Ordering uniform supertrees by their spectral radii

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Abstract

A connected and acyclic hypergraph is called a supertree. In this paper we mainly focus on the spectral radii of uniform supertrees. Li, Shao and Qi determined the first two k-uniform supertrees with large spectral radii among all the k-uniform supertrees on n vertices [H. Li, J. Shao, L. Qi, The extremal spectral radii of k-uniform supertrees, arXiv:1405.7257v1, May 2014]. By applying the operation of moving edges on hypergraphs and using the weighted incidence matrix method we extend the above order to the fourth k-uniform supertree.

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1 Introduction

Let G be an ordinary graph, and $A(G)$ be its adjacency matrix. Denote by $\rho(G)$ the spectral radius of graph G, i.e., the largest eigenvalue of $A(G)$. As usual, denote by S_n, P_n the star on n vertices, the path on n vertices, respectively.

We will take some notation from [9] and [11]. We denote the set $\{1, 2, \cdots, n\}$ by $[n]$. Hypergraph is a natural generalization of an ordinary graph (see [1]). A hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ on n vertices is a set of vertices say $V(\mathcal{H}) = \{1, 2, \cdots, n\}$ and a set of edges, say $E(\mathcal{H}) = \{e_1, e_2, \cdots, e_m\}$, where $e_i = \{i_1, i_2, \cdots, i_l\}, i_j \in [n]$, $j = 1, 2, \cdots, l$. If $|e_i| = k$ for any $i = 1, 2, \cdots, m$, then $\mathcal{H}$ is called k-uniform hypergraph. A vertex v is said to be incident to an edge e if $v \in e$. The degree $d(i)$ of vertex i is defined as $d(i) = |\{e_j : i \in e_j \in E(\mathcal{H})\}|$. A vertex of degree one is called a pendent vertex. For a k-uniform hypergraph $\mathcal{H}$, an edge $e \in E(\mathcal{H})$ is called a pendent edge if e contains exactly $k-1$ pendent vertices.

An order k dimension n tensor $\mathcal{A} = (\mathcal{A}_{i_1 i_2 \cdots i_k}) \in \mathbb{C}^{n\times n\times\cdots\times n}$ is a multidimensional array with n^k entries, where $i_j \in [n]$ for each $j = 1, 2, \cdots, k$. To study the properties of uniform hypergraph by algebraic methods, adjacency matrix of an ordinary graph is naturally generalized to adjacency tenor (it is called adjacency hypermatrix in [5]) of a hypergraph (see [5] [16]).

Definition 1 *Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a k-uniform hypergraph on n vertices. The adjacency tensor of $\mathcal{H}$ is defined as the k-th order n-dimensional tensor $\mathcal{A}(\mathcal{H})$ whose $(i_1 \cdots i_k)$-entry is:*

$$\mathcal{A}(\mathcal{H})_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \{i_1, i_2, \cdots, i_k\} \in E(\mathcal{H}) \\ 0 & \textit{otherwise.} \end{cases}$$

The following general product of tensors, is defined in [17] by Shao, which is a generalization of the matrix case.

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Definition 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be order $m \geq 2$ and $k \geq 1$ dimension n tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ and dimension n with entries:*

$$\mathcal{C}_{i\alpha_1\cdots\alpha_{m-1}} = \sum_{i_2,\cdots,i_m\in[n]} \mathcal{A}_{ii_2\cdots i_m}\mathcal{B}_{i_2\alpha_1}\cdots\mathcal{B}_{i_m\alpha_{m-1}}. \tag{1}$$

Where $i \in [n], \alpha_1, \cdots, \alpha_{m-1} \in [n] \times \cdots \times [n]$.

Let $\mathcal{A}$ be an order k dimension n tensor, let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$ be a column vector of dimension n. Then by (1) $\mathcal{A}x$ is a vector in $\mathbb{C}^n$ whose ith component is as the following

$$(\mathcal{A}x)_i = \sum_{i_2,\cdots,i_k=1}^{n} \mathcal{A}_{ii_2\cdots i_k}x_{i_2}\cdots x_{i_k}.$$

Let $x^{[k]} = (x_1^k, \cdots, x_n^k)^T$. Then (see [2] [16]) a number $\lambda \in \mathbb{C}$ is called an eigenvalue of the tensor $\mathcal{A}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ satisfying the following eigenequations

$$\mathcal{A}x = \lambda x^{[k-1]},$$

and in this case, x is called an eigenvector of $\mathcal{A}$ corresponding to eigenvalue λ.

