Matching in Gabriel Graphs*

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Abstract

Given a set P of n points in the plane, the order-k Gabriel graph on P, denoted by k-GG, has an edge between two points p and q if and only if the closed disk with diameter pq contains at most k points of P, excluding p and q. We study matching problems in k-GG graphs. We show that a Euclidean bottleneck perfect matching of P is contained in 10-GG, but 8-GG may not have any Euclidean bottleneck perfect matching. In addition we show that 0-GG has a matching of size at least $\frac{n-1}{4}$ and this bound is tight. We also prove that 1-GG has a matching of size at least $\frac{2(n-1)}{5}$ and 2-GG has a perfect matching. Finally we consider the problem of blocking the edges of k-GG.

1 Introduction

Let P be a set of n points in the plane. For any two points $p, q \in P$, let $D[p, q]$ denote the closed disk which has the line segment $\overline{pq}$ as diameter. Let $|pq|$ be the Euclidean distance between p and q. The *Gabriel graph* on P, denoted by $GG(P)$, is defined to have an edge between two points p and q if $D[p, q]$ is empty of points in $P \setminus \{p, q\}$. Let $C(p, q)$ denote the circle which has $\overline{pq}$ as diameter. Note that if there is a point of $P \setminus \{p, q\}$ on $C(p, q)$, then $(p, q) \notin GG(P)$. That is, (p, q) is an edge of $GG(P)$ if and only if

$$|pq|^2 < |pr|^2 + |rq|^2 \qquad \forall r \in P, \qquad r \neq p, q.$$

Gabriel graphs were introduced by Gabriel and Sokal [11] and can be computed in $O(n \log n)$ time [12]. Every Gabriel graph has at most $3n - 8$ edges, for $n \geq 5$, and this bound is tight [12].

A *matching* in a graph G is a set of edges without common vertices. A *perfect matching* is a matching which matches all the vertices of G. In the case that G is an edge-weighted graph, a *bottleneck matching* is defined to be a perfect matching in G in which the weight of the maximum-weight edge is minimized. For a perfect matching M, we denote the *bottleneck* of M, i.e., the length of the longest edge in M, by $\lambda(M)$. For a point set P, a *Euclidean bottleneck matching* is a perfect matching which minimizes the length of the longest edge.

In this paper we consider perfect matching and bottleneck matching admissibility of higher order Gabriel Graphs. The *order-k Gabriel graph* on P, denoted by k-GG, is the geometric graph which has an edge between two points p and q iff $D[p, q]$ contains at most k points of $P \setminus \{p, q\}$. The standard Gabriel graph, $GG(P)$, corresponds to 0-GG. It is obvious that 0-GG is plane, but k-GG may not be plane for $k \geq 1$. Su and Chang [13] showed that k-GG can be constructed in $O(k^2 n \log n)$ time and contains $O(k(n - k))$ edges. In [7], the authors proved that k-GG is $(k + 1)$-connected.

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1.1 Previous Work

For any two points p and q in P, the *lune* of p and q, denoted by $L(p, q)$, is defined as the intersection of the open disks of radius $|pq|$ centred at p and q. The *order-k Relative Neighborhood Graph* on P, denoted by k-RNG, is the geometric graph which has an edge (p, q) iff $L(p, q)$ contains at most k points of P. The *order-k Delaunay Graph* on P, denoted by k-DG, is the geometric graph which has an edge (p, q) iff there exists a circle through p and q which contains at most k points of P in its interior. It is obvious that

$$k\text{-}RNG \subseteq k\text{-}GG \subseteq k\text{-}DG.$$

The problem of determining whether a geometric graph has a (bottleneck) perfect matching is quite of interest. Dillencourt showed that the Delaunay triangulation (0-DG) admits a perfect matching [10]. Chang et al. [9] proved that a Euclidean bottleneck perfect matching of P is contained in 16-RNG.[1] This implies that 16-GG and 16-DG contain a (bottleneck) perfect matching of P. In [1] the authors showed that 15-GG is Hamiltonian which implies that 15-GG has a perfect matching.

Given a geometric graph $G(P)$ on a set P of n points, we say that a set K of points *blocks* $G(P)$ if in $G(P \cup K)$ there is no edge connecting two points in P, in other words, P is an independent set in $G(P \cup K)$. Aichholzer et al. [2] considered the problem of blocking the Delaunay triangulation (i.e. 0-DG) for P in general position. They show that $\frac{3n}{2}$ points are sufficient to block DT(P) and at least $n - 1$ points are necessary. To block a Gabriel graph, $n - 1$ points are sufficient, and $\frac{3}{4}n - o(n)$ points are sometimes necessary [3].

In a companion paper [6], we considered the matching and blocking problems in triangular-distance Delaunay (TD-Delaunay) graphs. The *order-k TD-Delaunay graph*, denoted by k-TD, on a point set P is the graph whose convex distance function is defined by a fixed-oriented equilateral triangle. Then, (p, q) is an edge in k-TD if there exists an equilateral triangle which has p and q on its boundary and contains at most k points of $P \setminus \{p, q\}$. We showed that 6-TD contains a bottleneck perfect matching and 5-TD may not have any. As for maximum matching, we proved that 1-TD has a matching of size at least $\frac{2(n-1)}{5}$ and 2-TD has a perfect matching (when n is even). We also showed that $\lceil \frac{n-1}{2} \rceil$ points are necessary and $n - 1$ points are sufficient to block 0-TD. In [4] it is shown that 0-TD has a matching of size $\lceil \frac{n-1}{3} \rceil$.

1.2 Our Results

In this paper we consider the following three problems: (a) for which values of k does every k-GG have a Euclidean bottleneck matching of P? (b) for a given value k, what is the size of a maximum matching in k-GG? (c) how many points are sufficient/necessary to block a k-GG? In Section 2 we review and prove some graph-theoretic notions. In Section 3 we consider the problem (a) and prove that a Euclidean bottleneck matching of P is contained in 10-GG. In addition, we show that for some point sets, 8-GG does not have any Euclidean bottleneck matching. In Section 4 we consider the problem (b) and give some lower bounds on the size of a maximum matching in k-GG. We prove that 0-GG has a matching of size at least $\frac{n-1}{4}$, and this bound is tight. In addition we prove that 1-GG has a matching of size at least $\frac{2(n-1)}{5}$ and 2-GG has a perfect matching. In Section 5 we consider the problem (c). We show that at least $\lceil \frac{n-1}{3} \rceil$ points are necessary to block a Gabriel graph and this bound is tight. We also show that at least $\lceil \frac{(k+1)(n-1)}{3} \rceil$ points are necessary and $(k + 1)(n - 1)$ points are sufficient to block a k-GG. The open problems and concluding remarks are presented in Section 6.

[1] They defined k-RNG in such a way that $L(p, q)$ contains at most $k - 1$ points of P.

2 Preliminaries

Let G be an edge-weighted graph with vertex set V and weight function $w : E \to \mathbb{R}^+$. Let T be a minimum spanning tree of G, and let $w(T)$ be the total weight of T.

Lemma 1. *Let $\delta(e)$ be a cycle in G which contains an edge $e \in T$. Let δ' be the set of edges in $\delta(e)$ which do not belong to T and let e'_{max} be the largest edge in δ'. Then, $w(e) \leq w(e'_{max})$.*

Proof. Let $e = (u, v)$ and let T_u and T_v be the two trees obtained by removing e from T. Let $e' = (x, y)$ be an edge in δ' such that one of x and y belongs to T_u and the other one belongs to T_v. By definition of e'_{max}, we have $w(e') \leq w(e'_{max})$. Let $T' = T_u \cup T_v \cup \{(x, y)\}$. Clearly, T' is a spanning tree of G. If $w(e') < w(e)$ then $w(T') < w(T)$; contradicting the minimality of T. Thus, $w(e) \leq w(e')$, which completes the proof of the lemma. □

For a graph $G = (V, E)$ and $S \subseteq V$, let $G - S$ be the subgraph obtained from G by removing all vertices in S, and let $o(G - S)$ be the number of odd components in $G - S$, i.e., connected components with an odd number of vertices. The following theorem by Tutte [14] gives a characterization of the graphs which have perfect matching:

Theorem 1 (Tutte [14])**.** *G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V$.*

Berge [5] extended Tutte's theorem to a formula (known as the Tutte-Berge formula) for the maximum size of a matching in a graph. In a graph G, the *deficiency*, $\text{def}_G(S)$, is $o(G - S) - |S|$. Let $\text{def}(G) = \max_{S \subseteq V} \text{def}_G(S)$.

