

## Hausdorff Dimension and Diophantine Approximation

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### 1. Introduction

Dimension provides an indication of the size and complexity of a set and various kinds, such as box counting, packing and Hausdorff dimensions, play an important role in the study of fractals [50]. For example, the Hausdorff dimension of the Cantor middle third set is  $\log 2 / \log 3$  (proved by Hausdorff in his seminal paper [60]), that of the Koch snowflake curve is  $\log 4 / \log 3$  and it has recently been shown that the boundary of the Mandelbrot set, a very complicated set of Lebesgue measure 0 or *null* set in the complex plane with topological dimension 1, has Hausdorff dimension 2 [122]. On the other hand, Diophantine approximation is a quantitative analysis of rational approximation and so, at least at first sight, is less geometrical. The purpose of this article is to show that Hausdorff dimension plays an important part in this theory too.

In order to keep the article accessible, the emphasis is on approximation of real numbers by rationals and the less well known topic of approximation of complex numbers by ratios of Gaussian integers. The more general theory, which recently has seen some spectacular advances, will be referred to and some applications sketched. The article is organised as follows. We begin with a brief treatment of Hausdorff measure and Hausdorff dimension. We then explain some of the principal results in Diophantine approximation and the Hausdorff dimension of related sets, originating in the pioneering work of Vojtěch Jarník [98]. We conclude with some applications of these results to the metrical structure of exceptional sets associated with some famous problems. It is not intended that all the recent developments be covered but they can be found in the references cited.

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## 2. Hausdorff measure and dimension

Felix Hausdorff introduced the notion of Hausdorff dimension in a remarkable and influential paper [60] that extended Carathéodory's approach to Lebesgue measure [24] in a simple but far-reaching way. (Kahane's Foreword to the book *Fractals* [29] includes a short and moving biography of Hausdorff.) Dimension had been taken to be a non-negative integer but by a simple observation, which Hausdorff described modestly as 'a small contribution', he modified Carathéodory's definition of measure to obtain a measure associated with a dimension that could be any non-negative real number. We shall assume a knowledge of Lebesgue measure and as usual, we shall often say *almost no* to indicate a null set – thus almost no numbers are rational – and we shall say *almost all* to indicate a set whose complement is null, so that almost all numbers are irrational.

For familiar sets such as the interval, circle and the plane, the Hausdorff dimension (defined below) coincides with the usual notion of dimension and is respectively 1, 1 and 2. However, a significant difference is that *any* set in Euclidean space has a Hausdorff dimension (a non-measurable set in  $\mathbb{R}^n$  has full Hausdorff dimension  $n$ ). In particular, null sets, such as Cantor's middle third set or the set of badly approximable numbers (see §3.2), have a Hausdorff dimension and this gives a way of discriminating between them. It is also natural to study the Hausdorff dimension of *exceptional sets* which are sets associated with the invalidity of some result, making it desirable that they be null. A brief and more or less self-contained account of Hausdorff measure and dimension is now given (more detailed expositions can be found in [16, 49, 50, 52, 89, 106]).

**2.1. Hausdorff measure.** Carathéodory's approach to the measure of a set  $E$  in  $\mathbb{R}^n$  was based on 'approximating'  $E$  by countable covers consisting of small 'simple' sets  $U$  in  $\mathbb{R}^n$ . Hausdorff's idea was to introduce for a given cover,  $\mathcal{C}$  say, of  $E$  the sum (sometimes termed the *s-length* of the cover  $\mathcal{C}$ )

$$\ell^s(\mathcal{C}) := \sum_{U \in \mathcal{C}} (\text{diam } U)^s,$$

where  $\text{diam } U = \sup\{|\mathbf{x} - \mathbf{y}|_2 : \mathbf{x}, \mathbf{y} \in U\}$  is the diameter of  $U$  ( $|\mathbf{x} - \mathbf{y}|_2$  is the usual Euclidean distance between  $\mathbf{x}$  and  $\mathbf{y}$ ) and where  $s$  is a non-negative real number that is not necessarily an integer. Hausdorff considered a monotonic function  $l$  which allows more discrimination but for simplicity we shall stick to the more familiar widely used special case  $l(\text{diam } U) = (\text{diam } U)^s$ , associated with what is now usually called the Hausdorff dimension but also sometimes called the Hausdorff-Besicovitch dimension [106]. The possibly infinite number  $\ell^s(\mathcal{C})$  gives an indication of the number of subsets  $U$  in  $\mathcal{C}$  needed to cover  $E$ . In order to effect the approximation, the diameter of the sets  $U$  in the cover is restricted to be at most  $\delta > 0$ .

Let

$$\mathcal{H}_\delta^s(E) := \inf_{\mathcal{C}_\delta} \sum_{U \in \mathcal{C}_\delta} (\text{diam } U)^s = \inf \ell^s(\mathcal{C}_\delta),$$

where the infimum is taken over all covers  $\mathcal{C}_\delta$  of  $E$  by sets  $U$  with  $\text{diam } U \leq \delta$ ; such covers are called  $\delta$ -covers. For a point  $\mathbf{x}$ ,  $\mathcal{H}_\delta^s(\{\mathbf{x}\}) = 1$  when  $s = 0$  and vanishes when  $s > 0$ . As  $\delta$  decreases,  $\mathcal{H}_\delta^s$  can only increase as there are fewer  $U$ 's available, *i.e.*, if  $0 < \delta < \delta'$ , then

$$\mathcal{H}_{\delta'}^s(E) \leq \mathcal{H}_\delta^s(E).$$

The set function  $\mathcal{H}_\delta^s$  is an outer measure on  $\mathbb{R}^n$  but the limit  $\mathcal{H}^s$  (which can be infinite) as  $\delta \rightarrow 0$ , given by

$$(2.1) \quad \mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E) = \sup_{\delta > 0} \mathcal{H}_\delta^s(E) \in [0, \infty],$$

is better behaved. From its construction by covers,  $\mathcal{H}^s(E) \leq \mathcal{H}^s(F)$  for any  $E \subset F$  and indeed is subadditive and a regular outer measure. The restriction to the  $\sigma$ -field of  $\mathcal{H}^s$  measurable sets (which includes open and closed sets, limsup and liminf sets and  $G_\delta$  and  $F_\sigma$  sets) is usually called the *Hausdorff  $s$ -dimensional measure*. Hausdorff 1-dimensional measure coincides with 1-dimensional Lebesgue measure and in higher dimensions, Hausdorff  $n$ -dimensional measure is comparable to  $n$ -dimensional Lebesgue measure, *i.e.*,

$$\mathcal{H}^n(E) \asymp |E|,$$

where  $|E|$  is the Lebesgue measure of  $E$  and where for  $a, b > 0$ ,  $a \asymp b$  means there exist constants  $c, c' > 0$  such that  $a \leq cb \leq c'a$  or  $a = O(b)$ ,  $b = O(a)$  in Landau's  $O$ -notation. Thus a set of positive  $n$ -dimensional Lebesgue measure has positive Hausdorff  $n$ -measure.

Because it is defined in terms of the diameter of the covering sets, Hausdorff  $s$ -measure is unchanged by restriction to closed, convex or open sets. It is also unchanged by isometries and so in particular by translations and rotations. It is, however, affected by scaling in the natural way (as are fractals): for any  $r \geq 0$ ,

$$\mathcal{H}^s(rE) = r^s \mathcal{H}^s(E).$$

**2.2. Hausdorff dimension.** Zero-dimensional Hausdorff measure  $\mathcal{H}^0(E)$  is simply counting measure; thus the Hausdorff  $s$ -measure of a set of  $k$  points is  $k$  when  $s = 0$  and 0 for  $s > 0$ . This pattern is typical. When the set  $E$  is infinite,  $\mathcal{H}^s(E)$  is either 0 or  $\infty$ , except for possibly one value of  $s$ . To see this, the definition of  $\mathcal{H}_\delta^s(E)$  implies that there is a  $\delta$ -cover  $\mathcal{C}_\delta$  of  $E$  such that

$$\sum_{C \in \mathcal{C}_\delta} (\text{diam } C)^s \leq \mathcal{H}_\delta^s(E) + 1 \leq \mathcal{H}^s(E) + 1 \leq \infty.$$

Suppose that  $\mathcal{H}^{s_0}(E)$  is finite and  $s = s_0 + \varepsilon$ ,  $\varepsilon > 0$ . Then for each member  $C$  of the cover  $\mathcal{C}_\delta$ ,  $(\text{diam } C)^{s_0 + \varepsilon} \leq \delta^\varepsilon (\text{diam } C)^{s_0}$ , so that the sum

$$\sum_{C \in \mathcal{C}_\delta} (\text{diam } C)^{s_0 + \varepsilon} \leq \delta^\varepsilon \sum_{C \in \mathcal{C}_\delta} (\text{diam } C)^{s_0}.$$

Hence

$$\mathcal{H}_\delta^{s_0 + \varepsilon}(E) \leq \sum_{C \in \mathcal{C}_\delta} (\text{diam } C)^{s_0 + \varepsilon} \leq \delta^\varepsilon \sum_{C \in \mathcal{C}_\delta} (\text{diam } C)^{s_0} \leq \delta^\varepsilon (\mathcal{H}^{s_0}(E) + 1)$$

and so

$$0 \leq \mathcal{H}^s(E) = \mathcal{H}^{s_0 + \varepsilon}(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^{s_0 + \varepsilon}(E) \leq \lim_{\delta \rightarrow 0} \delta^\varepsilon (\mathcal{H}^{s_0}(E) + 1) = 0.$$

On the other hand suppose  $\mathcal{H}^{s_0}(E) > 0$ . If for any  $\varepsilon > 0$ ,  $\mathcal{H}^{s_0 - \varepsilon}(E)$  were finite, then by the above  $\mathcal{H}^{s_0}(E) = 0$ , a contradiction, whence  $\mathcal{H}^{s_0 - \varepsilon}(E) = \infty$ .

Thus for each infinite set  $E$  in  $n$ -dimensional Euclidean space, there exists a unique non-negative exponent  $s_0$  such that

$$\mathcal{H}^s(E) = \begin{cases} \infty, & 0 \leq s < s_0, \\ 0, & s_0 < s < \infty, \end{cases}$$

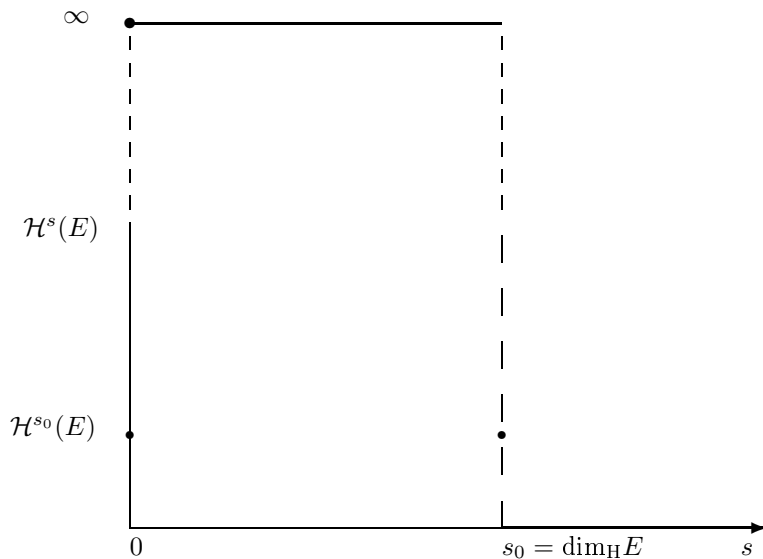


FIGURE 1. The graph of the Hausdorff measure of  $E$ . The Hausdorff dimension  $\dim_{\mathbb{H}} E = s_0$ , the point of discontinuity.

as shown in Figure 1 (reproduced with the permission of the Cambridge University Press from [16]). The critical exponent

$$(2.2) \quad s_0 = \inf\{s \in [0, \infty) : \mathcal{H}^s(E) = 0\}$$

where the Hausdorff  $s$ -measure crashes is called the *Hausdorff dimension* of the set  $E$  and is denoted by  $\dim_{\mathbb{H}} E$ . Thus the Hausdorff dimension of a finite set is 0, as it is for a countable set. It is clear that

$$(2.3) \quad \text{if } \mathcal{H}^s(E) = 0 \text{ then } \dim_{\mathbb{H}} E \leq s; \text{ and if } \mathcal{H}^s(E) > 0 \text{ then } \dim_{\mathbb{H}} E \geq s.$$

The Hausdorff dimension tells us nothing about the Hausdorff  $s$ -measure at the critical exponent  $s_0 = \dim_{\mathbb{H}} E$ , only that this is the appropriate exponent to investigate the measure. The sudden change in Hausdorff  $s$ -measure at  $s_0 = \dim_{\mathbb{H}} E$  can be compared to the focal length of a microscope. If the lens is too close, the image fills the eyepiece and cannot be resolved; if the lens is too far away, the image is invisible. At the focal length, the image is in focus and can be resolved.

The main properties of Hausdorff dimension for sets in  $\mathbb{R}^n$  are

- (i) If  $E \subseteq F$  then  $\dim_{\mathbb{H}} E \leq \dim_{\mathbb{H}} F$ .
- (ii)  $\dim_{\mathbb{H}} E \leq n$ .
- (iii) If  $|E| > 0$ , then  $\dim_{\mathbb{H}} E = n$ .
- (iv) The dimension of a point is 0.
- (v) If  $\dim_{\mathbb{H}} E < n$ , then  $|E| = 0$  (however  $\dim_{\mathbb{H}} E = n$  does not imply  $|E| > 0$ ).
- (vi)  $\dim_{\mathbb{H}}(E_1 \times E_2) \geq \dim_{\mathbb{H}} E_1 + \dim_{\mathbb{H}} E_2$
- (vii)  $\dim_{\mathbb{H}} \cup_{j=1}^{\infty} E_j = \sup\{\dim_{\mathbb{H}} E_j : j \in \mathbb{N}\}$ .

It can be shown that the Hausdorff dimension of any countable set is 0 and that of any open set in  $\mathbb{R}^n$  is  $n$  [50, p. 29]. The nature of the construction of Hausdorff measure ensures that the Hausdorff dimension of a set is unchanged by an invertible

transformation which is bi-Lipschitz. This implies that for any set  $S \subseteq \mathbb{R} \setminus \{0\}$ ,  $\dim_{\mathbb{H}} S^{-1} = \dim_{\mathbb{H}} S$ , where  $S^{-1} = \{s^{-1} : s \in S\}$ . Thus on the whole, Hausdorff dimension behaves as a dimension should, although the natural formula

$$\dim_{\mathbb{H}}(E_1 \times E_2) = \dim_{\mathbb{H}} E_1 + \dim_{\mathbb{H}} E_2$$

does *not* always hold [49, §5.3] (it does hold for certain sets, *e.g.*, cylinders, such as  $E \times I$ , where  $I$  is an interval:  $\dim_{\mathbb{H}}(E \times I) = \dim_{\mathbb{H}} E + \dim_{\mathbb{H}} I = \dim_{\mathbb{H}} E + 1$  by (iii), see [16]).

The general character of  $\delta$ -covers in the definition of Hausdorff outer measure can be difficult to work with and for many applications in higher dimensions, it is convenient to restrict the elements in the  $\delta$ -covers of a set to simpler sets such as balls or cubes. For example, covers consisting of hypercubes

$$H = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}|_{\infty} < \delta\},$$

where  $|\mathbf{x}|_{\infty} := \max\{|x_j| : 1 \leq j \leq n\}$  is the *height* of  $\mathbf{x} \in \mathbb{R}^n$ , centred at  $\mathbf{a} \in \mathbb{R}^n$  and with sides of length  $2\delta$  are used extensively. While outer measures corresponding to these more convenient restricted covers are not the same as Hausdorff measure, they are comparable and so have the same critical exponent [89, Chapter 5]. Thus there is no loss as far as dimension is concerned if the sets  $U$  are chosen to be balls or hypercubes. Of course, the two measures are identical for sets with Hausdorff  $s$ -measure which is either 0 or  $\infty$ . Such sets are said to obey a ‘0- $\infty$ ’ law, this being the appropriate analogue of the more familiar ‘0-1’ law in probability [76, p 339]. Sets which do not satisfy a 0- $\infty$  law, *i.e.*, sets which satisfy

$$(2.4) \quad 0 < \mathcal{H}^{\dim_{\mathbb{H}} E}(E) < \infty,$$

are called *s-sets*; these occur surprisingly often [50, p. 29] and enjoy some nice properties (see Chapters 2–4 of [49]). One example is the Cantor set which has Hausdorff  $s$ -measure 1 when  $s = \log 2 / \log 3$  [49, p. 14]. However it seems that  $s$ -sets are of less interest in Diophantine approximation where the sets that arise naturally, such as the set of badly approximable numbers or the set of numbers approximable to a given order (see next section), obey a 0- $\infty$  law. The first steps in this direction were taken by Jarník, who proved that the Hausdorff  $s$ -measure of set of numbers rationally approximable to order  $v$  (see §3.2) was 0 or  $\infty$  [67, 68]. This result turns on an idea related to density of Hausdorff measure.

