ON SYMMETRY OF EXTREMALS IN SEVERAL EMBEDDING THEOREMS

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We consider the best constant problem in the following embedding theorem:

$$\lambda(r,k,p,q) = \min \frac{\|f^{(r)}\|_{L_p[-1,1]}}{\|f^{(k)}\|_{L_q[-1,1]}}. \tag{1}$$

Here $r,k \in \mathbb{Z}_+$, $r>k$, $1 \le p,q \le \infty$, and minimum is taken over the set[1] $\overset{\circ}{W}{}^r_p(-1,1)$ that is

$$\left\{ f \in \mathcal{AC}^{r-1}[-1,1] \,\middle|\, f^{(r)} \in L_p(-1,1); \quad f^{(j)}(\pm 1) = 0,\ j=0,1,\dots,r-1 \right\}.$$

In the case of $r=1$, $k=0$ the problem (1) is well known. For $p=q=2$ it was solved by V.A. Steklov [13], for the arbitrary $p=q$ – by V.I. Levin [9] (see also [5, Sect. 7.6] for $p=q=2k$, $k\in\mathbb{N}$). Finally, E.Schmidt [14] obtained the following result for arbitrary p and q:

$$\lambda(1,0,p,q) = \frac{\mathfrak{F}\left(\frac{1}{q}\right)\mathfrak{F}\left(\frac{1}{p'}\right)}{2^{\frac{1}{q}+\frac{1}{p'}}\,\mathfrak{F}\left(\frac{1}{q}+\frac{1}{p'}\right)},$$

where $\mathfrak{F}(s) = \frac{\Gamma(s+1)}{s^s}$ and $p' = \frac{p}{p-1}$. Note that the extremal in this problem is even function.

For $r=2$, $k=1$ the problem (1) is reduced to the best constant problem in the Poincare inequality, which was also solved by Steklov [12] for $p=q=2$. However, the investigation of the general case was completed only at the beginning of the XXI century and required efforts of many authors ([3], [4], [2], [1], [8]; the final result was obtained in [10]). Namely, it turned out that for $q\le 3p$ the equality $\lambda(2,1,p,q) = 2\lambda(1,0,p,q)$ holds and the extremal is even function. However, for $q>3p$ we have $\lambda(2,1,p,q) < 2\lambda(1,0,p,q)$ and the extremal is asymmetrical.

This result, as well as some calculations, leads to the following conjecture.

Conjecture: *For $k \,\vdots\, 2$ the extremal in the problem (1) is even function for all admissible r,p,q. If $k \not\,\vdots\, 2$ then for all admissible r and p there exists $\widehat{q}(r,k,p) > p$ such that the extremal is even for $q \le \widehat{q}$ and is asymmetrical for $q > \widehat{q}$.*

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[1]The case of $p=1$ is special. In this case minimum should be taken over the set of f, which $(r-1)$-th derivative has bounded variation on $[-1,1]$.

For now, up to our knowledge, symmetry or asymmetry of the extremal is proved for the following parameters' values:

Article	Symmetry				Asymmetry			
	r	k	p	q	r	k	p	q
[14]	1	0	$\forall$	$\forall$				
[2]					2	1	$\forall$	$> 3p$
[10]	2	1	$\forall$	$\leq 3p$				
[11][2]	$k+1$	$\forall$	2	2				
[15]	$2,3$	0	$\forall$	∞				
[16]	$\forall$	0	2	∞				
[7]	$\forall$	$0,2$	2	∞	$\forall$	1	2	∞

In this paper we consider the case $p = 2$, $q = \infty$. The main result is the following.

Theorem 1. *Let $p = 2$, $q = \infty$.*

1. *If $k \not\vdots\, 2$ then for all $r > k$ the extremal in the problem (1) is asymmetrical.*

2. *If $k \,\vdots\, 2$ then for all $r > k$ even function provides local minimum to the functional (1).*

We didn't manage to obtain complete solution for even k. The following theorem is developing results of [7].

