ON LARGE TIME BEHAVIOR AND SELECTION PRINCIPLE FOR A DIFFUSIVE CARR-PENROSE MODEL

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Abstract. This paper is concerned with the study of a diffusive perturbation of the linear LSW model introduced by Carr and Penrose. A main subject of interest is to understand how the presence of diffusion acts as a selection principle, which singles out a particular self-similar solution of the linear LSW model as determining the large time behavior of the diffusive model. A selection principle is rigorously proven for a model which is a semi-classical approximation to the diffusive model. Upper bounds on the rate of coarsening are also obtained for the full diffusive model.

1. Introduction.

In [2] Carr and Penrose introduced a linear version of the Lifschitz-Slyozov-Wagner (LSW) model [12, 22]. In this model the density function $c_0(x,t),\ x>0, t>0$, evolves according to the system of equations,

$$\frac{\partial c_0(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[1-\frac{x}{\Lambda_0(t)}\right]c_0(x,t)\ , \quad x>0, \tag{1.1}$$

$$\int_0^\infty x c_0(x,t)dx = 1. \tag{1.2}$$

The parameter $\Lambda_0(t)>0$ in (1.1) is determined by the conservation law (1.2) and is therefore given by the formula,

$$\Lambda_0(t) = \int_0^\infty x c_0(x,t)dx \Big/ \int_0^\infty c_0(x,t)dx. \tag{1.3}$$

One can also see that the derivative of $\Lambda_0(t)$ is given by

$$\frac{d\Lambda_0(t)}{dt} = c_0(0,t)\Big/\left[\int_0^\infty c_0(x,t)dx\right]^2\ , \tag{1.4}$$

whence $\Lambda_0(\cdot)$ is an increasing function.

The system (1.1), (1.2) can be interpreted as an evolution equation for the probability density function (pdf) of random variables. Thus let us assume that the initial data $c_0(x)\ge 0,\ x>0$, for (1.1), (1.2) satisfies $\int_0^\infty c_0(x)\ dx<\infty$, and let X_0 be the non-negative random variable with pdf $c_0(\cdot)/\int_0^\infty c_0(x)\ dx$. The conservation law (1.2) implies that the mean $\langle X_0\rangle$ of X_0 is finite, and this is the only absolute requirement on the variable X_0. If for $t>0$ the variable X_t has pdf $c_0(\cdot,t)/\int_0^\infty c(x,t)\ dx$, then (1.1) with $\Lambda_0(t)=\langle X_t\rangle$ is an evolution equation for the pdf of X_t. Equation (1.4) now tells us that $\langle X_t\rangle$ is an increasing function of t.

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There is an infinite one-parameter family of self-similar solutions to (1.1), (1.2). Using the normalization $\langle X_0\rangle = 1$, the initial data for these solutions are given by (1.5)

$$P(X_0 > x) \ = \ \begin{cases} [1-(1-\beta)x]^{\beta/(1-\beta)} \ , \ 0<x<1/(1-\beta), & \text{if } \ 0<\beta<1, \\ e^{-x} & \text{if } \ \beta=1, \\ [1+(\beta-1)x]^{\beta/(1-\beta)} \ , \ 0<x<\infty, & \text{if } \ \beta>1. \end{cases}$$

The random variable X_t corresponding to the evolution (1.1), (1.2) with initial data (1.5) is then given by

$$X_t \ = \ \langle X_t\rangle X_0 \ , \quad \frac{d}{dt}\langle X_t\rangle \ = \beta \ . \tag{1.6}$$

The main result of [2] (see also [1]) is that a solution of (1.1), (1.2) converges at large time to the self-similar solution with parameter β, provided the initial data and the self similar solution of parameter β behave in the same way at the end of their supports. In §2 we give a simple proof of the Carr-Penrose convergence theorem using the *beta function* of a random variable introduced in [5].

The large time behavior of the Carr-Penrose (CP) model is qualitatively similar to the conjectured large time behavior of the LSW model [14], provided the initial data has compact support. In the LSW model there is a one-parameter family of self-similar solutions with parameter $\beta, \ 0<\beta\le 1$, all of which have compact support. The self-similar solution with parameter $\beta<1$ behaves in the same way towards the end of its support as does the CP self-similar solution with parameter β. It has been conjectured [14] that a solution of the LSW model converges at large time to the LSW self-similar solution with parameter β, provided the initial data and the self similar solution of parameter β behave in the same way at the end of their supports. A weak version of this result has been proven in [7].