Theorem 2 (Tutte-Berge formula; Berge [5])**.** *The size of a maximum matching in G is*

$$\frac{1}{2}(n - \text{def}(G)).$$

For an edge-weighted graph G we define the *weight sequence* of G, $\text{WS}(G)$, as the sequence containing the weights of the edges of G in non-increasing order. A graph G_1 is said to be less than a graph G_2 if $\text{WS}(G_1)$ is lexicographically smaller than $\text{WS}(G_2)$.

3 Euclidean Bottleneck Matching

Given a point set P, in this section we prove that 10-GG contains a Euclidean bottleneck matching of P. We also present a configuration of a point set P such that 8-GG does not contain any Euclidean bottleneck matching of P. We use a similar argument as in [1, 8]. First consider the following lemma of [1]:

Lemma 2 (Abellanas et al. [1])**.** *Let $0 < \theta \leq \pi/5$. Let $C(A, \theta, L, R)$ be a cone with apex A, bounding rays L and R emanating from A and angle θ computed clockwise from L to R. Given two points $x, y \in C(A, \theta, L, R)$ and a constant $r > 0$. If $|xA| > 2r$ and $|yA| > 2r$, then $|xy| < 2r$ or $|xy| < \max\{|xA| - r, |yA| - r\}$.*

Theorem 3. *For every point set P, 10-GG contains a Euclidean bottleneck matching of P.*

Proof. Let $\mathcal{M}$ be the set of all perfect matchings through the points of P. Define a total order on the elements of $\mathcal{M}$ by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order. Let $M^* = \{(a_1, b_1), \ldots, (a_{\frac{n}{2}}, b_{\frac{n}{2}})\}$ be a perfect matching in $\mathcal{M}$ with minimal weight sequence. It is obvious that M^* is a Euclidean bottleneck matching for P. We will show that all edges of M^* are in 10-GG. Consider any

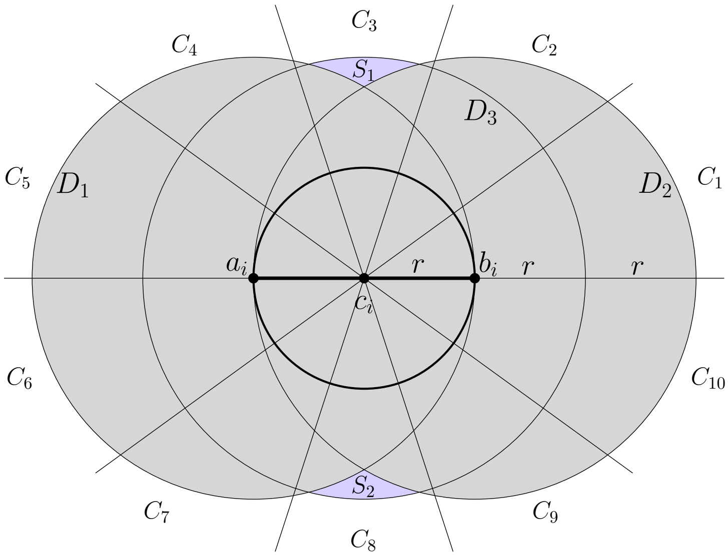

Figure 1: Illustration for Theorem 3.

edge $e = (a_i, b_i)$ in M^* and its corresponding disk $D[a_i, b_i]$. Suppose that $D[a_i, b_i]$ contains w points of $P \setminus \{a_i, b_i\}$. Let $U = \{u_1, u_2, \ldots, u_w\}$ represent the points inside $D[a_i, b_i]$, and $U' = \{r_1, r_2, \ldots, r_w\}$ represent the points where $(r_i, u_i) \in M^*$. We will show that $w \leq 10$. Let $r = |a_i b_i|/2$ be the radius of $D[a_i, b_i]$.

Claim 1: For each $r_j \in U'$, $\min\{|r_j a_i|, |r_j b_i|\} \geq 2r$. To prove this, assume that $|r_j a_i| < 2r$ and let M be the perfect matching obtained from M^* by deleting $\{(a_i, b_i), (r_j, u_j)\}$, and adding $\{(a_i, r_j), (b_i, u_j)\}$. The two new edges are smaller than the old ones. Thus, $\mathrm{WS}(M) < \mathrm{WS}(M^*)$ which contradicts the minimality of M^*.

Let D_1 and D_2 respectively be the open disks with radius $2r$ centered at a_i and b_i. By Claim 1, we may assume that no point of U' lies inside $D_1 \cup D_2$. In other words all points of U' are contained in $\overline{D_1 \cup D_2}$.

Claim 2: For each pair r_j and r_k of points in U', $|r_j r_k| \geq \max\{|a_i b_i|, |r_j u_j|, |r_k u_k|\}$. To prove this, assume that $|r_j r_k| < \max\{|a_i b_i|, |u_j r_j|, |u_k r_k|\}$. Let M be the perfect matching obtained from M^* by deleting $\{(u_j, r_j), (u_k, r_k), (a_i, b_i)\}$ and adding $\{(a_i, u_j), (b_i, u_k), (r_j, r_k)\}$. Since $\max\{|a_i u_j|, |b_i u_k|, |r_j r_k|\} < \max\{|u_j r_j|, |u_k r_k|, |a_i b_i|\}$, $\mathrm{WS}(M) < \mathrm{WS}(M^*)$ which contradicts the minimality of M^*.

Let c_i be the center of $D[a_i, b_i]$. Consider a decomposition of the plane into 10 cones $C_1, \ldots, C_{10}$ of angle $\pi/5$ with apex at c_i. See Figure 1. By contradiction, we will show that each cone $C_i, 1 \leq i \leq 10$, contains at most one point of U'. Suppose that a cone C_i where $1 \leq i \leq 10$ contains two points $r_j, r_k \in U'$. It is obvious that

$$|r_j u_j| \geq |c_i r_j| - r \qquad \text{and} \qquad |r_k u_k| \geq |c_i r_k| - r. \tag{1}$$

Claim 3: Each cone C_i where $1 \leq i \leq 10$ and $i \neq 3, 8$ contains at most one point of U'. Suppose that C_i contain two points $r_j, r_k \in U'$. By Claim 1, all points of U' are contained in $\overline{D_1 \cup D_2}$. Consider the disk D_3 with radius $2r$ centred at c_i, as shown in Figure 1. Since $D_3 \cap (\overline{D_1 \cup D_2}) = \emptyset$, r_j and r_k are outside D_3, i.e., $|r_j c_i| > 2r$ and $|r_k c_i| > 2r$. By Lemma 2, $|r_j r_k| < 2r$ or $|r_j r_k| < \max\{|r_j c_i| - r, |r_k c_i| - r\}$. By inequality (1), $|r_j r_k| < \max\{|a_i b_i|, |r_j u_j|, |r_k u_k|\}$ which contradicts Claim 2.