LEMMA 2.1. *Let  $E$  be a null set in  $\mathbb{R}$ . Suppose that for any interval  $(a, b)$  and  $s \in [0, 1]$ ,*

$$(2.5) \quad \mathcal{H}^s(E \cap (a, b)) \leq K(b - a)\mathcal{H}^s(E).$$

*Then  $\mathcal{H}^s(E) = 0$  or  $\infty$ .*

PROOF. Suppose the contrary, *i.e.*, suppose  $0 < \mathcal{H}^s(E) < \infty$ . Since  $E$  is null, given  $\varepsilon > 0$ , there exists a cover of  $E$  by open intervals  $(a_j, b_j)$  such that

$$\sum_j (b_j - a_j) < \varepsilon.$$

By (2.5), there exists a constant  $K > 0$  such that

$$0 < \mathcal{H}^s(E) = \mathcal{H}^s(\cup_j (a_j, b_j) \cap E) \leq K\mathcal{H}^s(E) \sum_j (b_j - a_j) < K\varepsilon\mathcal{H}^s(E) < \mathcal{H}^s(E)$$

for  $\varepsilon < 1/K$ , a contradiction. □

The proof for a general outer measure is essentially the same. The sets we encounter in Diophantine approximation are generally not  $s$ -sets and some satisfy this ‘quasi-independence’ property.

For other definitions of dimension, such as box-counting and packing dimension, and their relationship with Hausdorff dimension, see [50, Chapter 3].

**2.3. The determination of Hausdorff dimension.** Unless some general result is available, the Hausdorff dimension  $\dim_{\mathbb{H}} E$  of a null set  $E$  is usually determined in two steps, with the correct upward inequality  $\dim_{\mathbb{H}} E \leq s_0$  and downward inequality  $\dim_{\mathbb{H}} E \geq s_0$  being established separately.

2.3.1. *The upper bound.* In view of (2.3), an upper bound can be obtained by finding a value of  $s$  for which  $\mathcal{H}^s(E)$  vanishes. To find such a value, it suffices to exhibit a cover  $\{H\}$  of  $E$  ( $E \subseteq \cup H$ ) by hypercubes  $H$  of arbitrarily small sidelength and  $s$ -length. This can often be done by adapting the estimate involved in showing that Lebesgue measure is 0. When  $E$  is a limsup set, *i.e.*,

$$E = \limsup_{N \rightarrow \infty} E_N = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_k = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in E_k \text{ for infinitely many } k \in \mathbb{N}\}$$

for a sequence of sets  $E_n$ , a simple Hausdorff measure counterpart of the convergence case of the Borel-Cantelli lemma often gives the correct upper bound for  $\dim_{\mathbb{H}} E$ . This is useful in Diophantine approximation.

LEMMA 2.2. *Let*

$$E = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in E_k \text{ for infinitely many } k \in \mathbb{N}\}.$$

*If for some  $s > 0$ ,*

$$(2.6) \quad \sum_{k=1}^{\infty} \text{diam}(E_k)^s < \infty,$$

*then  $\mathcal{H}^s(E) = 0$  and  $\dim_{\mathbb{H}} E \leq s$ .*

PROOF. From the definition, for each  $N = 1, 2, \dots$ ,

$$E \subseteq \bigcup_{k=N}^{\infty} E_k,$$

so that the family  $\mathcal{C}^{(N)} = \{E_k : k \geq N\}$  is a cover for  $E$ . By (2.6),

$$\lim_{N \rightarrow \infty} \sum_{k=N}^{\infty} \text{diam}(E_k)^s = 0.$$

Hence  $\lim_{k \rightarrow \infty} \text{diam}(E_k) = 0$  and therefore given  $\delta > 0$ ,  $\mathcal{C}^{(N)}$  is a  $\delta$ -cover of  $E$  for  $N$  sufficiently large. But

$$\mathcal{H}_{\delta}^s(E) = \inf_{\mathcal{C}_{\delta}} \sum_{U \in \mathcal{C}_{\delta}} (\text{diam } U)^s \leq \ell^s(\mathcal{C}^{(N)}) = \sum_{k=N}^{\infty} \text{diam}(E_k)^s \rightarrow 0$$

as  $N \rightarrow \infty$ . Thus  $\mathcal{H}_{\delta}^s(E) = 0$  and by (2.1),  $\mathcal{H}^s(E) = 0$ , whence  $\dim_{\mathbb{H}} E \leq s$ .  $\square$

2.3.2. *The lower bound.* The lower bound is often harder (though by no means always, see [39]). It requires showing that given *any*  $s < s_0$  and *any* cover  $\{C\}$  of  $E$  with the diameters of the covering elements arbitrarily small, the  $s$ -length  $\sum_C (\text{diam } C)^s \geq \delta$  for some positive  $\delta$ . This can be very difficult and has led to the development of a variety of methods. In Diophantine approximation, the *regular systems* introduced by Baker and W. M. Schmidt [10] and the more general *ubiquitous systems* [45] depend on a good supply of approximating elements (*e.g.*, the rationals). These and related techniques have proved effective in obtaining lower bounds for the Hausdorff dimension of sets of number theoretic interest (see the survey article [13] for more details). A more fundamental approach is the so-called mass distribution principle.

LEMMA 2.3. *Let  $\mu$  be a measure supported on a bounded Borel set  $E$  in  $\mathbb{R}^n$ . Suppose that for some  $s \geq 0$ , there are strictly positive constants  $c$  and  $\delta$  such that  $\mu(B) \leq c(\text{diam } B)^s$  for any ball  $B$  in  $\mathbb{R}^n$  with  $\text{diam } B \leq \delta$ . Then  $\mathcal{H}^s(E) \geq \mu(E)/c$ .*

PROOF. The proof is short. Let  $\{B_k\}$  be a  $\delta$ -cover of  $E$  by balls  $B_k$ . Then

$$\mu(E) \leq \mu\left(\bigcup_k B_k\right) \leq \sum_k \mu(B_k) \leq c \sum_k (\text{diam } B)^s.$$

Taking infima over all such covers, we see that  $\mathcal{H}_\delta^s(E) \geq \mu(E)/c$ , whence on letting  $\delta \rightarrow 0$ ,

$$\mathcal{H}^s(E) \geq \mu(E)/c > 0.$$

□

This simple lemma is surprisingly useful and gives the easy part of Frostman's lemma [56] which is now stated in full. The Vinogradov notation  $a \ll b$  for  $a, b > 0$  means that  $a = O(b)$ .

LEMMA 2.4. *Let  $E$  be a Borel subset of  $\mathbb{R}^n$ . Then*

$$\mathcal{H}^s(E) > 0$$

*if and only if there exists a measure  $\mu$  on  $\mathbb{R}^n$  supported on  $E$  with  $\mu(E)$  finite such that  $\mu(B) \ll (\text{diam } B)^s$  for all sufficiently small balls  $B$ .*

Thus if  $E$  supports a probability measure  $\mu$  ( $\mu(E) = 1$ ) with  $\mu(B) \ll (\text{diam } B)^s$  for all sufficiently small balls  $B$ , then  $\dim_{\text{H}} E \geq s$ . The converse is more difficult but can be proved using net measures (see [25, 50, 89]).

### 3. Diophantine approximation

At its simplest level, Diophantine approximation is concerned with approximating real numbers by rationals. Hardy and Wright's classic *Introduction to the theory of numbers* [57] contains an excellent account while the more advanced [26, 119] are devoted wholly to Diophantine approximation. The theory extends to approximating vectors in  $\mathbb{R}^n$  (simultaneous Diophantine approximation) and to matrices (systems of linear forms). For simplicity, we will stick mainly to one particular direction in the one dimensional real and complex cases and treat the extensions to higher dimensions and other settings fairly briefly. Since the rationals  $\mathbb{Q}$  are a

dense subset of the real numbers  $\mathbb{R}$ , given any real number  $\alpha$  and any positive  $\varepsilon$ , there exists a rational  $p/q$  such that

$$(3.1) \quad \left| \alpha - \frac{p}{q} \right| < \varepsilon.$$

The numerator  $p$  is often of no interest and the size of the expression

$$(3.2) \quad \|q\alpha\| = \min\{|q\alpha - p| : p \in \mathbb{Z}\},$$

the distance of  $q\alpha$  from the integers  $\mathbb{Z}$ , is considered. Although convenient, it will not be used much here in order to keep the notational burden to a minimum.

In simultaneous Diophantine approximation, one considers the system of  $n$  inequalities

$$\left| \alpha_k - \frac{p_k}{q} \right| < \varepsilon, \quad k = 1, \dots, n.$$

This system can be expressed more concisely as a single vector inequality with  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ ,  $\mathbf{p} \in \mathbb{Z}^n$ ,  $q \in \mathbb{N}$ , by considering the expression

$$\left| \boldsymbol{\alpha} - \frac{\mathbf{p}}{q} \right|_{\infty},$$

where for  $\mathbf{x} \in \mathbb{R}^n$ ,  $|\mathbf{x}|_{\infty} = \max\{|x_1|, \dots, |x_n|\}$  is the *height* of  $\mathbf{x}$ , or, on multiplying by  $q$ , the expression

$$|q\boldsymbol{\alpha} - \mathbf{p}|_{\infty} = \|q\boldsymbol{\alpha}\| = \max\{\|q\alpha_j\| : j = 1, \dots, n\}.$$

The last inequality has a *dual* or linear form version: given  $\boldsymbol{\alpha} \in \mathbb{R}^n$  and  $\varepsilon > 0$ , one considers the inequality

$$|\mathbf{q} \cdot \boldsymbol{\alpha} - p| < \varepsilon,$$

where  $\mathbf{q} \in \mathbb{Z}^n$  and  $p \in \mathbb{Z}$ . The last two inequalities can be combined into a single general one. The system of  $n$  real linear forms

$$\xi_1 a_{1j} + \dots + \xi_m a_{mj}, \quad j = 1, \dots, n,$$

in  $m$  real variables  $\xi_1, \dots, \xi_m$ , can be written more concisely as  $\boldsymbol{\xi}A$ , where  $A = (a_{ij})$  and the system of  $n$  inequalities in  $m$  variables

$$|q_1 a_{11} + \dots + q_m a_{m1} - p_1| < \varepsilon$$

$$\vdots$$

$$|q_1 a_{1n} + \dots + q_m a_{mn} - p_n| < \varepsilon,$$

can be written  $|\mathbf{q}A - \mathbf{p}|_{\infty} < \varepsilon$ . Further details are in [26, 57, 119]. The theory extends naturally to the fields of  $p$ -adic numbers [27, 87] and formal power series [82, 84]. Less obviously, it also extends to discrete groups acting on hyperbolic space. This is relevant to Diophantine approximation over the Gaussian integers or rationals considered below in §4.3, so an outline is now given.

The hyperbolic space setting sprang from the observations that the the real axis is the set of limit points of the rationals and that the rationals can be characterised as the parabolic vertices of the modular group  $\mathrm{SL}(2, \mathbb{Z})$ , *i.e.*, as the orbit of the point at infinity under the linear fractional or Möbius transformations

$$(3.3) \quad z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1,$$



of the extended upper half plane  $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} : y \geq 0\} \cup \{\infty\}$ . For each element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in  $\mathrm{SL}(2, \mathbb{Z})$ , the point  $\infty$  in the extended real line evidently goes to  $a/c$  under the group action and is also a fixed point of the map  $z \mapsto z + 1$ . The maps  $g$  form the *modular* group,  $\mathrm{SL}(2, \mathbb{Z})$ , a discrete subgroup of  $\mathrm{SL}(2, \mathbb{R})$ , which is essentially (modulo the centre) the group of orientation preserving Möbius transformations of the upper half plane  $\mathbb{H}^2$  to itself. When the upper half plane is endowed with the hyperbolic metric derived from  $d\rho = |dz|/y$ , it is a model for two dimensional hyperbolic space  $(\mathbb{H}^2, \rho)$ . The Möbius group  $M(\mathbb{H}^2)$  is the group of isometries of  $(\mathbb{H}^2, \rho)$ . Because the group  $\mathrm{SL}(2, \mathbb{Z})$  is discrete, points in the orbit can accumulate only on the boundary  $\mathbb{R} \cup \{\infty\}$  of  $\mathbb{H}^2$  and because the group elements are isometries with respect to the hyperbolic metric, the limit set of any orbit is the extended real line  $\mathbb{R} \cup \{\infty\}$ . A discrete subgroup of  $M(\mathbb{H}^2)$  is called a *Fuchsian* group. Further details of this rich and beautiful theory are in [3, 4, 11, 95, 99] and there is a short survey in Chapter 7 of [16].

These observations allow the classical theory, including the metrical theory, of Diophantine approximation to be translated into Fuchsian groups acting on the hyperbolic plane and to the much more general setting of *Kleinian* groups acting on  $(n+1)$ -dimensional hyperbolic space  $(\mathbb{H}^{n+1}, \rho)$ ,  $n \geq 2$  (Kleinian groups are the discrete subgroups of the Möbius group  $M(\mathbb{H}^{n+1})$  of isometries of  $\mathbb{H}^{n+1}$ ; further details are in the references given). The *Picard* group  $\mathrm{SL}(2, \mathbb{Z}[i])$  consists of  $2 \times 2$  matrices over  $\mathbb{Z}[i]$  with determinant 1 and has an action on  $\mathbb{C}$  given by

$$(3.4) \quad z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{Z}[i], \quad ad - bc = 1.$$

The limit set of the Picard group is the extended complex plane (or Riemann sphere)  $\mathbb{C} \cup \{\infty\}$  and the orbit of the point at  $\infty$  under the group is the set of ratios of Gaussian integers [91]. Thus the Picard group plays a role precisely parallel to that played by the modular group, expressed by (3.3), in approximating real numbers by ratios of integers.

In the literature cited, hyperbolic space is usually taken in the equivalent Poincaré form of the open unit ball  $\mathbb{B}^{n+1} = \{\mathbf{x} \in \mathbb{R}^{n+1} : |\mathbf{x}|_2 < 1\}$ , where the ball is now endowed with the equivalent hyperbolic metric  $\rho$  given by  $d\rho = |d\mathbf{x}|_2 / (1 - |\mathbf{x}|_2^2)$ . To ease comparison, we shall adopt this viewpoint, even though the upper half plane model, as used by Sullivan in [130], is more natural for Diophantine approximation. We choose to consider the  $(n+1)$ -dimensional hyperbolic space, as the results in Diophantine approximation are results about the boundary of  $\mathbb{H}^{n+1}$ , which is  $n$ -dimensional.

The analogue  $\mathfrak{p}$  of the point at infinity for Kleinian groups is not quite straightforward. First of all the nature of the elements  $g$  of the Kleinian group  $G$  implies that each  $g$  has at most two fixed points on the boundary of the ball. The special point  $\mathfrak{p}$  is called a *parabolic* fixed point if it is the unique fixed point on the boundary of some element in  $G$ ; otherwise they are called *hyperbolic* fixed points. The orbit of a special point  $\mathfrak{p}$  under the action of a Kleinian group  $G$  corresponds to the rationals  $\mathbb{Q}$ . The limit set  $\Lambda(G)$  of the orbit under  $G$  of a point in  $\mathbb{H}^{n+1}$  lies in the

boundary  $\mathbb{S}^n$ . Given  $\alpha \in \Lambda(G) \subseteq \mathbb{S}^n$ , one considers the quantity

$$|\alpha - g(\mathbf{p})|_2,$$

where  $|\cdot|_2$  is the usual Euclidean metric in  $\mathbb{R}^{n+1}$ . Analogues of the principal theorems in Diophantine approximation have been obtained with relatively minor technical restrictions and will be discussed below; a brief survey is in Chapter 7 of [16]. There is a striking dynamical interpretation of the approximation in terms of flows on the associated quotient space  $\mathbb{H}^{n+1}/G$ ; more details are in [16, 53, 91, 130, 135]. We now return to the one-dimensional theory.

**3.1. Dirichlet's theorem.** It is not difficult to make (3.1) more precise: given any real number  $\alpha$  and any positive integer  $q$ , there exists an integer  $p$  such that  $|q\alpha - p| < 1$ , and indeed such that

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{2q}.$$

There are denominators  $q$  for which more can be said by using Dirichlet's celebrated 'box argument' (see [26, 57]).

