi) - b_0 \tanh^2(\xi) + (a_1 + b_1 - b_{-1}) \tanh(\xi) + a_0 + b_0 + \frac{a_{-1} + b_{-1}}{\tanh(\xi)}. \tag{3.4}$$

Substitute (3.4) into (3.2). Then, collect and set to zero the coefficients of each power of $\tanh^i(\xi)$. Solving the resulting system with respect to the parameters a_i, b_j, α, β, several solution sets are obtained leading to the following solutions for the equation (3.1): $\varepsilon \in \{-1, +1\}$

$$\begin{aligned}
u_1(t,x) &= 0, \\
u_2(t,x) &= 1, \\
u_3(t,x) &= \frac{1}{2} + \frac{\frac{\varepsilon}{2}}{\tanh\left(\frac{\varepsilon}{2}t\right)}, \\
u_4(t,x) &= \frac{1}{2} + \frac{\varepsilon}{2} \tanh\left(\frac{\varepsilon}{2}t\right), \\
u_5(t,x) &= \frac{1}{2} + \frac{\frac{\varepsilon}{2}}{\tanh\left[\frac{\varepsilon}{4}\left(\frac{5}{2}t + x\right)\right]}, \\
u_6(t,x) &= \frac{1}{2} + \frac{\varepsilon}{2} \tanh\left[\frac{\varepsilon}{4}\left(\frac{5}{2}t + x\right)\right], \\
u_7(t,x) &= \frac{1}{2} + \frac{\varepsilon}{4} \tanh\left(\frac{\varepsilon}{4}t\right) + \frac{\frac{\varepsilon}{4}}{\tanh\left(\frac{\varepsilon}{4}t\right)}, \\
u_8(t,x) &= \frac{1}{2} + \frac{\varepsilon}{4} \tanh\left[\frac{\varepsilon}{8}\left(\frac{5}{2}t + x\right)\right] + \frac{\frac{\varepsilon}{4}}{\tanh\left[\frac{\varepsilon}{8}\left(\frac{5}{2}t + x\right)\right]}.
\end{aligned}$$

(B) Case 2: Burger-Fisher equation

Consider now the double function expansion method for the analysis of the Burger-Fisher equation [32]

$$u_{xx} + uu_x - u_t + u - u^2 = 0, \tag{3.5}$$

where $u = u(t,x)$. We first substitute into (3.5) the variables $u(t,x) = v(\xi)$, $\xi = \alpha t + \beta x$, with $\alpha, \beta \in \mathbb{R}$ to obtain

$$\beta^2 v_{\xi\xi} + \beta v v_\xi - \alpha v_\xi + v - v^2 = 0. \tag{3.6}$$

We take the expansion (2.10) to look for the solution of the equation (3.6) assuming that the functions $F = F(\xi)$ and $G = G(\xi)$ satisfy the auxiliary equation $F'(\xi) = G(\xi)$ and $G'(\xi) = F(\xi)$. Balancing the degree of the nonlinear term vv_ξ with that of the linear term $v_{\xi\xi}$ yields

$m_1 = m_2 = \hat{m}_1 = \hat{m}_2 = \check{m}_1 = \check{m}_2 = \bar{m}_1 = \bar{m}_2 = 2$. This allows to take the ansatze

$$v = \sum_{i=-2}^{2} \sum_{j=-2}^{2} \left\{ a_{i,j} + b_{i,j} F_{(1)}[1] + c_{i,j} G_{(1)}[1] + d_{i,j} F_{(1)}[1] G_{(1)}[1] \right\} F^i G^j, \tag{3.7}$$

where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}$ are constants to determine. Note that $F(\xi) = \cosh(\xi)$ and $G(\xi) = \sinh(\xi)$. Substitute (3.7) into (3.6), collect and set to zero the coefficients of each power of $\cosh^i(\xi)\sinh^j(\xi)$. Solving the resulting system with respect to the parameters $a_{i,j}$, $b_{i,j}$, $c_{i,j}$, $d_{i,j}$, α, β, several solution sets are obtained leading, in addition to the solutions found in the Case 1, to the following solutions of equation (3.5), (a is an arbitrary constant):

$$\begin{aligned}
u_9(t,x) &= a\left[\sinh(2t+x) + \cosh(2t+x)\right], \\
u_{10}(t,x) &= 1 + a\left[\sinh(t+x) + \cosh(t+x)\right], \\
u_{11}(t,x) &= a\left[\sinh\left(t + \frac{1}{2}x\right) + \cosh\left(t + \frac{1}{2}x\right)\right]^2, \\
u_{12}(t,x) &= 1 + a\left[\sinh\left(\frac{1}{2}t + \frac{1}{2}x\right) + \cosh\left(\frac{1}{2}t + \frac{1}{2}x\right)\right]^2.
\end{aligned}$$