Claim 4: Each of C_3 and C_8 contains at most one point of U'. Let $\{S_1, S_2\}$ be the partition of $D_3 \cap (\overline{D_1 \cup D_2})$ which lies inside C_3 and C_8 as shown in Figure 1. Because of symmetry, we

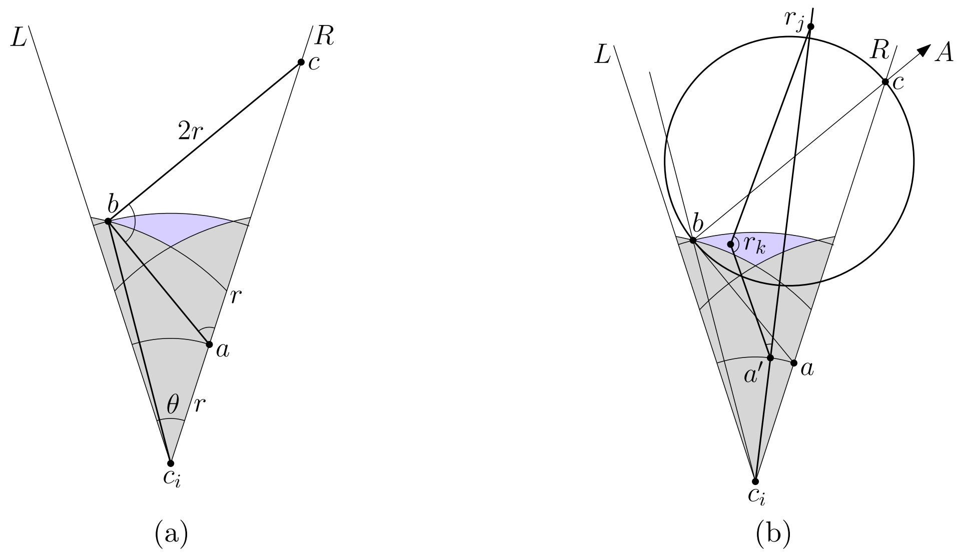

Figure 2: (a) The angle $\angle bac$ is smaller than the angle $\angle abc$, and hence (b) $\angle r_k a' r_j < \angle a' r_k r_j$.

only prove the claim for C_3. Suppose that C_3 contains two points $r_j, r_k \in U'$. For the rest of the proof, refer to Figure 2. W.l.o.g. assume that r_j is further from c_i than r_k and r_k is to the left of r_j (i.e., r_k is to the left of the line through c_i and r_j oriented from c_i to r_j). If $r_k \notin S_1$ then $|r_k c_i| > 2r$ and $|r_j c_i| > 2r$. Then, by Lemma 2 and Claim 2 we have a contradiction. Therefore, assume that $r_k \in S_1$. Let L and R be the two rays defining C_3. Let a be the intersection of R and $C(a_i, b_i)$. Let b be the intersection of the boundaries of D_1 and D_3 which is inside C_3. Define the point c on R such that $|bc| = 2r$ and $c \neq c_i$. See Figure 2(a). The triangle $\triangle cbc_i$ is isosceles, and hence $\angle bcc_i = \angle bc_i c < \frac{\pi}{5}$. This implies that $\angle cbc_i > \frac{3\pi}{5}$. On the other hand, in triangle $\triangle abc_i$, $|ab| > |ac_i|$, which implies that $\angle abc_i < \angle ac_i b < \frac{\pi}{5}$. Thus $\angle abc > \frac{2\pi}{5}$. In addition $\angle bac_i > \frac{3\pi}{5}$ and hence $\angle bac < \frac{2\pi}{5}$. Therefore in the triangle $\triangle abc$ we have

$$\angle abc > \frac{2\pi}{5} > \angle bac.$$

Let $C(b, c)$ be the circle with radius $2r$ having $\overline{bc}$ as diameter, and let A be the ray emanating from b which goes through c as shown in Figure 2(b). The intersection of C_3 with $\overline{D_1 \cup D_2}$ which lies to the right of A is completely inside $C(b, c)$. Thus, if r_j is to the right of A, $|r_j r_k| < 2r = |a_i b_i|$, which contradicts Claim 2. Therefore r_j lies to the left of A. If r_j is in the interior of C_3, rotate C_3 counter-clockwise around c_i until r_j lies on R. Since r_k is to the left of r_j, the point r_k is still in the interior of C_3. Let a' be the intersection of the new R with $C(a_i, b_i)$. Note that S_1 and hence r_k is contained in $\triangle abc$. In addition r_j and a' are outside $\triangle abc$ and to the left of the line through a and c. Therefore, $\angle a' r_k r_j \geq \angle abc > \angle bac \geq r_k a' r_j$ and hence

$$|r_j r_k| < |r_j a'| = |r_j c_i| - r \leq |r_j u_j|,$$

which contradicts Claim 2.

By Claim 3 and Claim 4 each cone C_i where $1 \leq i \leq 10$ contains at most one point of U'. Thus, $w \leq 10$, and $e = (a_i, b_i)$ is an edge of 10-GG. □

Now, we will show that for some point sets, 8-GG does not contain any Euclidean bottleneck matching. Consider Figure 3 which shows a configuration of a set P of 20 points. The closed disk $D[a, b]$ is centred at c and has diameter one, i.e., $|ab| = 1$. $D[a, b]$ contains 9 points $U = \{u_1, \ldots, u_9\}$ which lie on a circle with radius $\frac{1}{2} - \epsilon$ which is centred at c. Nine points in

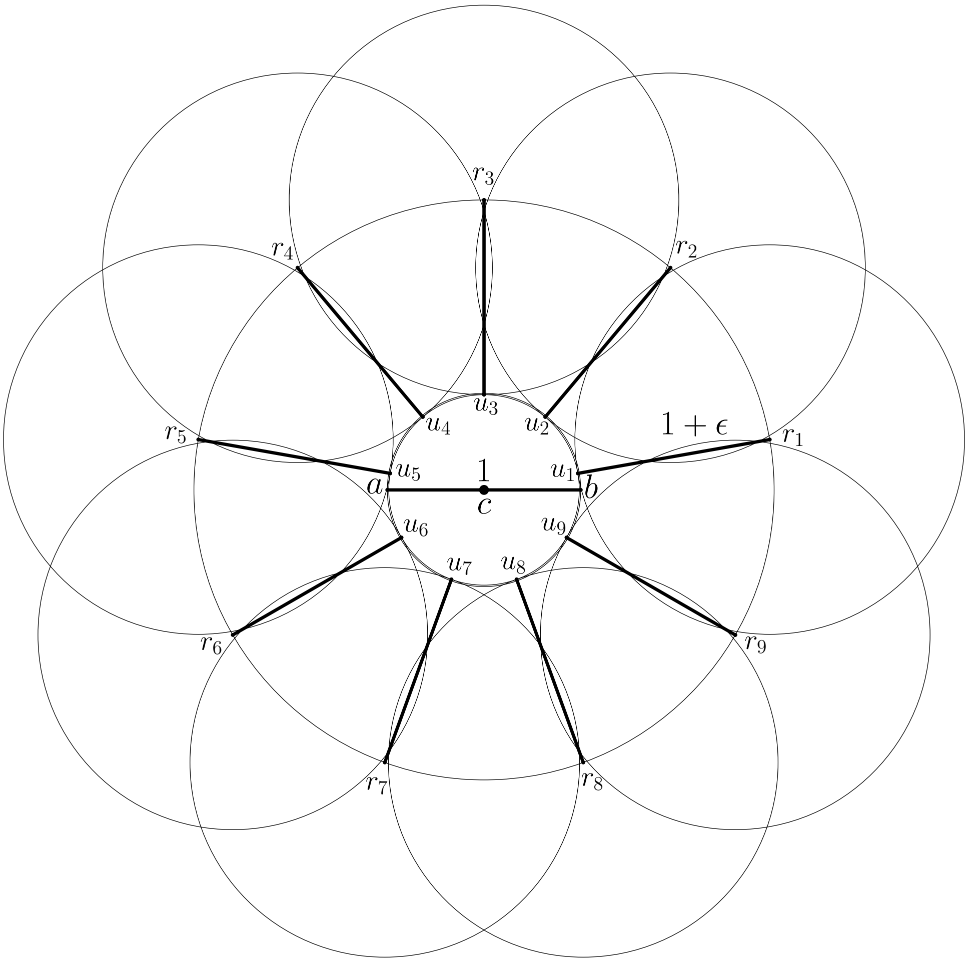

Figure 3: A set of 20 points such that 8-GG does not contain any Euclidean bottleneck matching.