It was already claimed in [12, 22] that the only physically relevant self-similar LSW solution is the one with parameter $\beta=1$. This has been explained in a heuristic way in several papers [13, 16, 20], by considering a model in which a second order diffusion term is added to the first order LSW equation. It is then argued that diffusion acts as a *selection principle*, which singles out the $\beta=1$ self-similar solution as giving the large time behavior. In this paper we study a diffusive version of the Carr-Penrose model, with the goal of understanding how a selection principle for the $\beta=1$ self-similar solution (1.5) operates.

In our diffusive CP model we simply add a second order diffusion term with coefficient $\varepsilon/2>0$ to the CP equation (1.1). Then the density function $c_\varepsilon(x,t)$ evolves according to a linear diffusion equation, subject to the linear mass conservation constraint as follows:

$$\frac{\partial c_\varepsilon(x,t)}{\partial t} \ = \ \frac{\partial}{\partial x}\left[1-\frac{x}{\Lambda_\varepsilon(t)}\right]c_\varepsilon(x,t)+\frac{\varepsilon}{2}\frac{\partial^2 c_\varepsilon(x,t)}{\partial x^2}, \quad x>0, \tag{1.7}$$

$$\int_0^\infty x c_\varepsilon(x,t)dx \ = \ 1. \tag{1.8}$$

We also need to impose a boundary condition at $x=0$ to ensure that (1.7), (1.8) with given initial data $c_\varepsilon(x,0)=c_0(x)\ge 0, \ x>0$, satisfying the constraint (1.8) has a unique solution. We impose the Dirichlet boundary condition $c_\varepsilon(0,t)=0, \ t>$

0, because in this case the parameter $\Lambda_\varepsilon(t) > 0$ in (1.7) is given by the formula

$$\Lambda_\varepsilon(t) = \int_0^\infty xc_\varepsilon(x,t)dx \Big/ \int_0^\infty c_\varepsilon(x,t)dx. \tag{1.9}$$

Hence the diffusive CP model is an evolution equation for the pdf $c_\varepsilon(\cdot,t)/\int_0^\infty c_\varepsilon(x,t)\,dx$ of a random variable $X_{\varepsilon,t}$ and $\Lambda_\varepsilon(t) = \langle X_{\varepsilon,t}\rangle$. Furthermore, it is easy to see from (1.7), (1.8) that

$$\frac{d\Lambda_\varepsilon(t)}{dt} \ = \ \frac{\varepsilon}{2}\frac{\partial c_\varepsilon(0,t)}{\partial x} \Big/ \left[\int_0^\infty c_\varepsilon(x,t)dx\right]^2 . \tag{1.10}$$

It follows from (1.10) and the maximum principle [15] applied to (1.7) that the function $t \to \Lambda_\varepsilon(t)$ is increasing.

In [17] Smereka studied a discretized CP model and rigorously established a selection principle for arbitrary initial data with finite support. He also proved that the *rate of convergence* to the $\beta = 1$ self-similar solution (1.5) is *logarithmic* in time. Since discretization of a first order PDE introduces an effective diffusion, one can just as well apply the discretization algorithm of [17] to (1.7). In the discretized model time t is left continuous and the x discretization Δx is required to satisfy the condition $\varepsilon = 2\Delta x$. In [17] the large time behavior of solutions to this discretized model is studied by using a Fourier method. The Fourier method cannot be implemented if the assumption $\varepsilon = 2\Delta x$ is dropped. In §2 we show that the discretized model is a 2 dimensional dynamical system if and only if $\varepsilon = 2\Delta x$, and that this dynamics is associated with the unique 2 dimensional non-Abelian Lie algebra. This places the Smereka discrete model in the same category as the CP model (1.1), (1.2) and the quadratic model introduced in [7], which have also been shown to be 2 dimensional with 2 dimensional non-Abelian Lie algebra.

