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On Loop States in Loop Quantum Gravity

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Abstract

We explicitly construct and characterize all possible independent loop states in 3+1 dimensional loop quantum gravity by regulating it on a 3-d regular lattice in the Hamiltonian formalism. These loop states, characterized by the (dual) angular momentum quantum numbers, describe SU(2) rigid rotators on the links of the lattice. The loop states are constructed using the Schwinger bosons which are harmonic oscillators in the fundamental (spin half) representation of SU(2). Using generalized Wigner Eckart theorem, we compute the matrix elements of the volume operator in the loop basis. Some simple loop eigenstates of the volume operator are explicitly constructed.

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1 Introduction

Spin networks were first constructed by Roger Penrose in 1970 to give a quantum mechanical interpretation of geometry of space [1]. Spin networks consist of a minimal subset of loop states in SU(2) gauge theories which solve the Mandelstam constraints and thus provide a complete and linearly independent basis in the gauge invariant Hilbert space. The loop states or gauge invariant states, in general, have been extensively studied in the context of gauge theories, both in the continuum [2] as well as on lattice [3, 4, 5] and topological field theories [6]. In particular, in [3] a complete labelling of the gauge invariant basis states of d+1 dimensional SU(N) lattice gauge theories was given in terms of the local observables. The SU(2) spin networks, in the context of loop quantum gravity (LQG) [7, 8], were first constructed in [9] by Rovelli and Smolin. They represent discrete quantum states of the gravitational field on a 3-d manifold and lead to interesting predictions of discrete length, area and volume [8, 10, 11, 12]. In the present work, we regulate Ashtekar's Hamiltonian formulation of gravity on a 3-d regular lattice [13, 14, 15] and present an alternative approach to construct the SU(2) spin networks in terms of the Schwinger bosons [18] which are harmonic oscillators in the fundamental representation of SU(2). The Schwinger boson creation operators create spin 1/2 fluxes and are the most natural objects to build the spin networks. We further use the resulting spin networks or the loop basis to analyze the spectrum of the volume operator. In particular, we explicitly construct some simple loop states which are eigenstates of the volume operator of LQG on lattice.

In general, for the case of gauge group G, spin networks are defined as graphs $\Gamma(\gamma, j, \mathcal{I})$ where:

- γ is a graph with finite set of edges e and a finite set of vertices v.
- each edge, attached with an irreducible representation of G, represents a parallel propagator or holonomy U in that representation.
- to each vertex v, we attach an intertwiner $\mathcal{I}$ to get gauge invariant operators at that vertex.

In the context of LQG, if we restrict ourselves to the real connections or holomorphic representation of the Hilbert space [8], the gauge group is SU(2). Therefore, each edge is characterized by spin j. Further, as any representation j of SU(2) can be obtained by taking symmetric direct product of $2j$ fundamental (spin half) representation, we also represent the edges as symmetric combination of $2j$ ropes with each rope representing the holonomy in the fundamental representation. These ropes form closed loops through the intertwiners at the vertices, thus relating the spin network graphs to the corresponding loop states. The ropes on the edges and the intertwiners at the vertices are the basic building blocks of the spin network. We find the Schwinger boson approach to spin network very natural as the ropes in the fundamental representation are created by the Schwinger boson creation operators (see section 2). Further, it leads to the following technical simplifications: a) as Schwinger boson creation operators commute amongst themselves there is no need to symmetrize them to get higher angular momentum states, b)the Schwinger bosons enable us to construct $k(k-1)/2$ manifestly SU(2) invariant intertwining operators at a k valent vertex (see section 2). Thus, we avoid the use of the Clebsch-Gordon coefficients in the standard spin network construction through the use of holonomies. Therefore, we completely bypass the problem of rapid proliferation of Clebsch-Gordan coefficients with increasing angular momenta at the spin network vertices with given valencies (see section 2.1). This proliferation is implicitly contained in the construction of spin networks [12, 19] when they are built out of holonomies in the fundamental representation.

Infact, using the representation theory of SU(2) group, Brunnemann and Thiemann [12] have completely characterized the spin network states at a N valent node by $(2N-3)$ angular momentum quantum numbers to study the spectrum of the volume operator at a vertex. The present paper compliments the work of Brunnemann and Thiemann as a) it provides a way to explicitly construct these spin network basis states in terms of Schwinger bosons, b) it also provides a way to interpret the $(2N-3)$ angular momentum quantum numbers in terms of the quantum numbers characterizing the corresponding loop states. The relationships between group theoretical angular momentum quantum numbers and the quantum numbers representing the corresponding loop states are made explicit in this work (see section 2.2). These relationships (see (19)), in turn, enable us to study the spectrum of volume operator in terms of entire loop states. We illustrate these points by explicitly constructing some simple loop eigenvectors of the volume operators involving 2 plaquettes on the lattice. We also derive the matrix elements of the volume operators using generalized Wigner Eckart theorem in the loop basis. These matrix elements have been already calculated in [12]. However, our method is geometrical and simpler as it exploits the tensor nature of the operators through the Wigner Eckart theorem to get the matrix elements directly.

The plan of the paper is as follows. In section 2, we briefly review the kinematical variables, their algebras and the constraints in lattice LQG [13, 14, 15] in the Hamiltonian formulation. This section is for the sake of completeness and also to emphasize the close connections between the kinematical variables in LQG and in the Hamiltonian lattice gauge theories. More details can be found in [13, 14, 20]. In this section, we also discuss the loop states or equivalently spin networks in terms of Schwinger bosons. This discussion is based on the work done by one of the authors [5] in the context of loop states in lattice gauge theories. In section 3, we investigate the volume operator in lattice LQG. Our volume operator on 3-d regular lattice involves only the 3 forward angular momenta. This choice coincides in form with the definition of the volume operator of Thiemann [12] for a 4-valent vertex. However, our vertices on the lattice sites can be up to 6-valent. Thus even for the most general states in the Hilbert space of lattice LQG we can compute the action of the volume operator. In section (3.1) we compute the matrix elements of the volume operator using generalized Wigner Eckart theorem. In section (3.2) we explicitly construct some of it's simple eigenvectors in terms of the Schwinger boson operators. We also exhibit their loop structure.