$U' = \{r_1, \dots, r_9\}$ are placed on a circle with radius 1.5 which is centred at c in such a way that $|r_j u_j| = 1 + \epsilon$, $|r_j a| > 1 + \epsilon$, $|r_j b| > 1 + \epsilon$, and $|r_j r_k| > 1 + \epsilon$ for $1 \leq j, k \leq 9$ and $j \neq k$. Consider a perfect matching $M = \{(a, b)\} \cup \{(r_i, u_i) : i = 1, \dots, 9\}$ where each point $r_i \in U'$ is matched to its closest point u_i. It is obvious that $\lambda(M) = 1 + \epsilon$, and hence the bottleneck of any bottleneck perfect matching is at most $1 + \epsilon$. We will show that any Euclidean bottleneck matching of P contains (a, b). By contradiction, let M^* be a Euclidean bottleneck matching which does not contain (a, b). In M^*, a is matched to a point $x \in U \cup U'$. If $x \in U'$, then $|ax| > 1 + \epsilon$. If $x \in U$, w.l.o.g. assume that $x = u_1$. Thus, in M^* the point r_1 is matched to a point y where $y \neq u_1$. Since u_1 is the closest point to r_1 and $|r_1 u_1| = 1 + \epsilon$, $|r_1 y| > 1 + \epsilon$. In both cases $\lambda(M^*) > 1 + \epsilon$, which is a contradiction. Therefore, M^* contains (a, b). Since $D[a, b]$ contains 9 points of $P \setminus \{a, b\}$, $(a, b) \notin$ 8-GG. Therefore 8-GG does not contain any Euclidean bottleneck matching of P.

4 Maximum Matching

Let P be a set of n points in the plane. In this section we will prove that 0-GG has a matching of size at least $\frac{n-1}{4}$; this bound is tight. We also prove that 1-GG has a matching of size at least $\frac{2(n-1)}{5}$ and 2-GG has a perfect matching (when n is even).

First we give a lower bound on the number of components that result after removing a set

S of vertices from k-GG. Then we use Theorem 1 and Theorem 2, respectively presented by Tutte [14] and Berge [5], to prove a lower bound on the size of a maximum matching in k-GG.

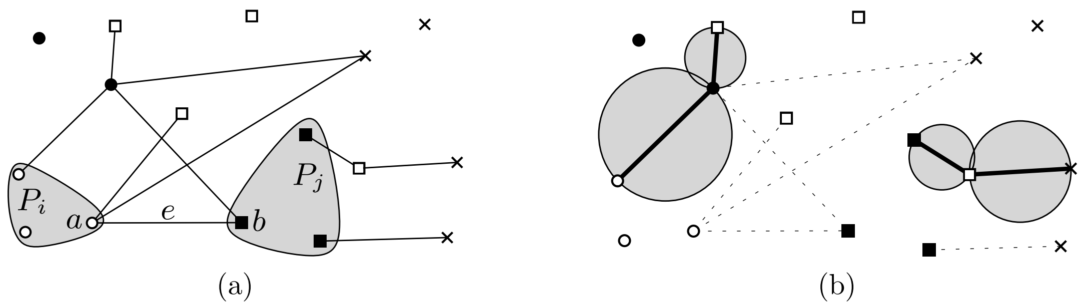

Figure 4: The point set P of 16 points is partitioned into open/closed disks, open/closed squares, and crosses. (a) The graph $G(\mathcal{P})$, (b) The set $\mathcal{T}$ of straight-line edges corresponding to $MST(G(\mathcal{P}))$ is in bold, and the set $\mathcal{D}$ of their corresponding disks.

Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a partition of the points in P. For two sets P_i and P_j in $\mathcal{P}$ define the distance $d(P_i, P_j)$ as the smallest Euclidean distance between a point in P_i and a point in P_j, i.e., $d(P_i, P_j) = \min\{|ab| : a \in P_i, b \in P_j\}$. Let $G(\mathcal{P})$ be the complete edge-weighted graph with vertex set $\mathcal{P}$. For each edge $e = (P_i, P_j)$ in $G(\mathcal{P})$, let $w(e) = d(P_i, P_j)$. This edge e is defined by two points a and b, where $a \in P_i$ and $b \in P_j$. Therefore, an edge $e \in G(\mathcal{P})$ corresponds to a straight line edge (a, b) in P; see Figure 4(a). Let $MST(G(\mathcal{P}))$ be a minimum spanning tree of $G(\mathcal{P})$. It is obvious that each edge e in $MST(G(\mathcal{P}))$ corresponds to a straight line edge (a, b) in P. Let $\mathcal{T}$ be the set of all these straight line edges. Let $\mathcal{D}$ be the set of disks which have the edges of $\mathcal{T}$ as diameter, i.e., $\mathcal{D} = \{D[a, b] : (a, b) \in \mathcal{T}\}$. See Figure 4(b).

Observation 1. *$\mathcal{T}$ is a subgraph of a minimum spanning tree of P, and hence $\mathcal{T}$ is plane.*

Lemma 3. *A disk $D[a, b] \in \mathcal{D}$ does not contain any point of $P \setminus \{a, b\}$.*

Proof. Let $e = (P_i, P_j)$ be the edge in $MST(G(\mathcal{P}))$ corresponding to $D[a, b]$. Note that $w(e) = |ab|$. By contradiction, suppose that $D[a, b]$ contains a point $c \in P \setminus \{a, b\}$. Three cases arise: (i) $c \in P_i$, (ii) $c \in P_j$, (iii) $c \in P_l$ where $l \neq i$ and $l \neq j$. In case (i) the edge (c, b) between $c \in P_i$ and $b \in P_j$ is smaller than (a, b); contradicting that $w(e) = |ab|$ in $G(\mathcal{P})$. In case (ii) the edge (a, c) between $a \in P_i$ and $c \in P_j$ is smaller than (a, b); contradicting that $w(e) = |ab|$ in $G(\mathcal{P})$. In case (iii) the edge (a, c) (resp. (c, b)) between P_i and P_l (resp. P_l and P_j) is smaller than (a, b); contradicting that e is an edge in $MST(G(\mathcal{P}))$. □

Lemma 4. *For each pair D_i and D_j of disks in $\mathcal{D}$, D_i (resp. D_j) does not contain the center of D_j (resp D_i).*

Proof. Let (a_i, b_i) and (a_j, b_j) respectively be the edges of $\mathcal{T}$ which correspond to D_i and D_j. Let C_i and C_j be the circles representing the boundary of D_i and D_j. W.l.o.g. assume that C_j is the bigger circle, i.e., $|a_i b_i| < |a_j b_j|$. By contradiction, suppose that C_j contains the center c_i of C_i. Let x and y denote the intersections of C_i and C_j. Let x_i (resp. x_j) be the intersection of C_i (resp. C_j) with the line through y and c_i (resp. c_j). Similarly, let y_i (resp. y_j) be the intersection of C_i (resp. C_j) with the line through x and c_i (resp. c_j).