Theorem 2. *Let $p = 2$, $q = \infty$. For $k = 4, 6$ and all $r > k$ the extremal in the problem (1) is even. Furthermore,*

$$\lambda(r, 4, 2, \infty) = \frac{1}{2^{r-2}(r-3)!}\sqrt{\frac{3(4r^2 - 24r + 39)}{2(2r-9)}};$$

$$\lambda(r, 6, 2, \infty) = \frac{1}{2^{r-2}(r-4)!}\sqrt{\frac{192r^4 - 3456r^3 + 23372r^2 - 70240r + 79065}{2(2r-13)}}.$$

Proof of the theorem 1. Following [7], we introduce the function

$$A_{r,k}(x) = \max\{|f^{(k)}(x)| : f \in \overset{\circ}{W}{}_2^r(-1,1), \|f^{(r)}\|_{L_2(-1,1)} \leq 1\}. \tag{2}$$

Obviously, $\max\limits_{[-1,1]} A_{r,k}(x) = \lambda^{-1}(r, k, 2, \infty)$.

We use the explicit formula, attained in [7]:

$$A_{r,k}^2(x) = \big(Q_{r-k-1}(x)\big)^2 \cdot \frac{1 - x^2}{2(2r - 2k - 1)} - \sum_{n=r-k}^{r-1} \big(Q_n^{(n+k-r)}(x)\big)^2 \big(n + \frac{1}{2}\big), \tag{3}$$

where

$$Q_n = \frac{1}{2^n n!} \cdot (1 - x^2)^n.$$

[2] see also [6].

Moreover, the function f providing the maximum in (2) is given by the following formula:

$$f(t) = \sum_{n\geq r} \big(n+\frac{1}{2}\big)\cdot Q_n^{(n+k-r)}(x) Q_n^{(n-r)}(t).$$

It is easy to see that this function is symmetrical (even for k even and odd for k odd) if and only if $x=0$.

Thus, to prove **Theorem 1** it is sufficient to prove the following lemma.

Lemma. *For odd k the point $x=0$ provides the local minimum to the function $A_{r,k}(x)$. For even k it provides the local maximum.*

Let us note that

$$Q_n^{(s)}(0) = \begin{cases} 0 & \text{for odd } s; \\ \frac{(-1)^k s!}{2^n n!} C_n^k & \text{for } s = 2k. \end{cases} \tag{4}$$

Since the function $A_{r,k}$ is even we have $A'_{r,k}(0)=0$. Taking into account (4) we have

$$\begin{aligned}
(A^2_{r,k})''(0) =& -\frac{2}{(r-k-1)!^2 2^{2r-2k-1}} \\
& - \sum_{s=0}^{k-1} 2\big(r-k+s+\frac{1}{2}\big)\Big(\big(Q_{r-k+s}^{(s+1)}(0)\big)^2 + Q_{r-k+s}^{(s)}(0) Q_{r-k+s}^{(s+2)}(0)\Big) \\
=& -\frac{2}{(r-k-1)!^2 2^{2r-2k-1}} \\
& - 2\sum_{t=0}^{\lfloor\frac{k-2}{2}\rfloor} \Big(\frac{(2t+2)! C_{r-k+2t+1}^{t+1}}{2^{r-k+2t+1}(r-k+2t+1)!}\Big)^2 \big(r-k+2t+\frac{3}{2}\big) \\
& + 2\sum_{t=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(2t)!(2t+2)! C_{r-k+2t}^{t} C_{r-k+2t}^{t+1}}{2^{2(r-k+2t)}(r-k+2t)!^2}\big(r-k+2t+\frac{1}{2}\big).
\end{aligned} \tag{5}$$

Let k be odd. In this case the second sum contains one additional term comparing to the first one. We separate the term corresponding to $t=0$ and get

$$\begin{aligned}
(A^2_{r,k})''(0) =& \frac{4(r-k)}{2^{2(r-k)}(r-k)!^2}\big(r-k+\frac{1}{2}\big) - \frac{2}{(r-k-1)!^2 2^{2r-2k-1}} \\
&+2\sum_{t=1}^{\frac{k-1}{2}} \Bigg(\frac{(2t)!(2t+2)! C_{r-k+2t}^{t} C_{r-k+2t}^{t+1}}{2^{2(r-k+2t)}(r-k+2t)!^2}\big(r-k+2t+\frac{1}{2}\big) \\
&\quad -\Big(\frac{(2t)! C_{r-k+2t-1}^{t}}{2^{r-k+2t-1}(r-k+2t-1)!}\Big)^2\big(r-k+2t-\frac{1}{2}\big)\Bigg).
\end{aligned}$$

