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Introductory Lectures on Contact Geometry

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1. Introduction

Though contact topology was born over two centuries ago, in the work of Huygens, Hamilton and Jacobi on geometric optics, and been studied by many great mathematicians, such as Sophus Lie, Elie Cartan and Darboux, it has only recently moved into the foreground of mathematics. The last decade has witnessed many remarkable breakthroughs in contact topology, resulting in a beautiful theory with many potential applications. More specifically, as a coherent – though sketchy – picture of contact topology has been developed, a surprisingly subtle relationship arose between contact structures and 3- (and 4-) dimensional topology. In addition, the applications of contact topology have extended far beyond geometric optics to include non-holonomic dynamics, thermodynamics and more recently Hamiltonian dynamics [**25, 40**] and hydrodynamics [**12**].

Despite it long history and all the recent work in contact geometry, it is not overly accessible to those trying to get into the field for the first time. There are a few books giving a brief introduction to the more geometric aspects of the theory. Most notably the last chapter in [**1**], part of Chapter 3 in [**34**] and an appendix to the book [**2**]. There have not, however, been many books or survey articles (with the notable exception of [**20**]) giving an introduction to the more topological aspects of contact geometry. It is this topological approach that has lead to many of the recent breakthroughs in contact geometry and to which this paper is devoted. I planned these lectures when asked to give an introduction to contact geometry at the Georgia International Topology Conference in the summer of 2001. My idea was to give an introduction to the "classical" theory of contact topology, in which the *characteristic foliation* plays a central roll, followed by a hint at the more modern trends, where specific foliations take a back seat to *dividing curves*. This was much too ambitious for the approximately one and a half hours I had for these lectures, but I nonetheless decided to follow this outline in preparing these lecture notes. These notes begin with an introduction to contact structures in Section 2, here all the basic definitions are given and many examples are discussed. In the following section we consider contact structures near a point and near a surface. It is in

this section that the fundamental notion of characteristic foliation on a surface first appears. In an appendix to Section 3, I briefly describe Moser's method, which is a technique for understanding families of contact structures. Section 4 is devoted to the all pervasive dichotomy in contact geometry: tight vs. overtwisted. Here we see that overtwisted contact structures are not so interesting from a topological point of view and that tight contact structures have and intimate and subtle relationship with topology. Then, in Section 5, we consider special knots in contact structures. The study of these knots sheds light on the tight vs. overtwisted dichotomy and allows us to prove a general existence theorem for contact structures. We end with a brief introduction to convex surfaces. Though this section is short we will be able to indicate the power of convex surfaces in contact geometry and point the interested reader to recent literature on the subject.

These lectures are written in an informal style with many exercises, which are usually not too difficult and copious hints are provided. The proofs of most results are left to the exercises, however for more complicated proofs the outline is given with the details left as exercises. I am assuming the reader is familiar with basic differential topology (manifolds, vector fields, Lie derivatives, forms, ..., see [**39**]) and has a passing knowledge of 3–manifold topology (as can be gleaned from a glance or two at [**37**] or [**24**]).

As these notes have a bias for topological techniques in contact geometry, many exciting and important recent developments have been left out, specifically in regards to the use of holomorphic curves in contact geometry. Here we refer the reader to [**7, 11, 23**]. For connections with Seiberg-Witten Theory see [**29, 30**]. Finally for an interesting historical overview the reader should consult [**18**].

2. Definitions and Examples

A *plane field* ξ on M is a subbundle of the tangent bundle TM such that $\xi_p = T_pM \cap \xi$ is a 2-dimensional subspace of T_pM for each $p \in M$.

Example 2.1 Consider the 3-manifold $M = \Sigma \times S^1$ where Σ is a surface. Then for each $p = (x, \theta) \in \Sigma \times S^1$ let $\xi_p = T_x\Sigma \subset T_pM$. Clearly ξ is a plane field on M.

Example 2.2 Let α be a 1-form on M. So at each point $p \in M$ we have a linear map

$$(1) \qquad\qquad\qquad \alpha_p \colon T_pM \to \mathbb{R}.$$

Thus $\ker \alpha_p$ is either a plane or all of T_pM. If we assume the 1-form never has all of T_pM as its kernel, then $\xi = \ker \alpha$ is a plane field. Note in the previous example the 1-form $\alpha = d\theta$ defines ξ.

It turns out that, locally, you can always represent a plane field as the kernel of a 1-form.

Exercise 2.3 Prove this. In other words, given a plane field ξ on M and a point $p \in M$ show you can find a neighborhood U of p and a 1-form α_U defined on the neighborhood such that $\xi|_U = \ker \alpha_U$.

Exercise 2.4 If M and ξ are both oriented show that you can find a 1-form α defined on all of M such that $\xi = \ker \alpha$.

A plane field ξ is called a *contact structure* if for any 1-form α with $\xi = \ker\alpha$ (α can be locally or globally defined) we have

$$(2) \qquad\qquad \alpha \wedge d\alpha \neq 0.$$

Exercise 2.5 Show that $\alpha \wedge d\alpha \neq 0$ if and only if $d\alpha|_\xi \neq 0$.

Before we look at some examples of contact structures note that our first example of a plane field is not a contact structure. Indeed the plane field is defined by the 1-form $\alpha = d\theta$ so $d\alpha = d(d\theta) = 0$.

Example 2.6 Consider the manifold $\mathbb{R}^3$ with standard Cartesian coordinates (x, y, z) and the 1-form

$$(3) \qquad\qquad \alpha_1 = dz + xdy.$$

Note that $d\alpha_1 = dx \wedge dy$ so $\alpha_1 \wedge d\alpha_1 = dz \wedge dx \wedge dy \neq 0$. Thus α_1 is a contact form and $\xi_1 = \ker\alpha_1$ is a contact structure. At a point (x, y, z) the contact plane ξ_1 is spanned by $\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\}$. So at any point in the yz-plane (*i.e.* where $x = 0$) ξ_1 is horizontal. If we move to the point $(1, 0, 0)$ then ξ_1 is spanned by $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z} - \frac{\partial}{\partial y}\}$. So the plane is tangent to the x-axis but has been tilted clockwise by 45%. In general, if we start at $(0, 0, 0)$ we have a horizontal plane and as we move out along the x-axis the plane will twist in a left handed manner (*i.e.* clockwise). The twist will be by 90% when x "gets to" ∞. There is similar behavior on all rays perpendicular to the yz-plane. See Figure 1.
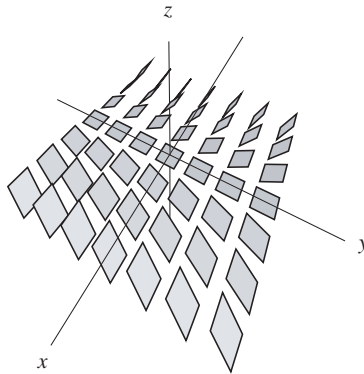

FIGURE 1. The contact structure $\ker(dz + xdy)$.

REMARK 2.7. Many authors prefer to use the form $dz - ydx$ to define the "standard" contact structure on $\mathbb{R}^3$. There is really no difference between these structures. (Rotating about the z-axis will take one of these structures to the other.)

Example 2.8 Consider $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and the 1-form

$$(4) \qquad\qquad \alpha_2 = dz + r^2 d\theta.$$

Since $\alpha_2 \wedge d\alpha_2 = 2rdr \wedge d\theta \wedge dz \neq 0$, $\xi_2 = \ker\alpha_2$ is a contact structure. At the point (r, θ, z) the contact plane ξ_2 is spanned by $\{\frac{\partial}{\partial r}, r^2\frac{\partial}{\partial z} - \frac{\partial}{\partial\theta}\}$. So when $r = 0$ (*i.e.* in the z-axis) ξ_2 is horizontal. As you move out on any ray perpendicular to

the z-axis the planes ξ_2 will twist in a clockwise manner. So this example is just like the previous one except that everything is symmetric about the z-axis.

Two contact structures ξ_0 and ξ_1 on a manifold M are called *contactomorphic* if there is a diffeomorphism $f\colon M \to M$ such that f send ξ_0 to ξ_1 :

$$f_*(\xi_0) = \xi_1.$$

Exercise 2.9 Show that a diffeomorphism $f\colon M \to M$ is a contactomorphism if and only if there are contact forms α_0 and α_1 for ξ_0 and ξ_1, respectively, and a non-zero function $g\colon M \to \mathbb{R}$ such that $f^*\alpha_1 = g\alpha_0$.

Exercise 2.10 Check that Examples 2.6 and 2.8 are contactomorphic. If you are having trouble coming up with the contactomorphism then first try to write down the contacto-morphism implied in Remark 2.7

Example 2.11 Once again consider $\mathbb{R}^3$ with cylindrical coordinates, but this time take the 1-form $\alpha_3 = \cos r\, dz + r \sin r\, d\theta$. One may compute that

$$(5) \qquad\qquad \alpha_3 \wedge d\alpha_3 = (1 + \frac{\sin r \cos r}{r})d\mathrm{vol}.$$

Thus to see that α_3 is a contact form you only have to check that

$$(6) \qquad\qquad 1 + \frac{\sin r \cos r}{r} > 0.$$

Note that $\xi_3 = \ker \alpha_3$ is horizontal along the z-axis and as you move out on any ray perpendicular to the z-axis the planes will twist in a clockwise manner. This time, however, the planes will twist 90% by the time you get to $r = \pi/2$. In fact, as you move out on any ray ξ_3 will make infinitely many full twists as r goes to ∞!

This example certainly looks different from our previous two examples, but it is not exactly obvious how one would actually show it is different. In the early 1980's Bennequin [**3**] did distinguish this example from the previous ones and in the process ushered in a new era in contact geometry. We will indicate Bennequin's proof in Section 5.

