On the stability of generalized mixed type quadratic and quartic functional equation in quasi-Banach spaces

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Abstract. In this paper, we establish the general solution of the functional equation

$$f(nx+y)+f(nx-y)=n^2f(x+y)+n^2f(x-y)+2(f(nx)-n^2f(x))-2(n^2-1)f(y)$$

for fixed integers n with $n \neq 0, \pm 1$ and investigate the generalized Hyers-Ulam-Rassias stability of this equation in quasi-Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [21] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, dose there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition dose there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D.H. Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x)-T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$, then T is linear. In 1978, Th. M. Rassias [17] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y), \tag{1.1}$$

is related to symmetric bi-additive function[1,2,11,13]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between real vector

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spaces is quadratic if and only if there exits a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [1,13]). The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)). \tag{1.2}$$

A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : A \longrightarrow B$, where A is normed space and B Banach space (see [19]). Cholewa [4] noticed that the Theorem of Skof is still true if relevant domain A is replaced an abelian group. In the paper [6] , Czerwik proved the Hyers-Ulam-Rassias stability of the equation (1.1). Grabiec [9] has generalized these result mentioned above.

In [14], Won-Gil Prak and Jea Hyeong Bae, considered the following quartic functional equation:

$$f(x + 2y) + f(x - 2y) = 4(f(x + y) + f(x - y) + 6f(y)) - 6f(x). \tag{1.3}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1.3) if and only if there exists a unique symmetric multi-additive function $D : X \times X \times X \times X \longrightarrow Y$ such that $f(x) = D(x, x, x, x)$ for all x. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.4), which is called a quartic functional equation (see also [5]).

In addition H. Kim [12], has obtained the generalized Hyers-Ulam-Rassias stability for the following mixed type of quartic and quadratic functional equation:

$$\biguplus_{x_2,\ldots,x_n}^{n-1} f(x_1) + 2^{n-1}(n-2)\sum_{i=1}^{n} f(x_i) = 2^{n-2}\sum_{1\leq i<j\leq n}[\biguplus_{x_j} f(x_i)] \tag{1.4}$$

for all n-variables $x_1, x_2, ..., x_n \in E_1$, where $n > 2$ and $f : E_1 \longrightarrow E_2$ be a function between two real linear spaces E_1 and E_2.

Also A. Najati and G. Zamani Eskandani [16], have established the general solution and the generalized Hyers-Ulam-Rassias stability for a mixed type of cubic and additive functional equation, whenever f is a mapping between two quasi-Banach spaces.

Now, we introduce the following functional equation for fixed integers n with $n \neq 0, \pm 1$:

$$\begin{aligned} f(nx + y) + f(nx - y) = {} & n^2 f(x + y) + n^2 f(x - y) + 2f(nx) \\ & - 2n^2 f(x) - 2(n^2 - 1)f(y) \end{aligned} \tag{1.5}$$

in quasi Banach spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of the functional equation (1.5). In the present paper we investigate the general solution of functional equation (1.5) when f is a function between vector spaces, and we establish the generalized Hyers-Ulam-Rassias stability of this functional equation whenever f is a function between two quasi-Banach spaces.

We recall some basic facts concerning quasi-Banach space and some preliminary results.

Definition 1.1. *(See [3, 18].) Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:*

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda.x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows from condition (3) that

$$\|\sum_{i=1}^{2m} x_i\| \leq M^m \sum_{i=1}^{2m} \|x_i\|, \qquad \|\sum_{i=1}^{2m+1} x_i\| \leq M^{m+1} \sum_{i=1}^{2m+1} \|x_i\|$$

for all $m \geq 1$ and all $x_1, x_2, ..., x_{2m+1} \in X$.

The pair $(X, \|.\|)$ is called a quasi-normed space if $\|.\|$ is a quasi-norm on X . The smallest possible M is called the modulus of concavity of $\|.\|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [18](see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. In [20], J. Tabor has investigated a version of Hyers-Rassias-Gajda theorem (see[7,17]) in quasi-Banach spaces.

2. General solution

Throughout this section, X and Y will be real vector spaces. We here present the general solution of (1.5).

Lemma 2.1. *If a function $f : X \longrightarrow Y$ satisfies the functional equation (1.5), then f is a quadratic and quartic function.*

Proof. By letting $x = y = 0$ in (1.5),we get $f(0) = 0$.Set $x = 0$ in (1.5) to get $f(y) = f(-y)$ for all $y \in X$.So the function f is even.We substitute $x = x + y$ in (1.5) and then $x = x - y$ in (1.5) to obtain that

$$f(nx + (n+1)y) + f(nx + (n-1)y) = n^2 f(x + 2y) + n^2 f(x) + 2f(nx + ny) \\ - 2n^2 f(x + y) - 2(n^2 - 1)f(y) \tag{2.1}$$

and

$$f(nx - (n-1)y) + f(nx - (n+1)y) = n^2 f(x) + n^2 f(x - 2y) + 2f(nx - ny) \\ - 2n^2 f(x - y) - 2(n^2 - 1)f(y) \tag{2.2}$$

for all $x, y \in X$.Interchanging x and y in (1.5) and using evenness of f to get the relation

$$f(x + ny) + f(x - ny) = n^2 f(x + y) + n^2 f(x - y) + 2f(ny) \\ - 2n^2 f(y) - 2(n^2 - 1)f(x) \tag{2.3}$$

for all $x, y \in X$.Replacing y by ny in (1.5) and then using (2.3),we have

$$f(nx + ny) + f(nx - ny) = n^4 f(x + y) + n^4 f(x - y) + 2f(ny) \\ + 2f(nx) - 2n^4 f(x) - 2n^4 f(y) \tag{2.4}$$

for all $x, y \in X$.If we add (2.1) to (2.2) and use (2.4),we have

$$f(nx + (n+1)y) + f(nx - (n+1)y) + f(nx + (n-1)y) + f(nx - (n-1)y) = \\ n^2 f(x + 2y) + n^2 f(x - 2y) + 2n^2(n^2 - 1)f(x + y) + 2n^2(n^2 - 1)f(x - y) \\ + 4f(ny) + 4f(nx) + (-4n^4 + 2n^2)f(x) + (-4n^4 - 4n^2 + 4)f(y) \tag{2.5}$$

for all $x, y \in X$.Substitute $y = x + y$ in (1.5) and then $y = x - y$ in (1.5) and using evenness of f to obtain that

$$f((n+1)x+y) + f((n-1)x-y) = n^2 f(2x+y) + n^2 f(y) + 2f(nx) - 2n^2 f(x) - 2(n^2-1)f(x+y) \tag{2.6}$$

and

$$f((n+1)x-y) + f((n-1)x+y) = n^2 f(2x-y) + n^2 f(y) + 2f(nx) - 2n^2 f(x) - 2(n^2-1)f(x-y) \tag{2.7}$$

for all $x, y \in X$.Interchanging x with y in (2.6) and (2.7) and using evenness of f,we get the relations

$$f(x+(n+1)y) + f(x-(n-1)y) = n^2 f(x+2y) + n^2 f(x) + 2f(ny) - 2n^2 f(y) - 2(n^2-1)f(x+y) \tag{2.8}$$

and

$$f(x-(n+1)y) + f(x+(n-1)y) = n^2 f(x-2y) + n^2 f(x) + 2f(ny) - 2n^2 f(y) - 2(n^2-1)f(x-y) \tag{2.9}$$

for all $x, y \in X$.With the substitution $y = (n+1)y$ in (1.5) and then $y = (n-1)y$ in (1.5), we have

$$f(nx+(n+1)y) + f(nx-(n+1)y) = n^2 f(x+(n+1)y) + n^2 f(x-(n+1)y) + 2f(nx) - 2n^2 f(x) - 2(n^2-1)f((n+1)y) \tag{2.10}$$

and

$$f(nx+(n-1)y) + f(nx-(n-1)y) = n^2 f(x+(n-1)y) + n^2 f(x-(n-1)y) + 2f(nx) - 2n^2 f(x) - 2(n^2-1)f((n-1)y) \tag{2.11}$$

