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# Extended navigability of small world networks: exact results and new insights

Cécile Caretta Cartozo and Paolo De Los Rios
*Institute of Theoretical Physics, SB, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015, Lausanne, Switzerland*
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Navigability of networks, that is the ability to find any given destination vertex starting from any other vertex, is crucial to their usefulness. In 2000 Kleinberg showed that optimal navigability could be achieved in *small-world* networks provided that a special recipe was used to establish long range connections, and that a greedy algorithm, that ensures that the destination will be reached, is used. Here we provide an exact solution for the asymptotic behavior of such a greedy algorithm as a function of the system's parameters. Our solution enables us to show that the original claim that only a very special construction is optimal can be relaxed depending on further criteria, such as, for example, cost minimization, that must be satisfied.

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By endowing nodes with both well-connected local neighborhoods and long-range shortcuts, that dramatically reduce the distances to any other node, transport on small-world (SW) networks is both locally and globally efficient [1]. Such feature has made SW networks appealing for several fields, such as social, computer and life sciences [2, 3, 4, 5, 6, 7, 8]. Unfortunately, taking full advantage of the small node-to-node distances requires a global knowledge of the system, that is, in general, not accessible. It is thus important to devise decentralized algorithms that rely only on local information and that are able to find good, although sub-optimal, routes from source to destination [9]. The analysis of a prototype decentralized algorithm showed that the precise recipe used to establish the long-range shortcuts affects the ability of decentralized algorithms to navigate the networks [10]. Here we provide an exact solution in any dimension for such problem, and, in light of recent insights into the properties of SW networks [11] and in contrast to the findings in [10], we show that there is a broad range of SW networks that are optimally navigable.

SW networks can be obtained from regular lattices in $d$ dimensions by adding to every node, with probability $q$, a long-range connection to another node taken at random over the whole lattice [12]. The key feature of SW networks is that the average distance between any two nodes grows at most logarithmically with the linear size $L$ of the lattice [12]. In order to analyze how different shortcut addition schemes affect the performances of a decentralized algorithm, Kleinberg [10] specified an additional rule for the selection of the long-range partnerships: the probability $s(l)$ that the shortcut added to a node ends at a node at euclidean (or lattice) distance $l$ is a decaying power-law, $s(l) = \mathcal{N} l^{-\alpha}$, where $\mathcal{N}$ is the normalization over the whole lattice. The two key ingredients for the construction of the SW network are thus the shortcut addition probability $q$ and the exponent $\alpha$.

The simple decentralized algorithm considered in [10] is of greedy nature: starting from a given node, at every step the algorithm chooses the edge with the end-node which is closer, in euclidean or lattice distance, to the selected destination; such scheme guarantees that the destination is always certainly reached. Using arguments from probability theory, it was possible to find, in $d = 2$, a lower bound for the average number of steps $\tau(L)$ that are necessary to connect nodes separated by a distance proportional to the lattice linear size $L$: $\tau(L) \leq L^{\beta}$, with $\beta = (2 - \alpha)/3$ if $0 \leq \alpha < 2$ and $\beta = (\alpha - 2)/(\alpha - 1)$ if $\alpha > 2$ (see Fig.1, dashed lines). In the case $\alpha = 2$ the upper bound $\tau(L) \leq (\ln L)^2$ was found instead. As a consequence, the best choice for an optimized navigability of a SW network using a decentralized algorithm would be $\alpha = 2$ in $d = 2$ and, more generally, $\alpha = d$ in $d$ dimensions [10, 13].

In what follows, we derive the exact asymptotic behavior of $\tau(L)$ in any dimensions. We resort to the same implicit assumption already used in the probabilistic approach in [13]: we treat the algorithm as a stochastic Markov process, by looking simultaneously at all possible network realizations [14]. If at a given stage the message is at site $i$, at lattice distance $d_i$ from the target, at the next step it will surely be at a site $j$, with distance $d_j < d_i$, due to the greediness of the algorithm. More in detail, if $j$ is a nearest neighbor of $i$, then their connecting edge is chosen only if there are no shortcuts from $i$ to sites $k$ with $d_k < d_j$. If, instead, such a useful shortcut exists, the greedy algorithm chooses it over the nearest neighbor connection. In general, we can write the following recursive relation for the average number of steps from a site $i$ to the destination

$$\tau(i) = \sum_j p_{i \to j} (\tau(j) + 1) \tag{1}$$

where the probability $p_{i \to j}$ depends on the presence and greedy-usefulness of shortcuts. In equation (1) we assume, following [10], that the time it takes to travel a network edge, be it a shortcut or a link of the underlying lattice, is 1.

Equation (1) can be easily solved numerically, on a lattice, by recursion. However, it is more instructive to

take its continuous space limit, where a lattice site $i$ is mapped onto a position $\vec{r}$ and the lattice spacing vanishes. Equation (1) in $d$ dimensions then becomes (for a complete derivation see the auxiliary material [15])

$$\tau'(r) = 1 - q\mathcal{N}\tau(r)\int \mathrm{d}\Omega \int_0^{2r\cos\theta} (\epsilon + l)^{-\alpha} l^{d-1} \mathrm{d}l + q\mathcal{N}\int \mathrm{d}\Omega \int_0^{2r\cos\theta} (\epsilon + l)^{-\alpha} l^{d-1} \tau\left(\sqrt{l^2 + r^2 - 2rl\cos\theta}\right) \mathrm{d}l \quad (2)$$

with $\epsilon$ a short-lengthscale cutoff that avoids divergences when $l \to 0$ ($\epsilon$ was clearly not necessary on a lattice). In equation (9), $\int \mathrm{d}\Omega$ is the integral over the $d$-dimensional hypersolid angle and it is parametrized, among others, by the azimuthal angle $\theta$, which covers only half of the hypersphere [15]; $q$ is the linear probability density of shortcuts. Clearly it takes no time to reach the destination starting from itself, and consequently $\tau(0) = 0$. Given the complete isotropy of the problem, $\tau(r)$ depends only on $r = |\vec{r}|$.