The expression in the first line is equal to $\frac{1}{2^{2r-2k-1}(r-k-1)!^2(r-k)} > 0$. We denote it by M and factor it out:

$$\frac{(A^2_{r,k})''(0)}{M} = 1 - 2\sum_{t=1}^{\frac{k-1}{2}} \frac{(2t)!^2 (r-k-1)!^2 (r-k)\big[(r-k)^2 + (r-k)(2t-1) - 2t - \frac{1}{4}\big]}{2^{4t-1} t!^2 (r-k+t-1)!^2 (r-k+t)}.$$

The term in square brackets equals $(r-k+t-1)(r-k+t)-(t+\frac{1}{2})^2$, and we obtain

$$\begin{aligned}\frac{(A^2_{r,k})''(0)}{M} &= 1-2\sum_{t=1}^{\frac{k-1}{2}}\Big(\frac{(2t)!^2(r-k-1)!^2(r-k)(r-k+t-1)}{2^{4t-1}t!^2(r-k+t-1)!^2} \\ &\qquad -\frac{(2t)!^2(r-k-1)!^2(r-k)(t+\frac{1}{2})^2}{2^{4t-1}t!^2(r-k+t-1)!^2(r-k+t)}\Big) \\ &= 1-2\sum_{t=1}^{\frac{k-1}{2}}\big(F(t)-F(t+1)\big) = 1-2F(1)+2F\big(\frac{k+1}{2}\big),\end{aligned}$$

where $F(t)=\frac{(2t)!^2(r-k-1)!^2(r-k)(r-k+t-1)}{2^{4t-1}t!^2(r-k+t-1)!^2}$. Obviously, $2F(1)=1$, and therefore

$$(A^2_{r,k})''(0) = 2MF\big(\frac{k+1}{2}\big) = \frac{\big(k+1\big)!^2\big(r-\frac{k+1}{2}\big)}{2^{2r-1}\big(\frac{k+1}{2}\big)!^2\big(r-\frac{k+1}{2}\big)!^2} > 0,$$

which proves the first part of Lemma.

Now let $k=2\ell$ be even. Then the number of summands in both sums in (5) equals $\ell-1$. Let us separate the term corresponding to $t=0$ from the second sum and add the term corresponding to $t=\ell$, which we subtract later. Then, similarly to the previous case, we get

$$(A^2_{r,k})''(0) = M\cdot\Big(1-2\sum_{t=1}^{\ell}\big(F(t)-F(t+1)\big)\Big)-R = 2MF\big(\ell+1\big)-R,$$

where $R=\frac{k!(k+2)!}{2^{2r-1}r!^2}C_r^\ell C_r^{\ell+1}\big(r+\frac{1}{2}\big)$.

After simplifying this expression we get

$$\begin{aligned}(A^2_{r,k})''(0) &= \frac{(2\ell+2)!^2(r-\ell)}{2^{2r+1}(\ell+1)!^2(r-\ell)!^2}-\frac{(2\ell)!(2\ell+2)!\big(r+\frac{1}{2}\big)}{2^{2r-1}\ell!(\ell+1)!(r-\ell)!(r-\ell-1)!} \\ &= -\frac{(2\ell)!(2\ell+2)!(r-\ell)}{2^{2r-1}\ell!(\ell+1)!(r-\ell)!(r-\ell-1)!} < 0,\end{aligned}$$

which proves the second part of the Lemma. Thus, Theorem 1 follows. □

Proof of the theorem 2. We use numerical-analytical method. Theoretically this scheme can be applied to any fixed k, but it requires more and more calculations when k is increasing.