Let $\mathcal{A}$ be a kth-order n-dimensional nonnegative tensor. The *spectral radius* of $\mathcal{A}$ is defined as

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}.$$

In this paper we call $\rho(\mathcal{A}(\mathcal{H}))$ the spectral radius of uniform hypergraph $\mathcal{H}$, denoted by $\rho(\mathcal{H})$. For more details on the eigenvalues of a uniform hypergraph one can refer to [5] [7] and [14].

In [6], the weak irreducibility of nonnegative tensors was defined. It was proved in [6] and [18] that a k-uniform hypergraph $\mathcal{H}$ is connected if and only if its adjacency tensor $\mathcal{A}(\mathcal{H})$ is weakly irreducible.

Theorem 3 *[2] If $\mathcal{A}$ is a nonnegative tensor, then $\rho(\mathcal{A})$ is an eigenvalue with a nonnegative eigenvector x corresponding to it. If furthermore $\mathcal{A}$ is weakly irreducible, then x is positive, and for any eigenvalue λ with nonnegative eigenvector, $\lambda = \rho(\mathcal{A})$. Moreover, the nonnegative eigenvector is unique up to a constant multiple.*

By Theorem 3, for a kth-order weakly irreducible nonnegative tensor $\mathcal{A}$, it has a unique positive eigenvector x corresponding to $\rho(\mathcal{A})$ with $||x||_k = 1$ and it is called the principal eigenvector of $\mathcal{A}$ ([11]).

Definition 4 *[11] A supertree is a hypergraph which is both connected and acyclic.*

A characterization of acyclic hypergraph has been given in Berge's textbook [1], and we just state a version for uniform hypergraphs.

Proposition 5 *[1] If $\mathcal{H}$ is a connected k-uniform hypergraph with n vertices and m edges, then it is acyclic if and only if $m(k-1) = n-1$.*

The concept of *power hypergraphs* was introduced in [9]. Let $G = (V, E)$ be an ordinary graph. For every $k \geq 3$, the kth power of G, $G^k := (V^k, E^k)$ is defined as the k-uniform hypergraph with the edge set

$$E^k := \{e \cup \{i_{e,1}, \cdots, i_{e,k-2}\} \mid e \in E\}$$

and the vertex set

$$V^k := V \cup (\cup_{e\in E}\{i_{e,1}, \cdots, i_{e,k-2}\}).$$

The kth power of an ordinary tree was called a *k-uniform hypertree* ([9] [11]). The following observations are clear. Any k-uniform hypertree is a supertree. A k-uniform supertree $\mathcal{T}$ with at least two edges is a k-uniform hypertree if and only if each edge of $\mathcal{T}$ contains at most two non-pendent vertices.

The kth power of S_n, denoted by S_n^k, is called *hyperstar* in [9]. Let $S(a, b)$ be the tree on $a + b + 2$ vertices obtained from an edge e by attaching a pendent edges to one end vertex of e, and attaching b pendent edges to the other end vertex of e. Let $S^k(a, b)$ be the kth power of $S(a, b)$.

In [11], it was proved that the hyperstar $S_{n'}^k$ attains uniquely the maximum spectral radius among all k-uniform supertrees on n vertices, and $S^k(1, n' - 3)$ attains uniquely the second largest spectral radius among all k-uniform supertrees on n vertices (where $n' = \frac{n-1}{k-1} + 1$).

Suppose that $m = \frac{n-1}{k-1}$, now we introduce a special class of supertrees with m edges, which are not hypertrees. Let $1 \le t_1 \le t_2 \le t_3$ be three integers such that $t_1 + t_2 + t_3 = m - 1$. Denote by $\mathcal{T}(t_1, t_2, t_3)$ the k-uniform supertree containing exactly three non-pendent vertices, say u_1, u_2, u_3, incident to one edge, and $d(u_i) = t_i + 1$ holding for each $i = 1, 2, 3$.