for all $x, y \in X$.Replacing x by y in (1.5), we obtain

$$f((n+1)y) + f((n-1)y) = n^2 f(2y) - 2(2n^2-1)f(y) + 2f(ny) \tag{2.12}$$

for all $y \in X$.Adding (2.10) with (2.11) and using (2.8), (2.9) and (2.12), we lead to

$$\begin{aligned} &f(nx+(n+1)y) + f(nx-(n+1)y) + f(nx+(n-1)y) + f(nx-(n-1)y) = \\ &n^4 f(x+2y) + n^4 f(x-2y) - 2n^2(n^2-1)f(x+y) - 2n^2(n^2-1)f(x-y) \\ &+ 4f(ny) + 4f(nx) - 2n^2(n^2-1)f(2y) + (2n^4-4n^2)f(x) \\ &+ (4n^4-12n^2+4)f(y) \end{aligned} \tag{2.13}$$

for all $x, y \in X$.By comparing (2.5) with (2.13), we arrive at

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) + 2f(2y) - 8f(y) - 6f(x) \tag{2.14}$$

for all $x, y \in X$.Interchange x with y in (2.14) and use evenness of f to get the relation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 2f(2x) - 8f(x) - 6f(y) \tag{2.15}$$

for all $x, y \in X$.

We would show that (2.15) is a quadratic and quartic functional equation.To get this, we show that the functions $g : X \longrightarrow Y$ defined by $g(x) = f(2x) - 16f(x)$ for all $x \in X$ and $h : X \longrightarrow Y$ defined by $h(x) = f(2x) - 4f(x)$ for all $x \in X$, are quadratic and quartic, respectively.

Replacing y by $2y$ in (2.15) and using evenness of f,we have

$$f(2x+2y)+f(2x-2y)=4f(2y+x)+4f(2y-x)+2f(2x)-8f(x)-6f(2y) \tag{2.16}$$

for all $x, y \in X$. By interchanging x with y in (2.16) and then using (2.15),we obtain by evenness of f

$$\begin{aligned} f(2x+2y)+f(2x-2y) &= 4f(2x+y)+4f(2x-y)+2f(2y)-8f(y)-6f(2x) \\ &= 16f(x+y)+16f(x-y)+2f(2x)+2f(2y) \\ &- 32f(x)-32f(y) \end{aligned} \tag{2.17}$$

for all $x, y \in X$.By rearranging (2.17), we have

$$[f(2x+2y)-16f(x+y)]+[f(2x-2y)-16f(x-y)]=$$
$$2[f(2x)-16f(x)]+2[f(2y)-16f(y)]$$

for all $x, y \in X$.This means that

$$g(x+y)+g(x-y)=2g(x)=2g(y)$$

for all $x, y \in X$.Therefore the function $g : X \longrightarrow Y$ is quadratic.

To prove that $h : X \longrightarrow Y$ is quartic,we have to show that

$$h(2x+y)+h(2x-y)=4h(x+y)+4h(x-y)+24h(x)-6h(y)$$

for all $x, y \in X$.Replacing x and y by $2x$ and $2y$ in (2.15), respectively,we get

$$f(4x+2y)+f(4x-2y)=4f(2x+2y)+4f(2x-2y)+2f(4x)-8f(2x)-6f(2y) \tag{2.18}$$

for all $x, y \in X$.Since $g(2x) = 4g(x)$ for all $x \in X$ where $g : X \longrightarrow Y$ is a quadratic function defined above, we have

$$f(4x)=20f(2x)-64f(x) \tag{2.19}$$

for all $x \in X$.Hence, it follows from (2.15) ,(2.18) and (2.19) that

$$\begin{aligned} h(2x+y)+h(2x-y) &= [f(4x+2y)-4f(2x+y)]+[f(4x-2y)-4f(2x-y)] \\ &= 4[f(2x+2y)-4f(x+y)]+4[f(2x-2y)-4f(x-y)] \\ &+ 24[f(2x)-4f(x)]-6[f(2y)-4f(y)] \\ &= 4h(x+y)+4h(x-y)+24h(x)-6h(y) \end{aligned}$$

for all $x, y \in X$.Therefore, $h : X \longrightarrow Y$ is a quartic function. □

Theorem 2.2. *A function $f : X \to Y$ satisfies (1.5) if and only if there exist a unique symmetric multi-additive function $D : X \times X \times X \times X \longrightarrow Y$ and a unique symmetric bi-additive function $B : X \times X \longrightarrow Y$ such that*

$$f(x)=D(x,x,x,x)+B(x,x)$$

for all $x \in X$.

Proof. We first assume that the function $f : X \longrightarrow Y$ satisfies (1.5).Let $g, h : X \to Y$ be functions defined by

$$g(x):=f(2x)-16f(x) \qquad h(x):=f(2x)-4f(x)$$

for all $x \in X$. Hence, by Lemma (2.1), we achieve that the functions g and h are quadratic and quartic, respectively, and

$$f(x):=\frac{1}{12}h(x)-\frac{1}{12}g(x)$$

for all $x \in X$. Therefore, there exist a unique symmetric multi-additive mapping $D : X \times X \times X \times X \longrightarrow Y$ and a unique symmetric bi-additive mapping $B : X \times X \longrightarrow Y$ such that $D(x, x, x, x) = \frac{1}{12}h(x)$ and $B(x, x) = -\frac{1}{12}g(x)$ for all $x \in X$(see[1, 14]). So

$$f(x) = D(x, x, x, x) + B(x, x)$$

for all $x \in X$.

Conversely assume that

$$f(x) = D(x, x, x, x) + B(x, x)$$

for all $x \in X$, where the function $D : X \times X \times X \times X \longrightarrow Y$ is symmetric multi-additive and $B : X \times X \longrightarrow Y$ is bi-additive defined above.By a simple computation, one can show that the functions D and B satisfy the functional equation (1.5),so the function f satisfies (1.5). □

3. Hyers-Ulam-Rassias stability of Eq.(1.5)

From now on, let X and Y be a quasi-Banach space with quasi-norm $\|.\|_X$ and a p-Banach space with p-norm $\|.\|_Y$,respectively.Let M be the modulus of concavity of $\|.\|_Y$.In this section using an idea of Găvruta[8] we prove the stability of Eq.(1.5) in the spirit of Hyers, Ulam and Rassias.For convenience we use the following abbreviation for a given function $f : X \longrightarrow Y$:

$$\triangle f(x, y) = f(nx+y)+f(nx-y)-n^2f(x+y)-n^2f(x-y)-2f(nx)+2n^2f(x)+2(n^2-1)f(y)$$

for all $x, y \in X$.We will use the following lemma in this section.

Lemma 3.1. *(see [15].) Let $0 < p \leq 1$ and let $x_1, x_2, \ldots, x_n$ be non-negative real numbers. Then*

$$(\sum_{i=1}^{n} x_i)^p \leq \sum_{i=1}^{n} {x_i}^p.$$

Theorem 3.2. *Let $\varphi_q : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} 4^m \varphi_q(\frac{x}{2^m}, \frac{y}{2^m}) = 0 \tag{3.1}$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 4^{pi} {\varphi_q}^p(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \tag{3.2}$$

for all $x \in X$ and for all $y \in \{x, 2x, 3x, nx, (n + 1)x, (n - 1)x, (n + 2)x, (n - 2)x, (n - 3)x\}$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\| \triangle f(x, y)\|_Y \leq \varphi_q(x, y) \tag{3.3}$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{m\to\infty} 4^m[f(\frac{x}{2^{m-1}}) - 16f(\frac{x}{2^m})] \tag{3.4}$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying

$$\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^{11}}{4}[\widetilde{\psi}_q(x)]^{\frac{1}{p}} \tag{3.5}$$