It is obvious from the formula (3) that $A^2_{r,k}(x)=P_{r,k}(x^2)\cdot(1-x^2)^{2r-2k-1}$, where $P_{r,k}$ is a polynomial of degree k. Therefore

$$\frac{d\big[A^2_{r,k}(\sqrt{x})\big]}{dx} = P^{(1)}_{r,k}(x)\cdot(1-x)^{2r-2k-2},$$

where $P^{(1)}_{r,k}$ is also a polynomial of degree k. It is easy to see that $P^{(1)}_{r,k} < 0$ in a left semi-neighborhood of one. From the second statement of Theorem 1 we deduce that $P^{(1)}_{r,k} < 0$ in a right semi-neighborhood of zero.

We construct a polynomial $\widetilde{P}_k$, such that all coefficients of the polynomial $\widetilde{P}_k(r\cdot)$ do not exceed the corresponding coefficients of $-P^{(1)}_{r,k}$. Thus, $-P^{(1)}_{r,k}(x) \geq \widetilde{P}_k(rx)$ for $x \geq 0$. Then we show that the polynomial $\widetilde{P}_k$ is positive outside the interval $[c_1(k), c_2(k)]$. This means that all roots of $P^{(1)}_{r,k}$ lye inside the interval $[\frac{c_1(k)}{r}, \frac{c_2(k)}{r}]$.

Now to proof the theorem it is sufficient to check that

$$\frac{P_{r,k}(x)}{P_{r,k}(0)} \cdot (1-x)^{2r-2k-1} < 1, \qquad x \in \Big[\frac{c_1(k)}{r}, \frac{c_2(k)}{r}\Big]. \tag{6}$$

First, we prove (6) for r big enough. To this end we rewrite $\frac{P_{r,k}(x)}{P_{r,k}(0)}$ as follows:

$$\frac{P_{r,k}(x)}{P_{r,k}(0)} = Q^+_{r,k}(x) - Q^-_{r,k}(x),$$

where $Q^+_{r,k}$ is even polynomial and $Q^-_{r,k}$ is odd one.

We construct polynomials $\widetilde{Q}^\pm_k$ with non-negative coefficients, such that for $r > r_0(k)$ all coefficients of the polynomial $\widetilde{Q}^+_k(r\cdot)$ are not less than the corresponding coefficients of $Q^+_{r,k}$ while the coefficients of $\widetilde{Q}^-_k(r\cdot)$ are not greater than the corresponding coefficients of $Q^-_{r,k}$. Then for $r > r_0(k)$ we have

$$\begin{aligned}\frac{P_{r,k}(x)}{P_{r,k}(0)} \cdot (1-x)^{2r-2k-1} &\leq (\widetilde{Q}^+_k(rx) - \widetilde{Q}^-_k(rx)) \cdot (1-x)^{2(r-k)-1} \\ &\leq (\widetilde{Q}^+_k(rx) - \widetilde{Q}^-_k(rx)) \cdot \exp(-\alpha(k)rx),\end{aligned}$$

where $\alpha(k) \leq 2 - \frac{2k+1}{r_0(k)}$.

Thus, the proof of (6) for $r > r_0(k)$ reduces to the proof of the following inequality:

$$(\widetilde{Q}^+_k(y) - \widetilde{Q}^-_k(y)) \cdot \exp(-\alpha(k)y) < 1, \qquad y \in \big[c_1(k), c_2(k)\big].$$

We prove this inequality by constructing suitable piecewise constant function f_k, which bounds the left-hand side of the inequality from above. To do so we note that the estimate

$$(\widetilde{Q}^+_k(y) - \widetilde{Q}^-_k(y)) \cdot \exp(-\alpha(k)y) \leq (\widetilde{Q}^+_k(y_1) - \widetilde{Q}^-_k(y_0)) \cdot \exp(-\alpha(k)y_0)$$

holds for $c_1(k) \leq y_0 \leq y \leq y_1 \leq c_2(k)$, since the coefficients of polynomials $\widetilde{Q}^\pm_k$ are non-negative.

The obtained estimator f_k was computed on a mesh fine enough. The inequality $f_k < 1$ proves (6) for $r > r_0(k)$.

We proceed similarly for $r \le r_0(k)$. Namely, for every fixed $r \le r_0(k)$ we rewrite the polynomial in the left-hand side of (6) as follows:

$$\frac{P_{r,k}(x)}{P_{r,k}(0)} = R^+_{r,k}(x) - R^-_{r,k}(x),$$

where $R^\pm_{r,k}$ are polynomials with non-negative coefficients.