2 Hamiltonian lattice gravity

In the Ashtekar formulation of Quantum Gravity [7, 8] the kinematical variables are Yang-Mills gauge field and the associated electric field. The difference from usual Yang-Mills theories lies in the constraint structure. In addition to the Hamiltonian and Gauss's law constraints one has to impose the additional diffeomorphism constraints. Here too one needs to regularise the theory. Either the regularisation scheme should manifestly maintain the symmetry and constraint structures of the theory, or there should be a mechanism to recover these in the continuum limit. As far as gauge theories are concerned, lattice regularisation is a non-perturbative regularisation that manifestly preserves gauge invariance, at least as far vector(as opposed to chiral) theories are concerned. Thus we can expect at least the non-Abelian gauge invariance of the Ashtekar formalism to be explicitly maintained. At this stage one can not say much about diffeomorphism invariance.

In the Ashtekar formulation there is a fiducial space-time on which the gauge fields and their canonical momenta live. This is not the physical space-time. So one possible way of regularising the Ashtekar formulation is to adopt a lattice discretisation of the spatial part of this fiducial space-time.

Renteln and Smolin [13, 14] were the first to put the Ashtekar's Hamiltonian formulation of gravity on lattice. We briefly review the nature of the kinematical variables involved in this formulation keeping in view the similarities with SU(2) Hamiltonian LGT [20]. We will use a regular cubic lattice as a regulator and denote the lattice sites by **n,m** and the 3 directions by **i,j,k** =1,2,3. The SU(2) color index will be denoted by a,b,c =1,2,3 and the SU(2) fundamental indices by the Greek alphabets $\alpha, \beta, \gamma = 1, 2$. One associates SU(2) link operators (or holonomies) $U_{\alpha\beta}(\mathbf{n}, \mathbf{i})$ with the link **(n,i)**. Infact, $U_{\alpha\beta}(n, i)$ can be thought of describing the orientation of a SU(2) rigid rotator from body fixed frame to space fixed frame. We further introduce the angular momentum operators $E^a_L(n, i)$ and $E^a_R(n, i)$ which produce the left (body fixed frame) and the right (space fixed frame) rotations respectively. Therefore, the canonical commutation relations involved are locally that of a rigid body [22, 20, 13] and are given by:

$$
\begin{aligned}
[U_{\alpha\beta}(l), U_{\gamma\delta}(l')] &= 0, \qquad \left[U_{\alpha\beta}(l), U^{\dagger}_{\gamma\delta}(l')\right] = 0, \\
\left[E^a_L(l), E^b_L(l')\right] &= i\delta_{l,l'}\epsilon^{abc}E_L(l), \qquad \left[E^a_R(l), E^b_R(m,j)\right] = i\delta_{l,l'}\epsilon^{abc}E_R(l), \\
[E^a_L(l), U_{\alpha\beta}(l')] &= i\delta_{l,l'}\left(\frac{\sigma^a}{2}\right)_{\alpha\gamma} U_{\gamma\beta}(l), \qquad [E^a_R(l), U_{\alpha\beta}(l')] = i\delta_{l,l'}U_{\alpha\gamma}(l)\left(\frac{\sigma^a}{2}\right)_{\gamma\beta}.
\end{aligned} \tag{1}
$$

In (1), $l = (n, i)$ and $l' = (m, j)$ and $\delta_{l,l'} = \delta_{n,m}\delta_{i,j}$. The commutation relations (1) clearly show that the $E_L(l)$ and $E_R(l)$ generate left (at site n) and right (at site $(n + i)$) gauge rotations on the

Figure 1: The assignment of the left and right electric fields on a link (n, i). The corresponding Schwinger bosons are also shown by the two • on the link.

holonomy $U(n, i)$. Therefore, it is convenient to attach them with the left and the right ends of the link (l) respectively as shown in Figure (1). We write them as as $E_L(l) = E_L(n, i) = \sum_{a=1}^{3} \sigma^a E^a_L(n, i)$ and $E_R(l) = E_R(n + i, i) = \sum_{a=1}^{3} \sigma^a E^a_R(n + i, i)$. The SU(2) gauge transformations are:

$$\begin{aligned} E_L(n,i) &\rightarrow \Lambda(n)E_L(n,i)\Lambda^{-1}(n), \\ E_R(n+i,i) &\rightarrow \Lambda(n+i)E_R(n+i,i)\Lambda^{-1}(n+i) \\ U(n,i) &\rightarrow \Lambda(n)U(n,i)\Lambda^{-1}(n+i) \end{aligned} \tag{2}$$

Note that the commutation relations (1) are invariant under (2) and $E_L(n, i)$ and $E_R(n + i, i)$ on the link (n, i) gauge transform by $\Lambda(n)$ and $\Lambda(n + i)$ respectively. The transformations (2) along with (1) imply that the generator of the gauge transformation at lattice site n is:

$$\mathcal{C}^a(n) \equiv \sum_{i=1}^{3} E^a_L(n,i) + \sum_{i=1}^{3} E^a_R(n-i,i) \tag{3}$$

Further, as in the case of rigid body, the body fixed and space fixed components of the angular momentum commute amongst themselves and their magnitudes are equal, i.e:

$$\sum_{a=1}^{3} E^a_L(n,i)E^a_L(n,i) = \sum_{a=1}^{3} E^a_R(n+i,i)E^a_R(n+i,i), \quad \forall \quad (n,i). \tag{4}$$