for all $x \in X$, where

$$\widetilde{\psi}_q(x) := \sum_{i=1}^{\infty} 4^{pi} \Big\{ \frac{1}{n^{2p}(n^2-1)^p} \Big[\varphi_q^p(\frac{x}{2^i}, \frac{(n+2)x}{2^i}) + \varphi_q^p(\frac{x}{2^i}, \frac{(n-2)x}{2^i})$$
$$+ 4^p\varphi_q^p(\frac{x}{2^i}, \frac{(n+1)x}{2^i}) + 4^p\varphi_q^p(\frac{x}{2^i}, \frac{(n-1)x}{2^i}) + 4^p\varphi_q^p(\frac{x}{2^i}, \frac{nx}{2^i}) + \varphi_q^p(\frac{2x}{2^i}, \frac{2x}{2^i})$$
$$+ 4^p\varphi_q^p(\frac{2x}{2^i}, \frac{x}{2^i}) + n^{2p}\varphi_q^p(\frac{x}{2^i}, \frac{3x}{2^i}) + 2^p(3n^2-1)^p\varphi_q^p(\frac{x}{2^i}, \frac{2x}{2^i})$$
$$+ (17n^2-8)^p\varphi_q^p(\frac{x}{2^i}, \frac{x}{2^i}) + \frac{n^{2p}}{(n^2-1)^p}(\varphi_q^p(0, \frac{x(n+1)x}{2^i}) + \varphi_q^p(0, \frac{(n-3)x}{2^i})$$
$$+ 10^p\varphi_q^p(0, \frac{(n-1)x}{2^i}) + 4^p\varphi_q^p(0, \frac{nx}{2^i}) + 4^p\varphi_q^p(0, \frac{(n-2)x}{2^i})) + \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_q^p(0, \frac{2x}{2^i})$$
$$+ \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_q^p(0, \frac{x}{2^i}) \Big] \Big\}. \tag{3.6}$$

Proof. Set $x = 0$ in (3.3) and then interchange x with y to get

$$\|(n^2-1)f(x) - (n^2-1)f(-x)\| \leq \varphi_q(0, x) \tag{3.7}$$

for all $x \in X$.Replacing y by x, $2x$, nx, $(n+1)x$ and $(n-1)x$ in (3.3),respectively,we get

$$\|f((n+1)x) + f((n-1)x) - n^2f(2x) - 2f(nx) + (4n^2-2)f(x)\| \leq \varphi_q(x, x) \tag{3.8}$$

and

$$\|f((n+2)x) + f((n-2)x) - n^2f(3x) - n^2f(-x) - 2f(nx) + 2n^2f(x)$$
$$+ 2(n^2-1)f(2x)\| \leq \varphi_q(x, 2x) \tag{3.9}$$

and

$$\|f(2nx) - n^2f((n+1)x) - n^2f((1-n)x) + 2(n^2-2)f(nx) + 2n^2f(x)\|$$
$$\leq \varphi_q(x, nx) \tag{3.10}$$

and

$$\|f((2n+1)x) + f(-x) - n^2f((n+2)x) - n^2f(-nx) - 2f(nx) + 2n^2f(x)$$
$$+ 2(n^2-1)f((n+1)x)\| \leq \varphi_q(x, (n+1)x) \tag{3.11}$$

and

$$\|f((2n-1)x) + f(x) - n^2f((2-n)x) - (n^2+2)f(nx) + 2n^2f(x)$$
$$+ 2(n^2-1)f((n-1)x)\| \leq \varphi_q(x, (n-1)x) \tag{3.12}$$

and

$$\|f(2(n+1)x) + f(-2x) - n^2f((n+3)x) - n^2f(-(n+1)x) - 2f(nx) + 2n^2f(x)$$
$$+ 2(n^2-1)f((n+2)x)\| \leq \varphi_q(x, (n+2)x) \tag{3.13}$$

and

$$\|f((2(n-1)x) + f(2x) - n^2f((n-1)x) - n^2f(-(n-3)x) - 2f(nx) + 2n^2f(x)$$
$$+ 2(n^2-1)f((n-2)x)\| \leq \varphi_q(x, (n-2)x) \tag{3.14}$$

and

$$\|f((n+3)x) + f((n-3)x) - n^2f(4x) - n^2f(-2x) - 2f(nx) + 2n^2f(x)$$
$$+ 2(n^2-1)f(3x)\| \leq \varphi_q(x, 3x) \tag{3.15}$$

for all $x \in X$.We combine (3.7) with (3.9), (3.10), (3.11), (3.12), (3.13), (3.14) and (3.15), respectively, to get the following inequalities:

$$\|f((n+2)x)+f((n-2)x)-n^2f(3x)-n^2f(x)-2f(nx)+2n^2f(x) \\ +2(n^2-1)f(2x)\| \le \varphi_q(x,2x)+\frac{n^2}{n^2-1}\varphi_q(0,x) \tag{3.16}$$

and

$$\|f(2nx)-n^2f((n+1)x)-n^2f((n-1)x)+2(n^2-2)f(nx)+2n^2f(x)\| \\ \le \varphi_q(x,nx)+\frac{n^2}{n^2-1}\varphi_q(0,(n-1)x) \tag{3.17}$$

and

$$\|f((2n+1)x)+f(x)-n^2f((n+2)x)-n^2f(nx)-2f(nx)+2n^2f(x) \\ +2(n^2-1)f((n+1)x)\| \le \varphi_q(x,(n+1)x) \\ +\frac{n^2}{n^2-1}\varphi_q(0,nx)+\frac{1}{n^2-1}\varphi_q(0,x) \tag{3.18}$$

and

$$\|f((2n-1)x)+f(x)-n^2f((n-2)x)-(n^2+2)f(nx)+2n^2f(x) \\ +2(n^2-1)f((n-1)x)\| \le \varphi_q(x,(n-1)x)+\frac{n^2}{n^2-1}\varphi_q(0,(n-2)x) \tag{3.19}$$

and

$$\|f(2(n+1)x)+f(2x)-n^2f((n+3)x)-n^2f((n+1)x)-2f(nx)+2n^2f(x) \\ +2(n^2-1)f((n+2)x)\| \le \varphi_q(x,(n+2)x) \\ +\frac{n^2}{n^2-1}\varphi_q(0,(n+1)x)+\varphi_q(0,2x) \tag{3.20}$$

and

$$\|f(2(n-1)x)+f(2x)-n^2f((n-1)x)-n^2f((n-3)x)-2f(nx)+2n^2f(x) \\ +2(n^2-1)f((n-2)x)\| \le \varphi_q(x,(n-2)x)+\frac{n^2}{n^2-1}\varphi_q(0,(n-3)x) \tag{3.21}$$

and

$$\|f((n+3)x)+f((n-3)x)-n^2f(4x)-n^2f(2x)-2f(nx)+2n^2f(x) \\ +2(n^2-1)f(3x)\| \le \varphi_q(x,3x)+\frac{n^2}{n^2-1}\varphi_q(0,2x) \tag{3.22}$$

for all $x \in X$.Replacing x and y by $2x$ and x in (3.3), respectively, we obtain

$$\|f((2n+1)x)+f((2n-1)x)-n^2f(3x)-2f(2nx)+2n^2f(2x) \\ +(n^2-2)f(x)\| \le \varphi_q(2x,x) \tag{3.23}$$

for all $x \in X$.Putting $2x$ and $2y$ instead of x and y in (3.3), respectively, we have

$$\|f(2(n+1)x)+f(2(n-1)x)-n^2f(4x)-2f(2nx)+2(2n^2-1)f(2x)\| \\ \le \varphi_q(2x,2x) \tag{3.24}$$

for all $x \in X$.It follows from (3.8), (3.16), (3.17), (3.18), (3.19) and (3.23) that