So far all our examples are on $\mathbb{R}^3$. We now give an example on a closed manifold.

Example 2.12 Consider the unit 3-sphere, S^3, in $\mathbb{R}^4$. Let

$$(7) \qquad\qquad \alpha = (x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)|_{S^3},$$

where (x_1, y_1, x_2, y_2) are standard Cartesian coordinates on $\mathbb{R}^4$ and set $\xi = \ker \alpha$.

Exercise 2.13 Check that $\alpha \wedge d\alpha \neq 0$ and thus ξ is a contact structure on S^3.
Hint: It might be helpful to read the following paragraph before trying attempting this exercise.

In anticipation of the next example it will be useful to describe ξ in another way. If we let $f(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2$ then $S^3 = f^{-1}(1)$. Moreover at a point (x_1, y_1, x_2, y_2) in S^3 the tangent space is given by

$$(8)\ \ T_{(x_1,y_1,x_2,y_2)}S^3 = \ker df_{(x_1,y_1,x_2,y_2)} = \ker(2x_1 dx_1 + 2y_1 dy_1 + 2x_2 dx_2 + 2y_2 dy_2).$$

Now we can think of $\mathbb{R}^4$ as $\mathbb{C}^2$. Under this identification we denote the complex structure (*i.e.* multiplication by i) by J. In other words, $Jx_i = y_i$, $Jy_i = -x_i$ for $i = 1, 2$. The complex structure J induces a complex structure on each tangent space: $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}$ and $J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}$ for $i = 1, 2$.

CLAIM. *The plane field ξ is the set of complex tangencies to S^3. By this we mean*

$$\xi = T_{(x_1,y_1,x_2,y_2)}S^3 \cap J(T_{(x_1,y_1,x_2,y_2)}S^3). \tag{9}$$

Indeed one may easily check that

$$J(T_{(x_1,y_1,x_2,y_2)}S^3) = \ker(df_{(x_1,y_1,x_2,y_2)} \circ J) \tag{10}$$

and

$$df_{(x_1,y_1,x_2,y_2)} \circ J = 2x_1 dy_1 - 2y_1 dx_1 + 2x_2 dy_2 - 2y_2 dx_2. \tag{11}$$

Thus we have $\alpha = (df \circ J)|_{S^3}$ and the claim is proved.

Exercise 2.14 Show that $(S^3 \setminus \{p\}, \xi|_{S^3 \setminus \{p\}})$ is contactomorphic to $(\mathbb{R}^3, \xi_2)$.
Hint: Pick the point p carefully and use stereographic coordinates.

It turns out that many contact structures can be described as the set of complex tangencies to a real hypersurface in a complex manifold.

Example 2.15 Let X be a complex manifold with boundary and denote the induced complex structure on TX by J. We can find a function ϕ defined in a neighborhood of the boundary such that $\phi^{-1}(0) = \partial X$. Now as in the previous example we can see that the complex tangencies to $M = \partial X$ are given by $\ker(d\phi \circ J)$. Thus the complex tangencies ξ to M form a contact structure if and only if $d(d\phi \circ J)$ is a non-degenerate 2-form on ξ.

A fruitful way to construct such manifolds has been through the use of Stein surfaces. To define Stein surfaces we need some preliminary notions. Let X be a complex manifold of complex dimension 2 (real dimension 4). Again let J denote the induced complex structure on TX. From a function $\phi \colon X \to \mathbb{R}$ we can define a 2-form $\omega = d(d\phi \circ J)$ and a symmetric form $g(v, w) = \omega(v, Jw)$. If this symmetric form is positive definite (*i.e.* defines a metric on X) the function ϕ is called *strictly plurisubharmonic*. The manifold X is a *Stein surface* if X admits a proper strictly plurisubharmonic function $\phi \colon X \to \mathbb{R}$. It is easy to see that in this situation the complex tangencies to $M_c = \phi^{-1}(c)$ form a contact structure whenever c is not a critical value. We will call such a contact structure *Stein fillable*. Later we will see that this implies ξ is a special type of contact structure. See [**22**] to learn how to construct many Stein surfaces and hence many contact structures.

3. Local Structure

In this section we discuss the nature of contact structures near a point (Darboux's Theorem) and near a surface. You can find further discussion of all these local theorems in [**1, 34**].

3.1. Darboux's Theorem. Darboux's Theorem essentially says that all contact structures look the same near a point. So contact structures do not have interesting local structure (this should be compared with Riemannian geometry, where the curvature is an obstruction to metrics being locally the same). This is an indication that any interesting phenomena in contact geometry should be of a global nature (*i.e.* be related to the global topology of the manifold supporting the contact structure).

THEOREM 3.1. *Let (M, ξ) be any contact 3-manifold and p any point in M. Then there exist neighborhoods N of p in M, and U of $(0, 0, 0)$ in $\mathbb{R}^3$ and a contactomorphism*

$$f \colon (N, \xi|_N) \to (U, \xi_1|_U).$$

The current modern proof of Darboux's Theorem uses "Moser's Method." We will discuss this more in the appendix to this section. The classical proofs of this theorem are more elementary in nature and we encourage the reader to try and come up with an elementary proof.

Exercise 3.2 Find an elementary proof of Darboux's theorem.

3.2. The Characteristic Foliations. Let Σ be an embedded oriented surface in a contact manifold (M, ξ). At each point x of Σ consider

$$l_x = \xi_x \cap T_x\Sigma.$$

For most x, the subspace l_x will be a line in $T_x\Sigma$, but at some points, which we call singular points, $l_x = T_x\Sigma$.

Exercise 3.3 Show that l_x cannot equal $T_x\Sigma$ for all x in some open subset of Σ.
Hint: If this is true, then you can show that the contact condition is violated. Consider two vectors fields v and w tangent to Σ defined along this open subset. Using the formula $d\alpha(v, w) = v\alpha(w) - w\alpha(v) + \alpha([v, w])$ compute $\alpha \wedge d\alpha$.

It is not hard to show (see the next exercise) that we may find a singular foliation $\mathcal{F}$ of Σ tangent to l_x at each x. By this we mean the complement of the singularities is the disjoint union of 1-manifolds, called leaves of $\mathcal{F}$, and the leaf through x is tangent to l_x. This singular foliation is called the *characteristic foliation* of Σ and is denoted Σ_ξ (some authors prefer $\xi\Sigma$).

Exercise 3.4 Show there is a singular foliation tangent to l_x.
Hint: Locally on the surface find a vector field tangent to l_x at the nonsingular points and 0 at the singular points. Then use basic existence results from ordinary differential equations to construct the singular foliation locally. Finally make sure your local foliations fit together to give a global foliation.

Example 3.5 Let Σ be the unit sphere in $(\mathbb{R}^3, \xi_2)$. The only singularities in Σ_{ξ_2} are at the north and south poles. See Figure 2.
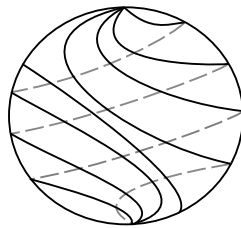

FIGURE 2. The characteristic foliation on S^2.

Example 3.6 Let Σ be the disk of radius π in the $r\theta$-plane in $(\mathbb{R}^3, \xi_3)$. As shown on the left hand side of in Figure 3, the center of Σ is a singular point and each point on the boundary of Σ is also a singular point. Let Σ' be Σ with its interior pushed up slightly. Now the only singularity in the characteristic foliation is at the center point. The boundary of Σ' is now a closed leaf in the foliation. See Figure 3.
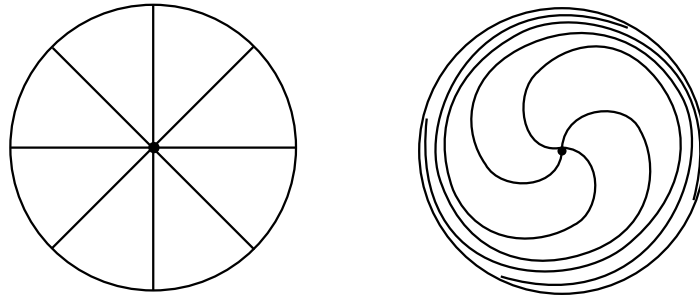
FIGURE 3. The degenerate and non-degenerate characteristic foliation on D^2.

This last example illustrates an important point: any surface Σ may be perturbed by a C^∞-small isotopy so that its characteristic foliation has only "generic" isolated singularities. A singularity is "generic" if it looks like one in Figure 4. On the left hand side is an elliptic point and on the right hand side is a hyperbolic point.
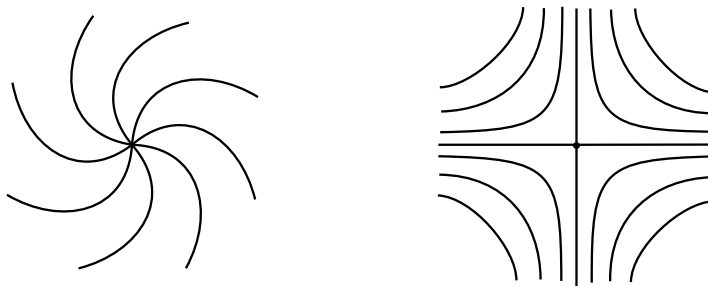

FIGURE 4. Generic singularities in the characteristic foliation.