By setting the scaling function form

$$\tau(r) = K^{-1} f(Kr) \quad (3)$$

(the value of $K$ to be discussed below case by case), it is possible to obtain, after some manipulations [15], an integral equation for the derivative of $f(x)$,

$$f'(x) = 1 - x^{d+1-\alpha}\int \mathrm{d}\Omega \int_0^{2\cos\theta} I(y, K\epsilon/x) \frac{y - \cos\theta}{\sqrt{y^2 + 1 - 2y\cos\theta}} f'\left(x\sqrt{y^2 + 1 - 2y\cos\theta}\right) \mathrm{d}y \quad (4)$$

where $I(y, K\epsilon/x) = \int_0^y (K\epsilon/x + z)^{-\alpha} z^{d-1} \mathrm{d}z$. To obtain the asymptotic behavior of $f(x)$ we rely on simple considerations. Since $\tau(r)$ cannot decrease with the distance $r$ from the target, $f(x)$ is non-decreasing and therefore $f'(x)$ is non-negative. Moreover, $\mathcal{I}(y, K\epsilon/x)$ is positive and thus from (4) we obtain $f'(x) \leq 1$. It is then possible to show that if $\alpha < d + 1$, asymptotically $f'(x)$ depends only on $\alpha$ and $f'(x) \sim x^{-(d+1-\alpha)}$. If instead $\alpha \geq d + 1$, $f'(x)$ is not universal anymore [15] and $f'(x) \to 1/[1 + c(\alpha, \epsilon, d)q]$, where $c(\alpha)$ is a constant depending only on $d$ and $\alpha$.

In the following we analyze the five cases $0 \leq \alpha < d$, $\alpha = d$, $d < \alpha < d+1$, $\alpha = d+1$ and $\alpha > d+1$ separately:

- $0 \leq \alpha < d$: In this case the normalization is $\mathcal{N} = (d - \alpha)/L^{d-\alpha}$ and $K = [q(d-\alpha)]^{1/(d+1-\alpha)} L^{-(d-\alpha)/(d+1-\alpha)}$ to the leading order in $L$. Since $f'(x) \sim 1/x^{(d+1-\alpha)}$, $f(x)$ converges asymptotically to a constant and consequently $\tau(L) \sim L^{(d-\alpha)/(d+1-\alpha)}/q^{1/(d+1-\alpha)}$ for large values of $L$.

- $\alpha = d$: The normalization constant is $\mathcal{N} = \ln L$ to the leading order in $L$. This implies $f'(x) \sim 1/x$, so that $\tau(r) \sim (\ln L)^2/q$ for large values of $L$. This results coincides with the upper bound found in [10].

- $d \leq \alpha < d + 1$: The normalization does not depend, to the leading order, on $L$. In this case $f'(x) \sim 1/x^{(d+1-\alpha)}$ is non-integrable and therefore $f(x)$ diverges as $x^{\alpha-d}$, and asymptotically $\tau(L) \sim L^{\alpha-d}/q$.

- $\alpha = d+1$: Again the normalization does not depend on $L$. After some calculations [15] it is possible to show that $\tau(r) \sim L/(q \ln L)$.

- $\alpha > d + 1$: Once more the normalization does not depend on $L$. It is easy to show that $f'(x) \to 1/[1+c(\alpha, \epsilon, d)q]$ and thus $\tau(L) \sim L/(1+c(\alpha, \epsilon, d)q)$ asymptotically.

The above results are summarized in Table I, and one major conclusion that can be drawn at this stage is that a greedy algorithm can significantly outperform the simple lattice distance only if the shortcut length distribution has diverging first moments in the $L \to \infty$ limit. The analytical predictions, shown in Fig.1 (red solid lines) for $d = 1, 2$, are compatible with the lower bounds found in [10]. We have also numerically solved equation (1) on a lattice, and using lattice distances, in $d = 1, 2$ and the estimates for the exponent $\beta$ agree with the analytical predictions (Fig.1, blue stars), apart from small discrepancies due to the not-yet fully achieved asymptotic limit, an effect that is expectedly more important in higher dimensions. Furthermore, in $d = 1$ we have verified that, as long as $\alpha < 2$, the $\tau(r)$ curves for different values

of $q$ and of $L$, but for the same value of $\alpha$, do collapse onto each other asymptotically once $K\tau(r)$ is plotted as a function of $Kr$ with the appropriate values of $K$ (see Fig.2). This result confirms the asymptotic universality of $f'(x)$ when $\alpha < d+1$.