As illustrated in Figure 5, the arcs $\widehat{x_i x}$, $\widehat{y_i y}$, $\widehat{x_j x}$, and $\widehat{y_j y}$ are the potential positions for the points a_i, b_i, a_j, and b_j, respectively. First we will show that the line segment $x_i x_j$ passes through x and $|a_i a_j| \leq |x_i x_j|$. The angles $\angle x_i x y$ and $\angle x_j x_y$ are right angles, thus the line segment $x_i x_j$

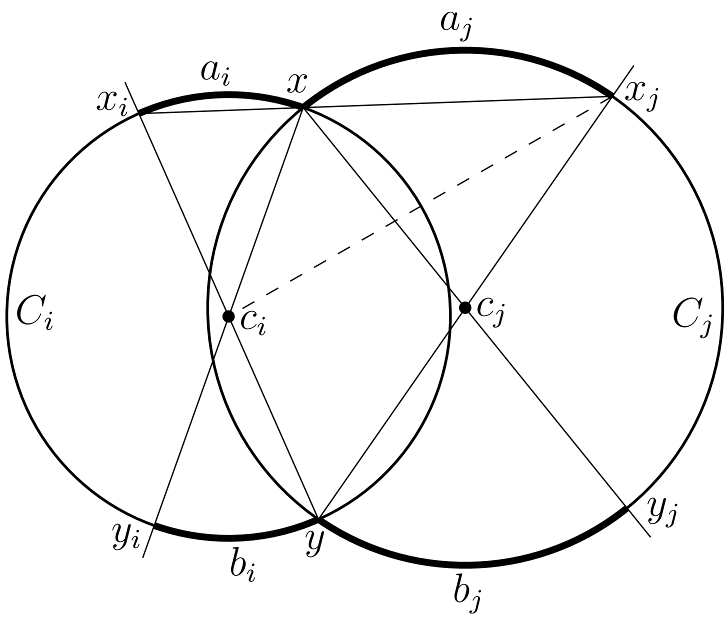

Figure 5: Illustration of Lemma 4: C_i and C_j intersect, and C_j contains the center of C_i.

goes through x. Since $\widehat{x_i x} < \pi$ (resp. $\widehat{x_j x} < \pi$), for any point $a_i \in \widehat{x_i x}, |a_i x| \leq |x_i x|$ (resp. $a_j \in \widehat{x_j x}, |a_j x| \leq |x_j x|$). Therefore,

$$|a_i a_j| \leq |a_i x| + |x a_j| \leq |x_i x| + |x x_j| = |x_i x_j|.$$

Consider triangle $\triangle x_i x_j y$ which is partitioned by segment $c_i x_j$ into $t_1 = \triangle x_i x_j c_i$ and $t_2 = \triangle c_i x_j y$. It is easy to see that $|x_i c_i|$ in t_1 is equal to $|c_i y|$ in t_2, and the segment $c_i x_j$ is shared by t_1 and t_2. Since c_i is inside C_j and $\widehat{y x_j} = \pi$, the angle $\angle y c_i x_j > \frac{\pi}{2}$. Thus, $\angle x_i c_i x_j$ in t_1 is smaller than $\frac{\pi}{2}$ (and hence smaller than $\angle y c_i x_j$ in t_2). That is, $|x_i x_j|$ in t_1 is smaller than $|x_j y|$ in t_2. Therefore,

$$|a_i a_j| \leq |x_i x_j| < |x_j y| = |a_j b_j|.$$

By symmetry $|b_i b_j| < |a_j b_j|$. Therefore $\max\{|a_i a_j|, |b_i b_j|\} < \max\{|a_i b_i|, |a_j b_j|\}$. In addition $\delta = (a_i, a_j, b_j, b_i, a_i)$ is a cycle and at least one of (a_i, a_j) and (b_i, b_j) does not belong to $\mathcal{T}$. This contradicts Lemma 1 (Note that by Observation 1, $\mathcal{T}$ is a subgraph of a minimum spanning tree of P). □

Now we show that four disks in $\mathcal{D}$ cannot intersect mutually. In other words, every point in the plane cannot lie in more than three disks in $\mathcal{D}$. In Section 4.1 we prove the following theorem, and in Section 4.2 we present the lower bounds on the size of a maximum matching in k-GG.

Theorem 4. *For every four disks $D_1, D_2, D_3, D_4 \in \mathcal{D}$, $D_1 \cap D_2 \cap D_3 \cap D_4 = \emptyset$.*

4.1 Proof of Theorem 4

Let $\mathcal{X} = D_1 \cap D_2 \cap D_3 \cap D_4$ and let x be a point in $\mathcal{X}$. Let (a_i, b_i) be the edge in $\mathcal{T}$ which corresponds to D_i, let c_i be the center of D_i, and let C_i denote the boundary of D_i, where $1 \leq i \leq 4$. Denote the angle $\angle a_i x b_i$ by α_i, where $1 \leq i \leq 4$. Since (a_i, b_i) is a diameter of D_i and x lies in D_i, $\alpha_i \geq \frac{\pi}{2}$. First we prove the following observation.

Observation 2. *For $1 \le i, j \le 4$ where $i \ne j$, the angles α_i and α_j are disjoint or one is completely contained in the other.*

Proof. The proof is by contradiction. Suppose that α_i and α_j share some part and w.l.o.g. assume that b_i is in the cone which is defined by α_j and b_j is in the cone which is defined by α_i. Three cases arise:

- $b_i \in \triangle x a_j b_j$. In this case b_i is inside D_j which contradicts Lemma 3.
- $b_j \in \triangle x a_i b_i$. In this case b_j is inside D_i which contradicts Lemma 3.
- $b_i \notin \triangle x a_j b_j$ and $b_j \notin \triangle x a_i b_i$. In this case (a_i, b_i) intersects (a_j, b_j) which contradicts Observation 1.

□

We call α_i a *blocked angle* if α_i is contained in an angle α_j where $j \ne i$, otherwise we call α_i a *free angle.*

Lemma 5. *At least one α_i, where $1 \le i \le 4$, is blocked.*

Proof. Suppose that all angles α_i, where $1 \le i \le 4$, are free. This implies that the α_is are pairwise disjoint and $\alpha = \sum_{i=1}^{4} \alpha_i \ge 2\pi$. If $\alpha > 2\pi$, we obtain a contradiction to the fact that the sum of the disjoint angles around x is at most 2π. If $\alpha = 2\pi$, then the four edges (a_i, b_i) where $1, \le i \le 4$, form a cycle which contradicts the fact that $\mathcal{T}$ is a subgraph of a minimum spanning tree of P. □

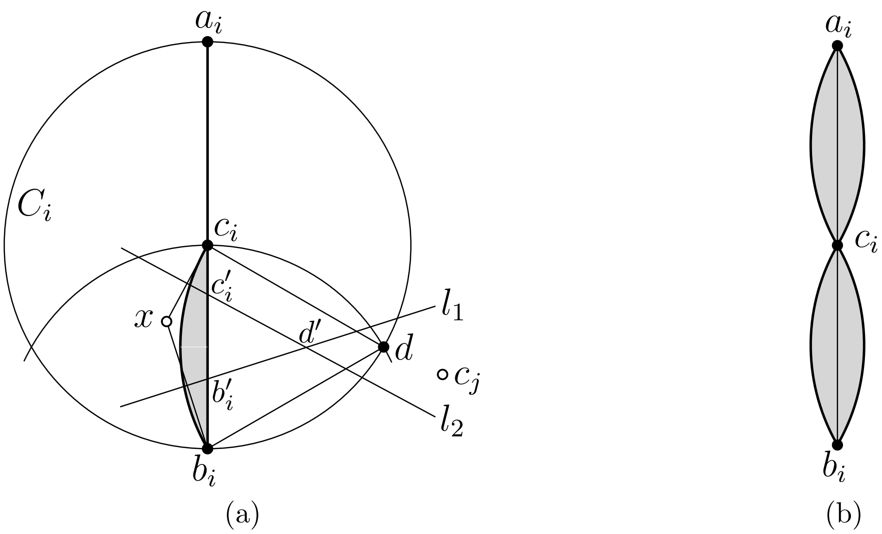

Figure 6: (a) The point x should be inside the arc $\widehat{c_i b_i}$. (b) The $trap(a_i, b_i)$ which consists of two almond-shaped regions known as $trap(a_i)$ and $trap(b_i)$.

By Lemma 5 at least one of the angles is blocked. Hereafter, assume that α_j is blocked by α_i where $1 \le i, j \le 4$ and $i \ne j$. W.l.o.g. assume that $a_i b_i$ is a vertical line segment and the point x (which belongs to $\mathcal{X}$) is to the left of $a_i b_i$. Thus, $a_j b_j$ and c_j are to the right of $a_i b_i$. This implies that $a_i b_i \cap D_j \ne \emptyset$. See Figure 6(a). By Lemma 4, c_i cannot be inside D_j, thus either $a_i c_i \cap D_j \ne \emptyset$ or $c_i b_i \cap D_j \ne \emptyset$, but not both. W.l.o.g. assume that $c_i b_i \cap D_j \ne \emptyset$. Let C' be the circle with radius $|c_i b_i|$ which is centred at b_i. Let d denote the intersection of C' with

C_i which is to the right of c_ib_i. Consider the circle C'' with radius $|xb_i|$ centred at d. Let $\widehat{c_ib_i}$ be the closed arc of C'' to the left of c_ib_i as shown in Figure 6(a).