Recall that Σ is oriented and we chose an orientation on ξ. We can orient $l_x = \xi_x \cap T_x\Sigma$ as follows: the vector $v \in l_x$ orients l_x if when we choose vectors $v_\xi \in \xi_x$ and $v_\Sigma \in T_x\Sigma$ such that (v, v_ξ) orients ξ_x and (v, v_Σ) orients $T_x\Sigma$ then (v, v_ξ, v_Σ) orients M. In this manner the leaves of the characteristic foliation inherit an orientation, so we can draw arrows on the leaves of the foliation and think of the foliation as a "flow." Moreover, at each singular point x we can assign a sign: the singularity is $+$ (respectively, $-$) if the orientation on $T_x\Sigma$ agrees (respectively, disagrees) with the one on ξ_x. With these conventions, a positive elliptic point is a source, a negative elliptic point is a sink. Note the sign if a hyperbolic point is not obvious at first glance as it is related to the ratio of the eigenvalues associated to the linearized flow at the singularity.

Exercise 3.7 Determine the sign of a hyperbolic point.

THEOREM 3.8. *Let (M_i, ξ_i) be a contact manifold and Σ_i an embedded surface for $i = 0, 1$. If there is a diffeomorphism $f\colon \Sigma_0 \to \Sigma_1$ that preserves the characteristic foliation:*

$$f((\Sigma_0)_{\xi_0}) = (\Sigma_1)_{\xi_1},$$

then f may be extended to a contactomorphism in some neighborhood of Σ_0. More-over, if f was already defined on a neighborhood of Σ_0 then we can isotop f so as to be a contactomorphism in some (possibly) smaller neighborhood.

So the characteristic foliation of on a surface determines (the germ of) the contact structure near the surface. Once again this theorem my be proved using "Moser's method".

Appendix to Section 3: Moser's Method

There are several good references for Moser's method and its many corollaries [**1, 34**]. One of the most general theorems one can prove using these techniques is

Theorem 3.9. *Let M be an oriented three manifold and $N \subset M$ a compact subset. Suppose ξ_0 and ξ_1 are contact structures on M for which $\xi_0|_N = \xi_1|_N$. Then there is a neighborhood U of N such that the identity map on a neighborhood of N is isotopic, rel. N, to a contactomorphism when restricted to U.*

Exercise 3.10 Show Theorems 3.1, 3.8 and 5.20 follow from this theorem.
Hint: Consider Darboux's theorem. Write down a diffeomorphism from a neighborhood N' of the point p in M to a neighborhood U' of $(0,0,0)$ in $\mathbb{R}^3$ so that the contact plane at p is sent to the contact plane at $(0,0,0)$. Push the contact structure ξ forward to U. Now you have two contact structures on U that agree on $(0,0,0)$, so use the above theorem to finish the proof. Theorem 5.20 is similar to this and Theorem 3.8 is similar but it is not so obvious you can write down the correct initial diffeomorphism.

The proof of this theorem follows essentially from the above mentioned references, but we will outline the proof in the following exercises.

Exercise 3.11 Let α_i be a contact form for $\xi_i, i = 0, 1$, that determines the orientation on $\xi_i|_N$. Let $\alpha_t = (1 - t)\alpha_0 + t\alpha_1$. Show that on some neighborhood U' of N all the $\xi_t = \ker \alpha_t$'s are contact structures.

Exercise 3.12 We now wish to find a family of diffeomorphisms $\phi_t \colon U \to U''$ (U and U'' are possibly smaller neighborhood of N) such that $\phi_t^*\xi_t = \xi_0$. (Here $\phi_t^* = (\phi_t^{-1})_*$.) This will of course finish the proof of the theorem. We will find the ϕ_t's as the flow of a vector field. Suppose v_t is a time dependent vector field whose flow generates the ϕ_t's. Show that if $v_t \in \xi_t$ then the ϕ_t's satisfy $\phi_t^*\xi_t = \xi_0$ if and only if $\iota_{v_t} d\alpha_t|_{\xi_t} = (h_t\alpha_t - \frac{d\alpha_t}{dt})|_{\xi_t}$, where $h_t = \frac{d\alpha_t}{dt}(X_t)$ and X_t is the unique vector field satisfying $\alpha_t(X_t) = 1$ and $\iota_{X_t} d\alpha_t = 0$. (Here ι_{v_t} means contraction with v_t).

Exercise 3.13 Given α_t above, prove there is a v_t as described in the previous exercise.

4. Tight and Overtwisted Contact Structures

There is a fundamental dichotomy in 3-dimensional contact geometry. A contact structure ξ on M is called *overtwisted* if there is an embedded disk D whose characteristic foliation is homeomorphic to the either one shown in Figure 3. Such a disk is called an *overtwisted disk*. A contact structure is called *tight* if it does not contain an overtwisted disk. Though tight vs. overtwisted is obviously a dichotomy, it is not clear that it is a useful one. Throughout the rest of these lectures we will indicate that overtwisted contact structures are somewhat "easy" to deal with, whereas tight contact structures are quite a bit more difficult to understand. Moreover, a tight contact structure is capable of detecting subtle properties of the manifold supporting it.

Later we will see directly that overtwisted contact structures are fairly simple to construct and work with. This is all reflected in the following theorem.

THEOREM 4.1 (Eliashberg [**6**]). *Given a closed compact 3-manifold M, let $\mathcal{H}$ be the set of homotopy classes of (oriented) plane fields on M and $\mathcal{C}_o$ be the set of isotopy classes of (oriented) overtwisted contact structures on M. The natural inclusion map $\mathcal{C}_o$ into $\mathcal{H}$ induces a homotopy equivalence.*

This theorem basically reduces the classification of overtwisted contact structures on a 3-manifold to the classification of homotopy classes of plane fields. This latter problem is algebraic in nature and can be understood through the Thom-Pontryagin construction, see [**35**]. In addition, see [**22**] for a discussion with contact geometry in mind.

One thing, among many, that this theorem implies is that any 3-manifold has an overtwisted contact structure on it! Moreover, any $c \in H^2(M, \mathbb{Z})$ that is the Euler class of an oriented plane field is also the Euler class of an overtwisted contact structure.

Exercise 4.2 Show that $c \in H^2(M, \mathbb{Z})$ is the Euler class of an oriented plane field if and only if its mod 2 reduction is 0. (You might need to review a few facts about characteristic classes to do this.)

Tight contact structures are not understood nearly as well and they do not always exist.

THEOREM 4.3 (Etnyre-Honda [**13**]). *There exists a closed compact 3-manifold that does not support any tight contact structure.*

Despite this theorem, it seems that in some sense "most" 3-manifolds do admit tight contact structures and when they do they reveal interesting things about the manifold, see Section 4.2 below. The easiest, and most common, way to construct tight contact structures is via symplectic geometry. Recall a closed two form ω on a 4–manifold X is a *symplectic form* if $\omega \wedge \omega \neq 0$. A compact symplectic 4–manifold (X, ω) is said to fill a contact 3–manifold (M, ξ) if $\partial X = M$ (as oriented manifolds!) and $\omega|_\xi$ is an area form on ξ. Note that all Stein fillable contact structures (Example 2.15) are filled by a symplectic 4–manifold (since $\omega = d(d\phi \circ J)$ is a symplectic form).

THEOREM 4.4 (Eliashberg, Gromov [**23, 7**]). *If a contact structure can be filled by a compact symplectic manifold then it is tight.*

We will not go into what is known about the classification of tight contact structures, see [**21, 26, 27**], but we do mention the method most commonly used to understand them. The key ingredient in all classification results is the following:

THEOREM 4.5 (Eliashberg [**8**]). *If $\mathcal{F}$ is a singular foliation on S^2 that is induced by some tight contact structure, then there is a unique (up to isotopy fixing the boundary) tight contact structure ξ on B^3 such that $(\partial B^3)_\xi = \mathcal{F}$.*

Now to understand tight contact structures on a manifold M one "merely" removes pieces from M on which you understand the contact structure (*e.g.* neighborhoods of surfaces on which the characteristic foliation is known) until all that is left of M is a collection of 3-balls. Then apply the previous theorem to conclude you understand the contact structure. This is, of course, quite vague but to understand the strategy better try the following exercise.

Exercise 4.6 By Theorem 4.4 the contact structure on S^3 described in Example 2.12 is tight. Use the above strategy to show there is only one tight contact structure on S^3. Specifically, fill in the details and understand the following argument: If you have two tight contact structures on S^3 use Darboux's Theorem to say they agree in a neighborhood of a point. Then use Theorem 4.5 to conclude that they agree in the complement of the neighborhood.

4.1. Manipulations of the Characteristic Foliations. Since any 3–manifold can be cut up along surfaces into a collection of 3–balls (in many ways, *e.g.* Heegaard decompositions, Haken decompositions, . . .) it is clear, from the strategy discussed above, that to understand tight contact structures on a 3–manifold we should understand tight contact structures in the neighborhood of surfaces better. A first step in this direction is to develop techniques to manipulate characteristic foliations. One of the most important theorems along these lines is:

LEMMA 4.7 (Elimination Lemma: Giroux, Fuchs [**9**]). *Suppose γ is a leaf in a characteristic foliation Σ_ξ connecting an elliptic and hyperbolic point of the same sign. Then given any neighborhood N of γ, we may find an isotopy, supported in N, of Σ to Σ' so that $\Sigma'_\xi \cap N$ contains no singularities and, of course, Σ_ξ and Σ'_ξ agree outside of N. See Figure 5.*
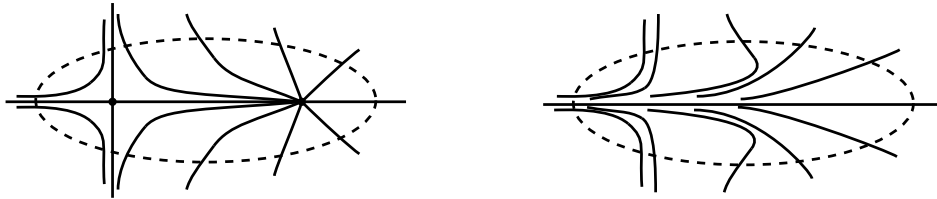

FIGURE 5. The cancellation of singularities with the same sign.