**THEOREM 3.1.** *For each real number  $\alpha$  and any positive integer  $N \geq 1$ , there exists a rational  $p/q$  with denominator satisfying  $1 \leq q \leq N$ , such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{qN} \leq \frac{1}{q^2}.$$

**PROOF.** Let  $[\alpha]$  be the integer part of  $\alpha$  and  $\{\alpha\}$  its fractional part, so that  $\alpha = [\alpha] + \{\alpha\}$ . Divide the interval  $[0, 1)$  into  $N$  subintervals  $[k/N, (k+1)/N)$ , where  $k = 0, 1, \dots, N-1$ , of length  $1/N$ . The  $N+1$  numbers  $\{r\alpha\}$ ,  $r = 0, 1, \dots, N$ , fall into the interval  $[0, 1)$  and so two,  $\{r\alpha\}$ ,  $\{r'\alpha\}$  say, must fall into the same subinterval,  $[k/N, (k+1)/N)$  say. Suppose that  $r > r'$ . Then

$$|\{r\alpha\} - \{r'\alpha\}| = |r\alpha - [r\alpha] - r'\alpha + [r'\alpha]| = |q\alpha - p| < \frac{1}{N},$$

where  $q = r - r'$ ,  $p = [r\alpha] - [r'\alpha] \in \mathbb{Z}$  and  $1 \leq q \leq N$ . Dividing by  $q$  gives the quantitative inequality

$$(3.5) \quad \left| \alpha - \frac{p}{q} \right| < \frac{1}{qN}$$

and since  $1 \leq q \leq N$ , the final inequality is immediate.  $\square$

A nice sharpening is in [58, p. 1]. When  $p, q$  are restricted to having highest common factor 1, the inequality

$$(3.6) \quad \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$$

has only finitely many solutions if and only if  $\alpha$  is a rational. Thus almost all real numbers satisfy (3.6) for infinitely many rationals  $p/q$ . Without this restriction, (3.6) holds infinitely often for all  $\alpha \in \mathbb{R}$ .

Dirichlet's theorem is one of the fundamental results in the theory of Diophantine approximation. It can be viewed as a result about covers and plays a central part in the Jarník–Besicovitch theorem, discussed below. The theorem generalises to the simultaneous Diophantine approximation of  $n$  real numbers  $\alpha_1, \dots, \alpha_n$  [57,

Theorem 200] and asserts that given  $N \in \mathbb{N}$ , there exists  $q \in \mathbb{N}$ , with  $q \leq N$  and  $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{Z}^n$  such that

$$\left| \boldsymbol{\alpha} - \frac{\mathbf{p}}{q} \right|_{\infty} < \frac{1}{qN^{1/n}}.$$

In particular, for simultaneous Diophantine approximation in the plane, given any  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$  and  $N \in \mathbb{N}$ , there exists  $q \in \mathbb{N}$  with  $q \leq N$  and  $p_1, p_2 \in \mathbb{Z}$  such that

$$(3.7) \quad \max \left\{ \left| \alpha_1 - \frac{p_1}{q} \right|, \left| \alpha_2 - \frac{p_2}{q} \right| \right\} < \frac{1}{qN^{1/2}} \leq q^{-3/2}.$$

There is a so-called *dual* version: given  $\boldsymbol{\alpha} \in \mathbb{R}^n$  and  $N \in \mathbb{N}$ , there exists  $\mathbf{q} \in \mathbb{Z}^n$  and a  $p \in \mathbb{Z}$  such that

$$|\mathbf{q} \cdot \boldsymbol{\alpha} - p| < N^{-n}.$$

These can be combined into a result for systems of  $n$  real linear forms [26, Chapter 1, Theorem VI]:

$$|\mathbf{q}A - \mathbf{p}|_{\infty} < N^{-m/n}.$$

In the setting of a Kleinian group acting on hyperbolic space, the analogue of the denominator in Theorem 3.1 corresponding to  $g(\mathbf{p})$  is defined to be

$$(3.8) \quad \lambda_g := |\det(Dg|_0)|^{-1} = \frac{1}{2} \cosh \rho(0, g(0)) \asymp e^{\rho(0, g(0))}$$

in the ball model, and so  $\lambda_g \rightarrow \infty$  as  $|g(0)| \rightarrow 1$ , *i.e.*, as the orbit of the origin moves towards the boundary. Here  $Dg|_0$  denotes the Jacobian of  $g$  evaluated at the origin. For finitely generated Fuchsian groups of the first kind taken to be acting on the closed unit disc  $\Delta$ , the elements  $g$  are of the form

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \Delta, |a|^2 - |b|^2 = 1$$

and  $\lambda_g = 2(|a|^2 + |b|^2)$ . Hedlund's lemma, which is a classical result in the theory of discontinuous groups, is a partial analogue of Dirichlet's theorem. Let  $\zeta \in \mathbb{S}^1$ , the unit circle. For any  $\xi \in \mathbb{S}^1$  which is not a parabolic point, there exist infinitely many  $g \in G$  such that

$$|\xi - g(\zeta)| < \frac{C}{\lambda_g},$$

for some  $C > 0$ . A complete analogue of Dirichlet's theorem, including the quantitative inequality (3.5), was obtained by Patterson [99] for Fuchsian groups and later he and others extended it to Kleinian groups [100, 128, 129, 135]. However, the statements in the Kleinian group setting differ for parabolic and hyperbolic fixed points and so for simplicity the result will be stated when  $G$  has a unique parabolic point  $\mathbf{p}$ . Let  $N \geq 2$ . Then for any  $\xi \in \Lambda(G)$ , there exists a  $g \in G$  with  $\lambda_g < N$  such that

$$|\xi - g(\mathbf{p})|_2 < \frac{C}{\sqrt{N\lambda_g}},$$

where  $C$  is a constant depending only on  $G$ .

**3.2. Types of approximation.** The equation (3.6) given by Dirichlet's theorem is essentially best possible as Hurwitz [64] showed that each  $\alpha \in \mathbb{R}$  satisfied the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}$$

for infinitely many positive integers  $q$  and that this was best possible in the sense that the constant  $1/\sqrt{5}$  cannot be reduced for numbers  $\alpha$  equivalent to the golden ratio  $(\sqrt{5} + 1)/2$  [26, 55, 57, 119].

3.2.1. *Badly approximable numbers.* A number  $\beta \in \mathbb{R}$  is called *badly approximable* or of *constant type* if there exists a  $K = K(\beta)$  such that

$$(3.9) \quad \left| \beta - \frac{p}{q} \right| \geq \frac{K}{q^2}$$

for all  $p/q \in \mathbb{Q}$ . Using (3.2), (3.9) can be written as  $q\|q\beta\| \geq K$ . In view of Hurwitz's theorem, the constant  $K < 1/\sqrt{5}$ . Quadratic irrationals, such as  $\sqrt{2}$  and the golden ratio  $(\sqrt{5} + 1)/2$ , are badly approximable. This is proved in [57, §11.4] using the fact that the partial quotients in the continued fraction expansion for a quadratic irrational are periodic. However, the proof relies only on the boundedness of the partial quotients, which therefore characterises the badly approximable numbers. The set of badly approximable numbers will be denoted by  $\mathfrak{B}$ . The notion extends naturally to higher dimensions and to the more general settings mentioned above.

Badly approximable numbers are important in applications, particularly in stability questions for certain dynamical systems [121]. For example the 'noble' numbers, which are equivalent to the golden ratio, have been conjectured to be the most robust in the breaking up of invariant tori [88]. One very practical application involved the design of rocket casings. These were made using ruled surfaces and vibrations from the motors were propagated along the generators. To reduce the effects of resonance and delay the onset of catastrophic vibration, the ratio of the circumference to the length of the casing was chosen to be a quadratic irrational (V. I. Arnol'd, personal communication). The desirable properties of badly approximable numbers (and in particular of the golden ratio) appear to be related to their occurrence in nature. It has recently been discovered that the ratio between two step heights on the surface of certain quasi-crystals is given by the golden ratio (see [30] for statements of this result and additional examples).

The notion of badly approximable numbers carries over to higher dimensions, including systems of linear forms [118],  $p$ -adics [1] and fields of formal power series [82] as well as to Kleinian groups acting on hyperbolic space [20, 53, 99, 101]. In the hyperbolic space setting, a point  $\beta$  in  $\Lambda(G)$  is said to be badly approximable with respect to  $\mathfrak{p}$  if there exists a positive constant  $K = K(\beta)$  such that

$$|\beta - g(\mathfrak{p})|_2 \geq K/\lambda_g$$

for all  $g \in G$ .

3.2.2. *Diophantine type.* The concept of a badly approximable number has extensions to restricted classes of real numbers and points in  $\mathbb{R}^n$  that are useful in connection with stability and other questions (see §7) and fortunately enjoys full measure. Let  $K > 0$ ,  $v > 1$ . The real number  $\alpha$  is said to be of *Diophantine type*

$(K, v)$  [6] (the definition has been altered slightly for consistency) if

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{K}{q^{v+1}}$$

for all rationals  $p/q$ ; the set of numbers of Diophantine type  $(K, v)$  is denoted by  $\mathcal{D}(K, v)$ . The union

$$\mathcal{D}_v = \bigcup_{K>0} \mathcal{D}(K, v)$$

consists of numbers of Diophantine type  $v$  (i.e., of type  $(K, v)$  for some  $K > 0$ ) and the union

$$\mathcal{D} = \bigcup_{v>1} \mathcal{D}_v = \bigcup_{K>0, v>1} \mathcal{D}(K, v)$$

is the set of numbers of Diophantine type  $v$  for some  $v > 1$ . Note that  $\mathcal{D}_1 = \mathfrak{B}$ , the set of badly approximable numbers. We will see in §4 that  $\mathfrak{B}$  is null but that when  $v > 1$ ,  $\mathcal{D}_v$  has full measure, which is pleasing since points of Diophantine type have desirable approximation properties for certain applications. Again, the notion of Diophantine type extends naturally to higher dimensions [46]. We will be particularly interested in the case of a single linear form and accordingly we extend the definition of Diophantine type for a real number to a point in  $\mathbb{R}^n$ . A point  $\beta \in \mathbb{R}^n$  is of (dual) Diophantine type  $(K, v)$  if for all  $p \in \mathbb{Z}$  and non-zero  $\mathbf{q} \in \mathbb{Z}^n$ ,

$$|\mathbf{q} \cdot \beta - p| \geq \frac{K}{|\mathbf{q}|_v^v}.$$

**3.2.3. Well approximable numbers.** In applications, we will be interested in points  $\alpha$  which are not of Diophantine type  $(K, v)$  for any  $K > 0$ , i.e., in one dimension with the set

(3.10)

$$E_v = \left\{ \alpha \in \mathbb{R} : \text{for any } K > 0, \left| \alpha - \frac{p}{q} \right| < \frac{K}{q^{v+1}} \text{ for some } \frac{p}{q} \in \mathbb{Q} \right\} = \mathbb{R} \setminus \mathcal{D}_v,$$

the complement of  $\mathcal{D}_v$ . These numbers are closely related to numbers which are *rationally approximable to order  $v + 1$*  [57, §11.4], i.e., to the set

(3.11)

$$R_v = \left\{ \alpha \in \mathbb{R} : \text{for some } K > 0, \left| \alpha - \frac{p}{q} \right| < \frac{K}{q^{v+1}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}$$

(the exponent  $v + 1$  in the denominator is a normalisation to keep the notation the same as elsewhere.) By Dirichlet's theorem, all real numbers are rationally approximable to order 2 and quadratic irrationals are rationally approximable to order exactly 2.

The constant  $K$  in the definitions of  $R_v$  and  $E_v$  is of less significance than the exponent, which motivates the next definition. A number  $\alpha$  which satisfies

(3.12)

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{v+1}}$$

for infinitely many  $p/q \in \mathbb{Q}$  will be called *v-approximable*; if  $v > 1$ ,  $\alpha$  is called *very well approximable*. Thus Liouville numbers, which satisfy (3.12) for any  $v$ , are very well approximable. We let  $W_v$  denote the limsup set of  $v$ -approximable numbers, i.e.,

$$W_v = \left\{ \alpha \in \mathbb{R} : \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{v+1}} \text{ for infinitely many } \frac{p}{q} \in \mathbb{Q} \right\}.$$

The sets  $E_v$ ,  $R_v$ ,  $W_v$  are related and decrease as  $v$  increases, as follows from the inclusions

$$(3.13) \quad R_{v+\varepsilon} \subset E_v \subset W_v \subset R_v,$$

where  $\varepsilon > 0$  is arbitrary. Clearly  $W_v \subset R_v$ . Next consider the complementary inclusion  $\mathbb{R} \setminus W_v \subset \mathcal{D}_v$  and let  $\alpha \notin W_v$ . Then for all but finitely many positive integers  $q$ ,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{q^{v+1}}, \quad j = 1, \dots, r.$$

Moreover, for the exceptional values of  $q$ , say  $q^{(1)}, \dots, q^{(r)}$ ,

$$K = \min \left\{ \left| q^{(j)} \alpha - p^{(j)} \right| \left( q^{(j)} \right)^v : j = 1, \dots, r \right\} > 0,$$

since  $q\alpha \in \mathbb{Z}$  implies  $kq\alpha \in \mathbb{Z}$  for each  $k \in \mathbb{Z}$ . Thus  $|\alpha - p/q| \geq K/q^{v+1}$  for all  $p/q \in \mathbb{Q}$  and so  $\alpha \in \mathcal{D}_v$ , i.e.,  $\alpha \notin E_v$ .

To establish the final inclusion, let  $\alpha \in R_{v+\varepsilon}$ , so that for some  $K > 0$ ,

$$\left| \alpha - \frac{p}{q} \right| < \frac{K}{q^{v+1+\varepsilon}}$$

for infinitely many  $p/q \in \mathbb{Q}$ . Given any  $K' > 0$ , choose as we may, a denominator  $q_0$  sufficiently large so that  $q_0^{-\varepsilon} \leq K'/K$ . Then there exists a  $p_0 \in \mathbb{Z}$  such that

$$\left| \alpha - \frac{p_0}{q_0} \right| < \frac{K'}{q_0^{v+1}}$$

and  $\alpha \in E_v$ . The Jarník–Besicovitch theorem (§4.2) gives us the Hausdorff dimension of  $W_v$  and allows us to deduce from (3.13) that all the sets have the same Hausdorff dimension.

**3.2.4.  $\Psi$ -approximable numbers.** Now we look at the set of points which enjoy a more general approximation by rationals. A function  $\Psi: \mathbb{N} \rightarrow \mathbb{R}^+$  such that

$$\lim_{q \rightarrow \infty} \Psi(q) = 0$$

will be called an *approximation function*; without loss of generality we can take  $\Psi(q) \leq 1/(2q)$  and later we shall also assume that  $q\Psi(q)$  is decreasing (by decreasing we mean non-increasing here and subsequently). A real number  $\alpha$  is said to be  $\Psi$ -*approximable* if  $\alpha$  satisfies the inequality

$$(3.14) \quad \left| \alpha - \frac{p}{q} \right| < \Psi(q)$$

for infinitely many  $p/q \in \mathbb{Q}$  (there should be no confusion with  $v$ -approximable numbers defined above). Note that this should not be confused with the inequality  $|q\alpha - p| < \psi(q)$  which is often considered, particularly in higher dimensions and that there are other definitions depending on the form of (3.14), see for example [17]. The set of  $\Psi$ -approximable numbers in  $\mathbb{R}$  will be denoted  $W(\Psi)$ .

Since for each  $k \in \mathbb{Z}$ ,  $(p + kq)/q \in \mathbb{Q}$  and

$$\left| \alpha + k - \frac{p + kq}{q} \right| = \left| \alpha - \frac{p}{q} \right|,$$

it follows that  $W(\Psi) \cap [k, k+1) = (W(\Psi) \cap [0, 1)) + k$ , so that  $W(\Psi)$  can be decomposed into a union over unit intervals:

$$W(\Psi) = \bigcup_{k \in \mathbb{Z}} (W(\Psi) \cap [k, k+1)) = \bigcup_{k \in \mathbb{Z}} (W(\Psi) \cap [0, 1)) + k.$$

As with the other types of approximation, the definitions of very well approximable,  $v$ -approximable and  $\Psi$ -approximable numbers extend naturally to systems of linear forms,  $p$ -adic numbers, formal power series and to the hyperbolic setting. We will be interested in the case of a single linear form: we say with some abuse of notation that a vector  $\alpha \in \mathbb{R}^n$  is  $v$ -approximable if

$$(3.15) \quad |\mathbf{q} \cdot \alpha - p| < \frac{1}{|\mathbf{q}|_\infty^v}$$

holds for infinitely many  $\mathbf{q} \in \mathbb{Z}^n$  and  $p \in \mathbb{Z}$ ;  $\alpha$  is very well approximable if  $v > n$ . In the hyperbolic space setting, a point  $\alpha$  in  $\mathbb{H}^n$  is  $\Psi$ -approximable with respect to  $G$  and  $\mathfrak{p}$  if

$$|\alpha - g(\mathfrak{p})|_2 < \Psi(\lambda_g)$$

for infinitely many  $g \in G$ . These and the other analogues of the real one-dimensional case will be discussed further when they arise below.