We show that x cannot be outside $\widehat{c_ib_i}$. By contradiction suppose that x is outside $\widehat{c_ib_i}$ (and to the left of c_ib_i). Let l_1 and l_2 respectively be the perpendicular bisectors of xb_i and xc_i. Let b_i' and c_i' respectively be the intersection of l_1 and l_2 with c_ib_i and let d' be the intersection point of l_1 and l_2. Since x is outside $\widehat{c_ib_i}$, the intersection point d' is to the left of (the vertical line through) d and inside triangle $\triangle b_ic_id$. If c_j is below l_1 then $|c_jb_i| < |c_jx|$ and D_j contains b_i which contradicts Lemma 4. If c_j is above l_2 then $|c_jb_i| < |c_jx|$ and D_j contains c_i which contradicts Lemma 4. Thus, c_j is above l_1 and below l_2, and (by the initial assumption) to the right of c_ib_i. That is, c_j is in triangle $\triangle b_i'c_i'd'$. Since $\triangle b_i'c_i'd' \subseteq \triangle b_ic_id \subseteq D_i$, c_j lies inside D_i which contradicts Lemma 4. Therefore, x is contained in $\widehat{c_ib_i}$.

By symmetry D_j can intersect a_ic_i and/or c_j can be to the left of a_ib_i as well. Therefore, if α_i blocks α_j, the point x can be in $\widehat{c_ib_i}$ or any of the symmetric arcs. For an edge a_ib_i we denote the union of these arcs by $trap(a_i, b_i)$ which is shown in Figure 6(b). For each disk D_i, let $trap(D_i) = trap(a_i, b_i)$ where (a_i, b_i) is the edge in $\mathcal{T}$ corresponding to D_i. Therefore x is contained in $trap(D_i)$ which implies that

$$\mathcal{X} \subseteq trap(D_i).$$

Note that $trap(D_i)$ consists of two almond-shaped symmetric regions; for simplicity we call them $trap(a_i)$ and $trap(b_i)$, i.e., $trap(D_i) = trap(a_i) \cup trap(b_i)$.

Lemma 6. *For any point* $x \in trap(a_i, b_i)$, $\angle a_ixb_i \geq 150°$.

Proof. See Figure 6(a). The angle $\angle b_idc_i = 60°$, which implies that $\widehat{c_ib_i} = 60°$. Thus, for any point x' on the arc $\widehat{c_ib_i}$, $\angle x'c_ib_i + \angle x'b_ic_i = 30°$, and hence for any point x in $\widehat{c_ib_i}$, $\angle xc_ib_i + \angle xb_ic_i \leq 30°$. This implies that in $\triangle xb_ic_i$, $\angle b_ixc_i \geq 150°$. On the other hand $\angle b_ixc_i \leq \angle b_ixa_i$, which proves the lemma. □

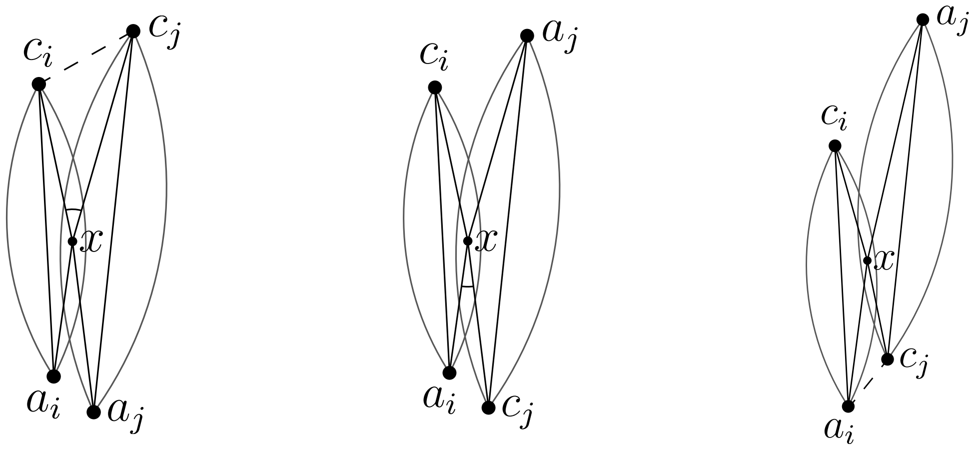

Figure 7: Illustration of Lemma 7.

Lemma 7. *For any two disks* D_i *and* D_j *in* $\mathcal{D}$, $trap(D_i) \cap trap(D_j) = \emptyset$.

Proof. We prove this lemma by contradiction. Suppose $x \in trap(D_i) \cap trap(D_j)$ and w.l.o.g. assume that $x \in trap(a_i) \cap trap(a_j)$ as shown in Figure 7. Connect x to a_i, c_i, a_j, and c_j (a_i may be identified with a_j). As shown in the proof of Lemma 6, $\min\{\angle a_ixc_i, \angle a_jxc_j\} > 150°$. Two configurations may arise:

- $\angle c_i x c_j \leq 60°$. In this case $|c_i c_j| \leq \max\{|xc_i|, |xc_j|\}$. W.l.o.g. assume that $|xc_i| \leq |xc_j|$ which implies that $|c_i c_j| \leq |xc_j|$; see Figure 7(a). Clearly $|xc_j| < |c_j a_j|$, and hence $|c_i c_j| < |c_j a_j|$. Thus, D_j contains c_i which contradicts Lemma 4.

- $\angle c_i x c_j > 60°$. In this case $\angle a_i x c_j \leq 60°$ and $\angle a_j x c_i \leq 60°$, hence $|a_i c_j| \leq \max\{|a_i x|, |c_j x|\}$ and $|a_j c_i| \leq \max\{|a_j x|, |c_i x|\}$. Three configurations arise:

  - $|a_i x| < |c_j x|$, in this case $|a_i c_j| < |c_j x| < |c_j a_j|$ and hence D_j contains a_i. See Figure 7(b).
  - $|a_j x| < |c_i x|$, in this case $|a_j c_i| < |c_i x| < |c_i a_i|$ and hence D_i contains a_j.
  - $|a_i x| \geq |c_j x|$ and $|a_j x| \geq |c_i x|$, in this case w.l.o.g. assume that $|a_i x| \leq |a_j x|$. Thus $|a_i c_j| \leq |a_i x| \leq |a_j x| < |a_j c_j|$ which implies that D_j contains a_i. See Figure 7(b).

  All cases contradict Lemma 3.

□

Recall that each blocking angle is representing a trap. Thus, by Lemma 5 and Lemma 7, we have the following corollary:

Corollary 1. *Exactly one α_i, where $1 \leq i \leq 4$, is blocked.*

Recall that α_j is blocked by α_i, $a_i b_i$ is vertical line segment, c_j is to the right of $a_i b_i$, and $x \in \widehat{c_i b_i}$. As a direct consequence of Corollary 1, α_i, α_k, and α_l are free angles, where $1 \leq i, j, k, l \leq 4$ and $i \neq j \neq k \neq l$. In addition, c_k and c_l are to the left of $a_i b_i$. It is obvious that

$$\mathcal{X} \subseteq trap(D_i) \cap D_k \cap D_l.$$

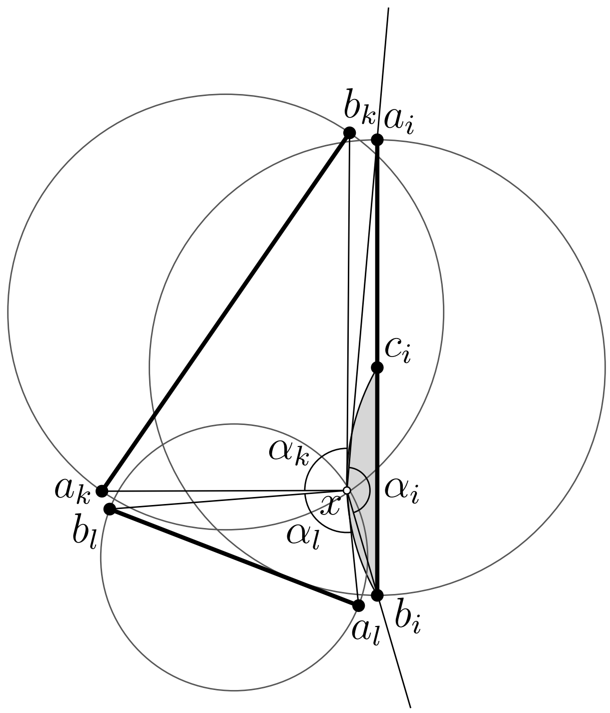

Figure 8: Illustration of Lemma 8.