In the current literature no attention seems to have been paid to these important constraints. These constraints are crucial at promoting discussion at a single site to the entire lattice. In the present work, we focus on the Gauss law constraint[3] (3), $\mathcal{C}^a = 0$, which implement SU(2) gauge invariance leading to the spin networks. To solve these constraints [5], we use the Schwinger boson representation of SU(2) Lie algebra [18] and write:

$$E^a_L(n,i) \equiv a^\dagger(n,i)\frac{\sigma^a}{2}a(n,i); \quad E^a_R(n+i,i) \quad \equiv \quad b^\dagger(n+i,i)\frac{\sigma^a}{2}b(n+i,i). \tag{5}$$

on every link (n,i). In (5), $\sigma^a(a = 1, 2, 3)$ are the Pauli matrices. The SU(2) gauge transformations (2) immediately imply that the Schwinger bosons $a_\alpha(n, i)$ and $b_\alpha(n + i, i)$ transform as fundamental representation of SU(2):

$$a_\alpha(n,i) \rightarrow \Lambda(n)_{\alpha\beta}a_\beta(n,i); \quad b_\alpha(n,i) \rightarrow \Lambda(n)_{\alpha\beta}b_\beta(n,i). \tag{6}$$

The LQG (like SU(2) lattice gauge theory) in terms of the Schwinger bosons also has a local U(1) gauge invariance [5] on the lattice links:

$$a_\alpha(n,i) \rightarrow exp(i\phi(n,i))a_\alpha(n,i); \quad b_\alpha(n+i,i) \rightarrow exp(-i\phi(n,i))b_\alpha(n+i,i). \tag{7}$$

The generator of this abelian gauge transformation is:

$$\mathcal{C}(n,i) = a^\dagger(n,i) \cdot a(n,i) - b^\dagger(n+i,i) \cdot b(n+i,i) \tag{8}$$

where $a^\dagger.a \equiv a^\dagger_1 a_1 + a^\dagger_2 a_2$. Thus, working with the fundamental spin half representation, we have $SU(2) \otimes U(1)$ gauge invariance. The corresponding Gauss law constraints are: $\mathcal{C}^a = \mathcal{C} = 0$.

[3]The Hamiltonian and diffeomorphism constraints are given in [7, 8, 13, 14].

This underlying $U(1)$ gauge invariance is important in our formulation. Without this the construction of the gauge invariant states is incomplete. With the choice of Hamiltonian regularisation on a regular 3-d lattice, all the vertices of any graph/loop state are at the most of valence 6 and this is a considerable simplification over the general situation. For the same reason, the lattice regulator also simplifies the the Mandelstam constraints considerably. Further, the present Schwinger boson approach to gauge theories also enables us to cast the Mandelstam constraints in local form and solve them explicitly [5] in terms of the loop states (18). We briefly discuss this issue in section (2.2). A detailed analysis is given in [21].

In this work we specifically look at the volume operator of Ashtekar and Lewandowski in loop quantum gravity [10, 12] on the lattice and study its matrix elements and properties of it's spectrum in the loop basis. Even within the lattice regularisation there are several choices for the volume operator that have been currently considered in the literature. The one originally considered by Loll [15] is where only the three 'forward' electric fields at each lattice site are involved in the volume operator at each site. We choose this definition for our present analysis. Technically this corresponds to the 4-valent vertex volume operator studied by Thiemann et. al. in [12]. If we extend the range of $\mathbf{i}$ from 3 to 6 where for $i \leq 3$ we have E_L at a site and for $i > 3$ we have E_R at that site, the other choices of the volume operator are

$$V(\Gamma) = \sum_{v\in\Gamma} \sqrt{\frac{1}{3!}\Big|\sum_{i,j,k=1}^{6} \epsilon(\hat{i}\hat{j}\hat{k})\epsilon_{abc}E^a[n,i]E^b[n,j]E^c[n,k]\Big|} \tag{9}$$

In (9), $\epsilon(\hat{i}\hat{j}\hat{k}) = sign(\hat{i}\times\hat{j}.\hat{k})$. Loll [16] has also used another definition which is

$$V(\Gamma) = \sum_{v\in\Gamma} \sqrt{\frac{1}{3!}\Big|\sum_{i,j,k=1}^{6} |\epsilon(\hat{i}\hat{j}\hat{k})|\epsilon_{abc}E^a[n,i]E^b[n,j]E^c[n,k]\Big|} \tag{10}$$

which amounts to replacing the 'forward' electric fields at each site by a symmetric sum of 'forward' and 'backward' electric fields.

All these definitions agree with each other in the naive continuum limit. According to Loll the difference between the symmetrised and original definitions is only an additive constant for 4-valent vertices. But for the 6-valent vertices considered here the matrix elements of the symmetrised expression of Loll, though they still involve only the equivalents of Thiemann's four valent operators, are more complicated but still calculable in closed form. We shall not carry out that here but postpone it to a later paper.

Giesel and Thiemann [17] have discussed some consistency conditions that volume operators have to satisfy but they do not seem to have compared the above alternatives from lattice discretisation.

2.1 The rigid rotator states on the spin network edges

An edge or a link (l) of the spin network with quantum number j describes the rigid rotator with angular momentum j. The rigid body constraints (4) or equivalently $\mathcal{C} = 0$ in terms of Schwinger bosons, imply that on any link the total occupation number of a- type oscillators is same as that of b- type oscillators: The states of the rigid rotator [22] on the link (n,i) are characterized by the eigenvalues of $E_L(n,i).E_L(n,i)(= E_R(n+,i).E_R(n+i,i))$, $E_L^{a=3}(n,i)$, $E_R^{a=3}(n+i,i)$. We denote the common eigenstate of these 3 operators by $|j(n,i), m(n,i), \tilde{m}(n+i,i) >$. These states are:

$$|j,m,\tilde{m} > \equiv \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}(b_1^\dagger)^{j+\tilde{m}}(b_2^\dagger)^{j-\tilde{m}}}{\sqrt{(j+m)!(j-m)!(j+\tilde{m})!(j-\tilde{m})!}}\Big|{}^{0\ \ 0}_{0\ \ 0}\Big\rangle. \tag{11}$$