Thus this theorem says we may eliminate singularities of the same sign that are connected by an arc!

Exercise 4.8 Visualize Figure 5 in $(\mathbb{R}^3, \xi_1)$ (recall $\xi_1 = \ker dz + x\,dy$) as follows. Start with a embedded rectangle containing the y-axis and tilted slightly out of the xy-plane (*e.g.* a piece of the graph of $f(x,y) = \epsilon x$). The characteristic foliation on this is nonsingular. Now create two singularities by rotating the middle part of the rectangle past the xy-plane (*e.g.* rotate a bit of the rectangle to agree with the graph of $-f(x,y)$). If you did this correctly then the characteristic foliation should look like the left hand side of Figure 5. From the construction we know how to remove the singularities in this example. Use Theorem 3.8 to prove Lemma 4.7. This argument is explicitly worked out in [**1**].

There is an important strengthening of the Elimination Lemma. Note that in the Elimination Lemma the arc γ is part of some leaf of the new characteristic foliation on Σ'. The strengthened lemma give some control over this new leaf.

LEMMA 4.9. *Suppose γ is as in the Elimination Lemma. Let γ' be any leaf (distinct from γ) that limits to the same elliptic point as γ. Then we may assume that after the cancellation of the singularities γ and γ' are on the same leaf of the new characteristic foliation.*

Note that there is no flexibility over which two leaves limiting to a hyperbolic point will end up on the same leaf after the cancellation.

As Exercise 4.8 indicates, it is much easier to create singularities that eliminate them. In particular we have

LEMMA 4.10. *Let γ be a segment of a leaf in Σ_ξ and N be a neighborhood of γ such that $\Sigma_\xi \cap N$ contains no singularities. Then we may find an isotopy, supported in N, of Σ to Σ' so that $\Sigma'_\xi \cap N$ contains an elliptic and hyperbolic singularity of the same sign and Σ_ξ and Σ'_ξ agree outside of N.*

Up to this point the careful reader might have been concerned that we discuss "elliptic" singularities as if there were only one type of elliptic singularity. (A similar discussion applies to hyperbolic singularities.) Topologically this is true (*i.e.* up to homeomorphism) but up to diffeomorphism this is not true and Theorem 3.8 needs a diffeomorphism!

Exercise 4.11 Show that any two elliptic sources singularities are topologically equivalent (and similarly for sinks).

Hint: This is a small extension of the Hartman-Grobman Theorem which you can find most books on dynamical systems [**36**].

Exercise 4.12 Show that (generically) up to (C^1) diffeomorphism an elliptic singularity is determined by the eigenvalues of its linearization (this is not so easy, you might want to consult [**36**]).

So how is it that we can ignore this subtlety? It turns out that we may perturb a surface near an elliptic singularity so that the singularity will be diffeomorphic to a preassigned elliptic singularity.

Exercise 4.13 Verify this statement.

Hint: Use Darboux's theorem to reduce the problem to considering disks in $(\mathbb{R}^3, \xi_1)$ which are tangent to the xy-plane at the origin. Such disk can be represented as graphs of functions $\{(x, y, f(x, y))\}$. Now use the previous exercise and perturbations of f to prove the statement.

So as long as we are willing to perturb our surfaces (by a C^∞-small isotopy) we may ignore this problem of smooth equivalence of elliptic singularities. More precisely, we actually have

LEMMA 4.14. *Suppose there is a homeomorphism from Σ_ξ and $\Sigma'_{\xi'}$ (both characteristic foliations should be generic), then there is a C^∞-small isotopy of Σ' to Σ'' such that Σ_ξ and $\Sigma''_{\xi'}$ are diffeomorphic by a diffeomorphism that is isotopic to the original homeomorphism.*

Thus we can just "look at" the characteristic foliation and do not need to worry about the subtleties of the singularities.

4.2. Tightness and Genus Bounds. We use the above manipulations of the characteristic foliation to show

THEOREM 4.15 (Eliashberg [**8**]). *Let (M, ξ) be a tight contact 3-manifold and Σ an embedded surface in M. If $e(\xi) \in H^2(M, \mathbb{Z})$ denotes the Euler class of ξ, then*

$$(12) \qquad |e(\xi)([\Sigma])| \leq \begin{cases} -\chi(\Sigma) & \text{if } \Sigma \neq S^2, \\ 0 & \text{if } \Sigma = S^2, \end{cases}$$

where $[\Sigma]$ denotes the homology class of Σ.

Though it may not be apparent at first, this theorem begins to indicate the delicacy of tight contact structures. For example, we have

COROLLARY 4.16. *There are only finitely many elements in $H^2(M, \mathbb{Z})$ that can be realized as the Euler class of a tight contact structure.*

PROOF. There is no torsion in $H_2(M, \mathbb{Z})$.

Exercise 4.17 Show this.
Hint: Use Poincaré Duality and the Universal Coefficients Theorem.

Now let $g_1, \ldots, g_n$ be generators for $H_2(M, \mathbb{Z})$

Exercise 4.18 Show that any element in $H_2(M, \mathbb{Z})$ can be represented by a surface.

Let $S_1, \ldots, S_n$ be embedded surfaces such that the homology class of S_i is g_i, for $i = 1, \ldots, n$. We can assume that none of the S_i are 2-spheres (Why?) and then for each of the S_i, Inequality (12) gives a region between two parallel hyperplanes in $H^2(M, \mathbb{Z})$ in which an Euler class for a tight contact structure can live.

Exercise 4.19 Show that all the hyperplanes coming from the S_i define a compact convex polytope in $H^2(M, \mathbb{Z})$.

There can clearly be only finitely many Euler classes of tight contact structures since they have to live in this polytope.   □

Note that this corollary clearly shows the difference between tight and overtwisted contact structures, since any element in $H^2(M, \mathbb{Z})$ whose mod 2 reduction equals 0 is the Euler class of an overtwisted contact structure by Theorem 4.1.

Inequalities like (12) have shown up in other places too. For example, Thurston [**38**] proved that the inequality in Theorem 4.15 is true for the Euler class of a taut foliation. Due in part to this inequality, and many interesting constructions, foliation theory has found a central place in 3–manifold topology.

PROOF OF THEOREM 4.15. Note that it suffices to prove the theorem when Σ is connected. We begin by trying to understand how to calculate $e(\xi)([\Sigma])$ and $\chi(\Sigma)$ in terms of Σ_ξ. First perturb Σ so that the characteristic foliation is generic. (By generic, we mean that the singularities are isolated elliptic or hyperbolic points and no two hyperbolic points are connected by a leaf in the foliation.) Then let $e_\pm$ be the number of $\pm$ elliptic points in Σ_ξ and $h_\pm$ be the number of $\pm$ hyperbolic points in Σ_ξ. We first have the following simple observation:

$$(13) \qquad \chi(\Sigma) = (e_+ + e_-) - (h_+ + h_-).$$

This should be clear since we may take a vector field v that directs the characteristic foliation (*i.e.* is tangent to Σ_ξ at non-singular points, is zero at the singularities and induces the orientation on Σ_ξ). The Poincaré-Hopf Theorem [**35**] now says that $\chi(\Sigma)$ can be computed in terms of the zeros of v.

Exercise 4.20 Check that the Poincaré-Hopf Theorem implies Equation (13).

We now claim that

$$(14) \qquad e(\xi)([\Sigma]) = (e_+ - h_+) - (e_- - h_-).$$

To see this recall the the Euler class of a bundle is the obstruction to finding a non-zero section of the bundle. Moreover, $e(\xi)([\Sigma])$ is just the Euler class of the

restriction of ξ to Σ (since everything behaves well with respect to pull back). Thus to compute the Euler class of $\xi|_\Sigma$ we just need to take a generic section of $\xi|_\Sigma$ and calculate the intersection of its graph (in $\xi|_\Sigma$) with the zero section. More specifically, take v from above as our section then the graph of v is

$$\Gamma = \{(x, p) \in \xi|_\Sigma : p = v(x)\},$$

where x is a point in Σ and $p \in \xi_x$. So Γ is a surface in the 4-manifold $\xi|_\Sigma$, this is a 4-manifold since it is the total space of a 2-dimensional vector bundle over a surface. The zero section, $\Gamma_0 = \{(x, 0) \in \xi|_\Sigma\}$, is another surface. Now the Euler class of $\xi|_\Sigma$ is just the (oriented) intersection number of these two surfaces.

Exercise 4.21 Show that the contribution to the intersection number of each zero of v is a $+1$ for a positive elliptic or negative hyperbolic point and a -1 for a negative elliptic or positive hyperbolic point.
Hint: It might be helpful to think about $\chi(\Sigma)$ in these terms and remember $\xi_x = \pm T_x\Sigma$ at the singularities.

Now to prove Equation (12) when $\Sigma \neq S^2$ we need to see that $\pm e(\xi)([\Sigma]) \leq -\chi(\Sigma)$. Adding Equations (13) and (14) we see that

(15) $$\chi(\Sigma) + e(\xi)([\Sigma]) = 2(e_+ - h_+).$$

So if we can show that, after isotoping Σ, $e_+ = 0$, then we will know $e(\xi)([\Sigma]) \leq -\chi(\Sigma)$. To this end, we first arrange that there are no closed leaves in Σ_ξ by using Lemma 4.9 to creating negative elliptic-hyperbolic pairs along any closed leaf. (Note we will of course have to isotop Σ to do this, but we still call the resulting surface Σ.) Now if there are any positive elliptic points x then let U_x be the set of all leaves in Σ_ξ that limit to x and B_x be the closure of U_x. Denote $B_x \setminus U_x$ by ∂B_x. Ultimately we will show that ∂B_x contains a positive hyperbolic point y (which is clearly connected to x by an arc) and use the Elimination Lemma to cancel x and y. To do this we need to understand the structure of B_x better.