(C) Case 3: Fifth-order KdV equations

Consider the well known fifth-order KdV (fKdV) equations [32] in its standard form:

$$u_t + \sigma u^2 u_x + \delta u_x u_{2x} + \rho u u_{3x} + u_{5x} = 0, \tag{3.8}$$

where σ, δ, ρ are arbitrary nonzero real parameters and $u = u(t,x)$ is a sufficiently smooth function. A variety of the fKdV equations can be retrieved from this equation by changing the real values of the parameters σ, δ and ρ. However, five well known forms of the fKdV equations are of particular interest in the literature. There are:

(C1) The Sawada-Kotera (SK) equation [29] given by

$$u_t + 5u^2 u_x + 5u_x u_{2x} + 5u u_{3x} + u_{5x} = 0; \tag{3.9}$$

(C2) The Caudrey-Dodd-Gibbon equation [8] given by

$$u_t + 180u^2 u_x + 30u_x u_{2x} + 30u u_{3x} + u_{5x} = 0; \tag{3.10}$$

(C3) The Lax equation [23] provided by

$$u_t + 30u^2 u_x + 20u_x u_{2x} + 10u u_{3x} + u_{5x} = 0; \tag{3.11}$$

(C4) The Kaup-Kupersmidt (KK) equation [22] expressed as

$$u_t + 20u^2 u_x + 25u_x u_{2x} + 10u u_{3x} + u_{5x} = 0. \tag{3.12}$$

(C5) The Ito equation [19] written as

$$u_t + 2u^2 u_x + 6u_x u_{2x} + 3u u_{3x} + u_{5x} = 0. \tag{3.13}$$

It is important to note that there is another significant fifth-order equation that appears in the literature in the form

$$u_t + \sigma u u_x + \delta u_{3x} - \rho u_{5x} = 0, \tag{3.14}$$

where σ, δ, ρ are constants. This equation is called the Kawahara equation. The standard form of the Kawahara equation [20] is a fifth order KdV equation of the form

$$u_t + 6uu_x + u_{3x} - u_{5x} = 0, \tag{3.15}$$

that describes a model for plasma waves, capillary-gravity water waves. The Kawahara equation appears in the theory of shallow water with surface tension and in the theory of magneto acoustic waves in a cold collision free plasma. In [13] we have studied the Kawahara equation (3.14) using the symmetry group analysis method.

In the new variables $y = \alpha t + \beta x$ and $v(y) = u(t,x)$, (3.8) is reduced to the ordinary differential equation

$$\alpha v_y + \beta^5 v_{5y} + \beta^3 \rho v v_{3y} + \beta \sigma v^2 v_y + \beta^3 \delta v_y v_{2y} = 0 \tag{3.16}$$

while equation (3.14) is reduced to

$$\alpha v_y + \beta \sigma v v_y + \beta^3 \delta v_{3y} - \beta^5 \rho v_{5y} = 0. \tag{3.17}$$

The single function expansion method suggests to take the ansatze

$$v(y) = \sum_{i=-2}^{2} a_i F^i(y) + F'(y) \sum_{j=-2}^{2} b_j F^j(y) \tag{3.18}$$

for the equation (3.16) and the ansatze

$$v(y) = \sum_{i=-4}^{4} a_i F^i(y) + F'(y) \sum_{j=-4}^{4} b_j F^j(y) \tag{3.19}$$

for the equation (3.17), where a_i, b_j are constants to determine and F is a sufficiently smooth function. Here, we assume that the function $F = F(y)$ satisfies the auxiliary equation $F'(y) = 1 - F^2(y)$, i.e. $F(y) = \tanh(y)$. Thus, (3.18) becomes

$$\begin{aligned} v(y) &= -b_2 \tanh^4(y) - b_1 \tanh^3(y) + (a_2 - b_0 + b_2) \tanh^2(y) + (a_1 - b_{-1} + b_1) \tanh(y) \\ &+ \frac{a_{-1} + b_{-1}}{\tanh(y)} + \frac{a_{-2} + b_{-2}}{\tanh^2(y)} \end{aligned} \tag{3.20}$$

and (3.19) becomes

$$\begin{aligned} v(y) &= -b_4 \tanh^6(y) - b_3 \tanh^5(y) + (a_4 - b_2 + b_4) \tanh^4(y) + (a_3 - b_1 + b_3) \tanh^3(y) \\ &+ (a_2 - b_0 + b_2) \tanh^2(y) + a_0 - b_{-2} + b_0 + \frac{a_{-1} - b_{-3} + b_{-1}}{\tanh(y)} \\ &+ \frac{a_{-2} - b_{-4} + b_{-2}}{\tanh^2(y)} + \frac{a_{-3} + b_{-3}}{\tanh^3(y)} + \frac{a_{-4} + b_{-4}}{\tanh^4(y)}. \end{aligned} \tag{3.21}$$