In this paper, we will determine the third and the fourth k-uniform supertree with the large spectral radii among all k-uniform supertrees on n vertices.

Theorem 6 *Let $\mathcal{T}$ be a k-uniform supertree on n vertices (with $m = n' - 1$ edges, where $n' = \frac{n-1}{k-1} + 1 \ge 5$). Suppose that $\mathcal{T} \notin \{S_{n'}^k, S^k(1, n' - 3)\}$. Then we have*

$$\rho(\mathcal{T}) \le \rho(S^k(2, n' - 4)),$$

with equality holding if and if $\mathcal{T} \cong S^k(2, n' - 4)$.

Theorem 7 *Let $\mathcal{T}$ be a k-uniform supertree on n vertices (with $m = n' - 1$ edges, where $n' = \frac{n-1}{k-1} + 1 \ge 5$). Suppose that $\mathcal{T} \notin \{S_{n'}^k, S^k(1, n' - 3), S^k(2, n' - 4)\}$. Then we have*

$$\rho(\mathcal{T}) \le \rho(\mathcal{T}(1, 1, m - 3)),$$

with equality holding if and if $\mathcal{T} \cong \mathcal{T}(1, 1, m - 3)$.

The operation of moving edges on hypergraphs introduced by Li, Shao and Qi ([11]) and the weighted incidence matrix method introduced by Lu and Man ([13]) are crucial for our proofs. In Section 2 we will show them and other useful tools. In Section 3, we will give the proofs of our main results.

2 Several tools to compare spectral radii

A novel method (we call it weighted incidence matrix method) for computing (or comparing) the spectral radii of hypergraphs was raised by Lu and Man.

Definition 8 *[13] A weighted incidence matrix B of a hypergraph $\mathcal{H} = (V, E)$ is a $|V| \times |E|$ matrix such that for any vertex v and any edge e, the entry $B(v, e) > 0$ if $v \in e$ and $B(v, e) = 0$ if $v \notin e$.*

Definition 9 *[13] A hypergraph $\mathcal{H}$ is called α-normal if there exists a weighted incidence matrix B satisfying*

(1). $\sum_{e: v \in e} B(v, e) = 1$, *for any $v \in V(\mathcal{H})$.*

(2). $\prod_{v: v \in e} B(v, e) = \alpha$, *for any $e \in E(\mathcal{H})$.*

Moreover, the weighted incidence matrix B is called consistent if for any cycle $v_0 e_1 v_1 e_2 \cdots v_l (v_l = v_0)$

$$\prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$

Definition 10 *[13] A hypergraph $\mathcal{H}$ is called α-subnormal if there exists a weighted incidence matrix B satisfying*

(1). $\sum_{e: v \in e} B(v, e) \le 1$, *for any $v \in V(\mathcal{H})$.*

(2). $\prod_{v:v\in e} B(v,e) \geq \alpha$*, for any* $e \in E(\mathcal{H})$*.*
Moreover, $\mathcal{H}$ *is called strictly* α*-subnormal if it is* α*-subnormal but not* α*-normal.*

Definition 11 *[13] A hypergraph* $\mathcal{H}$ *is called* α*-supernormal if there exists a weighted incidence matrix* B *satisfying*
(1). $\sum_{e:v\in e} B(v,e) \geq 1$*, for any* $v \in V(\mathcal{H})$*.*
(2). $\prod_{v:v\in e} B(v,e) \leq \alpha$*, for any* $e \in E(\mathcal{H})$*.*
Moreover, $\mathcal{H}$ *is called strictly* α*-supernormal if it is* α*-supernormal but not* α*-normal.*

For a fixed k-uniform hypergraph $\mathcal{H}$, $\rho(\mathcal{H})$ defined here times constant factor $(k-1)!$ is the value of $\rho(\mathcal{H})$ defined in [13]. While this is not essential. Remembering this difference we modify Lemma 3 and Lemma 4 of [13] as the following Theorem 12.

Theorem 12 *[13] Let* $\mathcal{H}$ *be a* k*-uniform hypergraph.*
(1). If $\mathcal{H}$ *is strictly* α*-subnormal, then we have* $\rho(\mathcal{H}) < \alpha^{-\frac{1}{k}}$*.*
(2). If $\mathcal{H}$ *is strictly and consistently* α*-supernormal, then* $\rho(\mathcal{H}) > \alpha^{-\frac{1}{k}}$*.*

The following result reveals the numerical relationship between $\rho(G^k)$ and $\rho(G)$, where G^k is the k-th power of an ordinary graph G.