$$\|f(3x) - 6f(2x) + 15f(x)\| \le \frac{M^5}{n^2(n^2-1)}[\varphi_q(x,(n+1)x) + \varphi_q(x,(n-1)x) + \varphi_q(2x,x) + 2\varphi_q(x,nx) + n^2\varphi_q(x,2x) + (4n^2-2)\varphi_q(x,x) + \frac{n^2}{n^2-1}(2\varphi_q(0,(n-1)x) + \varphi_q(0,nx) + \varphi_q(0,(n-2)x)) + \frac{n^4+1}{n^2-1}\varphi_q(0,x)] \tag{3.25}$$

for all $x \in X$.Also, from (3.8), (3.16), (3.17), (3.20), (3.21), (3.22) and (3.24), we conclude

$$\|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\| \le \frac{M^6}{n^2(n^2-1)}[\varphi_q(x,(n+2)x) + \varphi_q(x,(n-2)x) + \varphi_q(2x,2x) + 2\varphi_q(x,nx) + n^2(\varphi_q(x,3x) + \varphi_q(x,x)) + 2(n^2-1)\varphi_q(x,2x) + \frac{n^2}{n^2-1}(2\varphi_q(0,(n-1)x) + \varphi_q(0,(n-3)x) + \varphi_q(0,(n+1)x)) + \frac{n^4+1}{n^2-1}\varphi_q(0,2x) + 2n^2\varphi_q(0,x)] \tag{3.26}$$

for all $x \in X$.Finally, combining (3.25) and (3.26) yields

$$\|f(4x) - 24f(2x) + 64f(x)\| \le \frac{M^8}{n^2(n^2-1)}[\varphi_q(x,(n+2)x) + \varphi_q(x,(n-2)x) + 4\varphi_q(x,(n+1)x) + 4\varphi_q(x,(n-1)x) + 10\varphi_q(x,nx) + \varphi_q(2x,2x) + 4\varphi_q(2x,x) + n^2\varphi_q(x,3x) + 2(3n^2-1)\varphi_q(x,2x) + (17n^2-8)\varphi_q(x,x) + \frac{n^2}{n^2-1}(\varphi_q(0,(n+1)x) + \varphi_q(0,(n-3)x) + 10\varphi_q(0,(n-1)x) + 4\varphi_q(0,nx) + 4\varphi_q(0,(n-2)x)) + \frac{n^4+1}{n^2-1}\varphi_q(0,2x) + \frac{2(3n^4-n^2+2)}{n^2-1}\varphi_q(0,x)] \tag{3.27}$$

for all $x \in X$.By substituting

$$\psi_q(x) = \frac{1}{n^2(n^2-1)}[\varphi_q(x,(n+2)x) + \varphi_q(x,(n-2)x) + 4\varphi_q(x,(n+1)x) + 4\varphi_q(x,(n-1)x) + 10\varphi_q(x,nx) + \varphi_q(2x,2x) + 4\varphi_q(2x,x) + n^2\varphi_q(x,3x) + 2(3n^2-1)\varphi_q(x,2x) + (17n^2-8)\varphi_q(x,x) + \frac{n^2}{n^2-1}(\varphi_q(0,(n+1)x) + \varphi_q(0,(n-3)x) + 10\varphi_q(0,(n-1)x) + 4\varphi_q(0,nx) + 4\varphi_q(0,(n-2)x)) + \frac{n^4+1}{n^2-1}\varphi_q(0,2x) + \frac{2(3n^4-n^2+2)}{n^2-1}\varphi_q(0,x)] \tag{3.28}$$

(3.27) gives

$$\|f(4x) - 20f(2x) + 64f(x)\| \le M^8\psi_q(x) \tag{3.29}$$

for all $x \in X$.

Let $g : X \to Y$ be a function defined by $g(x) := f(2x) - 16f(x)$ for all $x \in X$.From (3.29), we conclude that

$$\|g(2x) - 4g(x)\| \le M^8\psi_q(x) \tag{3.30}$$

for all $x \in X$. If we replace x in (3.30) by $\frac{x}{2^{m+1}}$ and multiply both sides of (3.30) by 4^m, we get

$$\|4^{m+1}g(\frac{x}{2^{m+1}}) - 4^m g(\frac{x}{2^m})\|_Y \leq M^8 4^m \psi_q(\frac{x}{2^{m+1}}) \tag{3.31}$$

for all $x \in X$ and all non-negative integers m. Since Y is a p-Banach space, then inequality (3.31) gives

$$\begin{aligned}\|4^{m+1}g(\frac{x}{2^{m+1}}) - 4^k g(\frac{x}{2^k})\|_Y^p &\leq \sum_{i=k}^{m} \|4^{i+1}g(\frac{x}{2^{i+1}}) - 4^i g(\frac{x}{2^i})\|_Y^p \\ &\leq M^{8p} \sum_{i=k}^{m} 4^{ip} {\psi_q}^p(\frac{x}{2^{i+1}})\end{aligned} \tag{3.32}$$

for all non-negative integers m and k with $m \geq k$ and for all $x \in X$. Since $0 < p \leq 1$, then by Lemma 3.1, from (3.28), we conclude that

$$\begin{aligned}\psi_q^p(x) \leq{}& \frac{1}{n^{2p}(n^2-1)^p}[\varphi_q^p(x,(n+2)x) + \varphi_q^p(x,(n-2)x) \\ &+ 4^p\varphi_q^p(x,(n+1)x) + 4^p\varphi_q^p(x,(n-1)x) + 10^p\varphi_q^p(x,nx) + \varphi_q^p(2x,2x) \\ &+ 4^p\varphi_q^p(2x,x) + n^{2p}\varphi_q^p(x,3x) + 2^p(3n^2-1)^p\varphi_q^p(x,2x) + (17n^2-8)^p\varphi_q^p(x,x) \\ &+ \frac{n^{2p}}{(n^2-1)^p}(\varphi_q^p(0,(n+1)x) + \varphi_q^p(0,(n-3)x) + 10^p\varphi_q^p(0,(n-1)x) + 4^p\varphi_q^p(0,nx) \\ &+ 4^p\varphi_q^p(0,(n-2)x)) + \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_q^p(0,2x) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_q^p(0,x)]\end{aligned} \tag{3.33}$$

for all $x \in X$. Therefore, it follows from (3.2) and (3.33) that

$$\sum_{i=1}^{\infty} 4^{ip}{\psi_q}^p(\frac{x}{2^i}) < \infty \tag{3.34}$$

for all $x \in X$. Thus, we conclude from (3.32) and (3.34) that the sequence $\{4^m g(\frac{x}{2^m})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, then, the sequence $\{4^m g(\frac{x}{2^m})\}$ converges for all $x \in X$. So one can define the function $Q : X \to Y$ by

$$Q(x) = \lim_{m\to\infty} 4^m g(\frac{x}{2^m}) \tag{3.35}$$

for all $x \in X$. Letting $k = 0$ and passing the limit $m \to \infty$ in (3.32), we get

$$\|g(x) - Q(x)\|_Y^p \leq M^{8p} \sum_{i=0}^{\infty} 4^{ip}{\psi_q}^p(\frac{x}{2^{i+1}}) = \frac{M^{8p}}{4^p} \sum_{i=1}^{\infty} 4^{ip}{\psi_q}^p(\frac{x}{2^i}) \tag{3.36}$$

for all $x \in X$. Therefore, (3.5) follows from (3.2) and (3.36). Now we show that Q is quadratic. It follows from (3.1), (3.31) and (3.35) that

$$\begin{aligned}\|Q(2x) - 4Q(x)\|_Y &= \lim_{m\to\infty} \|4^m g(\frac{x}{2^{m-1}}) - 4^{m+1} g(\frac{x}{2^m})\|_Y \\ &= 4 \lim_{m\to\infty} \|4^{m-1} g(\frac{x}{4^{m-1}}) - 4^m g(\frac{x}{2^m})\|_Y \\ &\leq M^{11} \lim_{m\to\infty} 4^m \psi_q(\frac{x}{2^m}) = 0\end{aligned}$$

for all $x \in X$. So

$$Q(2x) = 4Q(x) \tag{3.37}$$

for all $x \in X$. On the other hand, it follows from (3.1), (3.3), (3.4) and (3.35) that