#### 4. Khintchine's theorem and metrical Diophantine approximation

Following earlier work of Borel [21], Khintchine gave an almost complete answer to the solubility of (3.14) in terms of the measure of  $W(\Psi)$ . In a series of papers in the 1920's on the Lebesgue measure of the sets  $W(\Psi)$  and  $\mathfrak{B}$  [72, 73, 74, 75], he laid the foundations of metrical Diophantine approximation. This theory, which is closely related to probability, measure and ergodic theory, considers sets of solutions to Diophantine inequalities in terms of Lebesgue and other measures. As a result, 0-1 laws are a feature of the Lebesgue part of the theory, as in Khintchine's theorem below (see also [58, §2.2]). In addition, because an exceptional set for which a result is invalid can be of measure zero, this can lead to theorems having a strikingly simple yet general character.

THEOREM 4.1 (Khintchine).

$$|W(\Psi) \cap [0, 1)| = \begin{cases} 0, & \text{if } \sum_{k=1}^{\infty} k\Psi(k) < \infty, \\ 1, & \text{if } k\Psi(k) \text{ is decreasing and } \sum_{k=1}^{\infty} k\Psi(k) = \infty. \end{cases}$$

Thus  $W(\Psi)$  is null when the series  $\sum_k k\Psi(k)$  converges and is full when  $k\Psi(k)$  decreases and the sum diverges. Subsequently Khintchine extended the result to simultaneous Diophantine approximation [75] (see also [26]) and Groshev extended it to systems of linear forms [126]. In particular the measure of the set of points  $(\alpha_1, \alpha_2) \in [0, 1)^2$  such that

$$\max \left\{ \left| \alpha_1 - \frac{p_1}{q} \right|, \left| \alpha_2 - \frac{p_2}{q} \right| \right\} < \Psi(q)$$

for infinitely many  $p_1, p_2 \in \mathbb{Z}$ ,  $q \in \mathbb{N}$  is 0 or 1 accordingly as  $\sum k^2\Psi(k)^2$  converges or as  $k\Psi(k)$  decreases and the sum diverges (*cf.* Theorem 4.6 below).

The convergence case follows readily from the Borel-Cantelli lemma [76] and the argument is closely related to that of Lemma 2.2. However, the case of divergence is much more difficult and relies on a crucial 'pairwise quasi-independence'

result combined with ‘mean and variance’ ideas or with results from ergodic theory. For further details of this remarkable improvement of the Borel-Cantelli lemma, see [31, 32, 58, 79, 105, 126, 131] (the books [58, 126] also include accounts of W. M. Schmidt’s important quantitative extension of Khintchine’s theorem [116]). In terms of probability, we recall that the events  $E_j$ ,  $j = 1, \dots, \infty$ , with corresponding probabilities  $P(E_j)$  are pairwise independent if for any  $j \neq k$ , the probability of the two events occurring is given by

$$P(E_j \cap E_k) = P(E_j)P(E_k).$$

As a result, the divergence half of the Borel-Cantelli Lemma holds for pairwise independence. Total independence, where the probability of any finite sequence of events is given by the product of the individual probabilities, is not necessary. Recently it has been shown that the divergence case is also related to the lower bound for Hausdorff dimension (see §6).

There has been dramatic progress in the metrical theory over the last decades, particularly in the theory of ‘dependent variables’ where the point  $\alpha$  lies on a manifold, so that coordinates of the point are functionally related. Sprindžuk’s solution [127] of Mahler’s conjecture in transcendence theory in terms of the Diophantine approximation of points on the Veronese curve  $\{(x, x^2, \dots, x^n) : x \in \mathbb{R}\}$  gave this topic an enormous impetus which has seen the recent proof of Sprindžuk’s conjectures and further results [12, 17, 78].

The  $p$ -adic analogue of Khintchine’s theorem was obtained by Jarník [70] and extended to systems of linear forms by Lutz [87]. For fields of formal power series, the analogue was obtained by de Mathan [37] and has recently been extended to systems of linear forms [84]. The complex analogue of Khintchine’s theorem, in which the approximation is by ratios of Gaussian integers, is discussed in §4.3 below.

Khintchine’s theorem corresponds to our intuition since if the approximation function  $\Psi$  is large, then there is a better chance of the inequality being satisfied. In particular, the set  $W_v$  is null for  $v > 1$ , since the series  $\sum_k k^{-v}$  converges, and full for  $v \leq 1$  (when  $W_v = [0, 1)$  by Dirichlet’s theorem).

COROLLARY 4.2.

$$|W_v \cap [0, 1)| = |R_v \cap [0, 1)| = |E_v \cap [0, 1)| = \begin{cases} 0, & \text{when } v > 1, \\ 1, & \text{when } v \leq 1. \end{cases}$$

The result for the other sets follows from (3.13). Less obviously, the theorem shows the Lebesgue measure of the set of  $\alpha \in [0, 1)$  such that (3.14) has infinitely many solutions is 1 when  $\Psi(k) = 1/(k^2 \log k)$  and 0 when  $\Psi(k) = 1/(k^2(\log k)^{1+\varepsilon})$  for any positive  $\varepsilon$ .

As a consequence of Jarník’s theorem on simultaneous Diophantine approximation to be discussed in §4.2, none of the sets  $W_v, R_v, E_v$  is an  $s$ -set (see (2.4) for the definition). However, using Lemma 2.1 and invariance under rational translates of  $R_v$ , Jarník had shown earlier that the set  $R_v$  of numbers rationally approximable to order  $v$  is not an  $s$ -set and obeys a ‘0- $\infty$ ’ law [67, 68]. Although superseded by the above theorem, the argument is very nice but the papers cited are not readily available, so the proof is repeated here. Let  $k$  be a positive integer and let



$\alpha \in R_v \cap [0, 1/k]$ , so that for some  $K = K(\alpha) > 0$ ,

$$\left| \alpha - \frac{p}{q} \right| < \frac{K}{q^{v+1}}$$

for infinitely many rationals  $p/q$ . For each  $j/k$ ,  $1 \leq j \leq k-1$ ,

$$\left| \alpha - \frac{p}{q} \right| = \left| \alpha + \frac{j}{k} - \frac{p}{q} - \frac{j}{k} \right| < \frac{Kk^{v+1}}{(kq)^{v+1}}.$$

Thus  $\alpha + j/k \in R_v \cap [j/k, (j+1)/k]$  and it follows that for each  $j = 0, 1, \dots, k-1$ ,

$$R_v \cap \left[ \frac{j}{k}, \frac{j+1}{k} \right] = R_v \cap \left[ 0, \frac{1}{k} \right] + \frac{j}{k}.$$

Moreover since Hausdorff measure is translation invariant,

$$\mathcal{H}^s(R_v \cap [0, 1]) = \sum_{j=0}^{k-1} \mathcal{H}^s\left(R_v \cap \left[ \frac{j}{k}, \frac{j+1}{k} \right]\right) = k\mathcal{H}^s\left(R_v \cap \left[ 0, \frac{1}{k} \right]\right),$$

whence for any  $j, k$  with  $1 \leq j \leq k-1$ ,

$$\mathcal{H}^s\left(R_v \cap \left[ \frac{j}{k}, \frac{j+1}{k} \right]\right) = \frac{1}{k}\mathcal{H}^s(R_v \cap [0, 1]).$$

Now every open interval  $(a, b)$  can be represented as a union of a countable set of intervals  $[j/k, (j+1)/k]$ , *i.e.*,

$$(a, b) = \bigcup_{j,k} \left[ \frac{j}{k}, \frac{j+1}{k} \right],$$

so that  $b - a = \sum_{j,k} 1/k$ . Hence

$$\mathcal{H}^s((a, b)) = \mathcal{H}^s\left(\bigcup_{j,k} \left[ \frac{j}{k}, \frac{j+1}{k} \right]\right)$$

and since  $\mathcal{H}^s(\cdot)$  is an outer measure,

$$\begin{aligned} \mathcal{H}^s((a, b) \cap R_v \cap [0, 1]) &= \mathcal{H}^s\left(\bigcup_{j,k} R_v \cap \left[ \frac{j}{k}, \frac{j+1}{k} \right]\right) \leq \sum_{j,k} \mathcal{H}^s\left(R_v \cap \left[ \frac{j}{k}, \frac{j+1}{k} \right]\right) \\ &\leq \mathcal{H}^s(R_v \cap [0, 1]) \sum_{j,k} \frac{1}{k} = (b - a)\mathcal{H}^s(R_v \cap [0, 1]). \end{aligned}$$

Thus the hypotheses of Lemma 2.1 are satisfied and so  $\mathcal{H}^s(R_v \cap [0, 1]) = \mathcal{H}^s(R_v)$  is 0 or  $\infty$ . The argument can be extended to show that the set of numbers approximable to order  $v$  by algebraic irrationals is not an  $s$ -set. The measure at the critical exponent was shown to be  $\infty$  by Bugeaud [23].

Khintchine's theorem also implies that the set  $\mathfrak{B}$  of badly approximable numbers is null. For given any  $K > 0$ , the sum  $\sum_q (K/q)$  diverges and so by Khintchine's theorem the set of real numbers  $\alpha$  satisfying  $|\alpha - p/q| < K/q^2$  for infinitely many  $p/q \in \mathbb{Q}$  is full. Thus the complementary set  $F(K)$  of the set of  $\alpha$  such that

$|\alpha - p/q| \geq K/q^2$  for all but finitely many  $p/q$  is null and evidently increases as  $K$  decreases. From its definition,

$$\mathfrak{B} \subset \bigcup_{K>0} F(K) = \bigcup_{N=1}^{\infty} F(1/N),$$

a countable union of null sets, whence  $\mathfrak{B}$  is null.

Since the set  $W_v$  of very well approximable numbers is null, the inclusions (3.13) imply that  $\mathcal{D}_v$  is full for  $v > 1$ , *i.e.*, that almost all real numbers are of Diophantine type  $(K, v)$  for some positive  $K$ . Thus almost all numbers are neither well or badly approximable. It is a remarkable fact, proved by Jarník [65] in 1928, the same year as Besicovitch's first paper on Hausdorff measure and dimension (on planar 1-sets) [18], that although the set  $\mathfrak{B}$  is null,  $\dim_{\mathbb{H}} \mathfrak{B} = 1$ , *i.e.*, its Hausdorff dimension is maximal in the sense that it coincides with that of the ambient space  $\mathbb{R}$ . This we now discuss.

**4.1. Jarník's theorem for badly approximable numbers.** Let  $\theta \in (0, 1)$  and let  $a_n$ ,  $n = 1, 2, \dots$ , denote the partial quotients of the continued fraction expansion for  $\theta$ . For each  $N \in \mathbb{N}$ , define

$$M_N := \{\theta : a_n \leq N\}.$$

Now a number is badly approximable if and only if it has bounded partial quotients  $a_n$  [57, §11.4], so that  $\mathfrak{B} = \lim_{N \rightarrow \infty} M_N$ . In a pioneering paper that was the first on Hausdorff dimension in Diophantine approximation, Jarník [65] showed that for each  $N \geq 8$ ,

$$1 - \frac{4}{N \log 2} \leq \dim_{\mathbb{H}} M_N \leq 1 - \frac{1}{8N \log N}.$$

It is of course immediate that  $\dim_{\mathbb{H}} \mathfrak{B} \leq 1$  since  $\mathfrak{B} \subset \mathbb{R}$ . We now state Jarník's theorem for badly approximable numbers, which follows from the above.

**THEOREM 4.3 (Jarník).**

$$\dim_{\mathbb{H}} \mathfrak{B} = 1.$$

Using  $(\alpha, \beta)$  games, W. M. Schmidt [117, 118] proved much more, extending Jarník's theorem to higher dimensions, so that the set of simultaneously badly approximable points in the plane has Hausdorff dimension 2. In fact,  $\mathfrak{B}$  is a *thick* set. This is a 'local' property in the sense that for each open interval  $I$ ,  $\mathfrak{B}$  has 'full' Hausdorff dimension, *i.e.*,  $\dim_{\mathbb{H}} \mathfrak{B} \cap I = 1$ . A very general inhomogeneous analogue has been proved using quite different ideas from dynamical systems [77].

The game in one dimension involves two players A and B, a non-empty set  $S \subset \mathbb{R}$  and two parameters  $\alpha \in (0, 1)$ , given to the player A, and  $\beta \in (0, 1)$ , given to the player B. Player B begins by picking a closed interval  $B_1$ . Then A chooses a closed subinterval  $A_1 \subset B_1$  with  $|A_1| = \alpha|B_1|$ . Then B picks an interval  $B_2 \subset A_1$  with  $|B_2| = \beta|A_1| = \beta\alpha|B_1|$  and then A chooses another subinterval  $A_2 \subset B_2$  with  $|A_2| = \alpha|B_2| = \alpha^2\beta|B_1|$  and so on. Clearly the intervals  $B_1, A_1, B_2, A_2, \dots$ , form a decreasing nested sequence so that their intersection is a point,  $\omega$  say, in  $B_1$ . Player A is called the winner if  $\bigcap_j A_j = \{\omega\} \subseteq S$ , otherwise B wins.

A fuller account of the game in  $\mathbb{C}$  will be given in §5, so we will simply say that Schmidt showed that when  $S = \mathfrak{B}$ , A can force  $\omega$  to be badly approximable, even though  $\mathfrak{B}$  is null, and deduced that  $\dim_{\mathbb{H}} \mathfrak{B} \geq 1$ . Since  $\mathfrak{B} \subset \mathbb{R}$ ,  $\dim_{\mathbb{H}} \mathfrak{B} = 1$ . Note

that since  $\mathcal{H}^1(\mathfrak{B}) = |\mathfrak{B}|$ , the Lebesgue measure of  $\mathfrak{B}$  and since  $|\mathfrak{B}| = 0$  (Khintchine's theorem), it follows that the Hausdorff measure of  $\mathfrak{B}$  vanishes at  $s = \dim_{\mathbb{H}} \mathfrak{B}$  and

$$\mathcal{H}^s(\mathfrak{B}) = \begin{cases} \infty, & 0 \leq s < 1, \\ 0, & s \geq 1. \end{cases}$$

Thus  $\mathfrak{B}$  is not an  $s$ -set (this can also be proved using Jarník's lemma in §2.2 above).

Jarník's theorem has also been extended to  $p$ -adic fields [1], to fields of formal power series [82] and to hyperbolic space. It follows from the hyperbolic space counterpart of Khintchine's theorem that for geometrically finite Kleinian groups  $G$ , the set  $\mathfrak{B}(G, \mathfrak{p})$  of the hyperbolic analogue of badly approximable points, has zero Patterson measure and the analogue of Jarník's theorem holds [53, 99, 101], *i.e.*,

$$\dim_{\mathbb{H}} \mathfrak{B}(G, \mathfrak{p}) = \dim_{\mathbb{H}} \Lambda(G).$$

In addition, the exponent of convergence of  $G$ ,

$$\delta(G) := \inf \{s > 0 : \sum_{g \in G} \lambda_g^{-s} < \infty\} = \dim_{\mathbb{H}} \Lambda(G)$$

[20, 100, 131]. In view of Lemma 2.2, this is perhaps not so surprising. In a striking parallel with continued fractions, badly approximable points correspond to bounded orbits of flows on manifolds [20, 34, 35, 53, 120].

**4.2. Jarník–Besicovitch theorem.** When  $v > 1$ , the set  $\mathcal{D}_v$  is complementary to the set  $R_v$  of points approximable to exponent  $v$ , which is related to the set  $W_v$  of  $v$ -approximable numbers (these statements also hold for the higher dimensional analogues). The Hausdorff dimension of  $W_v$  was determined by Jarník in 1929 [66] and independently by Besicovitch in 1934 [19].

**THEOREM 4.4 (Jarník–Besicovitch).** *When  $v \geq 1$ ,*

$$\dim_{\mathbb{H}} W_v = \frac{2}{v+1},$$

*and when  $v \leq 1$ ,  $W_v = \mathbb{R}$ .*

Establishing the upper bound is not difficult, since  $W_v$  is a limsup set. There is no loss of generality in working with the more convenient set  $W_v \cap [0, 1]$ , as  $W_v = \bigcup_{k \in \mathbb{Z}} (W_v \cap [0, 1] + k)$ . Consider the limsup set

$$W_v \cap [0, 1] = \bigcap_{N=1}^{\infty} \bigcup_{q=N}^{\infty} \bigcup_{p=0}^q B(p/q, q^{-v-1}),$$

where  $B(p/q, \varepsilon) = \{x \in [0, 1] : |x - p/q| < \varepsilon\}$ , so that  $\text{diam}(B(p/q, \varepsilon)) \leq 2\varepsilon$ . Then

$$\sum_{q=1}^{\infty} \sum_{p=0}^q \text{diam}(B(p/q, q^{-v-1}))^s \leq 2^s \sum_{q=1}^{\infty} q^{1-s(v+1)} < \infty$$

when  $s > 2/(v+1)$ . Hence by Lemma 2.2 and the properties of Hausdorff dimension,  $\dim_{\mathbb{H}} W_v \leq 2/(v+1)$  when  $v > 1$ . When  $v \leq 1$ , the theorem follows from Dirichlet's theorem.