Substitute (3.20) into (3.16) and (3.21) into (3.17). Then, collect and set to zero the coefficients of each power of $\tanh(y)$. Solving the resulting system of algebraic equations, several sets of parameters $\{a_i, b_j, \alpha, \beta\}$ are obtained leading to the following results:

(i) Analytical solutions to the Kawahara equation (3.15):

$$\begin{aligned}
u_1(t,x) &= a_0;\\
u_2(t,x) &= a_0 - \frac{35}{169}\left[\tanh^2(\varphi(t,x)) - \frac{1}{2}\tanh^4(\varphi(t,x))\right];\\
u_3(t,x) &= a_0 - \frac{35}{169}\left[\frac{1}{\tanh^2(\varphi(t,x))} - \frac{1}{2}\frac{1}{\tanh^4(\varphi(t,x))}\right];\\
u_4(t,x) &= a_0 + \frac{35}{5408}\left[\tanh^4(\psi(t,x)) + \frac{1}{\tanh^4(\psi(t,x))}\right]\\
&- \frac{35}{1352}\left[\tanh^2(\psi(t,x)) + \frac{1}{\tanh^2(\psi(t,x))}\right],
\end{aligned}$$

where $\varphi(t,x) = \frac{3\sqrt{13}(338a_0-23)}{4394}t - \frac{\sqrt{13}}{26}x$ and $\psi(t,x) = \frac{3\sqrt{13}(2704a_0-9)}{70304}t - \frac{\sqrt{13}}{52}x$.

(ii) Analytical solutions to the Sawada-Kotera equation (3.9):

$$\begin{aligned}
u_1(t,x) &= a_0;\\
u_2(t,x) &= 8\beta^2 - 12\beta^2\tanh^2(\varphi(t,x));\\
u_3(t,x) &= 8\beta^2 - 12\beta^2\frac{1}{\tanh^2(\varphi(t,x))};\\
u_4(t,x) &= a_0 - 6\beta^2\tanh^2\left(\overline{\varphi}(t,x)\right);\\
u_5(t,x) &= a_0 - 6\beta^2\frac{1}{\tanh^2\left(\overline{\varphi}(t,x)\right)};\\
u_6(t,x) &= 8\beta^2 - 12\beta^2\left[\tanh^2(\psi(t,x)) + \frac{1}{\tanh^2(\psi(t,x))}\right];\\
u_7(t,x) &= a_0 - 6\beta^2\left[\tanh^2\left(\overline{\psi}(t,x)\right) + \frac{1}{\tanh^2\left(\overline{\psi}(t,x)\right)}\right],
\end{aligned}$$

where

$$\varphi(t,x) = 16\beta^5 t - \beta x, \quad \psi(t,x) = 256\beta^5 t - \beta x,$$
$$\overline{\varphi}(t,x) = \beta\left(76\beta^4 + a_0^2 - 40\beta^2 a_0\right)t - \beta x, \quad \overline{\psi}(t,x) = \beta\left(16\beta^4 + 5a_0^2 - 40\beta^2 a_0\right)t - \beta x.$$

(iii) Analytical solutions to the Caudrey-Dodd-Gibbon equation (3.10):

$$\begin{aligned}
u_1(t,x) &= a_0;\\
u_2(t,x) &= \frac{4}{3}\beta^2 - 2\beta^2\tanh^2(\varphi(t,x));\\
u_3(t,x) &= \frac{4}{3}\beta^2 - 2\beta^2\frac{1}{\tanh^2(\varphi(t,x))};\\
u_4(t,x) &= a_0 - \beta^2\tanh^2\left(\psi(t,x)\right);\\
u_5(t,x) &= a_0 - \beta^2\frac{1}{\tanh^2\left(\psi(t,x)\right)};\\
u_6(t,x) &= \frac{4}{3}\beta^2 - 2\beta^2\left[\tanh^2\left(\overline{\varphi}(t,x)\right) + \frac{1}{\tanh^2\left(\overline{\varphi}(t,x)\right)}\right];\\
u_7(t,x) &= a_0 - \beta^2\left[\tanh^2\left(\overline{\psi}(t,x)\right) + \frac{1}{\tanh^2\left(\overline{\psi}(t,x)\right)}\right],
\end{aligned}$$

where

$$\varphi(t,x) = 16\beta^5 t - \beta x, \quad \overline{\varphi}(t,x) = 256\beta^5 t - \beta x,$$

$$\psi(t,x) = 4\beta\left(19\beta^4 + 45a_0^2 - 60\beta^2 a_0\right)t - \beta x, \quad \overline{\psi}(t,x) = 4\beta\left(4\beta^4 + 45a_0^2 - 60\beta^2 a_0\right)t - \beta x.$$