The results described above are necessary but not yet sufficient to thoroughly address the navigability of SW networks. Indeed, changing the value of $\alpha$ while keeping the value of $q$ fixed, as in [10], is only one of the possible ways to compare the navigability of SW networks. As a matter of fact, it has been shown that, as long as $0 \leq \alpha < 2d$, it is possible to make a $d$-dimensional lattice *small* by letting the shortcut probability $q$ depend on $\alpha$ [11]. In particular, the shortcut probability $q$ marking the crossover from the euclidean to the small world regime has the form $q \sim L^{-d}$ if $0 \leq \alpha < d$, $q \sim \ln L L^{-d}$ if $\alpha = d$ and $q \sim L^{\alpha-2d}$ if $d < \alpha < 2d$. Thus, keeping $q$ fixed for different values of $\alpha$ would pit against each other networks that are intrinsically differently *small*.

To alleviate such bias, a more appropriate comparison should take into account how $q$ must change with $\alpha$. In order to do so, we can start from a given value of the shortcut probability $q(d)$ at $\alpha = d$, where purportedly navigation is easier. Then, as $\alpha$ moves away from $d$, we let this value change according to $q(\alpha) = q(d)/\ln L$ if $\alpha < d$ and $q(\alpha) = q(d)L^{\alpha-d}/\ln L$ if $d < \alpha < 2d$. Since our derivation of the asymptotic behavior of $\tau(l)$ also provides the precise form of the $q$-dependent prefactors, we can obtain the correct expected asymptotics of $\tau(L)$ when $q$ is allowed to appropriately change: $\tau(L) \sim \left(L^{d-\alpha}\ln L\right)^{1/(d+1-\alpha)}$ if $0 \leq \alpha < d$, $\tau(L) \sim (\ln L)^2$ if $\alpha = d$, $\tau(L) \sim \ln L$ if $d < \alpha < d+1$, $\tau(L) \sim$ const if $\alpha = d+1$ and $\tau(L) \sim L^{-[\alpha-(d+1)]}$ if $d+1 < \alpha < 2d$. The last result is apparently paradoxical, because it predicts the navigation time to become smaller for larger systems. Actually, using the results obtained in [11], it is possible to show that if $q(\alpha) = q(d)L^{\alpha-d}/\ln L$, the average path length of a small-world actually decreases with $L$ for fixed $\alpha$, even faster than using a greedy algorithm [15]. It is thus not surprising that also within the greedy framework larger values of $\alpha$, with the corresponding increase of $q$, lead to shorter navigation times. Considering all the results summerized in Table I, we conclude that if we take the necessary change of $q$ into account any $d < \alpha < 2d$ outperforms the case $\alpha = d$.

To gain a definitive insight into the properties of such class of SW networks, we finally consider the total shortcut length per node $\mathcal{L} = q(\alpha, L)\bar{l}(\alpha, L)$ ($\bar{l}(\alpha, L)$ being the average shortcut length), for different values of $\alpha$. $\mathcal{L}$ can be associated to the amount of resources (cost) needed to set up such networks. We have $\mathcal{L} = q(d)L/\ln L$ for $0 < \alpha < d+1$ and $\mathcal{L} = q(d)\bar{l}L^{\alpha-d}/\ln L$ for $\alpha \geq d+1$. We observe that the value of $\mathcal{L}$ is the same for all $0 \leq \alpha < d+1$, and lower than that for $d+1 \leq \alpha < 2d$.

The results presented above reveal a new picture for the navigability of SW networks, which is richer and more variegate than previously outlined: navigability depends on several parameters, and the optimal choice depends on the criteria that have to be satisfied. If the average number $q$ of shortcuts per node must be kept constant, at the cost of making networks characterized by different values of $\alpha$ differently *small*, then our exact results confirm, while making more precise, the findings of [10]. If on the other hand a fairer comparison of small-world networks is desired, then the necessary change of $q$ as a function of $\alpha$ must be taken into account and any $\alpha > d$ outperforms $\alpha = d$, with optimal navigability reached for $\alpha = 2d$. If finally the amount of resources needed to set up the networks becomes a crucial factor, the whole range $\alpha \in (d, d+1)$ is where navigability is optimal and cheapest.

Outlining the dependence of navigability on various criteria, as highlighted in this work, is important in order to understand the architecture of networks where the small-world feature is believed to be crucial. Indeed, it can be expected that virtual connections, such as hyperlinks between web-pages, will bear almost no signature of cost-effects, but might be limited in number to a few units; contrary-wise, the number of connections per neuron in the brain can be extremely high, but establishing and maintaining them is surely resource expensive. Thus different systems might have achieved different layouts to enhance their navigability while respecting given sets of constraints. Keeping in mind all these criteria can be of paramount relevance given the growing interest in the greedy navigability of real networks [16].

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| | $\tau(L,q)$ | $\tau(L,q(L,\alpha))$ |
|---|---|---|
| $0 \leq \alpha < d$ | $L^{\frac{d-\alpha}{d+1-\alpha}}$ | $\left(\frac{L^{d-\alpha}}{\ln L}\right)^{\frac{1}{d+1-\alpha}}$ |
| $\alpha = d$ | $(\ln L)^2$ | $(\ln L)^2$ |
| $d < \alpha < d+1$ | $L^{\alpha-d}$ | $\ln L$ |
| $\alpha = d+1$ | $L/\ln L$ | const |
| $d+1 < \alpha < 2d$ | $L$ | $L^{-[\alpha-(d+1)]}$ |

TABLE I: **Asymptotic navigation times as a function of the exponent $\alpha$.** The column $\tau(L,q)$ reports the navigation time at constant shortcut density, whereas the column $\tau(L,q(L,\alpha))$ reports the navigation time when the shortcut density is allowed to change so to keep the system in the *small-world* regime.