Establishing the correct lower bound is much harder. Jarník's lengthy and complicated proof involved continued fractions and arithmetic arguments. Besicovitch's proof was simpler and more geometric and is the basis of regular and ubiquitous systems which have turned out to be very effective techniques in determining the

Hausdorff dimension of a variety of sets [10, 13, 45]. Indeed ubiquity can imply Khintchine’s theorem [14], so ideas developed for the study of the the Hausdorff dimension of the null sets also contribute substantially to our understanding of this theorem. The Jarník–Besicovitch theorem has been extended considerably, to higher dimensions, hyperbolic space and local fields. Jarník himself proved the Hausdorff measure analogue of Khintchine’s theorem for simultaneous Diophantine approximation [69] and deduced that the set of points in  $\mathbb{R}^n$  satisfying

$$\left| \alpha - \frac{\mathbf{p}}{q} \right| < q^{-v-1}$$

for infinitely many  $\mathbf{p} \in \mathbb{Z}^n$ ,  $q \in \mathbb{N}$  has Hausdorff dimension  $(n+1)/(v+1)$  when  $v \geq 1/n$  and  $n$  otherwise. He also showed that the Hausdorff  $s$ -measure of  $W_v$  at the critical dimension is infinite. In view of the inclusions (3.13), it follows that the same holds for  $E_v, R_v$ . The points in  $W_v$  form an uncountable totally disconnected subset of the line and so are in Mandelbrot’s picturesque language ‘fractal dust’, as is  $\mathfrak{B}$ .

We have seen that the notion of a very well approximable point extends naturally to systems of real and  $p$ -adic linear forms and to hyperbolic space (where they can be interpreted in terms of geodesic excursions [33]). The Jarník–Besicovitch theorem and the Hausdorff measure analogue of Khintchine’s theorem have been established in the real case [41, 42], the  $p$ -adic case [2, 40], the formal power series case [84] and the hyperbolic case [63, 91]. Other generalisations are to restricted sequences [110], inhomogeneous Diophantine approximation [86] and to small linear forms [38]. A further generalisation to ‘shrinking targets’ has revealed some unexpected connections with complex dynamics and ergodic theory [62].

We will be interested in the case of a single real linear form in §7. The set of  $v$ -approximable points  $\alpha \in \mathbb{R}^n$  (see (3.15)) will be denoted  $L_v$ , *i.e.*,

$$(4.1) \quad L_v = \{ \alpha : |\mathbf{q} \cdot \alpha - p| < |\mathbf{q}|_\infty^{-v} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n, p \in \mathbb{Z} \}$$

and by the dual or linear form version of the Jarník–Besicovitch theorem [22],

$$(4.2) \quad \dim_{\mathbb{H}} L_v = \begin{cases} n-1 + \frac{n+1}{v+1} & \text{when } v > n, \\ n & \text{when } v \leq n. \end{cases}$$

The  $n-1$  term arises from the dimension of the resonant hyperplanes

$$\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{q} \cdot \mathbf{x} = p \}$$

sets while the other represents ‘fractal dust’ normal to the hyperplanes.

**4.3. Approximation by ratios of Gaussian integers.** Recall that the Gaussian integers are defined as the set  $\mathbb{Z}[i] = \{p_1 + ip_2 \in \mathbb{C} : p_1, p_2 \in \mathbb{Z}\}$ . These form a ring (in fact a unique factorisation domain). The Gaussian rationals are defined as the set  $\mathbb{Q}(i) = \{a/b + ic/d : a/b, c/d \in \mathbb{Q}\}$ . Approximation by Gaussian rationals decouples into the independent approximation of the real and imaginary part respectively. Here we study the more interesting problem of approximation by ratios of Gaussian integers.

Approximation of complex numbers by ratios of Gaussian integers was studied by Hermite and Hurwitz in the 19th century [80, IV, §4] but, unlike in the real case, a continued fraction approach did not give the best possible analogue of Dirichlet’s

theorem. This was obtained in 1925 by Ford [54], who used additional geometrical ideas based on the Picard group  $\mathrm{SL}(2, \mathbb{Z}[i])$ . The Gaussian analogues of the theorems of Dirichlet, Khintchine, Jarník and Besicovitch treated below should be compared with those corresponding to simultaneous Diophantine approximation in the real plane  $\mathbb{R}^2$ .

In 1967 A. Schmidt introduced a theory of regular and dually regular chains for continued fractions, to treat approximation problems in complex numbers [111, 114]. Schmidt was concerned with the study of complex quadratic irrationals and the complex version of Pell's equation and with extensions to groups [112, 113]. Our interest, however, is with complex or Gaussian rational analogues of Dirichlet's theorem and with the observation that the extended complex plane is the limit set of the Picard group. In this connection, Patterson established analogues of Dirichlet's theorem for the less general Fuchsian groups [99, 134]. Later Stratmann and Velani obtained versions for Kleinian groups [128, 129, 135] and so for the Picard group. These results can be translated into complex versions of Dirichlet's theorem with an undetermined constant. Nevertheless, for completeness, a short geometry of numbers proof of the complex version of Dirichlet's theorem is given below. Although the constant here is not best possible, the result is all we need. Proofs of the complex analogues of Jarník's theorem on badly approximable numbers and the Jarník–Besicovitch theorem that do not use the hyperbolic space framework will be given in §5 and §6 below.

Complex Diophantine approximation has also been investigated from the point of view of the distribution of the values of polynomials with real integer coefficients but with complex variable  $z$  [15]; for another complex analogue see [59]. *For the rest of this section,  $p = p_1 + ip_2$ ,  $q = q_1 + iq_2$  will denote Gaussian integers with  $q \neq 0$ .*

**THEOREM 4.5.** *Given any  $z = x + iy \in \mathbb{C}$  and  $N \in \mathbb{N}$ , there exist Gaussian integers  $p = p_1 + ip_2$ ,  $q = q_1 + iq_2$  with  $0 < |q| \leq N$  such that*

$$(4.3) \quad \left| z - \frac{p}{q} \right| < \frac{2}{|q|N}.$$

Moreover for infinitely many  $p, q \in \mathbb{Z}[i]$ ,

$$(4.4) \quad \left| z - \frac{p}{q} \right| < \frac{2}{|q|^2}.$$

**PROOF.** The inequality (4.3) holds if and only if the inequality

$$(4.5) \quad \left| x + iy - \frac{p_1 + ip_2}{q_1 + iq_2} \right| < \frac{2}{|q_1 + iq_2|N}$$

holds, *i.e.*, if and only if

$$|(q_1x - q_2y - p_1) + i(q_2x + q_1y - p_2)| < \frac{2}{N}$$

holds, which is the case if

$$(4.6) \quad \max\{|q_1x - q_2y - p_1|, |q_2x + q_1y - p_2|\} < \frac{\sqrt{2}}{N}.$$

By Minkowski's linear forms theorem, the system of inequalities

$$\begin{aligned} |q_1x - q_2y - p_1| &< 2^{1/2} N^{-1} \\ |q_2x + q_1y - p_2| &< 2^{1/2} N^{-1} \\ |q_1| &\leq 2^{-1/2} N \\ |q_2| &\leq 2^{-1/2} N \end{aligned}$$

has a non-zero solution in integers  $p_1, p_2, q_1, q_2$ . Hence (4.6) has a solution with  $|q| = |q_1 + iq_2| \leq N$ , as claimed.  $\square$

Since the Gaussian rationals  $p/q$  are not required to be on lowest terms, (4.4) holds infinitely often. For if  $p/q$  fails to satisfy (4.4), then *a fortiori*,  $(\kappa p)/(\kappa q)$ , where  $\kappa$  is a non-zero Gaussian integer, will also fail. And if  $p/q$  satisfies (4.4), only a finite number of the fractions  $(\kappa p)/(\kappa q)$  can also satisfy it.

This result should be compared to (3.7). As in the real case, complex numbers for which Dirichlet's theorem cannot be significantly improved are called badly approximable and numbers for which it can be called very well approximable. More precisely, a complex number  $z$  is *badly approximable* if there exists a constant  $K = K(z)$  such that for all  $p, q \in \mathbb{Z}[i]$ ,  $q \neq 0$ ,

$$\left| z - \frac{p}{q} \right| \geq \frac{K}{|q|^2}.$$

Ford [54] showed that the complex quadratic irrationals  $(1 \pm i\sqrt{3})/2$  are the worst approximable numbers with  $K = 1/\sqrt{3}$  and thus correspond to the golden ratio  $(\sqrt{5} + 1)/2$  in the real case. Badly approximable numbers have been studied by A. Schmidt from the viewpoint of the Markoff spectrum [115].

As in the real case, a complex number  $z$  is *v-approximable* if

$$(4.7) \quad \left| z - \frac{p}{q} \right| < \frac{1}{|q|^{v+1}}$$

for infinitely many  $p, q \in \mathbb{Z}(i)$  and will be called  $\Psi$ -*approximable* if

$$(4.8) \quad \left| z - \frac{p}{q} \right| < \Psi(|q|),$$

where  $\Psi : [1, \infty) \rightarrow \mathbb{R}^+$ . The set of  $\Psi$ -approximable  $z$  will be written  $W^*(\Psi)$ . Thus  $W^*(\Psi)$  is the set of points  $z \in \mathbb{C}$  for which the inequality (4.8) holds for infinitely many Gaussian rationals  $p/q$ . The set of *v-approximable* complex numbers will be written  $W_v^*$ .

**4.4. Khintchine's theorem for complex numbers.** The complex analogue of Khintchine's theorem was proved in 1952 by LeVeque [85] who combined Khintchine's continued fraction approach with ideas from hyperbolic geometry. In 1976, Patterson proved slightly less sharp versions of Khintchine's theorem for Fuchsian groups acting on hyperbolic space [99, 134]. A little later, Sullivan [130] used Bianchi groups and some powerful hyperbolic geometry arguments to prove more general Khintchine theorems for real and for complex numbers. In the latter case, the result includes approximation of complex numbers by ratios  $a/b$  of integers  $a, b$  from the imaginary quadratic field  $\mathbb{R}(i\sqrt{d})$ , where  $d$  is a squarefree natural number. The case  $d = 1$  corresponds to the Picard group and approximation by ratios of

Gaussian integers. Stratmann and Velani extended Patterson's results with similar minor technical restrictions to Kleinian groups [129, 135]. These include the Bianchi groups and give a less precise and differently formulated version of Sullivan's result. We now state LeVeque's result in our notation.

**THEOREM 4.6.** *Suppose  $k^2\Psi(k)$  is decreasing. Then the Lebesgue measure of  $W^*(\Psi)$  is null or full according as the sum*

$$(4.9) \quad \sum_{k=1}^{\infty} k^3\Psi(k)^2$$

*converges or diverges.*

As well as being more general, Sullivan's result is more precise as the growth condition for the function  $\Psi$  is weakened to a 'comparability condition'. Instead of the sum, Sullivan and LeVeque use the equivalent integral  $\int x^3\Psi(x)^2 dx$  (their integrands are different owing to different forms of (4.8)). It is readily verified that  $W^*(\Psi)$  is invariant under translations by Gaussian integers  $p = p_1 + ip_2$ , so that

$$(4.10) \quad W^*(\Psi) = \bigcup_{p \in \mathbb{Z}[i]} V^*(\Psi) + p = \bigcup_{p_1, p_2 \in \mathbb{Z}} V^*(\Psi) + p_1 + ip_2,$$

where  $V^*(\Psi) = W^*(\Psi) \cap I^2$  and  $I^2 = [0, 1)^2 = \{x + iy : 0 \leq x, y < 1\}$ . We will work in the more convenient unit square  $I^2$  and consider the set  $V^*(\Psi)$ .

Before continuing, we need some more definitions and notation.

**4.5. Resonant sets and balls in  $I^2$ .** Let  $q = q_1 + iq_2 \in \mathbb{Z}[i] \setminus \{0\}$ . The set  $R(q) = R(q_1, q_2) \subset I^2$  where

$$\begin{aligned} R(q) &= \left\{ \frac{p}{q} : p \in \mathbb{Z}[i] \right\} \cap I^2 = R(q_1, q_2) \\ &= \left\{ \left( \frac{p_1 q_1 + p_2 q_2}{q_1^2 + q_2^2}, \frac{p_2 q_1 - p_1 q_2}{q_1^2 + q_2^2} \right) : p_1, p_2 \in \mathbb{Z} \right\} \cap I^2 \end{aligned}$$

is called a *resonant set*. This set is the analogue in the complex plane of the set  $\{p_1/q_1 : 0 \leq p_1 < q_1\}$  in the real line. The points in the resonant set form a lattice inclined at an angle  $\tan^{-1}(q_1/q_2)$  to the real axis and in which the side length of the fundamental region is  $|q| = (q_1^2 + q_2^2)^{-1/2}$ . Area and congruence considerations give that the number of points of  $R(q_1, q_2)$  in  $I^2$  is

$$(4.11) \quad \#R(q) = \#R(q_1, q_2) = |q|^2 = q_1^2 + q_2^2.$$

The disc

$$(4.12) \quad D(p/q; \varepsilon) = \{z \in \mathbb{C} : |z - p/q| < \varepsilon\},$$

with radius  $\varepsilon$  and centred at  $p/q$  where

$$\frac{p}{q} = \frac{p_1 + ip_2}{q_1 + iq_2} = \frac{(p_1 + ip_2)(q_1 - iq_2)}{q_1^2 + q_2^2} = \frac{p_1 q_1 + p_2 q_2}{q_1^2 + q_2^2} + i \frac{p_2 q_1 - p_1 q_2}{q_1^2 + q_2^2}$$

has area  $\pi\varepsilon^2$ . The set

$$(4.13) \quad B(q, \varepsilon) = \left\{ z \in I^2 : \left| z - \frac{p}{q} \right| < \varepsilon \text{ for some } p \in \mathbb{Z}[i] \right\} = \bigcup_{p \in \mathbb{Z}[i]} D(p/q, \varepsilon)$$

can be regarded as a neighbourhood of a resonant set and its measure

$$(4.14) \quad |B(q, \varepsilon)| = \pi\varepsilon^2 (|q|^2 + O(|q|)).$$

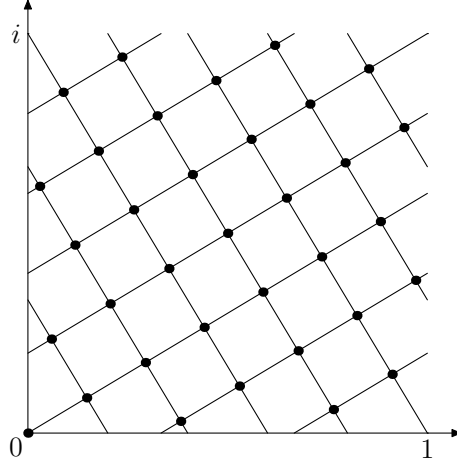


FIGURE 2. The lattice in  $I^2$  corresponding to  $q_1 = 5$ ,  $q_2 = 3$ .

Now  $V^*(\Psi)$  can be expressed in the form

$$(4.15) \quad V^*(\Psi) = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \bigcup_{|q|=k} B(q, \Psi(|q|)) = \limsup_{|q| \rightarrow \infty} B(q, \Psi(|q|)).$$

Thus  $V^*(\Psi)$  is a limsup set and for each  $N = 1, 2, \dots$  has a natural cover

$$(4.16) \quad \mathcal{C}_N(V^*(\Psi)) = \{B(q, \Psi(|q|) : |q| \geq N\}.$$

Hence for each  $N = 1, 2, \dots$ , the measure of  $V^*(\Psi)$  satisfies

$$|V^*(\Psi)| \leq \sum_{k=N}^{\infty} \sum_{k \leq |q| < k+1} |B(q, \Psi(|q|))| \ll \sum_{k=N}^{\infty} k^2 \Psi(k)^2 \sum_{k \leq |q| < k+1} 1.$$

Now,  $\sum_{1 \leq |q| \leq k} 1$  is the number of lattice points in the closed disc  $D(0, k)$  of radius  $k$ . By [28], given any  $\varepsilon > 0$ ,

$$\sum_{1 \leq |q| \leq k} 1 = \pi k^2 + O(k^{12/37+\varepsilon}).$$

Hence

$$(4.17) \quad \sum_{k \leq |q| < k+1} 1 = \pi(k+1)^2 + O(k^{1/3}) - \pi k^2 + O(k^{1/3}) = 2\pi k + O(k^{1/3}).$$

Therefore

$$|V^*(\Psi)| \ll \sum_{k=N}^{\infty} k^3 \Psi(k)^2.$$

Since  $N$  is arbitrary, the convergence of the series  $\sum_k k^3 \Psi(k)^2$  implies that

$$|V^*(\Psi)| = |W^*(\Psi)| = 0,$$

which is the convergence part of the complex analogue of Khintchine's theorem. The much more difficult case of divergence requires deeper arguments and the reader is referred to Sullivan's bold and highly geometrical paper [130].