Lemma 8. *For a blocking angle α_i and free angles α_k and α_l, $trap(D_i) \cap D_k \cap D_l = \emptyset$.*

Proof. Since α_i is a blocking angle and α_k, α_l are free angles, c_k and c_l are on the same side of a_ib_i. By contradiction, suppose that $x \in trap(D_i) \cap D_j \cap D_k$. See Figure 8. It is obvious that $\max\{|xa_i|, |xb_i|\} < |a_ib_i|$, $\max\{|xa_k|, |xb_k|\} < |a_kb_k|$, and $\max\{|xa_l|, |xb_l|\} < |a_lb_l|$. By Lemma 6, $\alpha_i \geq 150°$. In addition $\alpha_k, \alpha_l \geq 90°$. Thus, $\max\{\angle a_ixb_k, \angle a_kxb_l, \angle a_lxb_i\} \leq 30°$. Hence, $|a_ib_k| < \max\{|xa_i|, |xb_k|\}$, $|a_kb_l| < \max\{|xa_k|, |xb_l|\}$, and $|a_lb_i| < \max\{|xa_l|, |xb_i|\}$. Therefore, $\max\{|a_ib_k|, |a_kb_l|, |a_lb_i|\} < \max\{|a_ib_i|, |a_kb_k|, |a_lb_l|\}$. In addition $\delta = (a_i, b_i, a_l, b_l, a_k, b_k, a_i)$ is a cycle and at least one of (a_i, b_k), (a_k, b_l) and (a_l, b_i) does not belong to $\mathcal{T}$. This contradicts Lemma 1. □

Thus, $\mathcal{X} = \emptyset$; which complete the proof of Theorem 4.

4.2 Lower Bounds

In this section we present some lower bounds on the size of a maximum matching in 2-*GG*, 1-*GG*, and 0-*GG*.

Theorem 5. *For a set P of an even number of points, 2-GG has a perfect matching.*

Proof. First we show that by removing a set S of s points from 2-*GG*, at most $s+1$ components are generated. Then we show that at least one of these components must be even. Using Theorem 1, we conclude that 2-*GG* has a perfect matching.

Let S be a set of s vertices removed from 2-*GG*, and let $\mathcal{C} = \{C_1, \ldots, C_{m(s)}\}$ be the resulting $m(s)$ components, where m is a function depending on s. Actually $\mathcal{C} =$ 2-$GG - S$ and $\mathcal{P} = \{V(C_1), \ldots, V(C_{m(s)})\}$ is a partition of the vertices in $P \setminus S$.

Claim 1. $m(s) \leq s+1$. Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P}$ which is constructed as described above. Let $\mathcal{T}$ be the set of all edges in P corresponding to the edges of $MST(G(\mathcal{P}))$ and let $\mathcal{D}$ be the set of disks corresponding to the edges of $\mathcal{T}$. It is obvious that $\mathcal{T}$ contains $m(s)-1$ edges and hence $|\mathcal{D}| = m(s)-1$. Let $F = \{(p, D) : p \in S, D \in \mathcal{D}, p \in D\}$ be the set of all (point, disk) pairs where $p \in S$, $D \in \mathcal{D}$, and p is inside D. By Theorem 4 each point in S can be inside at most three disks in $\mathcal{D}$. Thus, $|F| \leq 3 \cdot |S|$. Now we show that each disk in $\mathcal{D}$ contains at least three points of S in its interior. Consider any disk $D \in \mathcal{D}$ and let $e = (a, b)$ be the edge of $\mathcal{T}$ corresponding to D. By Lemma 3, D does not contain any point of $P \setminus S$. Therefore, D contains at least three points of S, because otherwise (a, b) is an edge in 2-*GG* which contradicts the fact that a and b belong to different components in $\mathcal{C}$. Thus, each disk in $\mathcal{D}$ has at least three points of S. That is, $3 \cdot |\mathcal{D}| \leq |F|$. Therefore, $3(m(s)-1) \leq |F| \leq 3s$, and hence $m(s) \leq s+1$.

Claim 2: $o(\mathcal{C}) \leq s$. By Claim 1, $|\mathcal{C}| = m(s) \leq s+1$. If $|\mathcal{C}| \leq s$, then $o(\mathcal{C}) \leq s$. Assume that $|\mathcal{C}| = s+1$. Since $P = S \cup \{\bigcup_{i=1}^{s+1} V(C_i)\}$, the total number of vertices of P is equal to $n = s + \sum_{i=1}^{s+1} |V(C_i)|$. Consider two cases where (i) s is odd, (ii) s is even. In both cases if all the components in $\mathcal{C}$ are odd, then n is odd; contradicting our assumption that P has an even number of vertices. Thus, $\mathcal{C}$ contains at least one even component, which implies that $o(\mathcal{C}) \leq s$.

Finally, by Claim 2 and Theorem 1, we conclude that 2-*GG* has a perfect matching. □

Theorem 6. *For every set P of n points, 1-GG has a matching of size at least $\frac{2(n-1)}{5}$.*

Proof. Let S be a set of s vertices removed from 1-*GG*, and let $\mathcal{C} = \{C_1, \ldots, C_{m(s)}\}$ be the resulting $m(s)$ components. Actually $\mathcal{C} =$ 1-$GG - S$ and $\mathcal{P} = \{V(C_1), \ldots, V(C_{m(s)})\}$ is a partition of the vertices in $P \setminus S$. Note that $o(\mathcal{C}) \leq m(s)$. Let M^* be a maximum matching in 1-*GG*. By Theorem 2,

$$|M^*| = \frac{1}{2}(n - \text{def}(1\text{-}GG)), \tag{2}$$

where

$$\begin{aligned}\text{def}(1\text{-}GG) &= \max_{S \subseteq P}(o(\mathcal{C}) - |S|) \\ &\leq \max_{S \subseteq P}(|\mathcal{C}| - |S|) \\ &= \max_{0 \leq s \leq n}(m(s) - s). \end{aligned} \tag{3}$$

Define $G(\mathcal{P})$, $\mathcal{T}$, $\mathcal{D}$, and F as in the proof of Theorem 5. By Theorem 4, $|F| \leq 3 \cdot |S|$. By the same reasoning as in the proof of Theorem 5, each disk in $\mathcal{D}$ has at least two points of S in its interior. Thus, $2 \cdot |\mathcal{D}| \leq |F|$. Therefore, $2(m(s) - 1) \leq |F| \leq 3s$, and hence

$$m(s) \leq \frac{3s}{2} + 1. \tag{4}$$

In addition, $s + m(s) = |S| + |\mathcal{C}| \leq |P| = n$, and hence

$$m(s) \leq n - s. \tag{5}$$

By Inequalities (4) and (5),

$$m(s) \leq \min\{\frac{3s}{2} + 1, n - s\}. \tag{6}$$

Thus, by (3) and (6)

$$\begin{aligned}\text{def}(1\text{-}GG) &\leq \max_{0 \leq s \leq n}(m(s) - s) \\ &\leq \max_{0 \leq s \leq n}\{\min\{\frac{3s}{2} + 1, n - s\} - s\} \\ &= \max_{0 \leq s \leq n}\{\min\{\frac{s}{2} + 1, n - 2s\}\} \\ &= \frac{n+4}{5}, \end{aligned} \tag{7}$$

where the last equation is achieved by setting $\frac{s}{2}+1$ equal to $n-2s$, which implies $s = \frac{2(n-1)}{5}$. Finally by substituting (7) in Equation (2) we have

$$|M^*| \geq \frac{2(n-1)}{5}.$$

□

By similar reasoning as in the proof of Theorem 6 we have the following Theorem.