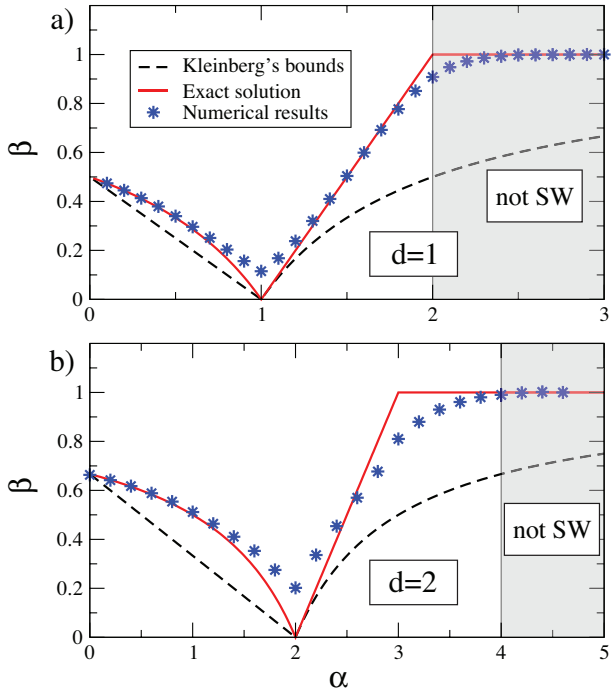


FIG. 1: **Asymptotic exponents of the greedy distance.** Comparison between the lower bounds found in the original reference (black dashed lines) and the exact solution presented here (red solid lines) in (a) $d = 1$ and (b) $d = 2$. The exact solutions are verified by numerical results (blue stars). The shadowed region ($\alpha > 2$ in $d = 1$ and $\alpha > 4$ in $d = 2$) is outside of the small-world regime.

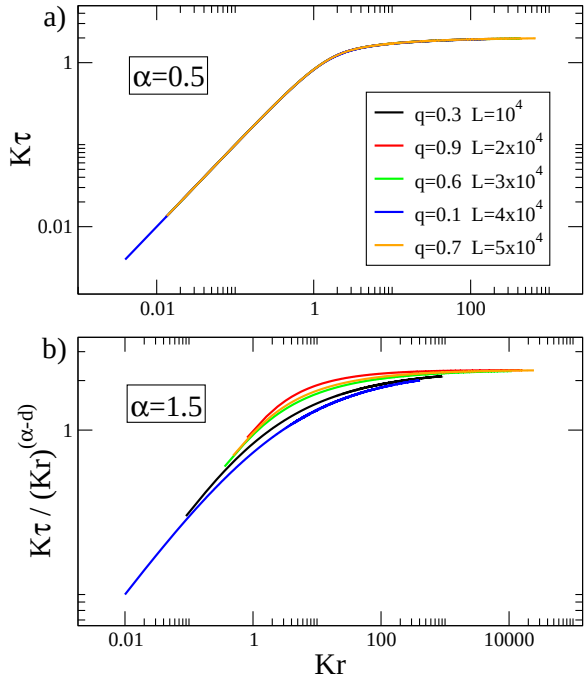


FIG. 2: **Universality of greedy distances.** The numerical results confirm the ansatz $\tau(r) = K^{-1} f(Kr)$ in $d = 1$. The collapses obtained once $K\tau(r)$ is plotted as a function of $Kr$ for a fixed $\alpha$ but for different $(q, L)$ pairs show that asymptotically ($Kr \to \infty$) $f(Kr)$ is a universal function that only depends on $\alpha$.

# Supplementary Material

## STOCHASTIC PROCESS FORMULATION

Kleinberg's problem can be formulated as a stochastic process when analyzed on the whole small-world network ensamble. We consider a walker that starts from site $i$ at distance $d_i$ from the destination. At the next step it will move differently on every network realization in the ensemble. Thus it will be on a site $j$ at distance $d_j$ from the destination with a probability $q$, that is the probability that site $i$ has a shortcut, times the probability that there is a shortcut from $i$ to $j$, times the probability such shortcut can be used by the greedy algorithm, *i.e.* that $d_j < d_i$. When $j$ is a nearest neighbor of $i$ (with $d_j = d_i - 1$), the probability of the walker to be in $j$ is given by the probability that $i$ does not have any shortcuts or that it does but that they end at distances from the destination larger than $d_j$. In that case the algorithm chooses an edge of the underlying lattice. It is then possible to formulate the evolution of the probability distribution of the walker's position as

$$P_k(t+1) = \sum_j p_{j\to k} P_j(t) \tag{5}$$

with initial conditions $P_k(0) = \delta_{i,k}$ if the walker starts from $i$. Given the nature of the greedy algorithm, the destination will be reached with certainity in a number of steps that is at most $d_i$.

Once the stochastic formulation is established, it is then easy to derive that the average trip time $\tau_i$ from site $i$ to the destination can be expressed recursively using the average trip time from every other node closer to the destination than $i$:

$$\tau_i = \sum_j p_{i\to j} \left(\tau_j + 1\right) \tag{6}$$

where the 1 in the parenthesis indicate the time it takes to go from site $i$ to site $j$ if there is a usable edge (shortcut or otherwise) from $i$ to $j$, which, as explained above, occurs with probability $p_{i\to j}$. Equation (6) can be easily solved

with the aid of the computer, but it is extremely demanding on time and memory resources for lattices with dimension $d \geq 2$ if the asymptotic limit has to be reached.