Refer to Figure 6 as we discuss B_x. First note that U_x does not contain any singularities (other than x), so all the singularities in B_x, except x, are in ∂B_x. Now if p is an elliptic point in ∂B_x then it must be negative (Why?). Also note that if p is a hyperbolic point in ∂B_x then its unstable manifolds (the two curves in Σ_ξ that limit to p in backwards time when we think if Σ_ξ as a flow) are also in ∂B_x and they limit (in forward time) to negative elliptic points in ∂B_x.
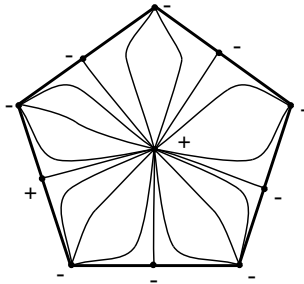

FIGURE 6. A typical B_x.

We claim that ∂B_x contains a positive hyperbolic point. To see this, we assume that there are no positive hyperbolic points in ∂B_x and derive a contradiction. Note that U_x is embedded in Σ and is diffeomorphic to an open disk.

Exercise 4.22 Show that if B_x is embedded then it is diffeomorphic to a closed disk, with piecewise smooth boundary, and the boundary of the disk contains only negative elliptic and hyperbolic points connected by arcs.

Thus if B_x is embedded then we may use the (strengthened) Elimination Lemma to cancel all the singularities in ∂B_x resulting in an overtwisted disk. Thus B_x cannot be embedded.

Exercise 4.23 If there are hyperbolic points in Σ_ξ, then convince yourself that we can think of B_x as the image of an immersed polygon $f \colon P \to M$ such that f is an embedding on the interior of P. Moreover, it can be arranged that each edge maps to the union of a hyperbolic point and its unstable manifolds and each vertex maps to an elliptic point. See Figure 6. Though $B_x \subset \Sigma$ we will sometimes talk as if $B_x = P$, this will simplify notation and (hopefully) clarify what is going on. Just remember that B_x can refer to the image of an immersed polygon or the polygon itself depending on context. If there are no hyperbolic singularities in Σ_ξ then convince yourself that $\Sigma = S^2$, x is the unique positive elliptic point in Σ_ξ and ∂B_x is the unique negative elliptic point in Σ_ξ.

If B_x is not embedded then f identifies vertices, or vertices and edges, of P. Suppose f identifies only vertices. In this case one may refine Lemma 4.10 to create a negative elliptic–hyperbolic pair near each non-embedded vertex, as shown in Figure 7, so as to make B_x embedded for this new characteristic foliation. Thus
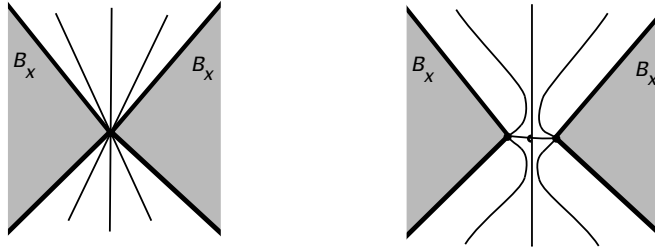

FIGURE 7. Making vertices disjoint.

we are back in the embedded case and can construct an overtwisted disk.

We are left to consider the case when f identifies edges of P. We consider the simplest case first. Suppose the image of f is as shown on the left hand side of Figure 8. If we cancel the hyperbolic point with the upper elliptic point then the new B_x will be related to the old B_x as shown in Figure 8. Thus the new B_x has only vertices identified, but we know from here we can get to an embedded B_x and thus an overtwisted disk!

Note the right hand side of Figure 8 is not correct if the top and bottom vertices on the left hand side are identified. In this case a periodic obit will be formed. If this happens then we loose the structure of B_x. To prevent this from happening always cancel edges with distinct vertices first (making vertices disjoint whenever possible). Since ∂B_x is connected we will eventually get to the situation where there is only one vertex in ∂B_x. If there are no edges left then $\Sigma = S^2$ as we discussed above.
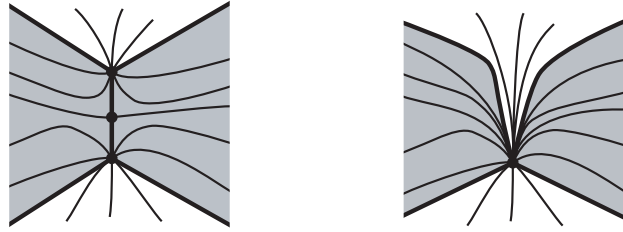
FIGURE 8. A possible B_x when edges are identified (right) and the resulting B_x when two singularities are canceled (left).

If there is only one edge then B_x is embedded and after canceling the boundary singularities we have an overtwisted disk. If ∂B_x has two or more edges that are not identified then we may use the move depicted in Figure 7 to create two distinct vertices on $\partial B_x \subset \Sigma$ and cancel more edges. So if we have simplified B_x as far as possible and have not found an overtwisted disk (or $\Sigma = S^2$) then $\partial B_x \subset \Sigma$ has only one vertex and all but possibly one edge is identified with some other edge. Thus the image of B_x in Σ is a closed surface or a subsurface with one boundary component. We show how to construct an overtwisted disk when $B_x = \Sigma = T^2$. In this case $\partial B_x \subset \Sigma$ has two edges E and F. Let N be a neighborhood of E in Σ and N' (respectively N'') a copy of N pushed slightly up off of Σ (respectively slightly down off of Σ). We can now cut Σ along E and push one side up to agree with N' and the other edge down to agree with N''. Note since the foliation on N is generic it is also stable so $N'_\xi = N''_\xi = N_\xi$. If we consider B_x sitting the new surface then we now have two copies of E and two vertices.

Exercise 4.24 Find an overtwisted disk associated to B_x on this new surface.

Even though this new B_x is not on Σ this is not a problem since we are arguing by contradiction — using our assumption that ∂B_x has no positive hyperbolic singularities to construct an overtwisted disk in M. It is irrelevant that the overtwisted disk is not actually on Σ.

Exercise 4.25 Expanding the above argument find an overtwisted disk when Σ is a surface of genus greater than one and when the image of $B_x \subset \Sigma$ has a boundary component.

So we can find an overtwisted disk unless there is some positive hyperbolic singularity on ∂B_x (or $\Sigma = S^2$ and $e(\xi)([\Sigma]) = 0$). Thus we can cancel x against a hyperbolic point. Continuing in this way we eventually show that $e(\xi)([\Sigma]) \leq -\chi(\Sigma)$. (Note you should be careful since when canceling x new closed leaf may be born. If this happens add another pair of negative singularities to break this closed leaf.)

Exercise 4.26 Finish the proof by showing that $-e(\xi)([\Sigma]) \leq -\chi(\Sigma)$. Note you can do this by showing that Σ may be perturbed so that $e_- = 0$. (Why is this sufficient?)

$\square$

Exercise 4.27 Using the ideas in the proof of Theorem 4.15 show: If there is an embedded disk D in (M, ξ) such that D_ξ contains a closed leaf, then ξ is overtwisted. The

original definition of *tight* was the absence of embedded disks D whose characteristic foliation contains closed leaves. So it was not clear that a contact structure must be tight or overtwisted. But this exercise shows that the original definition of tight is equivalent to not being overtwisted.

5. Legendrian and Transverse Knots

Just as studying surfaces in a contact 3-manifolds can illuminate the contact structure so can studying curves. Two particularly interesting types of curves to study are Legendrian curves and transverse curves. If (M, ξ) is a contact manifold then a curve $\gamma \colon S^1 \to M$ is called *Legendrian* (respectively *transverse*) if $\gamma(S^1)$ is always tangent (respectively transverse) to ξ, that is for every $x \in S^1$, $d\gamma(T_x S^1)$ is contained in (respectively, is transverse to) $\xi_{\gamma(x)}$. As is the custom in knot theory, we will frequently confuse γ with its image. When we try to classify Legendrian or transverse knots we will always be trying to classify them up to isotopies through knots of the same type.

Let's begin by considering Legendrian and transverse knots in the standard contact structure on $\mathbb{R}^3$. Recall, the contact structure is $\xi = \ker(dz + x\,dy)$. Now suppose γ is a Legendrian curve in $(\mathbb{R}^3, \xi)$. To picture γ we will project it to the yz-plane. This is called the *front projection* of γ. The projection of γ will "look like" Figure 9. What we mean by "look like" is two things:

(1) at all the crossings the strand of γ with the smaller slope lies in front of the strand with the larger slope, and
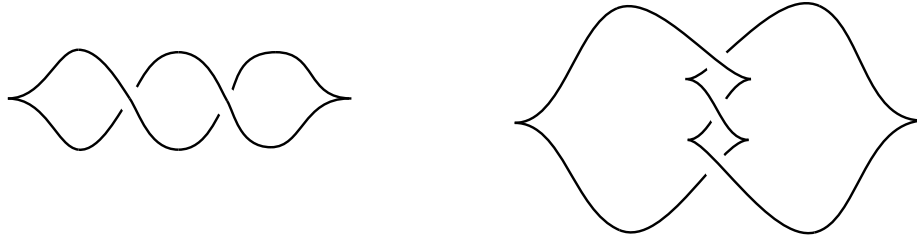(2) there are no vertical tangencies; instead there are cusps.