In (11), $a_\alpha^\dagger \equiv a_\alpha^\dagger(n,i)$ and $b_\alpha^\dagger \equiv b_\alpha^\dagger(n+i,i)$ are defined on the left and right end of the link (n,i) respectively. Note that a *rope* of the spin network represents the first excited state $(j = 1/2)$ of the SU(2) rigid rotator. The states $|j,m,\tilde{m} >$ can also be created by holonomies instead of Schwinger bosons:

$$|j,m,\bar{m}\rangle = \sum_{S_{2j}} U_{\alpha_1\beta_1}U_{\alpha_2\beta_2}...U_{\alpha_{2j}\beta_{2j}}|0\rangle \tag{12}$$

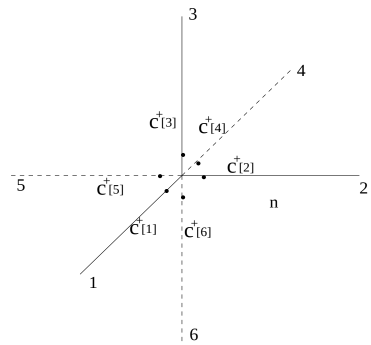

Figure 2: A graphical representation of the Schwinger boson operators $c^\dagger[n,i]$, i=1,2,...,6 which are associated with site n. They all transform as SU(2) doublets at site n and are shown by • on the corresponding axis.

where the magnetic quantum numbers $\alpha_i, \beta_i = \pm\frac{1}{2}$ should add upto m and $\bar{m}$ respectively, S_{2j} is the permutation group with $(2j)!$ elements which act on the indices $(\alpha_1, \alpha_2, ...\alpha_{2j})$ to symmetrize and produce higher angular momentum states. We note that the Schwinger boson construction (11) is much simpler as it does not require any permutation group or equivalently Clebsch-Gordan coefficients whose number will increase with increasing angular momentum quantum number j on each link.

2.2 The intertwining operators at the spin network vertices

A link with angular momentum j can also be thought of as being represented by $2j$ independent 'ropes'. Then at a given vertex we have ropes associated with each of the links. On the other hand a typical loop state consists of a set of closed loops. Thus to connect the angular momentum basis to loop states we must have rules on how the ropes on different links at a site are to be 'tied together' or 'intertwined'; these are given by quantum numbers which we call 'linking numbers' or 'intertwining numbers'. As we show below these quantum numbers arise naturally in the Schwinger boson formalism (see [5] for more details).

To define SU(2) gauge invariant intertwining operators at a lattice site n, we notice that the site n is associated with 6 Schwinger bosons: $a^\dagger(n,i)$ and $b^\dagger(n,i)$; i=1,2,3, all gauge transforming as SU(2) doublets. Therefore, we can label them collectively as $c^\dagger[n,i]$ with i=1,2,..,6. More explicitly, $c^\dagger[n,1] = a^\dagger(n,1), c^\dagger[n,2] = a^\dagger(n,2), c^\dagger[n,3] = a^\dagger(n,3), c^\dagger[n,4] = b^\dagger(n,1), c^\dagger[n,5] = b^\dagger(n,2), c^\dagger[n,6] = b^\dagger(n,3)$ as shown in Figure (2). We also relabel the corresponding angular momentum operators associated with lattice site n by $J[n,i] \equiv c^\dagger[n,i]\frac{\sigma^a}{2}c[n,i]$ with i =1,2,..,6. Note that $J[n,i] = E_L(n,i)$ and $J[n,3+i] = E_R(n-i,i)$ (now for $i = 1,2,3$) and the Gauss law (3) now takes the simple form:

$$\mathcal{C}^a(n) \equiv \sum_{i=1}^{6} J^a[n,i] = 0 \tag{13}$$

and simply states that the sum over the 6 angular momenta at any vertex is zero. The eigenvalues of $J[n,i] \cdot J[n,i]$ will be denoted by $j(n,i)(j(n,i)+1)$. Consider the intertwining of a single rope on the ith link with a single rope on the jth link. Such a state must be created from the oscillator vacuum by the action of an operator involving $c^\dagger_\alpha(n,i)c^\dagger_\beta(n,j)$. Any such operator must be $SU(2)$-invariant. We are thus led to the following intertwining operator at site n:

$$L_{ij}(n) \equiv \epsilon_{\alpha\beta} c^\dagger_\alpha[n,i] c^\dagger_\beta[n,j] = c^\dagger[n,i].\tilde{c}^\dagger[n,j], i,j = 1,2..,6. \tag{14}$$

In (14), $\epsilon_{\alpha\beta}$ is completely antisymmetric tensor ($\epsilon_{11} = \epsilon_{22} = 0, \epsilon_{12} = -\epsilon_{21} = 1$) and $\tilde{c}^\dagger_\alpha \equiv \epsilon_{\alpha\beta} c^\dagger_\beta$. Note that:

$$[\mathcal{C}^a(n), L_{ij}(n)] = 0 \tag{15}$$

and $L_{ij}(n) = -L_{ji}(n)$, $L_{ii} = 0$ implying self intertwining is not allowed. Therefore, we can choose the 15 operators $L_{ij}(n)$ with $i < j$ to span the space of SU(2) gauge invariant intertwining operators at site n. We conclude that any SU(2) gauge invariant state at site n must be of the form:

$$|\vec{l}(n) > \equiv \left| \begin{array}{ccccc} l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ & l_{23} & l_{24} & l_{25} & l_{26} \\ & & l_{34} & l_{35} & l_{36} \\ & & & l_{45} & l_{46} \\ & & & & l_{56} \end{array} \right\rangle = \prod_{\substack{i,j=1 \\ j>i}}^{6} \left(L_{ij}(n)\right)^{l_{ij}(n)} |0>, \quad l_{ij}(n) \in \mathcal{Z}_+. \tag{16}$$