### 5. Badly approximable complex numbers and Jarník's theorem

The results of [20] and of [53] on badly approximable points arising with the action of Kleinian groups and on bounded geodesics on Riemann surfaces respectively, specialised to the case of the Picard group, could be translated to give Jarník's theorem for complex numbers badly approximable by ratios of Gaussian integers. However, we will give a self-contained proof using an extension to the complex numbers of the  $(\alpha, \beta)$ -game introduced by W. M. Schmidt [117]. In the setting of the complex plane,  $S \subset \mathbb{C}$ ,  $\alpha, \beta \in (0, 1)$  and discs replace intervals. Thus, the game begins with player B choosing a disc  $B_1 = \{z \in \mathbb{C} : |z - b_1| \leq \rho_1\}$ . Next A chooses a disc  $A_1 \subset B_1$  with radius  $\alpha\rho_1$ . Then B chooses a disc  $B_2 \subset A_1$  with radius  $\beta\alpha\rho_1$  and so on ad infinitum, such that  $B_{n+1}$  has radius  $(\alpha\beta)^n\rho_1$  for any  $n \geq 0$ .

The discs  $B_1, A_1, B_2, A_2, \dots$  form a nested decreasing sequence of closed sets, so there is a unique intersection point,  $\omega$  say. Player A wins if this point is an element of the set  $S$ , *i.e.*, if  $\bigcap_j A_j = \{\omega\} \subset S$ . Otherwise, B wins. A set  $S$  is said to be  $(\alpha, \beta)$ -winning if A can win the game for the parameters  $\alpha$  and  $\beta$  no matter how well B plays. If for some  $\alpha \in (0, 1)$ , A can win the game for any  $\beta \in (0, 1)$ , the set  $S$  is said to be  $\alpha$ -winning.

Player A benefits from  $\alpha$  being small. Indeed, when  $\alpha$  gets smaller, A can limit the amount of choice B has in the next move. As in the real case, it may be shown that if  $\alpha' < \alpha$  and  $S$  is  $\alpha$ -winning, then  $S$  is  $\alpha'$ -winning. Hence, given  $S$ , there is a largest value of  $\alpha$  for which  $S$  is  $\alpha$ -winning so we may define

$$\alpha^*(S) := \sup \{ \alpha \in (0, 1) : S \text{ is } \alpha\text{-winning} \}.$$

In fact, for any  $S \neq \mathbb{C}$ , we easily see that  $\alpha^*(S) \leq \frac{1}{2}$ . To see this, note that for the parameters  $\alpha > \frac{1}{2}$  and  $\beta \in (0, 2\alpha - 1)$ , B may ensure that the centres of all the  $B_i$  are the same. Hence, by choosing the first disc  $B_1$  with centre  $b_1 \notin S$ , we have the result.

When  $S = \mathfrak{B}$ , A wins if she can force  $\omega$  to be badly approximable, *i.e.*, if  $\omega$  is in the set

$$\mathfrak{B} = \left\{ z \in \mathbb{C} : \exists K > 0 \forall p, q \in \mathbb{Z}[i] : \left| z - \frac{p}{q} \right| > \frac{K}{|q|^2} \right\}.$$

Since  $\mathfrak{B}$  is null, this seems unlikely to be the case, but in fact A can win the game whenever  $2\alpha < 1 + \alpha\beta$ . It immediately follows that

$$(5.1) \quad \alpha^*(\mathfrak{B}) = 1/2,$$

so in fact A may win this game almost as easily as she could win the game where  $S$  is the entire complex plane with one point removed. We take some time to prove (5.1).

We have already seen that  $\alpha^*(\mathfrak{B}) \leq 1/2$ , so we will only consider  $\alpha \leq 1/2$ , as this will simplify the proof. Let  $\alpha \in (0, 1/2]$  and  $\beta \in (0, 1)$  be fixed. Note that  $\gamma := 1 + \alpha\beta - 2\alpha > 0$ . We may assume without loss of generality that the radius of the initial ball  $B_1$  satisfies  $\rho_1 \leq \alpha\beta\frac{\gamma}{8}$ . Indeed, otherwise we would let the game continue in an arbitrary fashion until reaching a  $B_j$  for which  $\rho_j \leq \alpha\beta\frac{\gamma}{8}$  and then take this to be our starting point. The constant  $K$  in the definition of  $\mathfrak{B}$  will be  $\delta = \frac{\gamma}{4} \min(\rho, \alpha^2\beta^2\frac{\gamma}{8})$ , where  $\rho = \rho_1$ .

Let  $t \in \mathbb{N}$  be such that  $\alpha\beta\gamma \leq 2(\alpha\beta)^t < \gamma$ . Let  $R = (\alpha\beta)^{-t/2}$ . Clearly, it suffices to prove for any  $n \in \mathbb{N}$  that if

$$(5.2a) \quad \gcd(p, q) = 1,$$

$$(5.2b) \quad z \in B_{nt+1},$$

$$(5.2c) \quad 0 < |q| < R^n,$$

then  $\left|z - \frac{p}{q}\right| > \frac{\delta}{|q|^2}$ . This may be done using induction.

For  $n = 0$ , there is nothing to prove, as (5.2c) leaves us no  $q$  to consider. Hence, we may assume that we have  $B_1, \dots, B_{(k-1)t+1}$  such that the above holds for  $0 \leq n \leq k-1$ . In subsequent play, A thus only needs to worry about  $\frac{p}{q}$  with  $R^{k-1} \leq |q| < R^k$ , as the remaining problematic fractions have been sorted out in the preceding steps of the game.

In fact, there can be at most one such  $\frac{p}{q}$ . Indeed, suppose that there exists  $p, p', q, q' \in \mathbb{Z}[i]$  and  $z, z' \in B_{(k-1)t+1}$  with

$$\left|z - \frac{p}{q}\right| \leq \frac{\delta}{|q|^2} \quad \text{and} \quad \left|z' - \frac{p'}{q'}\right| \leq \frac{\delta}{|q'|^2}$$

with  $|q|, |q'| \in [R^{k-1}, R^k)$ . Then

$$\begin{aligned} \left|\frac{p}{q} - \frac{p'}{q'}\right| &\leq \left|z - \frac{p}{q}\right| + \left|z' - \frac{p'}{q'}\right| + |z - z'| \\ &\leq \frac{\delta}{|q|^2} + \frac{\delta}{|q'|^2} + 2\rho(\alpha\beta)^{(k-1)t} \leq 2\delta R^{2-2k} + 2\rho R^{2-2k} \\ &\leq 4\rho(\alpha\beta)^{-t} R^{-2k} \leq 4\frac{1}{8}\alpha\beta\gamma \frac{2}{\alpha\beta\gamma} R^{-2k} = R^{-2k}. \end{aligned}$$

On the other hand, since  $\gcd(p, q) = \gcd(p', q') = 1$  and since  $\mathbb{Z}[i]$  is a unique factorisation domain,

$$\left|\frac{p}{q} - \frac{p'}{q'}\right| = \left|\frac{pq' - p'q}{qq'}\right| \geq \frac{1}{|q||q'|} > R^{-2k}$$

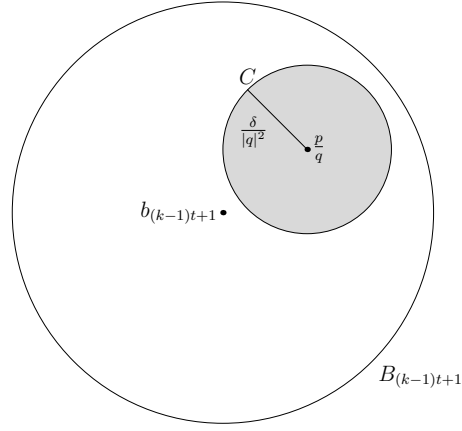
whenever  $p \neq p'$  and  $q \neq q'$ . Hence, there can be at most one problematic point.

Note that we have used the property of the ring of Gaussian integers being a unique factorisation domain. For other rings, this may not be the case and stronger tools are needed for proving the analogous result. However, for clarity of this exposition, we will take the simple route and use the unique factorisation property. Note also that for the corresponding result in simultaneous Diophantine approximation, the module  $\mathbb{Z}^2$  takes the place of the Gaussian integers. Even though the underlying sets are the same, the difference in algebraic structure prevents the method of this proof from working in the case of simultaneous Diophantine approximation.

As there can be at most one point in  $B_{(k-1)t+1}$  suitably close to a Gaussian rational  $p/q$ , we may devise strategies that avoids a disc around this point  $p/q$  of a suitable radius. It is clear that we need to avoid  $C = B\left(\frac{p}{q}, \frac{\delta}{|q|^2}\right)$ . We examine two possibilities in turn.

First, consider the case when  $\left|\frac{p}{q} - b_{(k-1)t+1}\right| > \delta R^{2-2k}$ . In this case,

$$\left|\frac{p}{q} - b_{(k-1)t+1}\right| > \delta R^{2-2k} \geq \frac{\delta}{|q|^2}.$$

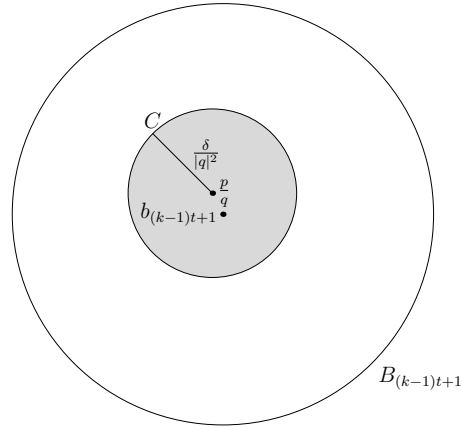

 FIGURE 3. A avoids  $C$  in one move

Hence,  $b_{(k-1)t+1} \notin C$ , so we are in the situation of figure 3. As  $\alpha \leq 1/2$ , we see that A can choose her next disc in such a way that it does not intersect with  $C$ .

This leaves the final case when

$$\left| \frac{p}{q} - b_{(k-1)t+1} \right| \leq \delta R^{2-2k}.$$

This corresponds to the case when  $b_{(k-1)t+1} \in C$  (figure 4). In this case, we clearly


 FIGURE 4. A avoids  $C$  in  $t$  steps

need to work harder to get an answer, as the right strategy of A is not immediately obvious. However, it turns out that picking a fixed direction and moving as far as possible in this direction at each turn will cause the disc chosen by B after  $t$  steps to have empty intersection with  $C$ . The following lemma formalises this.

**LEMMA 5.1.** *Let  $\alpha, \beta \in (0, 1)$  with  $\gamma = 1 + \alpha\beta - 2\alpha > 0$ . Let  $t \in \mathbb{N}$  be such that  $(\alpha\beta)^t < \gamma/2$ . Suppose that the disc  $B_k = (b_k, \rho_k)$  occurs at some stage in an  $(\alpha, \beta)$ -game. Then A can play in such a way that*

$$B_{k+t} \subseteq \left\{ z \in \mathbb{C} : |z - b_k| > \rho_k \frac{\gamma}{2} \right\}.$$

PROOF. We define a strategy for A in the following way: Suppose that the last disc chosen by B was  $B(c, \rho)$  for some  $\rho > 0$ . Choose some  $\hat{c} \in \mathbb{C}$  with  $|\hat{c}| = \rho(1 - \alpha)$ . We define a legal move for A by the map

$$B(c, \rho) \mapsto B(c + \hat{c}, \alpha\rho).$$

We need to convince ourselves that this move is in fact a legal move. First, we note that the radius is the right one. Hence, we need only prove the inclusion  $B(c + \hat{c}, \alpha\rho) \subseteq B(c, \rho)$ . But this follows since for any  $z \in B(c + \hat{c}, \alpha\rho)$ ,

$$|z - c| = |z - \hat{c} + \hat{c} - c| \leq |z - (c + \hat{c})| + |\hat{c}| \leq \alpha\rho + \rho(1 - \alpha) = \rho.$$

The strategy of player A will be to use the above move, no matter how B plays the game.

We denote the discs chosen by B by  $B_k = (b_k, \rho_k)$ , and the discs chosen by A by  $A_k = (a_k, \alpha\rho_k)$ . Note that

$$(5.3) \quad |a_k - b_k| = |b_k + \hat{c} - b_k| = \rho_k(1 - \alpha).$$

Also, from figure 5, we see that

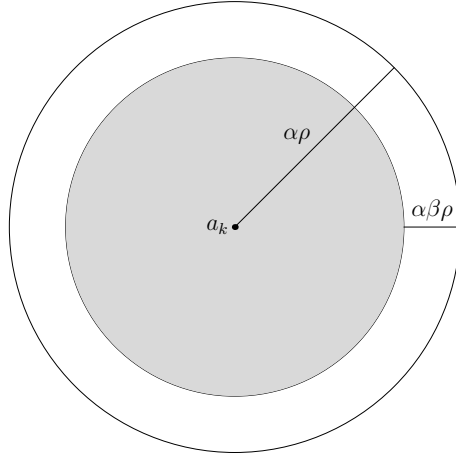


FIGURE 5. The point  $b_{k+1}$  must be chosen in the shaded area.

$$(5.4) \quad |b_{k+1} - a_k| \leq \alpha\rho_k - \alpha\beta\rho_k.$$

Hence, by (5.3) and (5.4),

$$\begin{aligned} |b_k - b_{k+1}| &= |(b_k - a_k) - (b_{k+1} - a_k)| \geq ||b_k - a_k| - |b_{k+1} - a_k|| \\ &= |\rho_k(1 - \alpha) - |b_{k+1} - a_k|| > |\rho_k(1 - \alpha) - \alpha\rho_k + \alpha\beta\rho_k| = \rho_k\gamma. \end{aligned}$$

Continuing as above, at each step choosing  $\hat{c}$  to point in the direction of the first one chosen, we obtain,

$$|b_{k+t} - b_k| > \rho_k\gamma.$$

But since  $\rho_{k+t} = (\alpha\beta)^t \rho_k < \rho_k \frac{\gamma}{2}$ , we have for any  $z \in B_{k+t}$ ,

$$|z - b_k| \geq ||z - b_{k+t}| - |b_{k+t} - b_k|| > \rho_k \frac{\gamma}{2}.$$

This completes the proof.  $\square$

With the induction step done, we have shown that A can play in such a way that (5.1) holds for the final intersection point. In fact, we have found an explicit lower bound on the constant  $K$  for which (5.1) holds.

From (5.1), we can get the Hausdorff dimension of the set  $\mathfrak{B}$ . By considering the amount of choice B has in the game and using this to find a lower bound on the  $s$ -length of appropriate covers, we obtain a lower bound on the dimension of any  $(\alpha, \beta)$ -winning set. This construction was carried out in considerable generality by W. M. Schmidt [117], who obtained a lower bound for the Hausdorff dimension of  $(\alpha, \beta)$ -winning sets in real Hilbert spaces. We sketch an approach to obtaining the dimension from the number  $\alpha^*(\mathfrak{B})$ . Because of the geometric nature of the problem,  $\mathfrak{B}$  will be regarded as a subset in  $\mathbb{R}^2$ . We will compute the usual planar Hausdorff dimension here and subsequently.

To obtain the Hausdorff dimension of  $\mathfrak{B}$ , we consider the game from B's point of view. Assume first without loss of generality that  $B_0$ , the first disc chosen, has radius 1. At any point of the game, B can choose to direct the game into a number of disjoint discs. While there may be a variety of ways in which these discs can be chosen, the maximum number  $N(\beta)$  is roughly equal to  $1/\beta^2$ , *i.e.*,  $N(\beta) \asymp 1/\beta^2$ .

We limit B's choice to these  $N(\beta)$  possible moves and assume that A is playing to win the game. This gives us a parametrisation of the sequence of discs chosen by B, so that  $B_k = B_k(j_1, \dots, j_k)$  with the  $j_i \in \{0, \dots, N(\beta) - 1\}$  for  $i = 1, \dots, k$ . For later use, note that the radius of each these discs is  $\rho_k = (\alpha\beta)^k$ . By simultaneously considering all the different ways, B may play the game, we obtain a function

$$f : \{0, \dots, N(\beta) - 1\}^{\mathbb{N}} \rightarrow \mathfrak{B}, \quad (\lambda_k)_{k \in \mathbb{N}} \mapsto \bigcap_{k \in \mathbb{N}} B_k(\lambda_1, \dots, \lambda_k) = \{x(\lambda)\}.$$

We define the set  $\mathfrak{B}^* \subseteq \mathfrak{B}$  to be the range of  $f$ . As every number in the interval  $[0, 1]$  has at least one expansion in base  $N(\beta)$ , we may map this set onto the unit interval by the map

$$g : \mathfrak{B}^* \rightarrow [0, 1], \quad x(\lambda) \mapsto 0.\lambda_1\lambda_2\dots$$

Note that these functions could well be multivalued, but this is of no concern to us. All we need is a cover of  $[0, 1]$ . We extend this function to subsets of the complex plane by defining  $g(Z) = g(Z \cap \mathfrak{B}^*)$  for any  $Z \subseteq \mathbb{C}$ , where by convention  $B(\emptyset) = 0$ .