$$\begin{aligned}\|\triangle Q(x,y)\|_Y &= \lim_{m\to\infty} 4^m\|\triangle g(\frac{x}{2^m},\frac{y}{2^m})\|_Y = \lim_{m\to\infty} 4^m\|\triangle f(\frac{x}{2^{m-1}},\frac{y}{2^{m-1}}) - 16\triangle f(\frac{x}{2^m},\frac{y}{2^m})\|_Y \\ &\le M\lim_{m\to\infty} 4^m\{\|\triangle f(\frac{x}{2^{m-1}},\frac{y}{2^{m-1}})\|_Y + 16\|\triangle f(\frac{x}{2^m},\frac{y}{2^m})\|_Y\} \\ &\le M\lim_{m\to\infty} 4^m\{\varphi_q(\frac{x}{2^{m-1}},\frac{y}{2^{m-1}}) + 16\varphi_q(\frac{x}{2^m},\frac{y}{2^m})\} = 0\end{aligned}$$

for all $x, y \in X$. Hence the function Q satisfies (1.5). By Lemma 2.1, the function $x \rightsquigarrow Q(2x) - 4Q(x)$ is quadratic. Hence, (3.37) implies that the function Q is quadratic.

It remains to show that Q is unique.Suppose that there exists another quadratic function $Q^{'} : X \to Y$ witch satisfies (1.5) and (3.5).Since $Q^{'}(\frac{x}{2^m}) = \frac{1}{4^m}Q^{'}(x)$ and $Q(\frac{x}{2^m}) = \frac{1}{4^m}Q(x)$ for all $x \in X$, we conclude from (3.5) that

$$\|Q(x)-Q^{'}(x)\|_Y^p = \lim_{m\to\infty} 4^{mp}\|g(\frac{x}{2^m}) - Q^{'}(\frac{x}{2^m})\|_Y{}^p \le \frac{M^{8p}}{4^p}\lim_{m\to\infty} 4^{mp}\widetilde{\psi}_q(\frac{x}{2^m}) \tag{3.38}$$

for all $x \in X$.On the other hand, since

$$\lim_{m\to\infty} 4^{mp}\sum_{i=1}^{\infty} 4^{ip}{\varphi_q}^p(\frac{x}{2^{m+i}},\frac{y}{2^{m+i}}) = \lim_{m\to\infty}\sum_{i=m+1}^{\infty} 4^{ip}{\varphi_q}^p(\frac{x}{2^i},\frac{y}{2^i}) = 0$$

for all $x \in X$ and for all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$, therefore

$$\lim_{m\to\infty} 4^{mp}\widetilde{\psi}_q(\frac{x}{2^m}) = 0 \tag{3.39}$$

for all $x \in X$.By using (3.39) in (3.38), we get $Q = Q^{'}$.

□

Theorem 3.3. *Let $\varphi_q : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{m\to\infty}\frac{1}{4^m}\varphi_q(2^m x, 2^m y) = 0$$

for all $x, y \in X$ and

$$\sum_{i=0}^{\infty}\frac{1}{4^{pi}}{\varphi_q}^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and for all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\triangle f(x,y)\|_Y \le \varphi_q(x,y)$$

for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{m\to\infty}\frac{1}{4^m}[f(2^{m+1}x) - 16f(2^m x)]$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying

$$\|f(2x) - 16f(x) - Q(x)\|_Y \le \frac{M^8}{4}[\widetilde{\psi}_q(x)]^{\frac{1}{p}}$$

for all $x \in X$, where

$$\widetilde{\psi}_q(x) := \sum_{i=0}^{\infty} \frac{1}{4^{pi}} \Big\{ \frac{1}{n^{2p}(n^2-1)^p} \Big[\varphi_q^p(2^i x, 2^i(n+2)x) + \varphi_q^p(2^i x, 2^i(n-2)x)$$
$$+ 4^p\varphi_q^p(2^i x, 2^i(n+1)x) + 4^p\varphi_q^p(2^i x, 2^i(n-1)x) + 10^p\varphi_q^p(2^i x, 2^i nx) + \varphi_q^p(2^i 2x, 2^i 2x)$$
$$+ 4^p\varphi_q^p(2^i 2x, 2^i x) + n^{2p}\varphi_q^p(2^i x, 2^i 3x) + 2^p(3n^2-1)^p\varphi_q^p(2^i x, 2^i 2x)$$
$$+ (17n^2-8)^p\varphi_q^p(2^i x, 2^i x) + \frac{n^{2p}}{(n^2-1)^p}(\varphi_q^p(0, 2^i(n+1)x) + \varphi_q^p(0, 2^i(n-3)x)$$
$$+ 10^p\varphi_q^p(0, 2^i(n-1)x) + 4^p\varphi_q^p(0, 2^i nx) + 4^p\varphi_q^p(0, 2^i(n-2)x))$$
$$+ \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_q^p(0, 2^i 2x) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_q^p(0, 2^i x) \Big] \Big\}.$$

Proof. The proof is similar to the proof of Theorem 3.2. □

Corollary 3.4. *Let θ, r, s be non-negative real numbers such that $r, s > 2$ or $s < 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|\triangle f(x,y)\|_Y \leq \begin{cases} \theta, & r = s = 0; \\ \theta\|x\|_X^r, & r > 0,\ s = 0; \\ \theta\|y\|_X^s, & r = 0,\ s > 0; \\ \theta(\|x\|_X^r + \|y\|_X^s), & r,\ s > 0. \end{cases} \tag{3.40}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)} \begin{cases} \delta_q, & r = s = 0; \\ \alpha_q(x), & r > 0,\ s = 0; \\ \beta_q(x), & r = 0,\ s > 0; \\ (\alpha_q^p(x) + \beta_q^p(x))^{\frac{1}{p}} & r,\ s > 0. \end{cases}$$

for all $x \in X$, where

$$\delta_q = \Big\{ \frac{1}{4^p - 1(n^2-1)^p}[(6n^2-2)^p(n^2-1)^p + (17n^2-8)^p(n^2-1)^p + (6n^4-2n^2+4)^p$$
$$+ n^{2p}(2 + 10^p + 2*4^p) + (n^4+1)^p + n^{2p}(n^2-1)^p + 3*4^p(n^2-1)^p + 10^p(n^2-1)^p$$
$$+ 3(n^2-1)^p] \Big\}^{\frac{1}{p}},$$

$$\alpha_q(x) = \Big\{ \frac{4^p(2+2^{rp}) + 10^p + (6n^2-2)^p + (17n^2-8)^p + 2^{rp} + n^{2p}}{|4^p - 2^{rp}|} \Big\}^{\frac{1}{p}} \|x\|_X^r$$

and

$$\beta_q(x) = \Big\{ \frac{1}{(n^2-1)^p|4^p - 2^{sp}|}[2^{sp}(6n^2-2)^p(n^2-1)^p + (17n^2-8)^p(n^2-1)^p$$
$$+ (6n^4-2n^2+4)^p + n^{2p}((n+1)^{sp} + (n-3)^{sp} + 10^p(n-1)^{sp}$$
$$+ 4^p n^{sp} + 4^p(n-2)^{sp}) + 2^{sp}(n^4+1)^p + 3^{sp}n^{2p}(n^2-1)^p + 4^p(n^2-1)^p$$
$$+ (n+2)^{sp}(n^2-1)^p + (n-2)^{sp}(n^2-1)^p + 4^p(n+1)^{sp}(n^2-1)^p$$
$$+ 4^p(n-1)^{sp}(n^2-1)^p + 10^p n^{sp}(n^2-1)^p] \Big\}^{\frac{1}{p}} \|x\|_X^s.$$

Proof. In Theorem 3.2, putting $\varphi_q(x,y) := \theta(\|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. □

Corollary 3.5. *Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r+s \neq 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|\triangle f(x,y)\|_Y \leq \theta \|x\|_X^r \|y\|_X^s, \tag{3.41}$$

for all $x, y \in X$. Then there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\begin{aligned}\|f(2x) - 16f(x) - Q(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)} \Big\{ \frac{1}{|4^p - 2^{\lambda p}|} [(n+2)^{sp} + (n-2)^{sp} + 4^p(n+1)^{sp} \\ + 4^p(n-1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p}3^{sp} \\ + 2^{sp}(6n^2-2)^p + (17n^2-8)^p] \Big\}^{\frac{1}{p}} \|x\|_X^\lambda\end{aligned}$$

for all $x \in X$.