Note that the states in (16) are the eigenstates of the angular momentum operators $J[n,i].J[n,i]|_{i=1,2..,6}$ with eigenvalues $j[n,i] = j_i(n)(j_i(n)+1)$ such that:

$$2j_i(n) = \sum_{k=1}^{6} l_{ik}(n) \tag{17}$$

This relation follows on counting the number of creation operators of the relevant type in eqn(16). It has the following very simple geometrical interpretation: if we draw $2j_i(n)$ lines on the link $[n,i]$ then the quantum numbers l_{ij} are the linking numbers connecting the i^{th} and j^{th} types of flux lines. Consequently they satisfy $l_{ij} = l_{ji}$. To get orthonormal loop states, we choose the complete set of commuting operators [3, 12, 5] to be $J[n,i].J[n,i]|_{i=1,2,..,6}, \left(J[n,1]+J[n,2]\right)^2, \left(J[n,1]+J[n,2]+J[n,3]\right)^2, ..., J_{total} = \left(J[n,1]+J[n,2]+..+J[n,6]\right)^2, J^{a=3}_{total}$. We denote their common eigenvector by $|j_i(n)|_{i=1,2,..,6}, j_{12}(n), j_{123}(n), j_{1234}(n), j_{12345},\ j_{total}, m_{total}\rangle$. Noting that SU(2) Gauss law implies $j_{total} = m_{total} = 0$ and $j_{12345} = j_6$. Thus the common eigenvectors are characterized by 9 quantum numbers at every lattice site. This characterization of the orthonormal basis by angular momentum operators has been done both in LGT [9] and LQG [12]. In [5] they were explicitly constructed and are given by

$$|j_i(n)|_{i=1,2,..,6}, j_{12}(n), j_{123}(n), j_{1234}(n)\rangle = N(j) \sum_{\{l(n)\}}{}' \prod_{\substack{i,j \\ i<j}} \frac{1}{l_{ij}(n)!} \left(L_{ij}(n)\right)^{l_{ij}(n)} |0> \tag{18}$$

The prime over the summation $\{l\}$ means that the 15 linking numbers l_{ij} are summed over such that the linking numbers $l_{ij}(n)$ satisfy (17) along with the following constraints:

$$\begin{aligned} l_{12} &= j_1 + j_2 - j_{12} \\ l_{13} + l_{23} &= j_{12} + j_3 - j_{123} \\ l_{14} + l_{24} + l_{34} &= j_{123} + j_4 - j_{1234} \\ l_{15} + l_{25} + l_{35} + l_{45} &= j_{1234} + j_5 - j_6 \end{aligned} \tag{19}$$

In (18), $N(j) = N(j_1, j_2, j_{12})N(j_{12}, j_3, j_{123})N(j_{123}, j_4, j_{1234})N(j_{1234}, j_5, j_{12345}(=j_6))N(j_{12345}(=j_6), j_6, 0)$ where $N(a,b,c) = \left[\frac{(2c+1)}{(a+b+c+1)!}\right]^{\frac{1}{2}} \left[(-a+b+c)!(a-b+c)!(a+b-c)!\right]^{\frac{1}{2}}$. The constraints (19) are easy to understand: Given $2j_1$ oscillators in direction $(n,1)$ and $2j_2$ oscillators in direction $(n,2)$, we need to intertwine (or antisymmetrize) l_{12} oscillators from each of these two directions to get a state with angular momentum j_{12}. Therefore, $(2j_1 - l_{12}) + (2j_2 - l_{12}) = 2j_{12}$, which is the first equation in (19). Similarly the other equations in (19) can be obtained. Thus we see the ease with which the intertwiners, so crucial to the construction of loop states both in lattice gauge theories and loop quantum gravity, emerges from the Schwinger boson construction.

We wish to point out here that the above orthonormal basis at every site in conjunction with the intertwiners satisfying the relations above completely eliminates the notorious problem of the Mandelstam constraints. In particular, the primed summation in eqn(18) picks a specific combination of loop states at each vertex that forms an orthonormal set. This is at the heart of the resolution of the Mandelstam constraints.

3 The volume operator

In this section we study the volume operator in the basis (18) and construct some simple loop eigenstates of the volume operator. In [10, 12] the volume operator associated with a spin network $\Gamma(\gamma, j, v)$ is defined as:

$$V(\Gamma) = \sum_{v\in\Gamma} \sqrt{\frac{1}{3!}\Big|\sum_{\hat{\imath},\hat{\jmath},\hat{k}=1}^{6} \epsilon(\hat{\imath}\hat{\jmath}\hat{k})\epsilon_{abc}E^a[n,\hat{\imath}]E^b[n,\hat{\jmath}]E^c[n,\hat{k}]\Big|} \tag{20}$$

In (20), $\epsilon(\hat{\imath}\hat{\jmath}\hat{k}) = sign(\hat{\imath}\times\hat{\jmath}.\hat{k})$. Our choice of the volume operator on lattice corresponds to choosing only the forward angular momenta at each lattice site [15], i.e:

$$\begin{aligned} V(\Gamma) &= \sum_n \sqrt{\frac{1}{3!}\Big|\sum_{i,j,k=1}^{3}\sum_{a,b,c=1}^{3} \epsilon_{ijk}\epsilon_{abc}J^a[n,i]J^b[n,j]J^c[n,k]\Big|} \\ &= \sum_n \sqrt{\Big|\sum_{a,b,c=1}^{3} \epsilon_{abc}J^a[n,1]J^b[n,2]J^c[n,3]\Big|} \equiv \sum_n \sqrt{|Q(n)|} \end{aligned} \tag{21}$$

In what follows we will study the spectrum of $Q(n) \equiv \epsilon_{abc}J^a[n,1]J^b[n,2]J^c[n,3]$. The local operator Q(n) has been extensively studied in the case of 4-valent vertex by Brunnemann and Thiemann [12] by writing it as $Q(n) = \frac{i}{4}\hat{q}_{123}(n)$ where $\hat{q}_{123}(n) \equiv \left[(J[n,1]+J[n,2])^2, (J[n,2]+J[n,3])^2\right]$. Note that we can also write the local operator $Q(n)$ in terms of the basic SU(2) gauge invariant intertwining operators (14) at site n as $Q(n) = \frac{i}{4}\sqrt{\left[L^{\dagger}_{12}(n)L_{12}(n), L^{\dagger}_{23}(n)L_{23}(n)\right]}$.