Theorem 13 *[19] Let* G^k *be the* k*th power of an ordinary graph* G*. Then we have*

$$\rho(G^k) = (\rho(G))^{\frac{2}{k}}.$$

Let F_n $(n \geq 5)$ be the tree obtained by coalescing the center of the star S_{n-4} and the center of the path P_5. Ordering the trees on n vertices according to their spectral radii was well studied in [10], [4] and [12]. We outline parts of the work in [10] as follows.

Theorem 14 *[10] Let* T *be a tree on* n *vertices* $(n \geq 5)$ *and* $T \notin \{S_n, S(1,n-3), S(2,n-4), F_n\}$*. Then we have*

$$\rho(S_n) > \rho(S(1,n-3)) > \rho(S(2,n-4)) > \rho(F_n) > \rho(T).$$

Combining Theorems 13 and 14, we have the following corollary.

Corollary 15 *Let* T^k *be the* k*th power of an ordinary tree* T*. Suppose that* T^k *has* n *vertices, and* $n' = \frac{n-1}{k-1} + 1 \geq 5$*. Suppose* $T \notin \{S_{n'}, S(1,n'-3), S(2,n'-4), F_{n'}\}$*, then we have*

$$\rho(S^k_{n'}) > \rho(S^k(1,n'-3)) > \rho(S^k(2,n'-4)) > \rho(F^k_{n'}) > \rho(T^k).$$

Definition 16 *[11] Let* $r \geq 1$*,* $\mathcal{G} = (V,E)$ *be a hypergraph with* $u \in V$ *and* $e_1, \cdots, e_r \in E$*, such that* $u \notin e_i$ *for* $i = 1, \cdots, r$*. Suppose that* $v_i \in e_i$ *and write* $e'_i = (e_i \backslash \{v_i\}) \cup \{u\} (i = 1, \cdots, r)$*. Let* $\mathcal{G}' = (V, E')$ *be the hypergraph with* $E' = (E \backslash \{e_1, \cdots, e_r\}) \cup \{e'_1, \cdots, e'_r\}$*. Then we say that* $\mathcal{G}'$ *is obtained from* $\mathcal{G}$ *by moving edges* $(e_1, \cdots, e_r)$ *from* $(v_1, \cdots, v_r)$ *to* u*.*

The effect on $\rho(\mathcal{G})$ of moving edges was studied by Li, Shao and Qi (see Theorem 17). The following fact was pointed out in [11]. If $\mathcal{G}$ is acyclic and there is an edge $e \in E(\mathcal{G})$ containing all the vertices $u, v_1, \cdots, v_r$, then the graph $\mathcal{G}'$ defined as above contains no multiple edges.

Theorem 17 *[11] Let* $r \geq 1$*,* $\mathcal{G}$ *be a connected hypergraph,* $\mathcal{G}'$ *be the hypergraph obtained from* $\mathcal{G}$ *by moving edges* $(e_1, \cdots, e_r)$ *from* $(v_1, \cdots, v_r)$ *to* u*, and* $\mathcal{G}'$ *contain no multiple edges. If* x *is the principal eigenvector of* $\mathcal{A}(\mathcal{G})$ *corresponding to* $\rho(\mathcal{G})$ *and suppose that* $x_u \geq max_{1\leq i\leq r}\{x_{v_i}\}$*, then* $\rho(\mathcal{G}') > \rho(\mathcal{G})$*.*

Denote by $N_2(\mathcal{T})$ the number of non-pendent vertices of $\mathcal{T}$. By using Theorem 17 (or modifying parts of the proof of Theorem 21 of [11]), we have the following observation.