Proof. In Theorem 3.2 putting $\varphi_q(x,y) := \theta\|x\|_X^r\|y\|_X^s$ for all $x, y \in X$. □

Theorem 3.6. *Let $\varphi_t : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} 16^m \varphi_t(\frac{x}{2^m}, \frac{y}{2^m}) = 0 \tag{3.42}$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 16^{pi} {\varphi_t}^p(\frac{x}{2^i}, \frac{y}{2^i}) < \infty \tag{3.43}$$

for all $x \in X$ and for all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\triangle f(x,y)\|_Y \leq \varphi_t(x,y) \tag{3.44}$$

for all $x, y \in X$. Then the limit

$$T(x) := \lim_{m\to\infty} 16^m [f(\frac{x}{2^{m-1}}) - 4f(\frac{x}{2^m})] \tag{3.45}$$

exists for all $x \in X$ and $T : X \to Y$ is a unique quartic function satisfying

$$\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8}{16}[\widetilde{\psi}_t(x)]^{\frac{1}{p}} \tag{3.46}$$

for all $x \in X$, where

$$\begin{aligned}\widetilde{\psi}_t(x) := \sum_{i=1}^{\infty} 16^{pi} \Big\{ \frac{1}{n^{2p}(n^2-1)^p} \Big[\varphi_t^p(\frac{x}{2^i}, \frac{(n+2)x}{2^i}) + \varphi_t^p(\frac{x}{2^i}, \frac{(n-2)x}{2^i}) \\ + 4^p\varphi_t^p(\frac{x}{2^i}, \frac{(n+1)x}{2^i}) + 4^p\varphi_t^p(\frac{x}{2^i}, \frac{(n-1)x}{2^i}) + 10^p\varphi_t^p(\frac{x}{2^i}, \frac{nx}{2^i}) + \varphi_t^p(\frac{2x}{2^i}, \frac{2x}{2^i}) \\ + 4^p\varphi_t^p(\frac{2x}{2^i}, \frac{x}{2^i}) + n^{2p}\varphi_t^p(\frac{x}{2^i}, \frac{3x}{2^i}) + 2^p(3n^2-1)^p\varphi_t^p(\frac{x}{2^i}, \frac{2x}{2^i}) \\ + (17n^2-8)^p\varphi_t^p(\frac{x}{2^i}, \frac{x}{2^i}) + \frac{n^{2p}}{(n^2-1)^p}(\varphi_t^p(0, \frac{x(n+1)x}{2^i}) + \varphi_t^p(0, \frac{(n-3)x}{2^i}) \\ + 10^p\varphi_t^p(0, \frac{(n-1)x}{2^i}) + 4^p\varphi_t^p(0, \frac{nx}{2^i}) + 4^p\varphi_t^p(0, \frac{(n-2)x}{2^i})) \\ + \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_t^p(0, \frac{2x}{2^i}) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_t^p(0, \frac{x}{2^i}) \Big] \Big\}.\end{aligned} \tag{3.47}$$

Proof. Similar to the proof Theorem 3.2, we have

$$\|f(4x) - 20f(2x) + 64f(x)\| \leq M^8\psi_t(x), \tag{3.48}$$

for all $x \in X$, where

$$\begin{aligned}\psi_t(x) = &\frac{1}{n^2(n^2-1)}[\varphi_t(x,(n+2)x) + \varphi_t(x,(n-2)x)\\ &+ 4\varphi_t(x,(n+1)x) + 4\varphi_t(x,(n-1)x) + 10\varphi_t(x,nx) + \varphi_t(2x,2x)\\ &+ 4\varphi_t(2x,x) + n^2\varphi_t(x,3x) + 2(3n^2-1)\varphi_t(x,2x) + (17n^2-8)\varphi_t(x,x)\\ &+ \frac{n^2}{n^2-1}(\varphi_t(0,(n+1)x) + \varphi_t(0,(n-3)x) + 10\varphi_t(0,(n-1)x) + 4\varphi_t(0,nx)\\ &+ 4\varphi_t(0,(n-2)x)) + \frac{n^4+1}{n^2-1}\varphi_t(0,2x) + \frac{2(3n^4-n^2+2)}{n^2-1}\varphi_t(0,x)].\end{aligned} \tag{3.49}$$

Let $h : X \to Y$ be a function defined by $h(x) := f(2x) - 4f(x)$. Then, we conclude that

$$\|h(2x) - 16h(x)\| \leq M^8\psi_t(x) \tag{3.50}$$

for all $x \in X$. If we replace x in (3.50) by $\frac{x}{2^{m+1}}$ and multiply both sides of (3.50) by 16^m, we get

$$\|16^{m+1}h(\frac{x}{2^{m+1}}) - 16^m h(\frac{x}{2^m})\|_Y \leq M^8 16^m\psi_t(\frac{x}{2^{m+1}}) \tag{3.51}$$

for all $x \in X$ and all non-negative integers m. Since Y is a p-Banach space, therefore, inequality (3.51) gives

$$\begin{aligned}\|16^{m+1}h(\frac{x}{2^{m+1}}) - 16^k h(\frac{x}{2^k})\|_Y^p &\leq \sum_{i=k}^{m}\|16^{i+1}h(\frac{x}{2^{i+1}}) - 16^i h(\frac{x}{2^i})\|_Y^p\\ &\leq M^{8p}\sum_{i=k}^{m}16^{pi}{\psi_t}^p(\frac{x}{2^{i+1}})\end{aligned} \tag{3.52}$$

for all non-negative integers m and k with $m \geq k$ and all $x \in X$. Since $0 < p \leq 1$, then by Lemma 3.1, we conclude from (3.49) that

$$\begin{aligned}\psi_t^p(x) \leq &\frac{1}{n^{2p}(n^2-1)^p}\ [\varphi_t^p(x,(n+2)x) + \varphi_t^p(x,(n-2)x)\\ &+ 4^p\varphi_t^p(x,(n+1)x) + 4^p\varphi_t^p(x,(n-1)x) + 10^p\varphi_t^p(x,nx) + \varphi_t^p(2x,2x)\\ &+ 4^p\varphi_t^p(2x,x) + n^{2p}\varphi_t^p(x,3x) + 2^p(3n^2-1)^p\varphi_t^p(x,2x) + (17n^2-8)^p\varphi_t^p(x,x)\\ &+ \frac{n^{2p}}{(n^2-1)^p}(\varphi_t^p(0,(n+1)x) + \varphi_t^p(0,(n-3)x) + 10^p\varphi_t^p(0,(n-1)x) + 4^p\varphi_t^p(0,nx)\\ &+ 4^p\varphi_t^p(0,(n-2)x)) + \frac{(n^4+1)^p}{(n^2-1)^p}\varphi_t^p(0,2x) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p}\varphi_t^p(0,x)\ ],\end{aligned} \tag{3.53}$$

for all $x \in X$. Therefore, it follows from (3.42) and (3.52) that

$$\sum_{i=1}^{\infty}16^{pi}{\psi_t}^p(\frac{x}{2^i}) < \infty \tag{3.54}$$

for all $x \in X$. Thus, we conclude from (3.52) and (3.54) that the sequence $\{16^m h(\frac{x}{2^m})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{16^m h(\frac{x}{2^m})\}$ converges for all $x \in X$. So one can define the function $T : X \to Y$ by

$$T(x) = \lim_{m\to\infty} 16^m h(\frac{x}{2^m}) \tag{3.55}$$

for all $x \in X$. Letting $k = 0$ and passing the limit $m \to \infty$ in (3.52), we get