Now, take some cover  $\mathcal{C} = \{C_l\}_{l \in \mathbb{N}}$  of  $\mathfrak{B}$  with discs of radius  $\rho(C_l) = \rho_l$ . We wish to find a lower bound on the  $s$ -length of this cover for appropriate  $s$  as  $\rho_l$  becomes smaller. This is where the function  $g$  comes in handy. As  $\mathcal{C}$  covers  $\mathfrak{B}^*$ , we see that  $g(\mathcal{C})$  covers  $[0, 1]$ . We let  $\bar{\mu}$  denote the outer Lebesgue measure. By sub-additivity,

$$(5.5) \quad \sum_{l=1}^{\infty} \bar{\mu}(g(C_l)) \geq \bar{\mu}\left(\bigcup_{l=1}^{\infty} g(C_l)\right) \geq 1.$$

Now let  $\omega > 0$  be sufficiently small so that any disc of radius  $\omega(\alpha\beta)^k$  intersects at most two of the discs  $B_k(j_1, \dots, j_k)$ . By [117, Lemma 20],  $\omega = 2/\sqrt{3} - 1 < 1$  has this property. We define integers

$$k_l = \left\lceil \frac{\log(2\omega^{-1}\rho_l)}{\log(\alpha\beta)} \right\rceil.$$

For  $\rho_l$  sufficiently small, we see that  $k_l > 0$  and  $\rho_l < \omega(\alpha\beta)^{k_l}$ . Hence, the disc  $C_l$  intersects at most two of the discs  $B_{k_l}(j_1, \dots, j_{k_l})$ . As  $g(B_{k_l}(j_1, \dots, j_{k_l}))$  is clearly an interval of length  $N(\beta)^{-k_l}$ , we have  $\bar{\mu}(g(C_l)) \leq 2N(\beta)^{k_l}$ , so by (5.5),

$$1 \leq \sum_{l=1}^{\infty} \bar{\mu}(g(C_l)) \leq \sum_{l=1}^{\infty} 2N(\beta)^{k_l} \leq 2(2\omega^{-1})^{\frac{\log(N(\beta))}{|\log(\alpha\beta)|}} \sum_{l=1}^{\infty} \rho_l^{\frac{\log(N(\beta))}{|\log(\alpha\beta)|}}.$$

Thus, for  $s = \log(N(\beta))/|\log(\alpha\beta)|$ , the  $s$ -length of the cover  $\mathcal{C}$  is strictly positive, so the Hausdorff dimension of  $\mathfrak{B}^*$  must be greater than this number.

Now, we fix  $\alpha \in (0, 1/2)$  and apply the above,

$$\dim_{\text{H}}(\mathfrak{B}) \geq \frac{\log \beta^{-2}}{|\log \alpha \beta|} = \frac{2|\log \beta|}{|\log \alpha| + |\log \beta|} \rightarrow 2$$

as  $\beta \rightarrow 0$ . We have thus proved the analogue of Jarník's theorem for the complex numbers:

**THEOREM 5.2.** *The set  $\mathfrak{B}$  is thick, i.e., for any disc  $B \subseteq \mathbb{C}$ ,*

$$\dim_{\text{H}}(B \cap \mathfrak{B}) = 2.$$

## 6. The complex Jarník–Besicovitch theorem

Let  $W_v^*$  be the set of  $v$ -approximable complex numbers which satisfy the inequality (4.7) for infinitely many Gaussian rationals  $p/q$  (recall that  $p, q \in \mathbb{Z}[i]$ ). The general Jarník–Besicovitch theorems in [63, 91] can be specialised to the Picard group to yield the Hausdorff dimension of  $W_v^*$  (see [91, Corollary 2]) but we give a more self-contained and direct proof. For convenience we consider

$$V_v^* := W_v^* \cap I^2 = \left\{ z \in I^2 : \left| z - \frac{p}{q} \right| < \frac{1}{|q|^{v+1}} \text{ for infinitely many } \frac{p}{q} \right\}.$$

**THEOREM 6.1.**

$$\dim_{\text{H}} V_v^* = \dim_{\text{H}} W_v^* = \begin{cases} \frac{4}{v+1} & \text{when } v \geq 1, \\ 2 & \text{when } v \leq 1. \end{cases}$$

When  $v \leq 1$ ,  $V_v^*$  is full by the complex form of Khintchine's theorem so the second equality holds by (iii) in § 2.2. As usual, the proof for  $v > 1$  falls into two parts. First, to obtain the upper bound for  $\dim_{\text{H}} V_v^*$ , consider the cover  $\mathcal{C}(V_v^*)$  given by (4.16) with  $N = 1$  and  $\Psi(k) = k^{-v-1}$ . By (4.17), this has  $s$ -length

$$\begin{aligned} \ell^s(\mathcal{C}(V_v^*)) &\ll \sum_{k=1}^{\infty} \sum_{k \leq |q| < k+1} |q|^2 \left( |q|^{-v-1} \right)^s \ll \sum_{k=1}^{\infty} k^{2-(v+1)s} \sum_{k \leq |q| < k+1} 1 \\ &\ll \sum_{k=1}^{\infty} k^{3-s(v+1)} < \infty \end{aligned}$$

for  $s > 4/(v+1)$ . It follows that when  $v \geq 1$ ,

$$(6.1) \quad \dim_{\text{H}} V_v^* \leq \frac{4}{v+1}.$$

To obtain the lower bound in the Jarník–Besicovitch theorem, we use ubiquity [16, 45]. Let  $S \subseteq I^2$  and let  $\rho: \mathbb{N} \rightarrow (0, \infty)$  be a function. Put

$$B(S; q, \varepsilon) := \left\{ z \in S : \left| z - \frac{p}{q} \right| < \varepsilon \text{ for some } p \in \mathbb{Z}[i] \right\}.$$

The set

$$\mathcal{R} = \bigcup_q R(q) \subset \mathbb{C},$$

where the union is over non-zero  $q \in \mathbb{Z}[i]$ , consists of discrete points and so has dimension 0. Let  $S$  be any open square in  $I^2$ . By definition,  $\mathcal{R}$  is *ubiquitous* in  $S \subseteq I^2 = [0, 1)^2$  with respect to  $\rho$  if

$$\left| \bigcup_q B(S; q, \rho(N)) \right| \rightarrow |S|$$

as  $N \rightarrow \infty$ . Now by the complex analogue of Dirichlet's theorem, for each  $N \in \mathbb{N}$ ,

$$\begin{aligned} & \left\{ z \in S : \left| z - \frac{p}{q} \right| < \frac{2}{|q|N} \text{ for some } p, q \in \mathbb{Z}[i], 1 \leq |q| \leq N \right\} \\ &= \bigcup_q B\left(q, \frac{2}{|q|N}\right) = S, \end{aligned}$$

so that

$$\left| \bigcup_q B\left(q, \frac{2}{|q|N}\right) \right| = |S|.$$

Consider the set  $S(N)$  of  $z \in S$  with 'small denominators':

$$\begin{aligned} S(N) &= \left\{ z \in S : \text{there exist } p, q \text{ such that } \left| z - \frac{p}{q} \right| < \frac{2}{|q|N}, 1 \leq |q| < \frac{N}{\log N} \right\} \\ &= \bigcup_{1 \leq |q| < N/\log N} B\left(q, \frac{2}{|q|N}\right). \end{aligned}$$

The measure  $|S(N)|$  of  $S(N)$  satisfies

$$\begin{aligned} |S(N)| &= \sum_{1 \leq |q| < N/\log N} |B(q, 2/|q|N)| \ll \sum_{1 \leq |q| < N/\log N} (|q|N)^{-2} |q|^2 \\ &\ll N^{-2} \sum_{1 \leq |q| < N/\log N} 1 \ll N^{-2} (N/\log N)^2 \ll (\log N)^{-2} \rightarrow 0 \end{aligned}$$

as  $N \rightarrow \infty$ . Choose  $\rho(N) = 2N^{-2} \log N$ . Then since  $|q| > N/\log N$  implies  $\rho(N) > 2/|q|N$ ,

$$\bigcup_{N/\log N < |q| \leq N} B(q, \rho(N)) \supset \bigcup_{N/\log N < |q| \leq N} B\left(q, \frac{2}{|q|N}\right) \rightarrow S$$

in measure as  $N \rightarrow \infty$  and so  $\mathcal{R}$  is ubiquitous with respect to  $\rho(N) = 2 \log N/N^2$  for any  $S \subseteq I^2$  and so for  $S = I^2$ . But since  $\Psi$  is decreasing, by [45],

$$(6.2) \quad \dim_{\mathbb{H}} V_v^* \geq \dim \mathcal{R} + \text{codim } \mathcal{R} \limsup_N \frac{\log \rho(N)}{\log \Psi(N)} = \frac{4}{v+1},$$

where  $\dim$  is the topological dimension, so that  $\dim \mathcal{R} = 0$ , the codimension  $\text{codim } \mathcal{R}$  in  $\mathbb{C}$  (regarded as  $\mathbb{R}^2$ ) of  $\mathcal{R}$  is 2 and  $\Psi(N) = N^{-v-1}$ .

The required result follows on combining the two complementary inequalities (6.1) and (6.2).

In fact, since  $S$  was an arbitrary open square,  $\mathcal{R}$  is *locally* ubiquitous and it has been shown that the local ubiquity of  $\mathcal{R}$  also implies the divergence case of Khintchine's theorem [14].

Note that even though one might expect Diophantine approximation in  $\mathbb{C}$  with respect to Gaussian integers to be similar to simultaneous Diophantine approximation in the real plane  $\mathbb{R}^2$ , this is not at all the case. The analogues of Dirichlet's theorem, Khintchine's theorem and the Jarník–Besicovitch theorem are quite different in the two cases. Indeed the complex Dirichlet's Theorem and the Jarník–Besicovitch theorem are closer to the real, one-dimensional case. Only the analogue of Jarník's theorem on badly approximable numbers remains unchanged and in this case there is a substantial difference in the proofs of the two theorems.

## 7. Applications

The connection between the physical phenomenon of resonance and Diophantine equations can give rise to the notorious problem of small denominators in which solutions to a variety of questions contain denominators that can become arbitrarily small. When these small denominators are related to very well approximable points, it is sometimes possible to impose appropriate Diophantine conditions which overcome the problem by excluding the offending denominators without significantly affecting the validity of the solution. The techniques developed in the metrical theory of Diophantine approximation lend themselves to this and in particular the Jarník–Besicovitch theorem allows the determination of the Hausdorff dimension of the associated exceptional sets. Some examples of problems involving small denominators and the associated exceptional sets are now discussed. We begin with a very simple example.

**7.1. Partial differential equations.** Diophantine approximation has been applied to the wave equation ([97] and more recently [51]), as well as to the Schrödinger equation [83]. For an extensive treatment of Diophantine problems related to partial differential equations, the reader is referred to [104].

We will illustrate the applications of Diophantine approximation by Gaussian rationals by considering the following innocuous first-order linear complex partial differential equation,

$$(7.1) \quad \alpha \frac{\partial f(z, t)}{\partial t} + \beta \frac{\partial f(z, t)}{\partial z} = g(z, t),$$

where  $z \in \{x + iy \in \mathbb{C} : x, y \geq 0\}$ ,  $t \geq 0$  and  $\alpha, \beta$  are non-zero complex numbers. That is, we are studying the partial differential equation on the interior of the set defined above under the additional assumption that the functions involved as well as all their derivatives may be extended to the whole set. Assume that  $g(z, t)$  is smooth (*i.e.*,  $C^\infty$ ) and can be expressed in the form

$$g(z, t) = \sum_{a, b, c, d \in \mathbb{Z}} g_{a, b, c, d} \exp((a + ib)z + (c + id)t), \quad g_{a, b, c, d} \in \mathbb{C}.$$

We seek smooth solutions to this equation of the same form, namely

$$(7.2) \quad f(z, t) = \sum_{a, b, c, d \in \mathbb{Z}} f_{a, b, c, d} \exp((a + ib)z + (c + id)t), \quad f_{a, b, c, d} \in \mathbb{C}.$$

Thus, we are not just looking for solutions but rather trying to solve the partial differential equation subject to boundary conditions.



As usual, we solve the problem formally by substituting these two expressions into (7.1) and identifying coefficients on either side of the equality. Isolating the coefficients of  $f$ , we get

$$(7.3) \quad f_{a,b,c,d} = \frac{\frac{1}{\alpha}}{\frac{\beta}{\alpha}(a+ib) + (c+id)} g_{a,b,c,d}.$$

We need the coefficients to decay fast enough so that both the series (7.2) and its derivatives are convergent. Since  $g$  is already smooth, the coefficients  $g_{a,b,c,d}$  decay rapidly and so are not obstructing this convergence. However, the denominator of the fraction may become small and cause the the coefficients  $f_{a,b,c,d}$  to become large enough to pose a problem. In order to avoid this, we see that it is certainly sufficient for the denominator to be bounded from below by some polynomial in  $a, b, c, d$ , *i.e.*, we require for some  $K, v > 0$  and for all  $(a, b, c, d) \in \mathbb{Z}^4 \setminus \{0\}$ ,

$$(7.4) \quad \left| \frac{\beta}{\alpha}(a+ib) + (c+id) \right| \geq K \max\{|a|, |b|, |c|, |d|\}^{-v}.$$

Since we are only concerned with small denominators, we can assume without loss of generality that  $|a+ib| \asymp |c+id|$ , so that after adjusting  $K$  we can drop the dependence on  $c, d$  on the right-hand side of (7.4) and require

$$\left| \frac{\beta}{\alpha} - \frac{p}{q} \right| \geq \frac{K}{|q|^{v+1}},$$

where  $p = a+ib$  and  $q = c+id$ . Thus we require  $\beta/\alpha$  to be of complex Diophantine type  $(K, v)$  for some  $K, v$ . The complement of this set is  $E^* = \bigcap_{v>1} E_v^*$ , where

$$E_v^* = \left\{ z \in \mathbb{C} : \text{for any } K > 0, \left| z - \frac{p}{q} \right| < \frac{K}{|q|^{v+1}} \text{ for some } p, q \in \mathbb{Z}[i] \right\}.$$

But it can be readily verified by an argument similar to that giving the inclusion (3.13), that for each  $\varepsilon > 0$ ,

$$W_{v+\varepsilon}^* \subset E_v^* \subset W_v^*,$$

whence by the properties of Hausdorff dimension given in §2.2 and the Jarník-Besicovitch theorem for Gaussian rational approximation (Theorem 6.1),

$$\dim_{\mathbb{H}} E_v^* = \frac{4}{v+1}$$

for  $v \geq 1$  and so  $\dim_{\mathbb{H}} E^* = \lim_{v \rightarrow \infty} \dim_{\mathbb{H}} E_v^* = 0$ . Thus the exceptional set associated with the inequality (7.4) failing to hold has Hausdorff dimension zero.

**7.2. The rotation number.** The rotation number  $\rho(f)$  is a measure of how far ‘on average’ a continuous, orientation preserving homeomorphism  $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  moves a point round the circle. We will not give the fairly lengthy definition which is explained in [16, 43, 71, 96] but content ourselves with the observation that the rotation number of a rotation  $r_\alpha$  by an angle  $2\pi\alpha$ , where  $0 \leq \alpha < 1$ , given by

$$r_\alpha(z) = ze^{2\pi i\alpha}$$

is, naturally enough,  $\alpha$ . The rotation number is a nice example of how Diophantine properties can arise in analysis as it can be shown that  $\rho(f)$  is irrational if and only if  $f$  has no periodic points (see [71, Chap. 11,12] or [96, Chap. 1] for more details).

If  $\rho(f)$  is irrational then for  $z \in \mathbb{S}^1$ , the closure  $A$  of the orbit

$$\omega(z) = \{f^n(z) : n \in \mathbb{N}\}$$

does not depend on  $z$  and either  $A$  is perfect and nowhere dense or  $A = \mathbb{S}^1$ . In the latter case  $f$  is transitive and is topologically conjugate to the rotation  $r_{\rho(f)}$  by the rotation number  $\rho(f)$  of  $f$ , *i.e.*, there exists an orientation preserving homeomorphism  $\varphi: \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that

$$f = \varphi^{-1} \circ r_{\rho(f)} \circ \varphi,$$

usually written  $f \sim r_{\rho(f)}$ . This can be regarded as obtaining a normal form for  $f$  and is analogous to diagonalising a matrix.