In §2 we begin by studying the CP model. If the initial data for (1.1), (1.2) is Gaussian, then it follows from [2] that the solution converges at large time to the $\beta = 1$ self-similar solution (1.5). We prove that the rate of convergence is logarithmic in time as in the Smereka model. Next we consider how to extend this result to the diffusive CP model (1.7), (1.8). We introduce a family of models with parameter ν, $0 \le \nu \le 1$, which interpolate between the diffusive CP model (1.7), (1.8) and a simpler model for which we can prove a selection principle. Each of these models is an evolution equation for the pdf of a non-negative random variable X_t, $t \ge 0$, in which the function $t \to \langle X_t\rangle$ is increasing. The evolution PDE is however now *non-linear* of viscous Burgers' type [9] with viscosity coefficient proportional to ν. The $\nu = 1$ model is identical to the diffusive CP model (1.7), (1.8), but the $\nu = 0$ model is not the same as the CP model (1.1), (1.2). We shall refer to it as the *inviscid* CP model since its evolution PDE is an inviscid Burgers' equation [18]. Similarly we refer to the model with $0 < \nu \le 1$ as the *viscous* CP model with viscosity ν.

In §3 we study the large time behavior of the inviscid CP model and obtain the following theorem:

Theorem 1.1. *Suppose the initial data for the inviscid CP model corresponds to the non-negative random variable X_0, and assume that X_0 satisfies*

$$\varepsilon < \langle X_0\rangle, \ \|X_0\|_\infty < \infty, \quad x \to E[X_0 - x \mid X_0 > x] \text{ is decreasing for } 0 \le x < \|X_0\|_\infty . \tag{1.11}$$

Then $\lim_{t\to\infty}\langle X_t\rangle/t = 1$. *If in addition the function* $x \to E[X_0 - x \mid X_0 > x]$ *is* C^1 *and convex for* x *close to* $\|X_0\|_\infty$ *with*

$$\liminf_{x\to\|X_0\|_\infty} \frac{\partial}{\partial x}\, \frac{1}{E[X_0 - x \mid X_0 > x]} \ > \ 0 \ , \tag{1.12}$$

then there exists $C, T > 0$ *such that*

$$1 - \frac{C}{\log t} \ \le \frac{d}{dt}\langle X_t\rangle \ \le \ 1 \quad \text{for } t \ge T \ . \tag{1.13}$$

Observe that a $\beta < 1$ self-similar solution (1.5) of the CP model has $\|X_0\|_\infty = 1/(1-\beta)$ and $E[X_0 - x \mid X_0 > x] = 1-(1-\beta)x,\ 0 \le x < \|X_0\|_\infty$. Hence the $\beta < 1$ self-similar solution satisfies all the conditions of Theorem 1.1 provided $\varepsilon < 1$. The condition $\varepsilon < 1$ is not crucial since for any $\varepsilon > 0$ one can rescale the initial data so that $\varepsilon < \langle X_0\rangle$. Therefore Theorem 1.1 proves a selection principle for the $\beta = 1$ self-similar solution (1.5) and establishes a rate of convergence which is logarithmic in time.

The remainder of the present paper is devoted to the study of the diffusive CP model (1.7), (1.8). Since existence and uniqueness has already been proven for a diffusive version of the LSW model [4] we do not revisit this issue, but concentrate on understanding large time behavior. In §7 we obtain the following:

Theorem 1.2. *Suppose the initial data for the diffusive CP model (1.7), (1.8) corresponds to the non-negative random variable* X_0 *with integrable pdf. Then* $\lim_{t\to\infty}\langle X_t\rangle = \infty$. *If in addition the function* $x \to E[X_0 - x \mid X_0 > x],\ 0 \le x < \|X_0\|_\infty$, *is decreasing, then there are constants* $C, T > 0$ *such that*

$$0 \ \le \frac{d}{dt}\langle X_t\rangle \ \le \ C \quad \text{for } t \ge T \ . \tag{1.14}$$

To establish the upper bound in (1.14) requires some delicate semi-classical analysis of the ratio of the Dirichlet Green's function for (1.7) on the half line $x > 0$ to the whole line Green's function. We carry this out in §5 by observing that the ratio of Green's functions is a probability for a generalized Brownian-bridge process, and obtaining a representation of the bridge process in terms of Brownian motion. In PDE terms this amounts to a boundary layer analysis of the solution to (1.7). To see why this is the case, observe that one can always rescale $\langle X_0\rangle$ to be equal to 1 in both the CP and diffusive CP models. Since the CP model is dilation invariant, the evolution PDE (1.1) remains the same. However for the diffusive CP model the diffusion coefficient in (1.7) changes from ε to $\varepsilon/\langle X_0\rangle$. Since $\lim_{t\to\infty}\langle X_t\rangle = \infty$ in the diffusive CP model, an analysis of large time behavior must therefore involve an analysis of solutions to (1.7), (1.8) as $\varepsilon \to 0$.