## CONTINUOUS SPACE FORMULATION OF EQUATION (6)

The continuous space formulation of equation (6) is easily obtained as follows. For reasons that will be clear in the following, shortcuts do not connect lattice sites, but rather the shortcut associated to a site go from the lattice edges connecting it to its neighbors to a lattice node. A small-world network obtained in this way is clearly as small-world as a network obtained with shortcuts from site to site.

Without affecting the generality of the results, from now on we shall assume that the destination is the origin. We also assume spatial isotropy of the process. Say that the walker starts from a point $\vec{r}$, at distance $r$ from the destination. Since it obeys a greedy scheme, it will always try to approach the destination as much as possible. Thus, in the absence of any useful shortcut, it will move along the ray defined by $\vec{r}$. Let's examine what may happen while it moves along the ray for an infinitesimal distance $\mathrm{d}r$. The point at distance $r - \mathrm{d}r$ will be reached only if 1) no shortcuts are encountered, with a probability $1 - qdr$ (where $q$ is now the linear density of shortcut probability, which also explains why the starting points of shortcuts are more appropriately associated to lattice edges in this context), or 2) if shortcuts are encountered (probability $q\mathrm{d}r$) but none of them are useful. The expected time to reach the destination, in $d$ dimensions, can then be written as

$$\tau(r) = \left[(1 - q\mathrm{d}r) + q\mathrm{d}r \left(1 - \int \mathrm{d}\Omega \int_0^{2r\cos\theta} \mathcal{N}(\epsilon + l)^{-\alpha} l^{d-1} \mathrm{d}l\right)\right] \left(\tau(r - \mathrm{d}r) + \frac{\mathrm{d}r}{v}\right) + \dots \tag{7}$$

where the dots indicate the contribution of *useful* shortcuts encountered along $\mathrm{d}r$, that we will discuss below. $\mathcal{N}(\epsilon+l)^{-\alpha}$ is the shortcut length probability distribution and $\epsilon$ is a short-distance cutoff introduces in order to avoid divergences when $l \to 0$ if $\alpha \geq d$.

The double integral in equation (7) is the probability for a shortcut encountered between $r$ and $r - \mathrm{d}r$ to fall inside a sphere of radius $r$, thus closer to the destination than $r$. It is composed of an angular integral over all the angles describing a (hyper)spherical parametrization of the space centered on $\vec{r}$ (with $\theta \in [0, \pi/2]$ in the integral over $\mathrm{d}\Omega$), and of a radial part $l$ centered on $\vec{r}$. The triangle defined by $\vec{r}$, $\vec{l}$ and $\vec{r} - \vec{l}$ identifies a two-dimensional plane in any dimension. Given the isotropy of the space, there is full rotational invariance around the ray $\vec{r}$, hence $\theta$ is the only angular variables of the integral over $\mathrm{d}\Omega$ that affects the radial integral. The reason for this peculiar choice of the parametrization will be explained below.

The term $\tau(r - \mathrm{d}r) + \frac{dr}{v}$ is the expected time to reach the destination from a distance $r - \mathrm{d}r$ augmented of the time to reach $r - \mathrm{d}r$ from $r$, which we assume to be proportional to $\mathrm{d}r$ and to depend on an intrinsic velocity $v$. In what follows, we fix $v = 1$.

Equation (7) is not complete yet. The term representing the contribution from useful shortcuts is given by:

$$\tau(r) = \dots + q\mathrm{d}r \int \mathrm{d}\Omega \int_0^{2r\cos\theta} \mathcal{N}(\epsilon + l)^{-\alpha} l^{d-1} \left(\tau\left(\sqrt{l^2 + r^2 - 2lr\cos\theta}\right) + \frac{\mathrm{d}r}{v}\right) \mathrm{d}l \tag{8}$$

In complete analogy with our assumption, in equation (6), that it takes exactly the same unitary time to travel along a lattice edge or a shortcut, the time to reach $r - \mathrm{d}r$ from $r$ is the same in (7) and (8) (with $v = 1$ in the latter).

By dividing both terms by $\mathrm{d}r$ and taking the limit $\mathrm{d}r \to 0$, we obtain the continuos space limit presented in the main text:

$$\tau'(r) = 1 - q\mathcal{N}\tau(r) \int \mathrm{d}\Omega \int_0^{2r\cos\theta} (\epsilon + l)^{-\alpha} l^{d-1} \mathrm{d}l + q\mathcal{N} \int \mathrm{d}\Omega \int_0^{2r\cos\theta} (\epsilon + l)^{-\alpha} l^{d-1} \tau\left(\sqrt{l^2 + r^2 - 2lr\cos\theta}\right) \mathrm{d}l \quad . \tag{9}$$