We construct piecewise constant function $g_{r,k}$, which bounds the left-hand side of (6) from above. Namely, for $\frac{c_1(k)}{r} \le x_0 \le x \le x_1 \le \frac{c_2(k)}{r}$ the following inequality holds:

$$(R^+_{r,k}(x) - R^-_{r,k}(x)) \cdot (1-x)^{2(r-k)-1} \le (R^+_{r,k}(x_1) - R^-_{r,k}(x_0)) \cdot (1-x_0)^{2(r-k)-1}.$$

The obtained estimators $g_{r,k}$ were calculated on a mesh fine enough. The inequalities $g_{r,k} < 1$ prove (6) for $r \le r_0(k)$, and the first statement of Theorem follows.

The values $\lambda(r,4,2,\infty) = \big(A_{r,4}(0)\big)^{-1}$ and $\lambda(r,6,2,\infty) = \big(A_{r,6}(0)\big)^{-1}$ are calculated by the formulae (3) and (4). □

Appendix

Here one can find the results of calculations, described in the proof of Theorem 2.

1. $k = 4$**.**

$$-P^{(1)}_{r,4}(x) = (16r^4 - 96r^3 + 200r^2 - 168r + 45)x^4 + (-128r^3 + 656r^2 - 1056r + 540)x^3 + (312r^2 - 1224r + 1134)x^2 + (-240r + 540)x + 45;$$

$$\widetilde{P}_4(rx) := (3r^4)x^4 + (-228r^3)x^3 + (112r^2)x^2 + (-350r)x + 45.$$

$$c_1(4) = 0.1; \qquad c_2(4) = 76.$$

Let us explain why the coefficients of $-P^{(1)}_{r,4}$ do not exceed the coefficients of $\widetilde{P}_4(r\cdot)$. Since $k = 4$ implies $r \ge 5$, it is sufficient to check that the difference of every corresponding coefficients pair is the polynomial in r having a positive leading coefficient and no roots greater or equal than 5. For instance,

$$(16r^4 - 96r^3 + 200r^2 - 168r + 45) - 3r^4 = r^2(13r^2 - 96r + 160) + (40r^2 - 168r + 45).$$

The roots of both quadratic polynomials in brackets are less than 5. For other coefficients the argument is similar.

In the same manner we deduce that $\widetilde{P}_4$ is positive outside the interval $[c_1(4), c_2(4)]$:

$$3x^4 - 228x^3 + 112x^2 - 350x + 45 = (3x^2 - 228x + 56)x^2 + (56x^2 - 350x + 45).$$

Further,

$$Q^+_{r,4}(x) = \Big(\frac{16}{9}r^4 - \frac{128}{9}r^3 + \frac{104}{3}r^2 - 32r + 9\Big)x^4 + \Big(\frac{56}{3}r^2 - 112r + 126\Big)x^2 + 1;$$

$$Q^-_{r,4}(x) = \Big(\frac{32}{3}r^3 - \frac{688}{9}r^2 + \frac{440}{3}r - 84\Big)x^3 + (8r - 36)x;$$

$$\widetilde{Q}_4^+(rx) := \Big(\frac{17}{9}r^4\Big)x^4 + \Big(\frac{57}{3}r^2\Big)x^2 + 1;$$
$$\widetilde{Q}_4^-(rx) := \Big(\frac{26}{3}r^3\Big)x^3 + (7r)x.$$
$$r_0(4) = 50; \qquad \alpha(4) = 1.8.$$

Let us explain, why the coefficients of $Q^-_{r,4}$ are not less than the coefficients of $\widetilde{Q}_4^-(r\cdot)$ for $r > r_0(4)$. The difference of every corresponding coefficients pair is the polynomial in r, equals to the sum of binomials, positive for large r. For instance,

$$\Big(\frac{32}{3}r^3 - \frac{688}{9}r^2 + \frac{440}{3}r - 84\Big) - \frac{26}{3}r^3 = \Big(2r^3 - \frac{688}{9}r^2\Big) + \Big(\frac{440}{3}r - 84\Big).$$

The value r_0 is chosen in the way that all the binomials are positive. One can show in the same way that coefficients of $\widetilde{Q}_4^+(r\cdot)$ are not less than the coefficients of $Q^+_{r,4}$.