Lemma 18 *Let $\mathcal{T}$ be a k-uniform supertree on n vertices with $N_2(\mathcal{T}) \geq 2$. Then there exists a k-uniform supertree $\mathcal{T}'$ on n vertices with $N_2(\mathcal{T}') = N_2(\mathcal{T}) - 1$ and $\rho(\mathcal{T}') > \rho(\mathcal{T})$.*

Lemma 19 *[11] Let a, b, c, d be nonnegative integers with $a + b = c + d$. Suppose that $a \leq b$, $c \leq d$ and $a < c$, then we have $\rho(S^k(a, b)) > \rho(S^k(c, d))$.*

Lemma 20 *Let $1 \leq t_1 \leq t_2 \leq t_3$ be three integers with $t_1 + t_2 + t_3 = m - 1$. Then we have*

$$\rho(\mathcal{T}(1, 1, m - 3)) \geq \rho(\mathcal{T}(t_1, t_2, t_3)),$$

with equality holding if and only if $t_2 = 1$.

Proof If $t_2 = 1$, the result is obvious. Now we suppose $t_2 > 1$, thus $t_3 > 1$. Let u_1, u_2 and u_3 be the (only) three non-pendent vertices of $\mathcal{T}(t_1, t_2, t_3)$ with $d(u_i) = t_i + 1$, $i = 1, 2, 3$. It is easy to see that u_i is incident to t_i pendent edges, $i = 1, 2, 3$. Let x be the principal eigenvector of $\mathcal{A}(\mathcal{T}(t_1, t_2, t_3))$ corresponding to $\rho(\mathcal{T}(t_1, t_2, t_3))$. Without loss of generality we suppose that $x_{u_3} = max_{1\leq i\leq 3}\{x_{u_i}\}$. Let $\mathcal{G}$ be obtained from $\mathcal{T}(t_1, t_2, t_3)$ by moving $t_1 - 1$ pendent edges from u_1 to u_3, and moving $t_2 - 1$ pendent edges from u_2 to u_3. Then $\mathcal{G}$ is isomorphic to $\mathcal{T}(1, 1, m - 3)$. Noting that $t_2 > 1$, by Theorem 17 we have $\rho(\mathcal{T}(1, 1, m - 3)) > \rho(\mathcal{T}(t_1, t_2, t_3))$. □

By Theorem 13 we know that $\rho(S^k(2, n' - 4))$ is determined by $\rho(S(2, n' - 4))$, and $\rho(F^k_{n'})$ is determined by $\rho(F_{n'})$. We will use the weighted incidence matrix method to compare $\rho(\mathcal{T}(1, 1, m - 3))$ with $\rho(S^k(2, n' - 4))$ and $\rho(F^k_{n'})$.

Lemma 21 *Suppose that $n' = \frac{n-1}{k-1} + 1$, $m = n' - 1 \geq 4$. We have*

$$\rho(S^k(2, n' - 4)) > \rho(\mathcal{T}(1, 1, m - 3)) > \rho(F^k_{n'}).$$

Proof Denote by u_1, u_2 and u_3 three non-pendent vertices of $\mathcal{T}(1, 1, m - 3)$. Label the m edges of $\mathcal{T}(1, 1, m - 3)$ as follows. The unique non-pendent edge (the edge containing u_1, u_2 and u_3) is numbered e_0, the pendent edge containing u_1 is numbered e_1, the pendent edge containing u_2 is numbered e_2, and the pendent edges containing u_3 are numbered $e_3, \cdots, e_{m-1}$. Now we construct an $n \times m$ matrix B. For any vertex v and any edge e of $\mathcal{T}(1, 1, m - 3)$, let $B(v, e) = 0$ if $v \notin e$. For any pendent vertex v in an edge e, let $B(v, e) = 1$. For the non-pendent vertices u_1, u_2 and u_3, let $B(u_1, e_1) = \alpha, B(u_1, e_0) = 1 - \alpha$; $B(u_2, e_2) = \alpha, B(u_2, e_0) = 1 - \alpha$; and let $B(u_3, e_i) = \alpha$, for $i = 3, \cdots, m - 1$, $B(u_3, e_0) = 1 - (m - 3)\alpha$. According to the above rules, we say that for any vertex v of $\mathcal{T}(1, 1, m - 3)$ we have

$$\sum_{e: v\in e} B(v, e) = 1. \tag{2}$$

For the pendent edge e_i $(i = 1, 2, \cdots, m - 1)$, we have

$$\prod_{v: v\in e_i} B(v, e_i) = \alpha. \tag{3}$$

For the unique non-pendent edge e_0 we have

$$\prod_{v: v\in e_0} B(v, e_0) = (1 - \alpha)^2[1 - (m - 3)\alpha],$$

and then

$$\prod_{v: v\in e_0} B(v, e_0) - \alpha = -(m - 3)\alpha^3 + (2m - 5)\alpha^2 - m\alpha + 1. \tag{4}$$