FIGURE 9. Examples of Legendrian knots.

Exercise 5.1 If γ is Legendrian then show that

$$(16) \qquad\qquad x = -\frac{dz}{dy}.$$

That is the x-coordinate of γ is determined by the slope of its front projection. Thus the Legendrian knot γ can be recovered from its front projection.

Exercise 5.2 Convince your self that the restrictions above on the front projection are the only restrictions on a Legendrian knot and that any projection satisfying these restrictions is the projection of a Legendrian knot.

From this exercise we see that the study of Legendrian knots in $\mathbb{R}^3$ reduces to the study of their front projections. In particular if two Legendrian knots are isotopic (through Legendrian knots) then you can get from the front projection of one to the front projection of the other by a sequence of Legendrian Reidemister

moves shown in Figure 10 (and the moves obtained from these by rotating the pictures $180°$ around the y or z-axes).
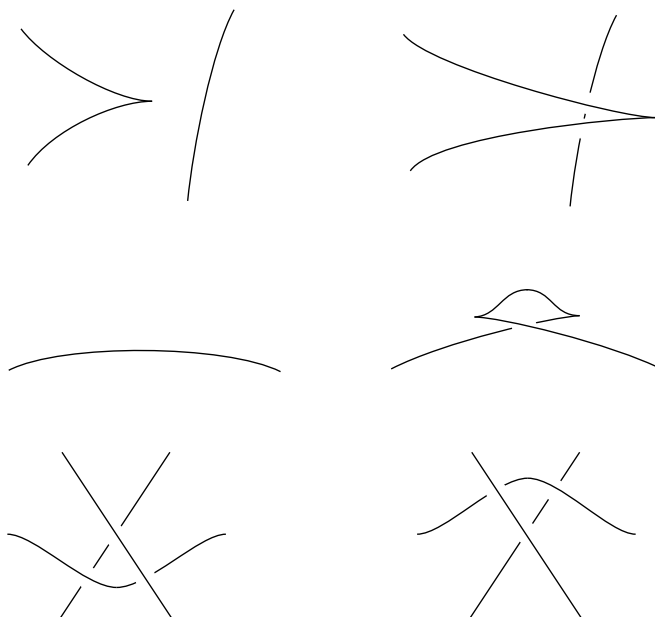

FIGURE 10. Legendrian Reidemister moves.

LEMMA 5.3. *Any knot in $\mathbb{R}^3$ can be C^0 approximated by a Legendrian knot.*

Exercise 5.4 Prove this lemma.
Hint: Consider the projection of the knot into the yz-plane. You will have to consider how the projection fails to satisfy the two condition discussed above for a front projection and how it fails to satisfy Equation (16). To fix this you can use "zig-zags."

Even though we have been only discussing knots in $(\mathbb{R}^3, \xi)$ we can actually use this and Darboux's Theorem to show

LEMMA 5.5. *Any curve in a contact manifold may be C^0 approximated by a Legendrian curve.*

Exercise 5.6 Prove this lemma.

Exercise 5.7 Try to carry out a discussion, similar to the one above, for transverse knots. In other words understand their "front projections" and prove the transverse versions of Lemmas 5.3 and 5.5. If you get stuck take a look at [**14**].

5.1. The Classical Invariants of Legendrian and Transverse Knots. The first step in trying to classify something is to find invariants that can help you distinguish the objects under consideration (*e.g.* the Euler characteristic for surfaces). For Legendrian knots there are two easily defined invariants. Let γ be a Legendrian knot and Σ a surface bounded by it. (If no such surface exists the situation is a bit more complicated, see [**13**].) Take a vector field v along γ that is

transverse to ξ, then form γ' by pushing γ in the direction of v. Now the *Thurston-Bennequin invariant* of γ, $\mathrm{tb}(\gamma)$, is the signed intersection number of γ' with Σ (*i.e.* the linking number of γ and γ'). If we orient γ then we can take a vector field u along γ that induces the chosen orientation on γ. Note that u is in ξ (since γ is Legendrian). The *rotation number* of γ, $r(\gamma)$, is Euler number $\xi|_\Sigma$ relative to u. By this we mean $r(\gamma)$ is the obstruction to extending u to a non-zero vector field in $\xi|_\Sigma$.

Exercise 5.8 Choose any trivialization of ξ over Σ. (Why can you always find such a trivialization?) Using this trivialization u rotates some number of times as we traverse γ positively (*i.e* in the direction of the orientation). Prove that this number of rotations is the rotation number of γ.

It is easy to compute these invariants in $\mathbb{R}^3$ using the front projection. Let γ be an oriented Legendrian knot in $\mathbb{R}^3$. Recall the *writhe* of a knot diagram is the sum (over the crossings in a diagram) of a ± 1 at each crossing, where the sign of the crossing is determined by its handedness. See Figure 11. Denote by $w(\gamma)$ the writhe of the front projection of γ. Let $c(\gamma), c_u(\gamma)$ and $c_d(\gamma)$ be the number
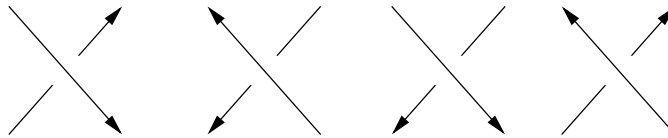

FIGURE 11. Right handed crossings (left) contribute $+1$ to the writhe while left handed crossings (right) contribute -1.

of cusps, upward oriented cusps and downward oriented cusp (respectively) in the front projection.

LEMMA 5.9. *With the notation above*

$$(17) \qquad \mathrm{tb}(\gamma) = w(\gamma) - \frac{1}{2}c(\gamma)$$

and

$$(18) \qquad r(\gamma) = \frac{1}{2}(c_d(\gamma) - c_u(\gamma)).$$

Exercise 5.10 Prove this Lemma.
Hint: A global trivialization of ξ is given by $\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\}$. Use this trivialization to compute the rotation number. Moreover, the writhe of a diagram is the difference between the "blackboard" framing of a knot (*i.e.* the obvious one coming from the diagram) and the framing coming from a Seifert surface. Now use the vector $\frac{\partial}{\partial z}$ to compute the Thurston-Bennequin invariant.

Exercise 5.11 Show tb and r are invariants of Legendrian knots in $\mathbb{R}^3$ by using the Legendrian Reidemister moves.

Now suppose γ is a transverse knot with Seifert surface Σ. We choose a nonzero vector field v in $\xi|_\Sigma$ and form a copy of γ' of γ by pushing γ in the direction of v. The *self-linking number* of γ is the signed intersection number of γ' with Σ (once again it is just the linking number of γ and γ'). The self-linking number of a knot in $\mathbb{R}^3$ may also be computed via its projection onto the yz-plane. Specifically, one can show

$$(19) \qquad l(\gamma) = w(\gamma).$$

5.2. The Bennequin Inequality. We may now state the fundamental Bennequin Inequality.

THEOREM 5.12. *If γ is a transverse knot in a tight contact structure then*

$$(20) \qquad\qquad\qquad l(\gamma) \leq -\chi(\Sigma),$$

where Σ is any Seifert surface for γ.

Exercise 5.13 Prove this theorem.
Hint: Since γ is oriented it induces an orientation on its Seifert surface Σ. With these orientations the characteristic foliation is oriented so that, thought of as a flow, it flows transversely out of $\partial\Sigma = \gamma$. Thus $\partial\Sigma$ "acts like a negative elliptic point." With this observation the proof of this theorem is very similar to the proof of Theorem 4.15. It will be helpful to interpret $l(\gamma)$ as a relative Euler class and then show (using notation from the proof of Theorem 4.15) that $l(\gamma) = -[(e_+ - h_+) - (e_- - h_-)]$.

This inequality provides a lower bound on the genus of a Seifert surface for γ. In general, it is difficult to determine the smallest possible genus of a Seifert surface for a given knot. (You should convince yourself that you can always find Seifert surfaces of arbitrarily large genus for a given knot.) The Bennequin inequality can sometimes help in determining this smallest genus.

Exercise 5.14 Look at the table of knots in [**37**] and see which of them have transverse realizations realizing the upper bound in Inequality (20).
Hint: It might be easier to consider Legendrian knots (see below). You will need to be able to construct Seifert surfaces for the knots. The most common algorithm for this can be found in [**37**].

It is interesting to note that Bennequin proved Inequality (20) for any transverse knot in the standard contact structure on $\mathbb{R}^3$. But he did it *without knowing that the contact structure was tight!* This, in fact, was the first hint that there was more than one type of contact structure, but it still took several years for the notions of "tight" and "overtwisted" to be developed. So, in modern language, Bennequin proved the standard contact structure on $\mathbb{R}^3$ was tight by proving Inequality (20). Indeed, being able to prove this inequality for a contact structure is equivalent to showing it is tight.

Exercise 5.15 Prove a contact structure is tight if and only if Inequality (20) is true. It might be better to read below about the Legendrian version of Bennequin's Inequality and then think in terms of Legendrian knots.

He did this by examining relations between transverse knots and braid theory. See [**4**] for more on this relationship. Nowadays, our understanding of the inequality is somewhat different. In stead of using it to prove a contact structure is tight, we usually prove the contact structure is tight using other techniques and then use the inequality to study transverse knots in the contact structure. The current preferred method to show contact structures are tight is to use Theorem 4.4

We now consider the Legendrian version of Bennequin's Inequality.

THEOREM 5.16. *Let γ be a Legendrian knot in a tight contact structure. Then*

$$(21) \qquad\qquad\qquad \mathrm{tb}(\gamma) + |r(\gamma)| \leq -\chi(\Sigma),$$

where Σ is a Seifert surface for γ.