The functions f_4 and $g_{r,4}$, $5 \le r \le 50$, were computed on the following mesh:

Function	Number of points
f_4	2^{11}
$g_{r,4}$, $5 \le r \le 9$	2^7
$g_{r,4}$, $10 \le r \le 21$	2^8
$g_{r,4}$, $22 \le r \le 44$	2^9
$g_{r,4}$, $45 \le r \le 50$	2^{10}

The computations were carried out with 17 significant digits. This gives the estimate $1 - f_4 \ge 10^{-5}$, $1 - g_{r,4} \ge 10^{-5}$.

2. $k = 6$**.**

$$\begin{aligned} -P^{(1)}_{r,6}(x) = {} & (64r^6 - 768r^5 + 3664r^4 - 8832r^3 + 11212r^2 - 6960r + 1575)x^6 \\ & + (-1152r^5 + 12768r^4 - 54528r^3 + 111792r^2 - 109560r + 40950)x^5 \\ & + (7440r^4 - 72960r^3 + 260760r^2 - 402240r + 225225)x^4 \\ & + (-21120r^3 + 170640r^2 - 449040r + 386100)x^3 \\ & + (26460r^2 - 156240r + 225225)x^2 + (-12600r + 40950)x + 1575; \\ \widetilde{P}(rx) := {} & (4r^6)x^6 + (-1200r^5)x^5 + (1000r^4)x^4 \\ & + (-22000r^3)x^3 + (8000r^2)x^2 + (-13000r)x + 1575. \end{aligned}$$

$$c_1(6) = 0.1; \qquad c_2(6) = 300.$$

Further,

$$
\begin{aligned}
Q^+_{r,6}(x) = {}& \Big(\frac{64}{225}r^6 - \frac{64}{15}r^5 + \frac{5296}{225}r^4 - \frac{1568}{25}r^3 + \frac{19228}{225}r^2 - \frac{836}{15}r + 13\Big)x^6 \\
&+ \Big(\frac{112}{5}r^4 - 288r^3 + \frac{6104}{5}r^2 - \frac{10584}{5}r + 1287\Big)x^4 \\
&+ (44r^2 - 396r + 715)x^2 + 1; \\
Q^-_{r,6}(x) = {}& \Big(\frac{64}{15}r^5 - \frac{1504}{25}r^4 + \frac{22496}{45}r^3 - \frac{17104}{25}r^2 + \frac{10852}{15}r - 286\Big)x^5 \\
&+ \Big(\frac{736}{15}r^3 - \frac{2736}{5}r^2 + \frac{26216}{15}r - 1716\Big)x^3 + (12r - 78)x.
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{Q}^+(r\cdot) &:= \Big(\frac{74}{225}r^6\Big)x^6 + \Big(\frac{202}{9}r^4\Big)x^4 + \Big(\frac{1982}{45}r^2\Big)x^2 + 1; \\
\widetilde{Q}^-(r\cdot) &:= \Big(\frac{172}{45}r^5\Big)x^5 + \Big(\frac{716}{15}r^3\Big)x^3 + \Big(\frac{104}{9}r\Big)x.
\end{aligned}
$$

$$
r_0(6) = 410; \qquad \alpha(6) = 1.95.
$$

All the arguments for $k = 4$ can be repeated word-by-word.

The functions f_6 and $g_{r,6}$, $7 \le r \le 410$, were computed on the following mesh:

Function	Number of points
f_6	2^{15}
$g_{r,6}$, $7 \le r \le 42$	2^{11}
$g_{r,6}$, $43 \le r \le 86$	2^{12}
$g_{r,6}$, $87 \le r \le 173$	2^{13}
$g_{r,6}$, $174 \le r \le 348$	2^{14}
$g_{r,6}$, $349 \le r \le 410$	2^{15}

The computations were carried out with 17 significant digits and gave the estimate $1 - f_6 \ge 10^{-5}$, $1 - g_{r,6} \ge 10^{-5}$.

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