(iv) Analytical solutions to the Lax equation (3.11):

$$\begin{aligned}
u_1(t,x) &= a_0; \\
u_2(t,x) &= 4\beta^2 - 6\beta^2 \tanh^2(\varphi(t,x)); \\
u_3(t,x) &= 4\beta^2 - 6\beta^2 \frac{1}{\tanh^2(\varphi(t,x))}; \\
u_4(t,x) &= a_0 - 2\beta^2 \tanh^2(\psi(t,x)); \\
u_5(t,x) &= a_0 - 2\beta^2 \frac{1}{\tanh^2(\psi(t,x))}; \\
u_6(t,x) &= 4\beta^2 - 2\beta^2 \left[\tanh^2(\phi(t,x)) + 3\frac{1}{\tanh^2(\phi(t,x))}\right]; \\
u_7(t,x) &= 4\beta^2 - 2\beta^2 \left[3\tanh^2(\phi(t,x)) + \frac{1}{\tanh^2(\phi(t,x))}\right]; \\
u_8(t,x) &= 4\beta^2 - 6\beta^2 \left[\tanh^2(\overline{\varphi}(t,x)) + \frac{1}{\tanh^2(\overline{\varphi}(t,x))}\right]; \\
u_9(t,x) &= a_0 - 2\beta^2 \left[\tanh^2(\overline{\psi}(t,x)) + \frac{1}{\tanh^2(\overline{\psi}(t,x))}\right],
\end{aligned}$$

where

$$\varphi(t,x) = 56\beta^5 t - \beta x, \quad \overline{\varphi}(t,x) = 896\beta^5 t - \beta x, \quad \phi(t,x) = 336\beta^5 t - \beta x,$$

$$\psi(t,x) = 2\beta\left(28\beta^4 + 15a_0^2 - 40\beta^2 a_0\right)t - \beta x, \ \overline{\psi}(t,x) = 2\beta\left(48\beta^4 + 15a_0^2 - 40\beta^2 a_0\right)t - \beta x.$$

(v) Analytical solutions to the Kaup-Kupersmidt equation (3.12):

$$\begin{aligned}
u_1(t,x) &= a_0; \\
u_2(t,x) &= \beta^2 - \frac{3}{2}\beta^2 \tanh^2(\varphi(t,x)); \\
u_3(t,x) &= \beta^2 - \frac{3}{2}\beta^2 \frac{1}{\tanh^2(\varphi(t,x))}; \\
u_4(t,x) &= 8\beta^2 - 12\beta^2 \tanh^2(\psi(t,x)); \\
u_5(t,x) &= 8\beta^2 - 12\beta^2 \frac{1}{\tanh^2(\psi(t,x))}; \\
u_6(t,x) &= \beta^2 - \frac{3}{2}\beta^2 \left[\tanh^2(\overline{\varphi}(t,x)) + \frac{1}{\tanh^2(\overline{\varphi}(t,x))}\right]; \\
u_7(t,x) &= 8\beta^2 - 12\beta^2 \left[\tanh^2(\overline{\psi}(t,x)) + \frac{1}{\tanh^2(\overline{\psi}(t,x))}\right],
\end{aligned}$$

where

$$\varphi(t,x) = \beta^5 t - \beta x, \quad \overline{\varphi}(t,x) = 16\beta^5 t - \beta x, \quad \psi(t,x) = 176\beta^5 t - \beta x, \quad \overline{\psi}(t,x) = 2816\beta^5 t - \beta x.$$

(vi) Analytical solutions to the Ito equation (3.13):

$$
\begin{aligned}
u_1(t,x) &= a_0; \\
u_2(t,x) &= 4\beta^2 - 6\beta^2 \tanh^2(\beta x); \\
u_3(t,x) &= 4\beta^2 - 6\beta^2 \frac{1}{\tanh^2(\beta x)}; \\
u_4(t,x) &= 20\beta^2 - 30\beta^2 \tanh^2\left(96\beta^5 t - \beta x\right); \\
u_5(t,x) &= 20\beta^2 - 30\beta^2 \frac{1}{\tanh^2\left(96\beta^5 t - \beta x\right)}; \\
u_6(t,x) &= 4\beta^2 - 6\beta^2 \left[\tanh^2(\beta x) + \frac{1}{\tanh^2(\beta x)}\right]; \\
u_7(t,x) &= 20\beta^2 - 30\beta^2 \left[\tanh^2\left(1536\beta^5 t - \beta x\right) + \frac{1}{\tanh^2\left(1536\beta^5 t - \beta x\right)}\right].
\end{aligned}
$$

Finally, let us emphasize that all these methods involve cumbersome computations and hence do require the use of powerful computers. Except for this disadvantage, our analysis can be straightforwardly extended to multiple function expansion methods and to the investigation of systems of partial differential equations.

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