(1). Write $\rho = \rho(S(2, n' - 4))$ for short. By Theorem 13, we have $\rho(S^k(2, n' - 4)) = \rho^{\frac{2}{k}}$. It is easy to check that the tree $S(2, n' - 4)$ contains m edges and the value ρ satisfies

$$\rho^4 - m\rho^2 + 2(m - 3) = 0. \tag{5}$$

As we all know that

$$\rho > \sqrt{\Delta(S(2, n' - 4))} = \sqrt{n' - 3} = \sqrt{m - 2},$$

where $\Delta(S(2, n' - 4))$ is the maximum degree of the tree $S(2, n' - 4)$.

Take $\alpha = \frac{1}{\rho^2}$. Then $\alpha < \frac{1}{m-2}$ and

$$1 - \alpha \geq 1 - (m - 3)\alpha > 1 - \frac{m - 3}{m - 2} > 0.$$

So $B(v, e) > 0$ for any vertex v and any edge e of $\mathcal{T}(1, 1, m - 3)$ when $v \in e$, i.e., the matrix B is a weighted incidence matrix of $\mathcal{T}(1, 1, m - 3)$ according to Definition 8. Now we will show $\mathcal{T}(1, 1, m - 3)$ is strictly α-subnormal with $\alpha = \frac{1}{\rho^2}$. Combining (2) and (3), we only need to show $\prod\limits_{v: v \in e_0} B(v, e_0) > \alpha$. In fact by (4) and (5) we have

$$\begin{aligned}
\prod_{v: v \in e_0} B(v, e_0) - \alpha &= -(m - 3)\alpha^3 + (2m - 5)\alpha^2 - m\alpha + 1 \\
&= \frac{1}{\rho^6}[\rho^6 - m\rho^4 + (2m - 5)\rho^2 - (m - 3)] \\
&= \frac{1}{\rho^6}[\rho^2 - (m - 3)] \\
&> 0.
\end{aligned}$$

So for the unique non-pendent edge e_0 we have

$$\prod_{v: v \in e_0} B(v, e_0) > \alpha. \tag{6}$$

By (1) of Theorem 12, we have

$$\rho(\mathcal{T}(1, 1, m - 3)) < \alpha^{-\frac{1}{k}} = \rho^{\frac{2}{k}} = \rho(S^k(2, n' - 4)).$$

(2). Write $\rho = \rho(F_{n'})$ for short. By Theorem 13, we have $\rho(F^k_{n'}) = \rho^{\frac{2}{k}}$. It is easy to see that the tree $F_{n'}$ contains m edges and the value ρ satisfies

$$\rho^4 - (m - 1)\rho^2 + (m - 4) = 0, \tag{7}$$

and

$$\rho > \sqrt{\Delta(F_{n'})} = \sqrt{n' - 3} = \sqrt{m - 2},$$

where $\Delta(F_{n'})$ is the maximum degree of the tree $F_{n'}$.

Take $\alpha = \frac{1}{\rho^2}$. Then $\alpha < \frac{1}{m-2}$ and

$$1 - \alpha \geq 1 - (m - 3)\alpha > 1 - \frac{m - 3}{m - 2} > 0.$$

So $B(v, e) > 0$ for any vertex v and any edge e of $\mathcal{T}(1, 1, m - 3)$ when $v \in e$, i.e., the matrix B is a weighted incidence matrix of the supertree $\mathcal{T}(1, 1, m - 3)$. Now we will show $\mathcal{T}(1, 1, m - 3)$ is strictly