3.1 The matrix elements

To calculate the matrix elements of $Q(n)$, we first define the angular momentum operator in terms of their tensor or spherical components:

$$J^{(1)}_{\pm 1} \equiv \mp\frac{1}{\sqrt{2}}(J^x \pm iJ^y), \qquad J^{(1)}_0 \equiv J^z. \tag{22}$$

The irreducible components of the direct product of two tensors $A^{(j_1)}_{m_1}$ and $B^{(j_2)}_{m_2}$ of rank j_1 and j_2 respectively with $m_1(m_2)$ varying from $-j_1(-j_2)$ to $j_1(j_2)$, are defined as [23]:

$$\left[A^{(j_1)} \times B^{(j_2)}\right]^{(j_{12})}_{m_{12}} \equiv \sum_{m_1,m_2} C^{j_{12}m_{12}}_{j_1,m_1;j_2,m_2} A^{(j_1)}_{m_1} B^{(j_2)}_{m_2}. \tag{23}$$

In (23), $C^{j_{12}m_{12}}_{j_1,m_1;j_2,m_2}$ are the Clebsch- Gordon coefficients [23] and m_{12} vary from $-j_{12}$ to $+j_{12}$. The scalar product of two tensors of the same rank is denoted by:

$$A^{(j)}\cdot B^{(j)} \equiv \left[A^{(j_1=j)} \times B^{(j_2=j)}\right]^{(0)}_0 = \sum_{m_1,m_2} C^{j_{12}=0 m_{12}=0}_{j,m_1;j,m_2} A^{(j)}_{m_1} B^{(j)}_{m_2} = \sum_{m=-j}^{+j} \frac{(-1)^{(j-m)}}{\sqrt{((j))}} A^{(j)}_m B^{(j)}_{-m}. \tag{24}$$

In (24), $((j)) = (2j+1)$ represents the multiplicity factor. Using the definitions (23) and (24), we write: $Q(n) \equiv \epsilon_{abc}J^a[n,1]J^b[n,2]J^c[n,3] = +i\sqrt{2((1))}(J^{(1)}[n,1]\times J^{(1)}[n,2])^{(1)}\cdot(J^{(1)}[n,3])^{(1)}$. Now the matrix elements of $Q(n)$ can be directly written down using generalized Wigner Eckart theorem [23]:

$$
\begin{aligned}
&< j_1, j_2, j_{12}, j_3; j_{123}, m_{123}|\Big[(J^{(1)}[n,1] \times J^{(1)}[n,2])^{(1)} \cdot J^{(1)}[n,3]\Big]|\bar{j}_1, \bar{j}_2, \bar{j}_{12}, \bar{j}_3; \bar{j}_{123}, \bar{m}_{123} > \\
&= (-1)^{j_{123}-m_{123}} \begin{pmatrix} j_{123} & 0 & \bar{j}_{123} \\ -m_{123} & 0 & \bar{m}_{123} \end{pmatrix} [((0))((j_{123}))((\bar{j}_{123}))]^{\frac{1}{2}} \begin{Bmatrix} j_{12} & \bar{j}_{12} & 1 \\ j_3 & \bar{j}_3 & 1 \\ j_{123} & \bar{j}_{123} & 0 \end{Bmatrix} \\
&\left(j_1, j_2, j_{12}|| \left(J^{(1)}[n,1] \times J^{(1)}[n,2]\right)^{(1)} ||\bar{j}_1, \bar{j}_2, \bar{j}_{12}\right) \left(j_3||J^{(1)}[n,3]||\bar{j}_3\right) \qquad (25)
\end{aligned}
$$

In (25), $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ and $\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix}$ are the Clebsch Gordan and $9j$ symbols respectively. The reduced matrix elements of the operator J are represented by $(j||J||\bar{j})$. Similarly, the reduced matrix element $(j_1, j_2, j_{12}||(J^{(1)}[n,1] \times J^{(1)}[n,2])^{(1)}||\bar{j}_1, \bar{j}_2, \bar{j}_{12})$ are given by:

$$
\begin{aligned}
\left(j_1, j_2, j_{12}|| \left(J^{(1)}[n,1] \times J^{(1)}[n,2]\right)^{(1)} ||\bar{j}_1, \bar{j}_2, \bar{j}_{12}\right) &= [((1))((j_{12}))((\bar{j}_{12}))]^{\frac{1}{2}} \begin{Bmatrix} j_1 & \bar{j}_1 & 1 \\ j_2 & \bar{j}_2 & 1 \\ j_{12} & \bar{j}_{12} & 1 \end{Bmatrix} \\
&\quad \left(j_1||J^{(1)}[n,1]||\bar{j}_1\right) \left(j_2||J^{(1)}[n,2]||\bar{j}_2\right) \qquad (26)
\end{aligned}
$$

Now using the values:

$$
\begin{aligned}
(j||J||\bar{j}) &= \delta_{j,\bar{j}}\,[j(j+1)(2j+1)]]^{\frac{1}{2}} \equiv \delta_{j,\bar{j}} x(j) \\
\begin{pmatrix} j_{123} & 0 & \bar{j}_{123} \\ -m_{123} & 0 & \bar{m}_{123} \end{pmatrix} &= (-1)^{j_{123}-m_{123}} \frac{1}{\sqrt{((j_{123}))}}\, \delta_{j_{123},\bar{j}_{123}} \delta_{m_{123},\bar{m}_{123}} \\
\begin{Bmatrix} j_1 & j_1 & 1 \\ j_2 & j_2 & 1 \\ j_{12} & \bar{j}_{12} & 1 \end{Bmatrix} &= (-1)^{j_1+j_2+j_{12}} \frac{[j_{12}(j_{12}+1) - \bar{j}_{12}(\bar{j}_{12}+1)]}{2\sqrt{2((1))}x(j_1)} \begin{Bmatrix} j_{12} & \bar{j}_{12} & 1 \\ j_2 & j_2 & j_1 \end{Bmatrix} \\
\begin{Bmatrix} j_{12} & \bar{j}_{12} & 1 \\ j_3 & \bar{j}_3 & 1 \\ j_{123} & \bar{j}_{123} & 0 \end{Bmatrix} &= \frac{(-1)^{j_{12}+j_3+j_{123}}}{\sqrt{((1))((j_{123}))}} \begin{Bmatrix} j_{12} & \bar{j}_{12} & 1 \\ j_3 & j_3 & j_{123} \end{Bmatrix} \qquad (27)
\end{aligned}
$$