$$\|h(x)-T(x)\|_Y^p \leq M^{8p} \sum_{i=0}^{\infty} 16^{pi} {\psi_t}^p(\frac{x}{2^{i+1}}) = \frac{M^{11p}}{16^p} \sum_{i=1}^{\infty} 16^{pi} {\psi_t}^p(\frac{x}{2^i}) \tag{3.56}$$

for all $x \in X$. Therefore (3.45) follows from (3.43) and (3.55). Now we show that T is quartic. According to (3.42), (3.51) and (3.55), it follows that

$$\begin{aligned}\|T(2x) - 16T(x)\|_Y &= \lim_{m\to\infty} \|16^m h(\frac{x}{2^{m-1}}) - 16^{m+1} h(\frac{x}{2^m})\|_Y \\ &= 16 \lim_{m\to\infty} \|16^{m-1} h(\frac{x}{16^{m-1}}) - 16^m h(\frac{x}{2^m})\|_Y \\ &\leq M^8 \lim_{m\to\infty} 16^m \psi_t(\frac{x}{2^m}) = 0\end{aligned}$$

for all $x \in X$. So

$$T(2x) = 16T(x) \tag{3.57}$$

for all $x \in X$. On the other hand, by (3.44), (3.54) and (3.55), we lead to

$$\begin{aligned}\|\triangle T(x,y)\|_Y &= \lim_{m\to\infty} 16^m \|\triangle h(\frac{x}{2^m}, \frac{y}{2^m})\|_Y = \lim_{m\to\infty} 16^m \|\triangle f(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}) - 4\triangle f(\frac{x}{2^m}, \frac{y}{2^m})\|_Y \\ &\leq M \lim_{m\to\infty} 16^m \{\|\triangle f(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}})\|_Y + 4\|\triangle f(\frac{x}{2^m}, \frac{y}{2^m})\|_Y\} \\ &\leq M \lim_{m\to\infty} 16^m \{\varphi_t(\frac{x}{2^{m-1}}, \frac{y}{2^{m-1}}) + 4\varphi_t(\frac{x}{2^m}, \frac{y}{2^m})\} = 0\end{aligned}$$

for all $x, y \in X$. Hence, the function T satisfies (1.5). By Lemma 2.1, the function $x \rightsquigarrow T(2x) - 16T(x)$ is quartic. Therefore (3.57) implies that the function T is quartic.

To prove the uniqueness property of T, let $T' : X \to Y$ be another quartic function satisfying (3.46). Since

$$\lim_{m\to\infty} 16^{mp} \sum_{i=1}^{\infty} 16^{pi} {\varphi_t}^p(\frac{x}{2^{m+i}}, \frac{x}{2^{m+i}}) = \lim_{m\to\infty} \sum_{i=m+1}^{\infty} 16^{pi} {\varphi_t}^p(\frac{x}{2^i}, \frac{x}{2^i}) = 0$$

for all $x \in X$ and for all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$, then

$$\lim_{m\to\infty} 16^{mp} \widetilde{\psi}_t(\frac{x}{2^m}) = 0 \tag{3.58}$$

for all $x \in X$. It follows from (3.46) and (3.58) that

$$\|T(x) - T'(x)\|_Y = \lim_{m\to\infty} 16^{mp} \|h(\frac{x}{2^m}) - T'(\frac{x}{2^m})\|_Y{}^p \leq \frac{M^{8p}}{16^p} \lim_{m\to\infty} 16^{mp} \widetilde{\psi}_t(\frac{x}{2^m}) = 0$$

for all $x \in X$. So $T = T'$.

□

Theorem 3.7. *Let* $\varphi_t : X \times X \to [0, \infty)$ *be a function such that*

$$\lim_{m\to\infty} \frac{1}{16^m} \varphi_t(2^m x, 2^m y) = 0$$

for all $x, y \in X$ *and*

$$\sum_{i=0}^{\infty} \frac{1}{16^{pi}} {\varphi_t}^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ *and for all* $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. *Suppose that a function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|\triangle f(x,y)\|_Y \leq \varphi_t(x,y)$$

for all $x, y \in X$. *Then the limit*

$$T(x) := \lim_{m\to\infty} \frac{1}{16^m}[f(2^{m+1}x) - 4f(2^m x)]$$

exists for all $x \in X$ *and* $T : X \to Y$ *is a unique quartic function satisfying*

$$\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8}{16}[\widetilde{\psi}_t(x)]^{\frac{1}{p}}$$

for all $x \in X$, *where*

$$\begin{aligned}
\widetilde{\psi}_t(x) :&= \sum_{i=0}^{\infty} \frac{1}{16^{pi}} \Big\{ \frac{1}{n^{2p}(n^2-1)^p} \Big[\varphi_t^p(2^i x, 2^i(n+2)x) + \varphi_t^p(2^i x, 2^i(n-2)x) \\
&+ 4^p \varphi_t^p(2^i x, 2^i(n+1)x) + 4^p \varphi_t^p(2^i x, 2^i(n-1)x) + 10^p \varphi_t^p(2^i x, 2^i nx) + \varphi_t^p(2^i 2x, 2^i 2x) \\
&+ 4^p \varphi_t^p(2^i 2x, 2^i x) + n^{2p} \varphi_t^p(2^i x, 2^i 3x) + 2^p(3n^2-1)^p \varphi_t^p(2^i x, 2^i 2x) \\
&+ (17n^2-8)^p \varphi_t^p(2^i x, 2^i x) + \frac{n^{2p}}{(n^2-1)^p}(\varphi_t^p(0, 2^i(n+1)x) + \varphi_t^p(0, 2^i(n-3)x) \\
&+ 10^p \varphi_t^p(0, 2^i(n-1)x) + 4^p \varphi_t^p(0, 2^i nx) + 4^p \varphi_t^p(0, 2^i(n-2)x)) \\
&+ \frac{(n^4+1)^p}{(n^2-1)^p} \varphi_t^p(0, 2^i 2x) + \frac{(2(3n^4-n^2+2))^p}{(n^2-1)^p} \varphi_t^p(0, 2^i x) \Big] \Big\}.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 3.6. □

Corollary 3.8. *Let* θ, r, s *be non-negative real numbers such that* $r, s > 4$ *or* $0 \leq r, s < 4$. *Suppose that a function* $f : X \to Y$ *with* $f(0) = 0$ *satisfies the inequality (3.40) for all* $x, y \in X$. *Then there exists a unique quartic function* $T : X \to Y$ *satisfying*

$$\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)} \begin{cases} \delta_t, & r = s = 0; \\ \alpha_t(x), & r > 0,\ s = 0; \\ \beta_t(x), & r = 0,\ s > 0; \\ (\alpha_t^p(x) + \beta_t^p(x))^{\frac{1}{p}}, & r,\ s > 0. \end{cases}$$

for all $x \in X$, *where*

$$\begin{aligned}
\delta_t = \Big\{ &\frac{1}{(16^p-1)(n^2-1)^p}[(6n^2-2)^p(n^2-1)^p + (17n^2-8)^p(n^2-1)^p + (6n^4-2n^2+4)^p \\
&+ n^{2p}(2+10^p+2*4^p) + (n^4+1)^p + n^{2p}(n^2-1)^p + 3*4^p(n^2-1)^p + 10^p(n^2-1)^p \\
&+ 3(n^2-1)^p] \Big\}^{\frac{1}{p}},
\end{aligned}$$

$$\alpha_t(x) = \Big\{ \frac{4^p(2+2^{rp}) + 10^p + (6n^2-2)^p + (17n^2-8)^p + 2^{rp} + n^{2p}}{|16^p - 2^{rp}|} \Big\}^{\frac{1}{p}} \|x\|_X^r$$

and

$$\beta_t(x) = \Big\{ \frac{1}{(n^2-1)^p|16^p - 2^{sp}|}[2^{sp}(6n^2-2)^p(n^2-1)^p + (17n^2-8)^p(n^2-1)^p + (6n^4-2n^2+4)^p + n^{2p}((n+1)^{sp} + (n-3)^{sp} + 10^p(n-1)^{sp} + 10^p n^{sp} + 4^p(n-2)^{sp}) + 2^{sp}(n^4+1)^p + 3^{sp}n^{2p}(n^2-1)^p + 4^p(n^2-1)^p + (n+2)^{sp}(n^2-1)^p + (n-2)^{sp}(n^2-1)^p + 4^p(n+1)^{sp}(n^2-1)^p + 4^p(n-1)^{sp}(n^2-1)^p + 4^p n^{sp}(n^2-1)^p] \Big\}^{\frac{1}{p}} \|x\|_X^s.$$