Theorem 7. *For every set P of n points, 0-GG has a matching of size at least $\frac{n-1}{4}$.*

The bound in Theorem 7 is tight, as can be seen from the graph in Figure 9, for which the maximum matching has size $\frac{n-1}{4}$. Actually this is a Gabriel graph of maximum degree four which is a tree. The dashed edges do not belong to 0-GG because any closed disk which has one of these edges as diameter has a point on its boundary. Observe that each edge in any matching is adjacent to one of the vertices of degree four.

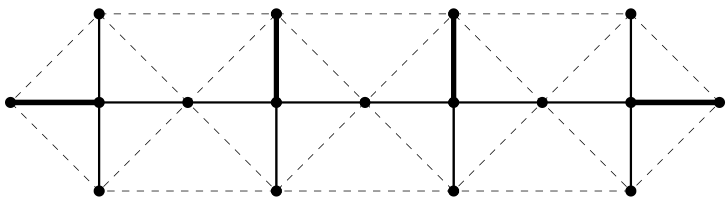

Figure 9: A 0-GG of $n = 17$ points with a maximum matching of size $\frac{n-1}{4} = 4$ (bold edges). The dashed edges do not belong to the graph because any of their corresponding closed disks has a point on its boundary.

Note: For a point set P, let $\nu_k(P)$ and $\alpha_k(P)$ respectively denote the size of a maximum matching and a maximum independent set in k-GG. For every edge in the maximum matching, at most one of its endpoints can be in the maximum independent set. Thus,

$$\alpha_k(P) \leq |P| - \nu_k(P).$$

By combining this formula with the results of Theorems 7, 6, 5, respectively, we have $\alpha_0(P) \leq \frac{3n+1}{4}$, $\alpha_1(P) \leq \frac{3n+2}{5}$, and $\alpha_2(P) \leq \lceil \frac{n}{2} \rceil$. The 0-$GG$ graph in Figure 9 has an independent set of size $\frac{3n+1}{4} = 13$, which shows that this bound is tight for 0-GG. On the other hand, 0-GG is planar and every planar graph is 4-colorable; which implies that $\alpha_0(P) \geq \lceil \frac{n}{4} \rceil$. There are some examples of 0-GG in [12] such that $\alpha_0(P) = \lceil \frac{n}{4} \rceil$, which means that this bound is tight as well.

5 Blocking Higher-Order Gabriel Graphs

In this section we consider the problem of blocking higher-order Gabriel graphs. Recall that a point set K blocks k-$GG(P)$ if in k-$GG(P \cup K)$ there is no edge connecting two points in P.

Theorem 8. *For every set P of n points, at least $\lceil \frac{n-1}{3} \rceil$ points are necessary to block* 0-$GG(P)$.

Proof. Let K be a set of m points which blocks 0-$GG(P)$. Let $G(\mathcal{P})$ be the complete graph with vertex set $\mathcal{P} = P$. Let $\mathcal{T}$ be a minimum spanning tree of $G(\mathcal{P})$ and let $\mathcal{D}$ be the set of closed disks corresponding to the edges of $\mathcal{T}$. It is obvious that $|\mathcal{D}| = n - 1$. By Lemma 3 each disk $D[a, b] \in \mathcal{D}$ does not contain any point of $P \setminus \{a, b\}$, thus, $\mathcal{T} \subseteq$ 0-$GG(P)$. To block each edge of $\mathcal{T}$, corresponding to a disk in $\mathcal{D}$, at least one point is necessary. By Theorem 4 each point in K can lie in at most three disks of $\mathcal{D}$. Therefore, $m \geq \lceil \frac{n-1}{3} \rceil$, which implies that at least $\lceil \frac{n-1}{3} \rceil$ points are necessary to block all the edges of $\mathcal{T}$ and hence 0-$GG(P)$. □

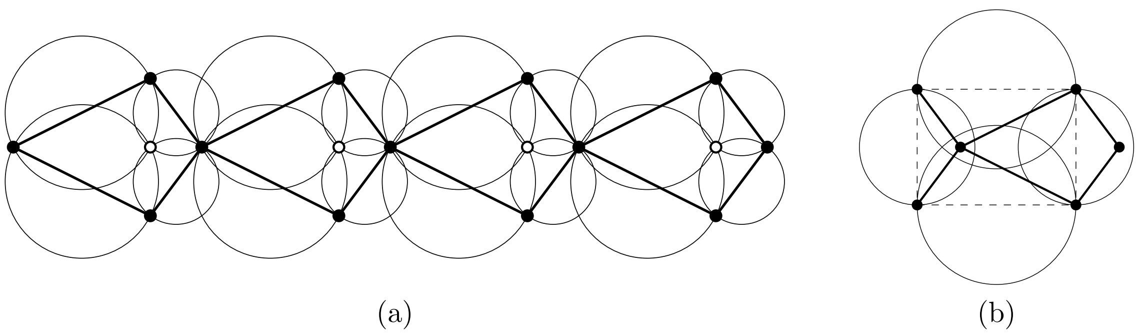

Figure 10: (a) 0-GG graph of $n = 13$ points (in bold edges) which is blocked by $\lceil \frac{n-1}{3} \rceil = 4$ white points, (b) dashed edges do not belomg to 0-GG.

Figure 10(a) shows a 0-GG with $n = 13$ (black) points which is blocked by $\lceil \frac{n-1}{3} \rceil = 4$ (white) points. Note that all the disks, corresponding to the edges of every cycle, intersect at the same point in the plane (where we have placed the white points). As shown in Figure 10(b), the dashed edges do not belong to 0-GG. Thus, the lower bound provided by Theorem 8 is tight. It is easy to generalize the result of Theorem 8 to higher-order Gabriel graphs. Since in a k-GG we need at least $k + 1$ points to block an edge of $\mathcal{T}$ and each point can be inside at most three disks in $\mathcal{D}$, we have the following corollary:

Corollary 2. *For every set P of n points, at least $\lceil \frac{(k+1)(n-1)}{3} \rceil$ points are necessary to block k-$GG(P)$.*

In [3] the authors showed that every Gabriel graph can be blocked by a set K of $n-1$ points by putting a point slightly to the right of each point of P, except for the rightmost one. Every disk with diameter determined by two points of P will contain a point of K. Using a similar argument one can block a k-GG by putting $k + 1$ points slightly to the right of each point of P, except for the rightmost one. Thus,

Corollary 3. *For every set P of n points, there exists a set of $(k + 1)(n - 1)$ points that blocks k-$GG(P)$.*

Note that this upper bound is tight, because if the points of P are on a line, the disks representing the minimum spanning tree are disjoint and each disk needs $k + 1$ points to block the corresponding edge.

6 Conclusion

In this paper, we considered the bottleneck and perfect matching admissibility of higher-order Gabriel graphs. We proved that

- 10-GG contains a Euclidean bottleneck matching of P and 8-GG may not have any.
- 0-GG has a matching of size at least $\frac{n-1}{4}$ and this bound is tight.
- 1-GG has a matching of size at least $\frac{2(n-1)}{5}$.
- 2-GG has a perfect matching.
- $\lceil \frac{n-1}{3} \rceil$ points are necessary to block 0-GG and this bound is tight.
- $\lceil \frac{(k+1)(n-1)}{3} \rceil$ points are necessary and $(k + 1)(n - 1)$ points are sufficient to block k-GG.

We leave a number of open problems:

- Does 9-GG contain a Euclidean bottleneck matching of P?
- What is a tight lower bound on the size of a maximum matching in 1-GG?

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