To prove this we just need to notice a simple relation between Legendrian and transverse knots. Let γ be a Legendrian knot and $A = [-\epsilon, \epsilon] \times S^1$ an embedded annulus with $\gamma = \{0\} \times S^1$ and twisting so as never to be tangent to ξ along γ. Note that γ is a closed leaf in A_ξ and for a generic choice of A there will be no singularities and no other closed orbits. In fact, we can assume (Why?) that away from $\{0\} \times S^1$ the curves $\gamma_\pm = \{\pm x\} \times S^1$, where $0 < x < \epsilon$, are transverse to ξ. Furthermore, one can show that

$$(22) \qquad l(\gamma_\pm) = \mathrm{tb}(\gamma) \mp r(\gamma).$$

Exercise 5.17 Understand the relation between γ and $\gamma_\pm$ in the front projection. Use this to prove Equation (22) for knots in the standard contact structure on $\mathbb{R}^3$. You might want to try to prove the equation in general, or see [**14**].

Exercise 5.18 Show how Theorem 5.16 follows from Equations (20) and (22).

We have seen that the study of Legendrian and transverse knots can illuminate the nature of contact structures (such as the tight vs. overtwisted dichotomy), but their study is also quite interesting in its own right. Legendrian and transverse unknots [**10**], torus knots and figure eight knots [**14**] have been classified and are essentially determined by their knot types and the invariants described above. However, there are Legendrian knots that are topologically isotopic, have the same Thurston-Bennequin invariants and rotation numbers but are not Legendrian isotopic. Such examples were first found in a tight contact structure on $S^2 \times S^1 \# S^2 \times S^1$ (see [**16**]). Here a geometric argument very specific to the situation was used to distinguish the knots. Shortly after these examples were found an exciting new invariant was discovered [**5, 11, 15**] that allowed one to find many such "non-simple" Legendrian knots in the standard tight contact structure on S^3. The situation for transverse knots is not so well understood: it is unknown whether transverse knots are determined by their topological knot type and their self-linking number.

5.3. Transverse Knots and the Existence of Contact structures. Dehn surgery is an important tool in understanding topological 3-manifolds. We wish to show that Dehn surgery can be used in the world of contact 3-manifolds too. First let us recall the relevant definition. If γ is a knot in a 3-manifold M then it has a neighborhood, N, diffeomorphic to $S^1 \times D^2$. Fix an embedded curve α on $\partial N \subset \partial \overline{M \setminus N}$. Now choose any diffeomorphism f of $T^2 = \partial(S^1 \times D^2)$ that sends the meridian, $\{p\} \times \partial D^2$, to α and define the α *Dehn surgery along* γ to be the manifold obtained from $\overline{M \setminus N}$ by gluing in a solid torus via f :

$$(23) \qquad M(\gamma, \alpha) = (\overline{M \setminus N}) \cup_f (S^1 \times D^2).$$

Exercise 5.19 Show that any choice of f sending the meridian to α will produce the same 3-manifold (up to diffeomorphism).

We would now like to consider doing Dehn surgery on a transverse knot. To this end we observe that another application of Moser's method yields

LEMMA 5.20. *Let γ_i be a transverse knot in (M_i, ξ_i) for $i = 1, 2$. Then any smooth map from γ_1 to γ_2 may be extended to a contactomorphism from a neighborhood of γ_1 to a neighborhood of γ_2.*

Let's construct a standard model for the neighborhood of a transverse curve. For this consider the contact structure $\xi = \ker(\cos r d\phi + r \sin r d\theta)$ on $S^1 \times \mathbb{R}^2$, where ϕ is the coordinate on S^1 and (r, θ) are polar coordinates on $\mathbb{R}^2$. (Note this contact structure is just the one in Example 2.11 with the z-axis wrapped around the S^1. Said another way, $\mathbb{R}^3$ is the universal cover of $S^1 \times \mathbb{R}^2$ and the contact structure in Example 2.11 is just the pull back of this one under the covering map.) Note that $T_a = \{(\phi, r, \theta)|r = a\}$ is a torus, and $(T_a)_\xi$ is a non-singular foliation by lines of slope $-a \tan a$. Lemma 5.20 implies that any transverse knot γ has a neighborhood N contactomorphic to $S_a = \{(\phi, r, \theta)|r \leq a\}$ for some a.

Now if γ is some transverse knot in S^3, with the standard contact structure, then it has a neighborhood N contactomorphic to S_a for some a. If we remove N from S^3 and then glue in a solid torus $S^1 \times D^2$ via a map f, the resulting manifold M has a contact structure defined on all but the $S^1 \times D^2$ part. Note that on $\partial(S^1 \times D^2) \subset M$ we have a characteristic foliation. This foliation is a linear foliation with some slope s (when measured with respect to the $S^1 \times D^2$ product structure).

Exercise 5.21 Determine what s is in terms of a and the slope of the curve α. Is s uniquely determined? If not what are the possible s's.

Now we can find a model contact structure on $S^1 \times D^2$ whose characteristic foliation is also linear with slope s.

Exercise 5.22 Check that this model contact structure on $S^1 \times D^2$ and the contact structure on $\overline{S^3 \setminus N}$ induced from S^3 define a contact structure on M.

We can clearly perform this construction on a link in S^3. Thus since any 3-manifold can be obtained from S^3 by Dehn surgery on a link we have proved:

THEOREM 5.23 (Martinet [**33**]). *All closed compact 3-manifolds support a contact structure.*

Note that there are many choices for a so that S_a has the appropriate slope to be used in the above construction. However, it is clear that if we choose any a except the smallest possible a then we automatically get an overtwisted structure. (Find the overtwisted disk!) Even choosing the smallest possible a we will frequently get an overtwisted structure on the surgered manifold. If you are sufficiently careful with this construction you can show

THEOREM 5.24 (Lutz [**31**]). *In every homotopy class of oriented plane fields on a closed compact 3-manifold there is an overtwisted contact structure.*

Exercise 5.25 Try to prove this theorem on S^3.
Hint: There are $\mathbb{Z}$ homotopy classes of oriented plane fields. (To see this trivialize the tangent bundle and choose a metric. Now given a plane field you can use the unit vector orthogonal to the plane field to get a map to S^2, well defined up to homotopy. Thus homotopy classes of plane fields are in one-to-one correspondence with homotopy classes of maps $S^3 \to S^2$. That is $\pi_3(S^2) = \mathbb{Z}$.) The standard contact structure on S^3 is orthogonal to the Hopf fibration of S^3. So if γ is a fiber in the Hopf fibration then it has a neighborhood contactomorphic to S_a for some a. Now replace S_a with S_{a+b}, where b is chosen so that S_a and S_{a+b} have characteristic foliations with the same slope. Note that when we do this we are still on S^3 but the contact structure is (possibly) different.

Unfortunately it is much harder to construct tight contact structures.

Exercise 5.26 Try to understand how the constructions above relate to "Legendrian surgery" in which tight contact structures are produced. See [**22**] for a discussion of Legendrian surgery and [**17**] for part of its relation to the surgery described above.

6. Introduction to Convex Surfaces

In the previous sections we have been discussing a classical approach to contact geometry. By classical, I mean concentrating on specific characteristic foliations. In [**19**], Giroux initiated the use of convex surfaces in contact geometry. Using the theory of convex surfaces one can ignore specific characteristic foliations when studying surfaces in a contact structure and concentrate on a few curves on the surface (the so called "dividing curves"). In this section we will indicate how to use convex surfaces in the study of contact geometry. For applications of this to the classification of contact structures see [**26**] and [**21**], to the classification of Legendrian knots see [**14**] and to the nature of tightness see [**13**].

Given a contact manifold (M, ξ) a vector field is called *contact* if its flow preserves the contact structure. A surface Σ is called *convex* if there is a contact vector field transverse to it.

Exercise 6.1 Show a surface Σ is convex if and only if there is a neighborhood $N = \Sigma \times I$ such that $\xi|_N$ is invariant in the I direction. (Note this exercise implies that convex is a not such a great term for such a surface but we are stuck with it.)

The first question one should ask is: Are there any convex surfaces? In [**19**] it was shown that any closed surface is C^∞-close to a convex surface. Moreover, in [**28**] it was shown that this is also true for a surface with boundary so long as the surface has Legendrian boundary and the twisting of the contact planes relative to the surface is not positive.

Now let Γ be the set of points on a convex surface Σ where the contact vector field is tangent to ξ. In [**19**] it was shown that (generically) Γ is a multi-curve, that is collection of curves, on Σ. The multi-curve Γ satisfies:

(1) $\Sigma \setminus \Gamma = \Sigma_+ \coprod \Sigma_-$,
(2) Σ_ξ is transverse to Γ, and
(3) there is a vector field w and volume form ω on Σ such that
    (a) w is directs Σ_ξ (*i.e.* w is tangent to Σ_ξ where it is nonsingular and is zero where it is singular),
    (b) the flow of w expands ω on Σ_+ and contracts ω on Σ_-,
    (c) w points transversely out of Σ_+.

Exercise 6.2 Verify these properties for Γ.
Hint: Use Exercise 6.1 to show that in a neighborhood of Σ, ξ is the kernel of $\beta + udt$ where β is a 1–form on Σ and u is a function on Σ. Now try to understand Γ and Σ_ξ in terms of this 1–form.

If $\mathcal{F}$ is any singular foliation of Σ then a multi-curve Γ on Σ is said to *divide* $\mathcal{F}$ if they satisfy the above conditions where Σ_ξ is replaced by $\mathcal{F}$. Moreover, the curves Γ are called the *dividing curves* for $\mathcal{F}$. We now have the first major theorem about convex surfaces.