α-supernormal with $\alpha = \frac{1}{\rho^2}$. Combining (2) and (3), we only need to show $\prod_{v:v\in e_0} B(v,e_0) < \alpha$. In fact by (4) and (7) we have

$$\begin{aligned}\prod_{v:v\in e_0} B(v,e_0) - \alpha &= -(m-3)\alpha^3 + (2m-5)\alpha^2 - m\alpha + 1\\ &= \frac{1}{\rho^6}[\rho^6 - m\rho^4 + (2m-5)\rho^2 - (m-3)]\\ &= \frac{1}{\rho^6}[-\rho^4 + (m-1)\rho^2 - (m-3)]\\ &= -\frac{1}{\rho^6}\\ &< 0.\end{aligned}$$

So for the unique non-pendent edge e_0 we have

$$\prod_{v:v\in e_0} B(v,e_0) < \alpha. \tag{8}$$

Clearly, the weighted incidence matrix B of $\mathcal{T}(1,1,m-3)$ is consistent, since the supertree $\mathcal{T}(1,1,m-3)$ is acyclic. By (2) of Theorem 12, we have

$$\rho(\mathcal{T}(1,1,m-3)) > \alpha^{-\frac{1}{k}} = \rho^{\frac{2}{k}} = \rho(F^k_{n'}).$$

The proof is complete. □

3 The proofs of the main results

Suppose that $n' = \frac{n-1}{k-1} + 1$, and $m = n' - 1$. Recall that $N_2(\mathcal{T})$ is the number of non-pendent vertices of a supertree $\mathcal{T}$. For a k-uinform supertree $\mathcal{T}$ on n vertices we have the following observations.

(1). $N_2(\mathcal{T}) = 1$ if and only if $\mathcal{T} \cong S^k_{n'}$;

(2). $N_2(\mathcal{T}) = 2$ if and only if $\mathcal{T} \cong S^k(a,b)$ for some integers a, b, where $b \geq a \geq 1$ and $a + b = n' - 2$;

(3.1). $N_2(\mathcal{T}) = 3$ and three non-pendent vertices incident to one edge if and only if $\mathcal{T} \cong \mathcal{T}(t_1,t_2,t_3)$ for some interges t_1, t_2, t_3, where $t_1 + t_2 + t_3 = m - 1$.

(3.2). $N_2(\mathcal{T}) = 3$ and three non-pendent vertices not incident to one edge, if and only if $\mathcal{T} \cong T^k$ for some ordinary tree T and T containing three non-pendent vertices.

Proof of Theorem 6 Since $\mathcal{T} \ncong S^k_{n'}$, we have $N_2(\mathcal{T}) \geq 2$.

If $N_2(\mathcal{T}) = 2$, then $\mathcal{T} \cong S^k(a,b)$ for some integers a, b, where $b \geq a \geq 1$ and $a + b = n' - 2$. Since $\mathcal{T} \ncong S^k(1, n'-3)$, by Lemma 19, we have

$$\rho(\mathcal{T}) \leq \rho(S^k(2, n'-4)),$$

with equality holding if and if $\mathcal{T} \cong S^k(2, n'-4)$.

If $N_2(\mathcal{T}) = 3$ and $\mathcal{T} \cong \mathcal{T}(t_1,t_2,t_3)$, then combining Lemmas 20 and 21 we have

$$\rho(\mathcal{T}) \leq \rho(\mathcal{T}(1,1,m-3)) < \rho(S^k(2, n'-4)).$$

If $N_2(\mathcal{T}) = 3$ and $\mathcal{T} \cong T^k$ for some ordinary tree T, then T contains three non-pendent vertices and then $T \notin \{S_{n'}, S(a,b)\}$. From Corollary 15, we have

$$\rho(\mathcal{T}) < \rho(S^k(2, n'-4)).$$

If $N_2(\mathcal{T}) \geq 4$, then there exists a k-uinform supertree $\mathcal{T}'$ with $N_2(\mathcal{T}') = 3$ and $\rho(\mathcal{T}') > \rho(\mathcal{T})$ by Lemma 18. Thus we have

$$\rho(\mathcal{T}) < \rho(\mathcal{T}') < \rho(S^k(2, n' - 4)).$$

The proof is complete. □

Proof of Theorem 7 Since $\mathcal{T} \ncong S^k_{n'}$, we have $N_2(\mathcal{T}) \geq 2$.