Proof. In Theorem 3.6, putting $\varphi_t(x,y) := \theta(\|x\|_X^r + \|y\|_X^s)$ for all $x, y \in X$. □

Corollary 3.9. *Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r+s \neq 4$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.41) for all $x, y \in X$. Then there exists a unique quartic function $T : X \to Y$ satisfying*

$$\|f(2x) - 4f(x) - T(x)\|_Y \leq \frac{M^8\theta}{n^2(n^2-1)}\Big\{ \frac{1}{|16^p - 2^{\lambda p}|}[(n+2)^{sp} + (n-2)^{sp} + 4^p(n+1)^{sp} + 4^p(n-1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p}3^{sp} + 2^{sp}(6n^2-2)^p + (17n^2-8)^p] \Big\}^{\frac{1}{p}} \|x\|_X^\lambda$$

for all $x \in X$.

Proof. In Theorem 3.6, putting $\varphi_t(x,y) := \theta\|x\|_X^r\|y\|_X^s$ for all $x, y \in X$. □

Theorem 3.10. *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} 4^m \varphi(\frac{x}{2^m}, \frac{y}{2^m}) = 0 = \lim_{m\to\infty} \frac{1}{16^m}\varphi(2^m x, 2^m y) \tag{3.59}$$

for all $x, y \in X$ and

$$\sum_{i=1}^{\infty} 4^{pi}\varphi^p(\frac{x}{2^i}, \frac{y}{2^i}) < \infty$$

and

$$\sum_{i=0}^{\infty} \frac{1}{16^{pi}}\varphi^p(2^i x, 2^i y) < \infty$$

for all $x \in X$ and for all $y \in \{x, 2x, 3x, nx, (n+1)x, (n-1)x, (n+2)x, (n-2)x, (n-3)x\}$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|\triangle f(x,y)\|_Y \leq \varphi(x,y), \tag{3.60}$$

for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $T : X \to Y$ such that

$$\|f(x) - Q(x) - T(x)\|_Y \leq \frac{M^9}{192}\ (4[\widetilde{\psi}_q(x)]^{\frac{1}{p}} + [\widetilde{\psi}_t(x)]^{\frac{1}{p}}) \tag{3.61}$$

for all $x \in X$, where $\widetilde{\psi}_q(x)$ and $\widetilde{\psi}_t(x)$ have been defined in Theorems 3.2 and 3.7, respectively, for all $x \in X$.

Proof. By Theorems 3.2 and 3.7, there exist a quadratic function $Q_0 : X \to Y$ and a quartic function $T_0 : X \to Y$ such that

$$\|f(2x) - 16f(x) - Q_0(x)\|_Y \le \frac{M^8}{4}[\widetilde{\psi}_q(x)]^{\frac{1}{p}}, \qquad \|f(2x) - 4f(x) - T_0(x)\|_Y \le \frac{M^8}{16}[\widetilde{\psi}_t(x)]^{\frac{1}{p}}$$

for all $x \in X$. Therefore, it follows from the last inequalities that

$$\|f(x)+\frac{1}{12}Q_0(x)-\frac{1}{12}T_0(x)\|_Y \le \frac{M^9}{192}\,(4[\widetilde{\psi}_q(x)]^{\frac{1}{p}}+[\widetilde{\psi}_t(x)]^{\frac{1}{p}})$$

for all $x \in X$. So we obtain (3.61) by letting $Q(x) = -\frac{1}{12}Q_0(x)$ and $T(x) = \frac{1}{12}T_0(x)$ for all $x \in X$.

To prove the uniqueness property of Q and T, we first show the uniqueness property for Q_0 and T_0 and then we conclude the uniqueness property of Q and T. Let $Q_1, T_1 : X \to Y$ be another quadratic and quartic functions satisfying (3.61) and let $Q_2 = \frac{1}{12}Q_0$, $T_2 = \frac{1}{12}T_0$, $Q_3 = Q_2 - Q_1$ and $T_3 = T_2 - T_1$. So

$$\begin{aligned}\|Q_3(x) - T_3(x)\|_Y &\le M\{\|f(x) - Q_2(x) - T_2(x)\|_Y + \|f(x) - Q_1(x) - T_1(x)\|_Y\} \\ &\le \frac{M^{10}}{96}\,(4[\widetilde{\psi}_q(x)]^{\frac{1}{p}} + [\widetilde{\psi}_t(x)]^{\frac{1}{p}})\end{aligned} \tag{3.62}$$

for all $x \in x$. Since

$$\lim_{m\to\infty} 4^{mp}\widetilde{\psi}_q(\frac{x}{2^m}) = \lim_{m\to\infty}\frac{1}{16^{mp}}\widetilde{\psi}_t(2^m x) = 0$$

for all $x \in X$, then (3.62) implies that $\lim_{m\to\infty}\|4^m Q_3(\frac{x}{2^m}) + \frac{1}{16^m}T_3(2^m x)\|_Y = 0$ for all $x \in X$. Thus, $T_3 = Q_3$. But T_3 is only a quartic function and Q_3 is only a quadratic function. Therefore, we should have $T_3 = Q_3 = 0$ and this complete the uniqueness property of Q and T. The other results proved similarly. □

Corollary 3.11. *Let θ, r, s be non-negative real numbers such that $r, s > 4$ or $2 < r, s < 4$ or $0 \le r, s < 2$. Suppose that a function $f : X \to Y$ satisfies the inequality (3.40) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $T : X \to Y$ such that*

$$\|f(x)-Q(x)-T(x)\|_Y \le \frac{M^9\theta}{12n^2(n^2-1)}\begin{cases}\delta_q + \delta_t, & r = s = 0;\\ \alpha_q(x) + \alpha_t(x), & r > 0,\ s = 0;\\ \beta_q(x) + \beta_t(x), & r = 0,\ s > 0;\\ (\alpha_q^p(x) + \beta_q^p(x))^{\frac{1}{p}} + (\alpha_t^p(x) + \beta_t^p(x))^{\frac{1}{p}}, & r,\ s > 0.\end{cases}$$

for all $x \in X$, where $\delta_q, \delta_t, \alpha_q(x), \alpha_t(x), \beta_q(x)$ and $\beta_t(x)$ are defined as in Corollaries 3.4 and 3.8.

Corollary 3.12. *Let $\theta \ge 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \in (0, 2) \cup (2, 4) \cup (4, \infty)$. Suppose that a function $f : X \to Y$ satisfies the inequality (3.41) for all $x, y \in X$. Then there exist a unique quadratic function $Q : X \to Y$ and a unique quartic function $T : X \to Y$ such that*

$$\begin{aligned}\|f(x) - Q(x) - T(x)\|_Y \le \frac{M^9\theta}{12n^2(n^2-1)}\Big\{&\frac{1}{|4^p - 2^{\lambda p}|}[(n+2)^{sp} + (n-2)^{sp} + 4^p(n+1)^{sp} \\ &+ 4^p(n-1)^{sp} + 10^p n^{sp} + 2^{(r+s)p} + 4^p 2^{rp} + n^{2p}3^{sp} \\ &+ 2^{sp}(6n^2-2)^p + (17n^2-8)^p]\Big\}^{\frac{1}{p}}\ \|x\|_X^{\lambda}\end{aligned}$$

for all $x \in X$.

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