The simplest problem to understand concerning $\varepsilon \to 0$ behavior of the diffusive CP model is the problem of proving convergence to the solution of the CP model (1.1), (1.2) over some fixed time interval $0 \le t \le T$. Thus we assume that the CP and diffusive CP models have the same initial data corresponding to a random variable X_0. If $X_t,\ t > 0$, is the random variable corresponding to the solution of (1.1), (1.2) and $X_{\varepsilon,t},\ t > 0$, the random variable corresponding to the solution of (1.7), (1.8) then $X_{\varepsilon,t}$ converges in distribution as $\varepsilon \to 0$ to X_t, uniformly in the interval $0 \le t \le T$. Boundary layer analysis becomes necessary in proving the convergence of the diffusive coarsening rate (1.10) as $\varepsilon \to 0$ to the CP coarsening rate (1.4). In the diffusive model there exists a boundary layer with length of order ε so that $c_\varepsilon(x,t) \simeq c_0(x,t)$ for $x/\varepsilon >> 1$. Since $c_\varepsilon(0,t) = 0$ one has that $\partial c_\varepsilon(0,t)/\partial x \simeq 1/\varepsilon$, whence the RHS of (1.10) remains bounded above 0 as $\varepsilon \to 0$, and in fact converges to the RHS of (1.4).

In §6 we prove using the estimates of §5 the convergence results over a finite time interval described in the previous paragraph. Analogous results for the diffusive LSW model have already been proven in [4]. However the semi-classical estimates obtained in [4] to prove convergence are not strong enough to prove a uniform in time upper bound on the rate of coarsening as given in (1.14). Although there is a formal analogy between the problem of understanding large time behavior of the diffusive CP model and the problem of $\varepsilon \to 0$ convergence of the diffusive CP model over a fixed time interval, the former problem is considerably more difficult than the latter.

2. The Carr-Penrose Model and Extensions

The analysis of the CP model [2] is based on the fact that the characteristics for the first order PDE (1.1) can be easily computed. Thus let $b : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$ be given by $b(y,s) = A(s)y - 1,\ y \in \mathbf{R},\ s \geq 0$, where $A : \mathbf{R}^+ \to \mathbf{R}^+$ is a continuous non-negative function. We define the mapping $F_A : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$ by setting

$$F_A(x,t) = y(0), \quad \text{where } \frac{dy(s)}{ds} = b(y(s),s),\ 0 \leq s \leq t, \quad y(t) = x. \tag{2.1}$$

From (2.1) we see that the function F_A is given by the formula

$$F_A(x,t) = \frac{x + m_{2,A}(t)}{m_{1,A}(t)}\ , \quad \text{where} \tag{2.2}$$

$$m_{1,A}(t) = \exp\left[\int_0^t A(s)\ ds\right]\ , \quad m_{2,A}(t) = \int_0^t \exp\left[\int_s^t A(s')\ ds'\right]\ ds\ .$$

If we let $w_0 : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be the function

$$w_0(x,t) \ = \ \int_x^\infty c_0(x',t)\ dx'\ , \quad x,t \geq 0, \tag{2.3}$$

where $c_0(\cdot,\cdot)$ is the solution to (1.1), then from the method of characteristics we have that

$$w_0(x,t) \ = \ w_0(F_{1/\Lambda_0}(x,t),0)\ , \quad x,t \geq 0. \tag{2.4}$$

The conservation law (1.2) can also be expressed in terms of w_0 as

$$\int_0^\infty w_0(x,t)\ dx \ = \ \int_0^\infty w_0(F_{1/\Lambda_0}(x,t),0)\ dx \ = \ 1. \tag{2.5}$$

Observe now that the functions $m_{1,A},\ m_{2,A}$ of (2.2) are related by the differential equation

$$\frac{d}{dt}\left[\frac{m_{2,A}(t)}{m_{1,A}(t)}\right] \ = \ \frac{1}{m_{1,A}(t)}\ . \tag{2.6}$$

It follows from (2.5), (2.6) that if we define variables $[u(t),v(t)],\ t \geq 0$, by

$$u(t) \ = \ \frac{1}{m_{1,1/\Lambda_0}(t)}\ , \quad v(t) \ = \ \frac{m_{2,1/\Lambda_0}(t)}{m_{1,1/\Lambda_0}(t)}\ , \tag{2.7}$$

then the CP model (1.1), (1.2) with given initial data $c_0(\cdot,0)$ is equivalent to the 2 dimensional dynamical system

$$\frac{dv(t)}{dt} = u(t) \ , \quad \frac{d}{dt}\int_0^\infty w_0\left(u(t)x+v(t),0\right)\ dx \ = \ 0. \tag{2.8}$$

Note however that the *dynamical law* for the 2 dimensional evolution depends on the initial data for (1.1), (1.2), whereas the initial condition is always $u(0)=1,\ v(0)=0$.