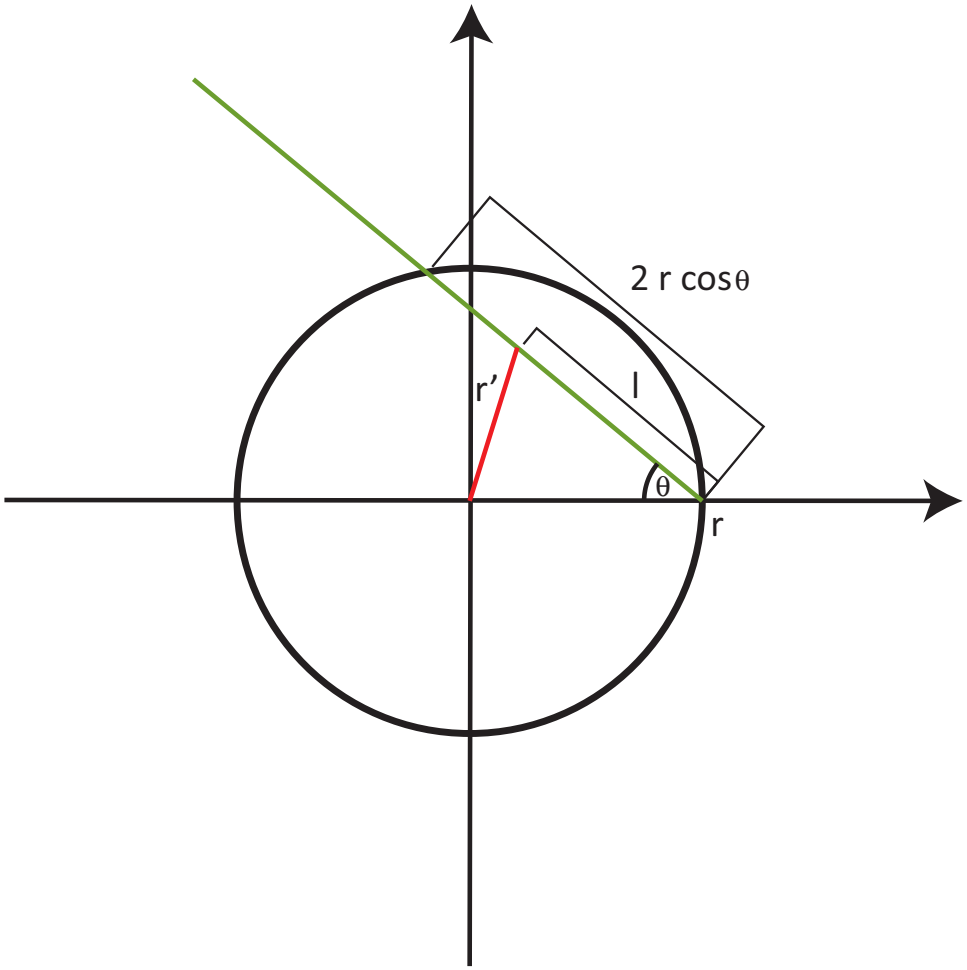


FIG. 3: Graphical representation of the relations between $\vec{r}$, $\vec{r'}$ and $l$.

## RESCALING OF EQUATION (9)

When we introduce the assumption $\tau(r) = K^{-1} f(Kr)$ the left hand side of equation (9) becomes $\tau'(r) = f'(Kr)$. After some manipulations, we can write:

$$\begin{aligned} f'(Kr) &= 1 - q\mathcal{N}K^{-(d+1-\alpha)} f(Kr) \int \mathrm{d}\Omega \int_0^{2r\cos\theta} (K\epsilon + Kl)^{-\alpha} (Kl)^{d-1} \mathrm{d}(Kl) + \\ &+ q\mathcal{N}K^{-(d+1-\alpha)} \int \mathrm{d}\Omega \int_0^{2r\cos\theta} (K\epsilon + Kl)^{-\alpha} f\left(\sqrt{K^2l^2 + K^2r^2 - 2(Kl)(Kr)\cos\theta}\right) (Kl)^{d-1} \mathrm{d}(Kl) \quad . \end{aligned} \tag{10}$$

A simple change of variable $y = Kl/Kr$ in the integral leads to

$$\begin{aligned} f'(Kr) &= 1 - q\mathcal{N}K^{-(d+1-\alpha)} (Kr)^{d-\alpha} f(Kr) \int \mathrm{d}\Omega \int_0^{2\cos\theta} \left(\frac{K\epsilon}{Kr} + y\right)^{-\alpha} y^{d-1} \mathrm{d}y + \\ &+ q\mathcal{N}K^{-(d+1-\alpha)} (Kr)^{d-\alpha} \int \mathrm{d}\Omega \int_0^{2\cos\theta} \left(\frac{K\epsilon}{Kr} + y\right)^{-\alpha} f\left(Kr\sqrt{y^2 + 1 - 2y\cos\theta}\right) y^{d-1} \mathrm{d}y \quad . \end{aligned} \tag{11}$$

Now we can set the value of $K$ according to $q\mathcal{N}K^{-(d+1-\alpha)} = 1$. Once the corresponding expression of $K$ is introduced,

equation (11) depends only on $Kr$. We can, then, change the general variable to $x = Kr$ and obtain the equation

$$\begin{aligned} f'(x) &= 1 - x^{d-\alpha} f(x) \int \mathrm{d}\Omega \int_0^{2\cos\theta} \left(\frac{K\epsilon}{x} + y\right)^{-\alpha} y^{d-1} \mathrm{d}y + \\ &+ x^{d-\alpha} \int \mathrm{d}\Omega \int_0^{2\cos\theta} \left(\frac{K\epsilon}{x} + y\right)^{-\alpha} f\left(x\sqrt{y^2+1-2y\cos\theta}\right) y^{d-1} \mathrm{d}y \quad . \end{aligned} \tag{12}$$