If $N_2(\mathcal{T}) = 2$, then $\mathcal{T} \cong S^k(a, b)$ for some interges a, b, where $b \geq a \geq 1$, and $a + b = n' - 2$. Since Since $\mathcal{T} \notin \{S^k(1, n' - 3), S^k(2, n' - 4)\}$, by Lemma 19, Corollary 15 and Lemma 21, we have

$$\rho(\mathcal{T}) \leq \rho(S^k(3, n' - 5)) < \rho(F^k_{n'}) < \rho(\mathcal{T}(1, 1, m - 3)).$$

If $N_2(\mathcal{T}) = 3$ and $\mathcal{T} \cong \mathcal{T}(t_1, t_2, t_3)$, then from Lemma 20 we have

$$\rho(\mathcal{T}) \leq \rho(\mathcal{T}(1, 1, m - 3)),$$

with equality holding if and only if $\mathcal{T} \cong \mathcal{T}(1, 1, m - 3)$.

If $N_2(\mathcal{T}) = 3$ and $\mathcal{T} \cong T^k$, then $T \notin \{S_{n'}, S(a, b)\}$. From Corollary 15, Lemma 21 we have

$$\rho(\mathcal{T}) \leq \rho(F^k_{n'}) < \rho(\mathcal{T}(1, 1, m - 3)).$$

If $N_2(\mathcal{T}) \geq 4$, then there exists a k-uinform supertree $\mathcal{T}'$ with $N_2(\mathcal{T}') = 3$ and $\rho(\mathcal{T}') > \rho(\mathcal{T})$ by Lemma 18. Thus we have

$$\rho(\mathcal{T}) < \rho(\mathcal{T}') \leq \rho(\mathcal{T}(1, 1, m - 3)).$$

The proof is complete. □

References

[1] C. Berge, Hypergraph. Combinatorics of Finite sets, third edition, North-Holland, Amsterdam, 1973.

[2] K.C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6(2008)507-520.

[3] K.C. Chang, L. Qi and T. Zhang, A survey on the spectral theory of nonnegative tensors, Numer. Linear Alg. Appl., 20 (2013) 891-912.

[4] A. Chang, Q. Huang, Ordering trees by their largest eigenvalues, Linear Algebra Appl. 370 (2003) 175-184.

[5] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, Linear Algebra Appl. 436 (2012) 3268-3292.

[6] S. Friedland, A. Gaubert, L. Han, Perron-Frobenius theorems for nonnegative multilinear forms and extensions, Linear Algebra Appl.,438 (2013) 738-749.

[7] S. Hu, Z. Huang, C. Ling, L. Qi, On determinants and eigenvalue theory of tensors, J. Symbolic Comput. 50 (2013) 508-531.

[8] S. Hu, L. Qi, J. Shao, Cored hypergraphs and their Laplacian eigenvalues, Linear Algebra Appl. 439 (2013) 2980-2998.

[9] S. Hu, L. Qi and J. Shao, Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues, Linear Alg. Appl., 439 (2013) 2980-2998.

[10] M. Hofmeister, On the two largest eigenvalues of trees, Linear Algebra Appl. 260 (1997) 43-59.

[11] H. Li, Y.Shao, L. Qi, The extermal spectral radii of k-uniform supertrees, arXiv:1405.7257v1, May 2014.

[12] W. Lin, X. Guo, Ordering trees by their largest eigenvalues, Linear Algebra Appl. 418 (2006) 450-456.

[13] L. Lu, S. Man, Connected hypergraphs with small spectral radius, arXiv:1402.5402v3, Mar 2014.

[14] V. Nikiforov, Analytic methods for uniform hypergraphs, Linear Algebra Appl. 457 (2014) 455-535.

[15] L. Qi, Eigenvalues of a real supersymmetric tensor, Journal of Symbolic Computation 40 (2005) 1302-1324.

[16] L. Qi, H^+-eigenvalue of Laplacian and signless Laplacian tensors, Commun. Math. Sci. 12(2014)1045-1064.

[17] Y. Shao, A general product of tensors with applications, Linear Algebra Appl. 439 (2013) 2350-2366.

[18] Y. Yang, Q. Yang, On some properties of nonegative weakly irreducible tensors, arXiv: 1111.0713 v3, 2011.

[19] J. Zhou, L. Sun, W. Wang, C. Bu, Some spectral properties of uniform hypergraphs, arXiv: 1407.5193 v1, Jul. 2014.