We can understand the 2 dimensionality of the CP model and relate it to some other models of coarsening by using some elementary Lie algebra theory. Thus observe that for operators $\mathcal{A}_0,\mathcal{B}_0$ defined by

$$\mathcal{A}_0 \ = \ \frac{d}{dx} \ , \quad \mathcal{B}_0 = \frac{d}{dx}x \ , \quad \text{then } \mathcal{A}_0\mathcal{B}_0-\mathcal{B}_0\mathcal{A}_0 \ = \ \mathcal{A}_0 \ . \tag{2.9}$$

The initial value problem (1.1) can be written in operator notation as

$$\frac{\partial c_0(\cdot,t)}{dt} \ = \ \left[\mathcal{A}_0-\frac{\mathcal{B}_0}{\Lambda_0(t)}\right]c_0(\cdot,t) \text{ for } t>0, \quad c_0(\cdot,0) \ = \ \text{given}. \tag{2.10}$$

It follows from (2.9) that the Lie Algebra generated by $\mathcal{A}_0,\mathcal{B}_0$ is the unique two dimensional non-Abelian Lie algebra. The corresponding 2 dimensional Lie group is the affine group of the line (see Chapter 4 of [19]). That is the Lie group consists of all transformations $z\to az+b,\ z\in\mathbf{R}$, with $a>0,\ b\in\mathbf{R}$. The solutions of equation (2.10) are a flow on this group. Hence solutions of (2.10) for all possible functions $\Lambda_0(\cdot)$ lie on a two dimensional manifold.

Next we consider the discretized version of the CP model studied by Smereka [17]. Letting Δx denote space discretization, then a standard discretization of (1.7) with Dirichlet boundary condition is given by

$$\begin{aligned} &\frac{\partial c_\varepsilon(x,t)}{\partial t}+\frac{J_\varepsilon(x,t)-J_\varepsilon(x-\Delta x,t)}{\Delta x} \\ &= \ \frac{\varepsilon}{2}\frac{c_\varepsilon(x+\Delta x,t)+c_\varepsilon(x-\Delta x,t)-2c_\varepsilon(x,t)}{(\Delta x)^2} \ , \quad x=(n+1)\Delta x, \quad n=0,1,2,.., \end{aligned} \tag{2.11}$$

where

$$J_\varepsilon(x,t) \ = \ \left[\frac{x}{\Lambda_\varepsilon(t)}-1\right]c_\varepsilon(x,t) \ , \quad c_\varepsilon(0,t) \ = \ 0. \tag{2.12}$$

The backward difference approximation for the derivative of $J_\varepsilon(x,t)$ is chosen in (2.11) to ensure stability of the numerical scheme for large x. Let D,D^* be the discrete derivative operators acting on functions $u:(\Delta x)\mathbf{Z}\to\mathbf{R}$ defined by

$$Du(x) \ = \ \frac{u(x+\Delta x)-u(x)}{\Delta x} \ , \quad D^*u(x) \ = \ \frac{u(x-\Delta x)-u(x)}{\Delta x} \ . \tag{2.13}$$

Then using the notation of (2.13) we can rewrite (2.11) as

$$\frac{\partial c_\varepsilon(x,t)}{\partial t}-D^*J_\varepsilon(x,t) \ = \ \frac{\varepsilon}{2\Delta x}\left[(D+D^*)c_\varepsilon(x,t)\right] \ . \tag{2.14}$$

Observe that for operators $\mathcal{A}_{\Delta x},\mathcal{B}_{\Delta x}$ defined by

$$\mathcal{A}_{\Delta x} \ = \ D \ , \quad \mathcal{B}_{\Delta x}=-D^*x \ , \quad \text{then } \mathcal{A}_{\Delta x}\mathcal{B}_{\Delta x}-\mathcal{B}_{\Delta x}\mathcal{A}_{\Delta x} \ = \ \mathcal{A}_{\Delta x} \ . \tag{2.15}$$