## TRANSFORMING THE INTEGRO-DIFFERENTIAL EQUATION INTO AN INTEGRAL EQUATION

By calling $I(y, K\epsilon/x)$ the primitive of $\left(\frac{K\epsilon}{x} + y\right)^{-\alpha} y^{d-1}$, equation (12) becomes

$$\begin{aligned} f'(x) &= 1 - x^{d-\alpha} f(x) \int \mathrm{d}\Omega \left[I(2\cos\theta, K\epsilon/x) - I(0, K\epsilon/x)\right] + \\ &+ x^{d-\alpha} \int \mathrm{d}\Omega \int_0^{2\cos\theta} \left(\frac{K\epsilon}{x} + y\right)^{-\alpha} y^{d-1} f\left(x\sqrt{y^2+1-2y\cos\theta}\right) \mathrm{d}y \quad . \end{aligned} \tag{13}$$

Integration by parts the last integral in equation (13) gives

$$\begin{aligned} f'(x) &= 1 - x^{d-\alpha} f(x) \int \mathrm{d}\Omega \left[I(2\cos\theta, K\epsilon/x) - I(0, K\epsilon/x)\right] + \\ &+ x^{d-\alpha} \int \mathrm{d}\Omega \left[I(2\cos\theta, K\epsilon/x) - I(0, K\epsilon/x)\right] f(x) + \\ &- x^{d+1-\alpha} \int \mathrm{d}\Omega \int_0^{2\cos\theta} I(y, K\epsilon/x) \frac{y - \cos\theta}{\sqrt{y^2+1-2y\cos\theta}} f'\left(x\sqrt{y^2+1-2y\cos\theta}\right) \mathrm{d}y \end{aligned} \tag{14}$$

which simplifies to

$$f'(x) = 1 - x^{d+1-\alpha} \int \mathrm{d}\Omega \int_0^{2\cos\theta} I(y, K\epsilon/x) \frac{y - \cos\theta}{\sqrt{y^2+1-2y\cos\theta}} f'\left(x\sqrt{y^2+1-2y\cos\theta}\right) \mathrm{d}y \quad . \tag{15}$$

This is an integral equation for $f'(x)$. The properties of the kernel $I(y, K\epsilon/x)$ are simple: it is always positive because it is the integral of a positive function, and $I(0, K\epsilon/x) \sim (K\epsilon/x)^{d-\alpha}$. Thus, in the asymptotic limit $x \to \infty$, $I(y, K\epsilon/x)$ develops an integrable singularity for $\alpha < d+1$ and it converges to a well-behaved kernel. In what follows we can safely disregard the short-distance cutoff. For $\alpha \geq d+1$, the singularity is non-integrable and it will require a special treatment.

Equation (15) shows that, since we can safely take the $K\epsilon/x \to 0$ limit when $\alpha < d+1$, $f'(x)$ must obey an asymptotically universal form that depends only on $\alpha$. We have numerically verified that this is indeed the case (see main text). In the case $\alpha \geq d+1$, instead, the $K\epsilon/x$ term dominates and $f'(x)$ will not be universal any more.

## ASYMPTOTIC BEHAVIOR OF $f'(x)$

We want to determine the asymptotic behavior of $f'(x)$. Clearly $\tau(r)$ must be a non decreasing function of $r$, since it obviously takes longer to reach the destination starting from increasingly farther distances. Thus, $f'(x)$ is non negative. Consequently, equation (15) tells us that $f'(x) \leq 1$. We can, then, rewrite equation (15) as

$$f'(x) = 1 - \int \mathrm{d}\Omega \int_0^{2\cos\theta} I(y, K\epsilon/x) \frac{y - \cos\theta}{\left(\sqrt{y^2+1-2y\cos\theta}\right)^{d+2-\alpha}} \left[\left(x\sqrt{y^2+1-2y\cos\theta}\right)^{d+1-\alpha} f'\left(x\sqrt{y^2+1-2y\cos\theta}\right)\right] \mathrm{d}y \tag{16}$$

**The case $\alpha < d+1$**

If $\alpha < d+1$ it is easy to show that the only ansatz for the behavior of $x^{d+1-\alpha}f'(x)$ that does not lead to contradicting conclusions is that

$$x^{d+1-\alpha}f'(x) \xrightarrow{x\to\infty} \mathcal{C} = \left\{ \int \mathrm{d}\Omega \int_0^{2\cos\theta} I(y, K\epsilon/x) \frac{y-\cos\theta}{\left(\sqrt{y^2+1-2y\cos\theta}\right)^{d+2-\alpha}} \mathrm{d}y \right\}^{-1} \tag{17}$$

so that $f'(x) \sim \mathcal{C}/x^{d+1-\alpha}$. Indeed, should $f'(x)$ decrease faster than that, then the whole integral would become increasingly negligible and $f'(x)$ would be asymptotically equal to 1, against its fast decay assumption. Should $f'(x)$ decay slower than $\mathcal{C}/x^{d+1-\alpha}$, then the integral would asymptotically diverge and $f'(x)$ would become negative, implying a non-physical decrease of $\tau(r)$ for larger distances.