Denjoy showed that when  $f$  is  $C^2$  and  $\rho(f)$  is irrational, then  $f$  is topologically conjugate to the rotation by  $\rho(f)$ . More subtle aspects arise when additional differentiability conditions are imposed on the conjugation. For example, every  $C^\infty$  diffeomorphism  $f$  of the circle is  $C^\infty$  conjugate to a rotation if and only if the rotation number  $\rho(f)$  of  $f$  is of Diophantine type [136]. In the analytic case, the rotation numbers of real analytic diffeomorphisms form a set lying strictly between  $\mathcal{D}$  and the set of Bruno numbers [137, p. 92]. As in the preceding example, the Diophantine condition arises from the denominator  $1 - e^{2\pi i \rho k}$  in the coefficients for a Fourier series solution of a linearised auxiliary equation in an iterative Newton's tangent method argument, modified at each step to retain convergence. In order to guarantee convergence of the iterative argument and of the Fourier series, the inequalities

$$|1 - e^{2\pi i \rho k}| \geq 2 \left| \sin \left( \frac{\rho k - j}{2} \right) \right| \geq \frac{2|\rho k - j|}{\pi},$$

where  $\rho k - j \in [0, 2\pi)$ , must be set against the very rapid decay of the corresponding Fourier coefficient  $f_k$  ( $\ll k^{-N}$  for any  $N > 0$ ) in the numerator. It suffices that  $\rho$  is of Diophantine type  $(K, v)$  for some  $K > 0$ ,  $v > 1$ , since then

$$|k\rho - j| \geq \frac{K}{k^v}, \quad \text{i.e.,} \quad \left| \rho - \frac{j}{k} \right| \geq \frac{K}{k^{v+1}},$$

for all  $j/k \in \mathbb{Q}$ .

Now we saw in §3.2.2 that when  $v > 1$ , almost all real numbers are of Diophantine type  $(K, v)$  for some positive  $K$ , *i.e.*, the set

$$\mathcal{D}_v = \bigcup_{K>0} \{ \alpha \in \mathbb{R} : |\alpha - p/q| \geq Kq^{-1-v} \text{ for each } p/q \in \mathbb{Q} \}$$

is of full Lebesgue measure. Thus the complementary set  $E_v$

$$E_v = \bigcap_{K>0} \{ \alpha \in \mathbb{R} : |\alpha - p/q| < Kq^{-2-v} \text{ for some } p/q \in \mathbb{Q} \}$$

is null for  $v > 1$  (see §3.2.2). As in the preceding section, its Hausdorff dimension can be determined using the inclusions (3.13) and the Jarník–Besicovitch theorem (Theorem 4.4)

$$\dim_{\mathbb{H}} E_v = \dim_{\mathbb{H}} W_v = \frac{2}{v+1}$$

for  $v \geq 1$ . If the rotation number of the smooth circle function  $f$  does not lie in  $E = \bigcap_{v>1} E_v = \lim_{v \rightarrow \infty} E_v$  ( $E_v$  decreases as  $v$  increases), then  $f$  is smoothly conjugate to a rotation. The Hausdorff dimension of the exceptional set is

$$(7.5) \quad \dim_{\mathbb{H}} E = \lim_{v \rightarrow \infty} \frac{2}{v+1} = 0,$$

*i.e.*, the complement of  $\mathcal{D}$  has Hausdorff dimension 0.

**7.3. The structure of Julia and Fatou sets.** Let  $R(z) = P(z)/Q(z)$  be a rational map on the Riemann sphere  $\mathbb{C}_\infty$ . A famous result due to Sullivan [90, 132, 133] (see also [102]) states that the Fatou set  $F_R = \mathbb{C}_\infty \setminus J_R$  of such a map has countably many periodic connected components. These connected components were further classified according to the type of periodic cycles they are associated with. As the Julia set is the complementary set of the Fatou set, this classification also deals with the structure of the Julia set on which the dynamics of  $Q$  is chaotic.

Cycles may be classified according to the value of  $dR/dz$  on the points of the cycle. By the chain rule, this value remains constant,  $\lambda$  say. The cycle is attracting (resp. repelling) as  $|\lambda| < 1$  (resp.  $|\lambda| > 1$ ). Repelling cycles are part of the Julia set  $J_R$ , so no parts of the Fatou set corresponds to this case. For attracting cycles, an associated periodic connected component of the Fatou set is in fact the immediate basin of attraction of this cycle, *i.e.* the union of the connected components of the Fatou set containing the points of the cycle in question.

When  $|\lambda| = 1$ , Diophantine properties of  $\lambda$  determine the behaviour of the dynamics of  $R$  and hence the structure of the Julia and Fatou sets. In this case,  $\lambda = \exp(2\pi i\alpha)$ . When  $\alpha$  is rational, the cycle is said to be parabolic and the associated component of the Fatou set is again the immediate basin of attraction.

In the case when  $\alpha$  is irrational, one is interested in the situation when the domain of the Fatou set corresponding to the periodic point is a collection of Siegel discs associated with the cycle. That is, we are looking for sets on which the dynamics are topologically conjugate to a rotation of a disc. Such rotation numbers were studied in the preceding section, where the exceptional set associated with the failure of this condition was shown to have Hausdorff dimension zero.

**7.4. Linearising diffeomorphisms.** Suppose the complex analytic diffeomorphism  $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$  has a fixed point. Without loss of generality, this can be taken to be the origin, so that  $f(0) = 0$ . If in a neighbourhood of 0,  $f$  is analytically (biholomorphically) conjugate to its linear part or Jacobian  $Df|_0 = A$  say, the function  $f$  is said to be linearisable. The linearising transformation  $\phi$  is given by the solution to the functional equation

$$f = \phi^{-1} \circ A \circ \phi,$$

known as Schröder's equation when  $n = 1$ . Thus linearisation is similar to conjugating a circle map to a rotation, discussed above in §7.2. And problems of small denominators arise when the eigenvalues  $\alpha_1, \dots, \alpha_n$  of  $A$  are close to being resonant in the sense that they are close to satisfying the equation

$$\alpha_k = \prod_{r=1}^n \alpha_r^{j_r}$$

for all  $\mathbf{j} = (j_1, \dots, j_n)$  with  $j_r \in \mathbb{N} \cup \{0\}$ ,  $r = 1, \dots, n$  and  $|\mathbf{j}|_1 = \sum_r |j_r| \geq 2$ . Linearisation is well understood when  $n = 1$  and the diffeomorphism  $f : \mathbb{C} \rightarrow \mathbb{C}$  with  $f(0) = 0$  can be linearised when  $|(Df|_0)| = |f'(0)| \neq 1$ . The interesting case when  $|f'(0)| = 1$  is closely related via lifts to the conjugacy of a circle map to a rotation, discussed in §7.2 above, and necessary and sufficient conditions for the linearisation of a diffeomorphism  $f : \mathbb{C} \rightarrow \mathbb{C}$  are known [137]. On the other hand, when  $n \geq 2$ , the problem of finding which functions can be linearised is very difficult but Siegel's normal forms theorem [123, 124] gives sufficient conditions on  $Df|_0$

for the existence of a linearising transformation  $\phi$ . The point  $(\alpha_1, \dots, \alpha_n)$  in  $\mathbb{C}^n$  is said to be of *multiplicative type*  $(K, v)$  [7, p. 191] if

$$(7.6) \quad \left| \alpha_k - \prod_{r=1}^n \alpha_r^{j_r} \right| \geq K |\mathbf{j}|_1^{-v}$$

for all  $\mathbf{j} \in (\mathbb{N} \cup \{0\})^n$  with  $|\mathbf{j}|_1 \geq 2$ . Siegel showed that if the vector  $(\alpha_1, \dots, \alpha_n)$  of eigenvalues of  $Df|_0$  is of multiplicative type  $(K, v)$  for some  $K > 0$  and  $v > 0$ , then  $f$  can be linearised in a neighbourhood (further details are in [7, 61, 93]). To stop Siegel's condition being too restrictive, one chooses  $v > (n-1)/2$ , since then the set of points of multiplicative type  $(K, v)$  has full measure for any  $K > 0$ . However, the neighbourhood of linearisation decreases as  $v$  increases (it also depends on  $K$  but less significantly) and so we do not want  $v$  to be too large.

Let  $\mathcal{E}_v$  denote the exceptional set of points in  $\mathbb{C}^n$  (regarded as  $\mathbb{R}^{2n}$ ) which for a given exponent  $v$ , fail to be of multiplicative type  $(K, v)$  for any  $K > 0$  and so fail to satisfy the conditions of Siegel's theorem and suppose if  $v > (n-1)/2$ . Then  $\mathcal{E}_v$  is null and its Hausdorff dimension is given by (4.2); namely,

$$(7.7) \quad \dim \mathcal{E}_v = 2(n-1) + \frac{n+1}{v+1}.$$

This result is established by means of an exponential map which preserves the Hausdorff dimension and allows  $\mathcal{E}_v$  to be replaced by a set involving a more general kind of additive type (see next section) with a simpler structure [46].

**7.5. Lyapunov stability of vector fields.** Consider the differential equation

$$(7.8) \quad \dot{\mathbf{z}} = A\mathbf{z} + Q(\mathbf{z}) \in \mathbb{C}^n,$$

where  $A$  is a  $n \times n$  complex matrix and the holomorphic functions  $Q: \mathbb{C}^n \rightarrow \mathbb{C}^n$  and  $\partial Q_k / \partial z_j$  vanish at the origin 0. The obvious solution  $\mathbf{z}_0(t) = 0$  is said to be *future (resp. past) stable* if points near 0 remain there under evolution by (7.8) to the future (resp. past). More precisely, the solution  $\mathbf{z}(t)$  is future (resp. past) stable if for every neighbourhood  $N$  of 0, there exists a subneighbourhood  $N'$  with  $0 \in N' \subset N$  such that  $\mathbf{z}(0) \in N'$  guarantees that  $\mathbf{z}(t) \in N$  for all  $t > 0$  (resp.  $t < 0$ ). By a well known theorem of Lyapunov [93], the solution is future stable if the real parts of the eigenvalues of  $A$  are at most 0 and past stable if the real parts are at least 0. Thus for the solution  $\mathbf{z}(t)$  to be future and past stable or simply stable, the eigenvalues must have zero real part and so must be purely imaginary. The stability of the solution  $\mathbf{z}(t)$  is determined by a remarkable theorem due to Carathéodory and Cartan [93] which asserts that stability is equivalent to  $A$  being diagonalisable with purely imaginary eigenvalues and the vector field being holomorphically linearisable in a neighbourhood of the origin. By Siegel's normal form theorem [123, 124], also used in the preceding section, this last condition holds if the eigenvalues  $\gamma_k = i\lambda_k$ ,  $\lambda_k \in \mathbb{R}$ , satisfy for some  $K > 0$ ,  $v > n-1$ ,

$$(7.9) \quad \left| \lambda_k - \sum_{r=1}^n \lambda_r j_r \right| \geq K |\mathbf{j}|_1^{-v}$$

for all  $\mathbf{j} \in (\mathbb{N} \cup \{0\})^n$  with  $|\mathbf{j}|_1 \geq 2$  (note that this is the additive form of (7.6)). The complement of this set of points  $\boldsymbol{\lambda} \in \mathbb{R}^n$  of additive type is the set of  $\boldsymbol{\alpha} \in \mathbb{R}^n$

such that for any  $K > 0$ ,

$$(7.10) \quad \left| \alpha_k - \sum_{r=1}^n \alpha_r j_r \right| < K |\mathbf{j}|_1^{-v}$$

for some  $\mathbf{j} \in \mathbb{Z}^n$ . This set,  $\widehat{E}_v$  say, is related to and has the same metrical character as the set

$$(7.11) \quad \widehat{L}_v = \{ \boldsymbol{\alpha} : |\mathbf{q} \cdot \boldsymbol{\alpha}| < |\mathbf{q}|_\infty^{-v} \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n \}$$

and in fact by an argument similar to that giving (3.13), for any  $\varepsilon > 0$ ,

$$\widehat{L}_{v+\varepsilon} \subset \widehat{E}_v \subset \widehat{L}_v$$

(see [16, Sect. 7.5.2]). This inclusion implies that the two sets  $\widehat{L}_v$  and  $\widehat{E}_v$  have the same Hausdorff dimension. The set  $\widehat{L}_v$  is also related to  $L_v$  and roughly speaking, has one degree of freedom less. The Hausdorff dimension of  $\widehat{L}_v$  is a special case of an ‘absolute value’ analogue, proved by Dickinson [38], of the general form of the Jarník–Besicovitch theorem:

$$(7.12) \quad \dim \widehat{L}_v = \dim \widehat{E}_v = n - 1 + \frac{n}{v+1}$$

when  $v > n - 1$  (see also [47]); note that  $\widehat{L}_v = \mathbb{R}^n$  otherwise. Thus the exceptional set of eigenvalues for which the solution to (7.8) cannot be shown to be stable has Hausdorff dimension 0.

**7.6. Kolmogorov-Arnol’d-Moser theory.** The stability of the solar system is one of the oldest problems in mechanics [94]. It is of course a special case of the  $N$  body problem of understanding the motion of  $N$  point masses subject only to gravitational attraction, with all other forces neglected. When  $N = 2$ , the solution is well known and the periodic solutions in which the bodies move in an ellipse about their centre of mass persist forever. For  $N \geq 3$ , however, the situation is extraordinarily complicated and is far from being fully understood, even for solar systems where the mass  $m_N$  of the sun is much greater than the masses of the  $n = N - 1$  planets. If, as a first approximation, the centre of mass of the system is assumed to coincide with that of the sun and if the gravitational interactions between the planets and other effects are neglected, the system decouples into  $n$  two-body problems, in which each planet describes an elliptical orbit around the sun, with period  $T_j$  say and frequency  $\omega_j = 2\pi/T_j$ ,  $j = 1, \dots, n$ .

For each vector  $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$  of frequencies in the  $n$ -dimensional torus  $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ , the map  $\varphi_\omega: \mathbb{R} \rightarrow \mathbb{T}^n$  given by

$$\varphi_\omega(t) = \varphi_\omega(0) + t\boldsymbol{\omega}$$

is a quasi-periodic flow on the torus.

The case  $n = 1$  corresponds to uniform motion around a circle and so is periodic. When the frequencies are all rational, the flow is periodic. If the frequencies are not all rational, then by Kronecker’s theorem [57], the flow winds round the torus, densely filling a subspace of dimension given by the number of rationally independent frequencies. Thus when the frequencies are independent, the closure of  $\varphi_\omega(\mathbb{R})$  is the torus  $\mathbb{T}^n$  and solutions will persist for ever. Gravitational interactions between the planets are represented by a small perturbation of the original Hamiltonian describing the system. Stability then reduces to the solutions of the perturbed Hamiltonian system continuing to wind round a perturbed invariant

torus. Of course this model is idealised and takes no account of the final fate of the universe.

Details of the history of the solution to this problem are in [93, Chap. 1]. Siegel's success in overcoming the related 'small denominator' problem in the linearisation of complex diffeomorphisms (see §7.4, §7.5) was followed by Kolmogorov's conjecture that quasi-periodic solutions for a perturbed analytic Hamiltonian system not only existed but were relatively abundant in the sense that they formed a complicated Cantor type set of positive Lebesgue measure [81]. This was proved completely in 1962 by Arnol'd [5] and independently Moser proved an analogous result for sufficiently smooth 'twist' maps [92, 125]. The results imply that for planets very much smaller than the sun and for the majority of initial conditions in which the orbits are close to co-planar circles, distances between the bodies will remain perpetually bounded, *i.e.*, the planets will never collide, escape or fall into the sun. Further details can be found in [8] and [36].

The differentiability and Diophantine conditions were relaxed substantially by Rüssmann in [107, 108, 109] (see also [9, Sect. 6.3], [48], [93, Chap. 1], [103]). Another approach is to use 'averaging' methods [7]; this can involve Diophantine approximation on manifolds [44], see also [109].

As in the above examples, it turns out that in order to ensure convergence of a Fourier series and an infinite dimensional extension of Newton's iterative tangent method, the frequencies  $\omega = (\omega_1, \dots, \omega_n)$  must satisfy a Diophantine condition, which in this case is

$$(7.13) \quad |\mathbf{q} \cdot \boldsymbol{\omega}| = |q_1\omega_1 + \dots + q_n\omega_n| \geq K|\mathbf{q}|_1^{-v},$$

for some positive constants  $K = K(\boldsymbol{\omega})$  and  $v = v(\boldsymbol{\omega})$  for all non-zero  $\mathbf{q} = (q_1, \dots, q_n)$  vectors in  $\mathbb{Z}^n$ . The exponent  $v$  is subject to two conflicting requirements. It should be large enough ( $v > n - 1$ ) to ensure that the Diophantine condition above is not too restrictive, but small enough to ensure that the perturbation has physical significance and that the stability is robust. The proof breaks down when the frequencies lie in the complementary exceptional set  $E_v$ , say, of frequencies which are close to resonance in the sense that, given any  $K > 0$ , there exists a  $\mathbf{q} \in \mathbb{Z}^n$  such that

$$|\mathbf{q} \cdot \boldsymbol{\alpha}| < K|\mathbf{q}|_1^{-v}.$$

The set  $E_v$  is related to the set  $\widehat{L}_v$  discussed in the preceding section and allows us to deduce that

$$\dim_{\mathbb{H}} E_v = \dim_{\mathbb{H}} \widehat{L}_v = n - 1 + \frac{n}{v + 1}$$

when  $v > n - 1$ .

Thus Hausdorff dimension plays its part in other branches of mathematics and mechanics, as well as in number theory. It has even been of use to Mandelbrot in arousing the interest of mathematicians in his profoundly original and imaginative ideas [29, Chapter 2].

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