Choosing $\varepsilon = 2\Delta x$ in (2.14), we see that the equation can be expressed in terms of $\mathcal{A}_{\Delta x}, \mathcal{B}_{\Delta x}$ as

$$\frac{\partial c_\varepsilon(\cdot,t)}{\partial t} \;=\; \left[\mathcal{A}_{\Delta x} - \frac{\mathcal{B}_{\Delta x}}{\Lambda_\varepsilon(t)}\right] c_\varepsilon(\cdot,t) \; . \tag{2.16}$$

Comparing (2.9), (2.10) to (2.15), (2.16), we see that we can obtain a representation for the solution to (2.15), (2.16) by using the fact that the solution to (2.9), (2.10) is given by (2.4). To see this we use the fact that for $\mathcal{A}_0, \mathcal{B}_0$ as in (2.9) then

$$e^{\mathcal{A}_0 s} f(x) \;=\; f(x+s) \; , \quad e^{\mathcal{B}_0 s} f(x) \;=\; e^s f(e^s x) \; , \quad x, s \in \mathbf{R}. \tag{2.17}$$

From (2.4), (2.7), (2.17) it follows that the solution to (2.9), (2.10) is given by

$$c_0(\cdot,t) \;=\; u(t)^{\mathcal{B}_0} e^{\mathcal{A}_0 v(t)} c_0(\cdot,0) \; , \tag{2.18}$$

where $u(t), v(t)$ are given by (2.7). Hence the solution to (2.15), (2.16) is given by

$$c_\varepsilon(\cdot,t) \;=\; u(t)^{\mathcal{B}_{\Delta x}} e^{\mathcal{A}_{\Delta x} v(t)} c_\varepsilon(\cdot,0) \; , \tag{2.19}$$

where $\mathcal{A}_{\Delta x}, \mathcal{B}_{\Delta x}$ are given by (2.15), and $u(t), v(t)$ are given by (2.7) with Λ_ε in place of Λ_0.

The operator $\mathcal{A}_{\Delta x}$ of (2.15) is the generator of a Poisson process. Thus

$$\int_{(\Delta x)\mathbf{Z}} dx \; g(x) e^{\mathcal{A}_{\Delta x} s} f(x) \;=\; \int_{(\Delta x)\mathbf{Z}} dx \; E[g(X_s) \mid X_0 = x] f(x) \; , \tag{2.20}$$

where X_s is the discrete random variable taking values in $(\Delta x)\mathbf{Z}$ with pdf

$$P\left(X_s = y \;\middle|\; X_0 = x\right) \;=\; \frac{(s/\Delta x)^n}{n!} \exp\left[-\frac{s}{\Delta x}\right] \; , \quad n \;=\; \frac{x-y}{\Delta x} \; , \; n = 0,1,2,... \tag{2.21}$$

If $f, g : \mathbf{R} \to \mathbf{R}$ are continuous functions of compact support then it is easy to see from (2.21) that

$$\lim_{\Delta x \to 0} \int_{(\Delta x)\mathbf{Z}} dx \; g(x) e^{\mathcal{A}_{\Delta x} s} f(x) \;=\; \int_{-\infty}^{\infty} dx \; g(x-s) f(x) \; , \tag{2.22}$$

as we expect from (2.17). The operator $\mathcal{B}_{\Delta x}$ of (2.15) is the generator of a Yule process [11]. Thus

$$\int_{(\Delta x)\mathbf{Z}} dx \; g(x) e^{-\mathcal{B}_{\Delta x} s} f(x) \;=\; \int_{(\Delta x)\mathbf{Z}} dx \; E[g(Y_s) \mid Y_0 = x] f(x) \; , \tag{2.23}$$

where Y_s is a discrete random variable taking values in $(\Delta x)\mathbf{Z}$. The pdf of Y_s conditioned on $Y_0 = \Delta x$ is given by

$$P\left(Y_s = y \;\middle|\; Y_0 = \Delta x\right) \;=\; e^{-s} \left\{1 - e^{-s}\right\}^{n-1} \; , \quad n \;=\; \frac{y}{\Delta x} \; , \; n = 1,2,... \tag{2.24}$$