**The case $\alpha = d+1$**

In this case $K$ is not set by any equation, hence we can choose $K = 1$. This implies that in the rescaling of the continuos limit we can simply set $\tau(r) = f(x)$ and $r = x$. The integral equation becomes (already in the limit when $x \to \infty$)

$$f'(x) \;=\; 1 - q\mathcal{N} \int \mathrm{d}\Omega \int_0^{2\cos\theta} \delta(y) \frac{y-\cos\theta}{\sqrt{y^2+1-2y\cos\theta}} f'\left(x\sqrt{y^2+1-2y\cos\theta}\right) \mathrm{d}y \quad . \tag{18}$$

Since asymptotically $\int_0^{2\cos\theta} I(y, \epsilon/x)\mathrm{d}y \sim \ln x/\epsilon$, we have $f'(x) = 1/[1 + qc(\alpha, d)\ln x]$. Thus $\tau(r) \sim \frac{1}{q\mathcal{N}c(\alpha,d)} \frac{x}{\ln x}$.

**The case $\alpha > d+1$**

If $\alpha \geq d+1$, the kernel diverges as $(x/K\epsilon)^{\alpha-d}$ for $x \to \infty$. We thus divide and multiply it for its integral between 0 and $2\cos\theta$ to obtain

$$f'(x) \;=\; 1 - x^{d+1-\alpha} \left[\int_0^{2\cos\theta} I(y, K\epsilon/x)\mathrm{d}y\right] \int \mathrm{d}\Omega \int_0^{2\cos\theta} \mathcal{I}(y, K\epsilon/x) \frac{y-\cos\theta}{\sqrt{y^2+1-2y\cos\theta}} f'\left(x\sqrt{y^2+1-2y\cos\theta}\right) \mathrm{d}y \tag{19}$$

where $\mathcal{I}(y, K\epsilon/x) = I(y, K\epsilon/x)/\left[\int_0^{2\cos\theta} I(y, K\epsilon/x)\mathrm{d}y\right]$. Interestingly, $\mathcal{I}(y, K\epsilon/x) \to \delta(y)$ as $x \to \infty$ and thus equation (19) reduces to

$$f'(x) = 1 - x^{d+1-\alpha} \left[\int_0^{2\cos\theta} I(y, K\epsilon/x)\mathrm{d}y\right] f'(x) \quad . \tag{20}$$

Since $\int_0^{2\cos\theta} I(y, K\epsilon/x)\mathrm{d}y \sim (x/K\epsilon)^{\alpha-d-1}$, from equation (20) we obtain that $f'(x) \sim 1/(1 + c(\alpha, \epsilon, d)q)$, where $c(\alpha, \epsilon, d)$ is a constant that depends only on $\alpha$, $\epsilon$ and $d$. Thus, $f(x) \sim x/(1 + c(\alpha, \epsilon, d)q)$ and $\tau(r) \sim r/(1 + c(\alpha, \epsilon, d)q)$.

This result can be derived on a lattice by simple arguments. When $\alpha > d+1$, the average shortcut length $\bar{l}$ is finite, so that shortcuts simply redefine the elementary length-scale of the system. The typical step length is, then, $l_{typ} = 1 \cdot \{(1-q) + q[1 - \tilde{c}(\alpha, \epsilon, d)]\} + \bar{l} \cdot q\tilde{c}(\alpha, \epsilon, d) = 1 + (\bar{l} - 1)q\tilde{c}(\alpha, \epsilon, d)$, where $\tilde{c}(\alpha, \epsilon, d)$ is the probability that, if a shortcut is present, it can be used by the greedy algorithm. $\bar{l} > 1$ because on a lattice there are no shortcuts of unitary length. The average time to reach the destination, given that it increases linearly, will be $r/l_{typ}$.

**GREEDY AND GLOBAL PATH LENGTH IN THE CASE $\alpha > d+1$ AND $q(\alpha) = q(d)L^{\alpha-d}/\ln L$**

As outlined and numerically verified in [Petermann and De Los Rios, Phys. Rev. E **73**, 026114 (2006)], the average path length $< L >$ for small world networks of linear size $L$ built according to the present recipe (given $\alpha$ and $q$)

obeys the scaling relation

$$< L >= L^* \mathcal{F}_\alpha \left( \frac{L}{L^*} \right) \tag{21}$$

where $L^* = q^{-1/d}$ if $(\alpha < d)$ and $L^* = q^{-1/(2d-\alpha)}$ if $(d < \alpha < 2d)$. The function $\mathcal{F}_\alpha(x)$ is

$$\mathcal{F}_\alpha(x) = \begin{cases} x & \text{if } x \ll 1 \\ (\ln x)^\gamma & \text{if } x \gg 1 \end{cases} \tag{22}$$

where the $x >> 1$ asymptotic behavior is a power of a logarithm (see the above mentioned reference and also [Kosmidis, Havlin and Bunde, Europhys. Lett. **82**, 48005 (2008)]).

Using $q = q(d) L^{\alpha - d} / \ln L$ (for $\alpha > d$) in equation (21) it is possible to obtain

$$< L >= L^{-\frac{\alpha-d}{2d-\alpha}} (\ln L)^{\frac{1}{2d-\alpha}} \mathcal{F}_\alpha \left( \frac{L^{\frac{d}{2d-\alpha}}}{(\ln L)^{\frac{1}{2d-\alpha}}} \right) \quad . \tag{23}$$

Thus, since $\mathcal{F}_\alpha$ increases at most logarithmically, the average path length decreases asymptotically as a power of $L$ as $\alpha$ increases. The logarithmic decrease of the greedy path length for $\alpha > d+1$ is thus compatible with this behavior and, expectedly, the greedy path length remains larger than the global one.