THEOREM 6.3 (Giroux [**19**]). *Suppose $\mathcal{F}$ and Σ_ξ are both divided by the same multi-curve Γ. Then inside any neighborhood N of Σ there is an isotopy $\Phi_t \colon \Sigma \to N, t \in [0, 1]$ of Σ such that*

(1) $\Phi_0 = $ *inclusion of Σ into N*,
(2) $\Phi_t(\Sigma)$ *is a convex surface for all t*,
(3) Φ_t *does not move Γ*,
(4) $(\Phi_1(\Sigma))_\xi = \Phi_1(\mathcal{F})$.

This theorem basically says that given a convex surface we can assume the characteristic foliation is anything we wish it to be as long as it is divided by the appropriate curves. Or said another way, it is really the dividing curves that carry the essential information about the contact structure in a neighborhood of a convex surface and not the specific characteristic foliation. One needs to be very careful with this heuristic statement but it is a useful way to think about convex surfaces.

Exercise 6.4 Show that if $\Sigma \neq S^2$ is any convex surface in (M, ξ) and it has a closed contractible dividing curve then ξ is overtwisted.
Hint: Try to write down some foliation respecting the dividing curves in which it is easy to see an overtwisted disk. Why is it important that $\Sigma \neq S^2$?

Example 6.5 Consider $\mathbb{R}^3$ with the contact structure $\xi = \ker(dz + x\,dy)$. If we quotient $\mathbb{R}^3$ by $z \mapsto z + 1$ and $y \mapsto y + 1$ we will get $M = \mathbb{R} \times T^2$ and since the contact structure is preserved by this action ξ will induce a contact structure on M. The characteristic foliation on $\{0\} \times T^2$ is by horizontal lines (*i.e.* by the lines $z = $ constant). You can check that this is *not* a convex surface, but it is easy to perturb into a convex surface. Let $f\colon [0,1] \to \mathbb{R}$ be the function whose graph is given in Figure 12, then set $\Sigma = \{(f(z), y, z)\}$. Clearly Σ is a small perturbation of
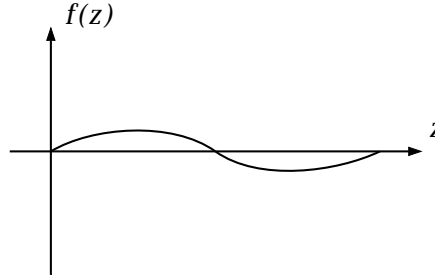

FIGURE 12. The graph of f.

$T^2 \times \{0\}$, moreover, the characteristic foliation on Σ is as shown on the left hand side of Figure 13. It is easy to check that the dotted lines in the figure give a set of dividing curves for Σ_ξ, and thus Σ is convex. Now using Theorem 6.3 we can perturb Σ so as to realize any characteristic foliation that respects these dividing curves. In particular, we can arrange for the foliation to look like the one the right hand side of Figure 13. This foliation has two lines of singularities along $\{z = 0\} \cup \{z = \frac{1}{2}\}$ and all of the nonsingular leaves have slope $s \neq 0$. The nonsingular leaves are called *ruling curves* and the singular curves are called *Legendrian divides*. Note the Legendrian divides must be parallel to the dividing curves, but we may choose the ruling curves to have any slope except 0, the slope of the Legendrian divides. Any torus with a foliation like this will be said to be in *standard form*.

The power of convex surfaces is contained largely in Theorem 6.3 in conjunction with
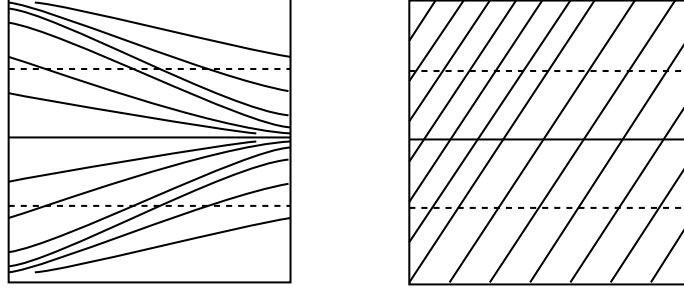
FIGURE 13. The characteristic foliation on Σ (left), with divid-
ing curves (dotted lines). Another foliation on Σ with the same
dividing set (right).

LEMMA 6.6 ([**28, 26**]). *Suppose that Σ and Σ' are convex surfaces, with dividing
curves Γ and Γ', and $\partial\Sigma' \subset \Sigma$ is Legendrian. Let $S = \Gamma \cap \partial\Sigma'$ and $S' = \Gamma' \cap \partial\Sigma'$.
Then between each two adjacent points in S there is one point in S' and vice verse.
See Figure 14. (Note the sets S and S' are cyclically ordered since they sit on $\partial\Sigma'$)*
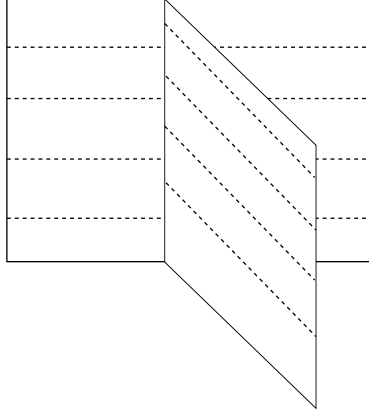

FIGURE 14. Transferring information about dividing curves from
one surface to another. The top and bottom of the picture are
identified.

In this lemma clearly Σ' is not a closed surface. All of our previous discussion
goes through for surface with boundary as long as the boundary is Legendrian and
the twisting of the contact planes relative to the surface in not positive. See [**28**].

We can now give a simple proof of the following result which is essentially due
to Makar-Limanov [**32**], but for the form presented here see Kanda [**28**]. Though
this theorem seems easy, it has vast generalizations which we indicate below.

THEOREM 6.7. *Suppose $M = D^2 \times S^1$ and $\mathcal{F}$ is a singular foliation on ∂M that
is divided by two parallel curves with slope $\frac{1}{n}$ (here slope $\frac{1}{n}$ means that the curves
are homotopic to $n[\partial D^2 \times \{p\}] + [\{q\} \times S^1]$ where $p \in S^1$ and $q \in \partial D^2$). Then there
is a unique tight contact structure on M whose characteristic foliation on ∂M is
$\mathcal{F}$.*

PROOF. Suppose we have two tight contact structures ξ_0 and ξ_1 on M inducing $\mathcal{F}$ as the characteristic foliation on ∂M. We will find a contactomorphism from ξ_0 to ξ_1 (in fact this contactomorphism will be isotopic to the identity). Let $f\colon M \to M$ be the identity map. By Theorem 3.8 we can isotop f rel. ∂M to be a contacto-morphism in a neighborhood N of ∂M. Now let T be a convex torus in N isotopic to ∂M. Moreover we can assume that the characteristic foliation on T is in stan-dard form. We know the slope of the Legendrian divides is $\frac{1}{n}$ and we choose the slope of the ruling curves to be 0. Let D be a meridianal disk whose boundary is a ruling curve. We can perturb D so that it is convex and using Lemma 6.6 we know that the dividing curves for D intersect the boundary of D in two points. More-over, since there are no closed dividing curves on D (since the contact structure is tight, see Exercise 6.4) we know that Γ_D consists of one arc. We may isotop $f(D)$ (rel. boundary) to D' so that all of this is true for D' with respect to ξ_1. Now using Theorem 6.3 we can arrange that the characteristic foliations on D and D' agree; and further, we can isotop f (rel. N) so that f takes D to D' and preserves the characteristic foliation on D. Thus another application of Theorem 3.8 says we can isotop f so as to be a contactomorphism on $N' = N \cup U$, where U is a neighborhood of D. Note that $B = \overline{M \setminus N'}$ is a 3–ball, so Theorem 4.5 tells us that we can isotop f on B so that it is a contactomorphism on B too. Thus f is a contactomorphism on all of M and we are done with the proof.   □

Exercise 6.8 Suppose that $\mathcal{F}$ is a convex foliation on $\partial(S^1 \times D^2)$ with $2n$ dividing curves of slope $\frac{p}{q}$. Find an upper bound on the number of tight contact structures on $S^1 \times D^2$ which induce this foliation. For $\frac{p}{q} = \frac{2}{q}$ or $\frac{3}{q}$ (and $n = 1$) classify the corresponding tight contact structures. If you are feeling bold you might want to try and prove the upper bound you found is not in general sharp and then actually find the sharp upper bound. This second part is not particularly easy; if you would like to see the answer consult [**26**].

Exercise 6.9 Try to generalize Exercise 6.8 to a genus g-handle body. Which configu-rations of dividing curves correspond to a unique contact structure? Can their ever be infinitely many tight structures with a fixed foliation?

Exercise 6.10 Prove the well known folk theorem of Eliashberg: There is a unique positive tight contact structure on $S^1 \times S^2$.
Hint: The argument is similar to the one in the proof of Theorem 6.7. First see that you can normalize the contact structure in a neighborhood of $\{p\} \times S^2$. The complement of this neighborhood is $I \times S^2$, where I is an interval. Now normalize the contact structure in the neighborhood of an annulus $A = I \times S^1$, for an appropriately chosen $S^1 \subset S^2$. Warning: Be careful here, you need to find a way to deal with the fact that the dividing curves on A can spin around the S^1 factor many times. The complement of the normalized regions is a 3–ball which has a unique tight contact structure (with given boundary data).

These exercises only give a hint at the power of convex surfaces. For further developments see [**13, 14, 21, 26, 27**].

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