and putting them in (25), we get:

$$
\begin{aligned}
&< \bar{j}_i|_{i=1,..,6}, \bar{j}_{12}, \bar{j}_{123}, \bar{j}_{1234}|Q(n)|j_i|_{i=1,..,6}, j_{12}, j_{123}, j_{1234} >= \frac{i}{2}(-1)^{(j_1+j_2+j_{12}+\bar{j}_{12}+j_3+j_{123})} \\
&\left(\prod_{\hat{i}=1}^{6} \delta_{\bar{j}_{\hat{i}},j_{\hat{i}}}\right) \delta_{\bar{j}_{123},j_{123}} \delta_{\bar{j}_{1234},j_{1234}} x(j_2,j_3) y(j_{12},\bar{j}_{12}) \begin{Bmatrix} \bar{j}_{12} & j_{12} & 1 \\ j_2 & j_2 & j_1 \end{Bmatrix} \begin{Bmatrix} \bar{j}_{12} & j_{12} & 1 \\ j_3 & j_3 & j_{123} \end{Bmatrix} \\
&\equiv \left(\prod_{\hat{i}=1}^{6} \delta_{\bar{j}_{\hat{i}},j_{\hat{i}}}\right) \delta_{\bar{j}_{123},j_{123}} \delta_{\bar{j}_{1234},j_{1234}} < \bar{j}_{12}|Q(n)|j_{12} > \qquad (28)
\end{aligned}
$$

where,

$$
\begin{aligned}
x(j_2,j_3) &\equiv x(j_2)x(j_3) = \sqrt{j_2(j_2+1)(2j_2+1)j_3(j_3+1)(2j_3+1)} \\
y(j_{12},\bar{j}_{12}) &\equiv \sqrt{((j_{12}))((\bar{j}_{12}))}.\,[j_{12}(j_{12}+1) - \bar{j}_{12}(\bar{j}_{12}+1)]\,. \qquad (29)
\end{aligned}
$$

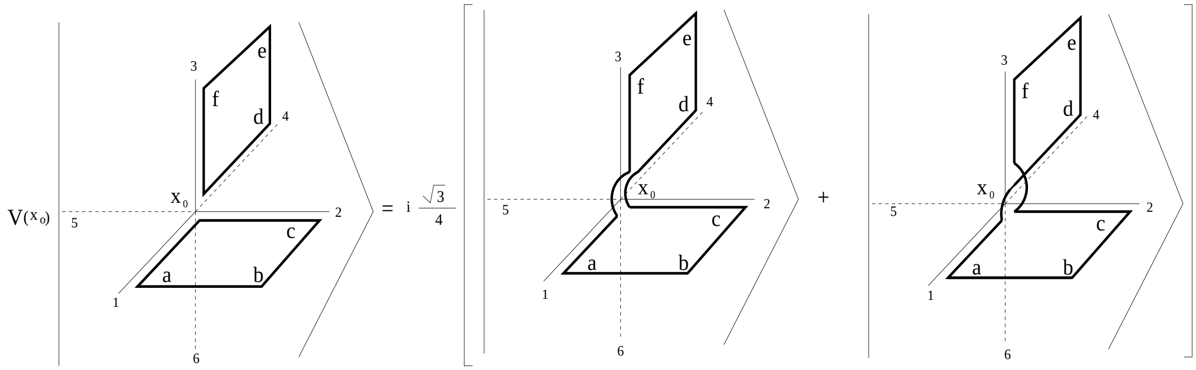

Figure 3: Graphical representation of the action of the operator Q(n) (32) on the simple loop state $|j_{12} = 0>_{x_0} \otimes|abcdef>$ in (31). The loop state on the right hand side represents: $|j_{12} = 1>_{x_0} \otimes|abcdef> \equiv |1/2,1/2,1/2,1/2,0,0,1,1/2,0>_{x_0} \otimes|abcdef>$. Note that the operator Q(n) changes the linking number: $l_{12} \rightarrow l_{12} \pm 1$ at x_0. The other linking numbers are changed in accordance with eqn(19).

We note that the matrix elements (28) are antisymmetric and imaginary as the operator $Q(n)$ is hermitian. Putting the explicit values of the 6-j symbols we get:

$$
\begin{aligned}
< j_{12}+1|Q(n)||j_{12}> &= \frac{+i}{4\sqrt{((j_{12}))((j_{12}+1))}}\Big[(j_1+j_{12}+j_2+2)(j_{123}+j_{12}+j_3+2) \\
&\quad (j_1+j_{12}-j_2+1)(j_{123}+j_{12}-j_3+1)(j_1-j_{12}+j_2)(j_{123}-j_{12}+j_3) \\
&\quad (-j_1+j_{12}+j_2+1)(-j_{123}+j_{12}+j_3+1)\Big]^{\frac{1}{2}} \\
&= -< j_{12}|Q(n)||j_{12}+1> . \qquad (30)
\end{aligned}
$$

The last result follows because the matrix elements of Q(n) are antisymmetric and purely imaginary. As mentioned in the introduction, the matrix elements in (28) and (30) are already obtained in [12] by writing $Q(n) \equiv \frac{i}{4}\left[(J[n,1]+J[n,2])^2, (J[n,2]+J[n,3])^2\right]$. This computation required a change of basis from $|j_1, j_2, j_{12}, j_3, j_{123}, m_{123}>$ to $|j_1, j_2, j_3, j_{23}, j_{123}, m_{123}>$ and use of Elliot-Biedenharn identity. The analysis of this section, based on the Wigner-Eckart theorem, is direct and also quite general.

3.2 An example

In this section, we use an example to illustrate the simplest but non trivial action of the volume operator on an entire loop state. We also work out the corresponding loop eigenstates of the volume operator. As all the planar loops are trivially annihilated by the volume operator and therefore have zero volume, we need to consider loops spread over all the 3 forward directions. This implies that the simplest graph with non-zero volume will involve 2 plaquettes in two different planes. Further, we associate j= 1/2 with all the links of the two plaquettes centered at a lattice site x_0 as shown in the Figure (3). The corresponding loop state $|LS>$ is direct product of the states $|j_1, j_2, j_3, j_4, j_5, j_6, j_{12}, j_{123}, j_{1234}>$ in (18) at the vertices (x_0, a, b, c, d, e, f).

$$
\begin{aligned}
|LS> &= |1/2,1/2,1/2,1/2,0,0,0,1/2,0>_{x_0} \otimes|0,1/2,0,1/2,0,0,1/2,1/2,0>_a \\
&\quad \otimes|0,0,0,1/2,1/2,0,0,0,1/2>_b \otimes|1/2,0,0,0,1/2,0,1/2,1/2,1/2>_c \\
&\quad \otimes|1/2,0,1/2,0,0,0,1/2,0,0>_d \otimes|1/2,0,0,0,0,1/2,1/2,1/2,1/2>_e \\
&\quad \otimes|0,0,0,1/2,0,1/2,0,0,1/2>_f \equiv |j_{12}=0>_{x_0} \otimes|abcdef> \qquad (31)
\end{aligned}
$$

Using (30) we find that:

$$
\begin{aligned}
Q(x_0)|1/2,1/2,1/2,1/2,0,0,0,1/2,0>_{x_0} &= i\,\frac{\sqrt{3}}{4}\,|1/2,1/2,1/2,1/2,0,0,1,1/2,0>_{x_0} \\
Q(x_0)|1/2,1/2,1/2,1/2,0,0,1,1/2,0>_{x_0} &= -i\frac{\sqrt{3}}{4}\,|1/2,1/2,1/2,1/2,0,0,0,1/2,0>_{x_0} \qquad (32) \\
Q(n)|j_1,j_2,j_3,j_4,j_5,j_6,j_{12},j_{123},j_{1234}>_n &= 0; \quad n \neq x_0.
\end{aligned}
$$

Note that the states at a,b,c,d,e and f are all annihilated by the volume operator and $Q = -\frac{\sqrt{3}}{4}\sigma_2$ in the loop space of this example where σ_2 is the Pauli matrix. It is also instructive to use this example to clarify the meaning of l_{ij}. First consider the state occurring on the lhs of the first equation of eqn(32); it is easy to check on using eqn(17) and eqn(19) that for this state $l_{12} = l_{34} = 1$ and all other l_{ij}'s are zero. Likewise, for the state occurring on the rhs of this equation there are two possibilities: $l_{13} = l_{24} = 1$ all the rest being zero, and $l_{14} = l_{23} = 1$ and all the rest being zero. It also follows from eqn(18) that both these states occur with equal weight and they have non-zero overlap. Thus the top of eqn(32) at x_0 can be graphically represented as shown in Figure 3.

Thus the two simplest loop eigenstates of the volume operator are:

$$
\begin{aligned}
|+\frac{\sqrt{3}}{4}> &= \frac{1}{\sqrt{2}}\Big(\,|j_{12}=0>_{x_0}\otimes|abcdef> +i\,|j_{12}=1>_{x_0}\otimes|abcdef>\Big) \\
|-\frac{\sqrt{3}}{4}> &= \frac{1}{\sqrt{2}}\Big(\,|j_{12}=0>_{x_0}\otimes|abcdef> -i\,|j_{12}=1>_{x_0}\otimes|abcdef>\Big) \qquad (33)
\end{aligned}
$$

with the property: $V|\pm\frac{\sqrt{3}}{4}> = \Big[\sum_n \sqrt{|Q(n)|}\Big]|\pm\frac{\sqrt{3}}{4}> = \sqrt{\frac{\sqrt{3}}{4}}|\pm\frac{\sqrt{3}}{4}>$. These degenerate loop eigenstates can be explicitly constructed in terms of the Schwinger bosons using (18).

4 Summary and Discussion

The main objective of the present work is to emphasize the utility of spin half Schwinger bosons in explicitly constructing and characterizing all the $SU(2)\otimes U(1)$ gauge invariant loop states in loop quantum gravity. In particular, the reformulation in terms of Schwinger boson enables us to interpret the SU(2) gauge invariant states of [3, 4, 12] in the (dual) angular momentum representation in terms of the geometrical loops. This approach is also technically useful as it is completely in terms of gauge invariant intertwining quantum numbers defined in section 2.2 and also bypasses the problem of rapid proliferation of Clebsch Gordan coefficients in constructing the spin networks using holonomies. Further, we have given a simple derivation of the matrix elements of the volume operator in LQG by using generalized Wigner Eckart theorem. The simplest loop eigenstates of the volume operator are explicitly constructed.

We note that in the present approach the SU(2) Gauss law constraints (13) and their independent solutions (18) are defined at the lattice *sites*. Next are the the U(1) Gauss law (8) constraints which act on the *links*. In this logical sequence, the next set of constraints are the diffeomorphism and the Hamiltonian constraints which involve the holonomies and electric field operators over the *plaquettes* [13, 14] of the lattice. Therefore, our next objective is to study the diffeomorphism and the Hamiltonian constraints in the loop basis (18) over the entire lattice. It will also be interesting to cast the Hamilitonian and the diffeomorphism constraints in terms of Schwinger bosons or equivalently in terms of the invariant intertwiners discussed in section (2.2). These issues are currently under investigation.

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