Hence Y_s conditioned on $Y_0 = \Delta x$ is a geometric variable. More generally, the variable Y_s conditioned on $Y_0 = m\Delta x$ with $m \ge 2$ is a sum of m independent geometric variables with distribution (2.24) and is hence negative binomial. It follows that if $f(\cdot)$ is supported in the set $\{x \in (\Delta x)\mathbf{Z} \; : \; x > 0\}$ then

$$\int_{(\Delta x)\mathbf{Z}} dx \; g(x) e^{-\mathcal{B}_{\Delta x} s} f(x) \;=\; \int_{(\Delta x)\mathbf{Z}} dx \; E\left[g\left(Y_s^1 + \cdots + Y_s^m\right) \;\middle|\; m = x/\Delta x\right] f(x) \; , \tag{2.25}$$

where the Y_s^j, $j = 1, 2, ..,$ are independent and have the distribution (2.24). Since the mean of Y_s is $e^s \Delta x$, it follows from (2.25) that

$$\lim_{\Delta x \to 0} \int_{(\Delta x)\mathbf{Z}} dx\ g(x) e^{-\mathcal{B}_{\Delta x} s} f(x) \ = \ \int_0^\infty dx\ g(e^s x) f(x) \ , \tag{2.26}$$

as we expect from (2.17).

The Smereka model consists of the evolution determined by (2.11) with $\varepsilon = 2\Delta x$ and the conservation law

$$\int_{\{(\Delta x)\mathbf{Z} \ : \ x>0\}} x c_\varepsilon(x,t)\ dx \ = \ 1. \tag{2.27}$$

We see from (2.19) that the model is equivalent to a two dimensional dynamical system with dynamical law depending on the initial data. The first differential equation in this system is given by the first equation in (2.8). The second differential equation is determined by differentiating the expression on the LHS of (2.27) and setting it equal to zero. Using (2.21), (2.24) we can write the LHS of (2.27) in terms of $u(t), v(t)$. In the case when the initial data is given by

$$c_\varepsilon(x,0) \ = \ 0 \text{ if } x \neq \Delta x \ , \quad c_\varepsilon(\Delta x, 0) \ = \ \frac{1}{(\Delta x)^2} \ , \tag{2.28}$$

it has a simple form. Thus from (2.19), (2.21), (2.24) we have that

$$c_\varepsilon(\cdot,t) \ = \ u(t)^B e^{Av(t)} c_\varepsilon(\cdot,0) \ = \ u(t)^B \exp\left[-\frac{v(t)}{\Delta x}\right] c_\varepsilon(\cdot,0) \ , \tag{2.29}$$

$$\text{so} \ \ c_\varepsilon(x,t) \ = \ u(t)\left[1-u(t)\right]^{n-1} \exp\left[-\frac{v(t)}{\Delta x}\right] \frac{1}{(\Delta x)^2} \ , \quad n = \frac{x}{\Delta x} \ .$$

From (2.29) we see that the conservation law (2.27) becomes in this case

$$\frac{1}{u(t)} \exp\left[-\frac{v(t)}{\Delta x}\right] \ = \ 1. \tag{2.30}$$

Hence from the first equation of (2.8) and (2.30) we conclude that $v(\cdot)$ is the solution to the initial value problem

$$\exp\left[-\frac{v(t)}{\Delta x}\right] \ = \ \frac{dv(t)}{dt}, \quad v(0) = 0. \tag{2.31}$$

The initial value problem (2.31) was derived in §3 of [17] by a different method. It can be solved explicitly, and so we obtain the formulas

$$u(t) \ = \ \frac{1}{1 + t/\Delta x} \ , \quad v(t) \ = \ \Delta x \log\left[1 + \frac{t}{\Delta x}\right] \ , \tag{2.32}$$

when the initial data is given by (2.28). Hence from (2.29), (2.32) we have an explicit expression for $c_\varepsilon(\cdot,t)$, and it is easy to see that this converges as $t \to \infty$ to the self-similar solution corresponding to the $\beta = 1$ random variable defined by (1.5). It was also shown in [17] that if the initial data has finite support then $c_\varepsilon(\cdot,t)$ converges as $t \to \infty$ to the $\beta = 1$ self-similar solution.

The large time behavior of the CP model can be easily understood using the *beta function* of a random variable introduced in [5]. If X is a random variable with pdf $c_X(\cdot)$, we define functions $w_X(\cdot)$, $h_X(\cdot)$ by

$$w_X(x) \ = \int_x^\infty c_X(x')\ dx' \ , \quad h_X(x) \ = \ \int_x^\infty w_X(x')\ dx' \ , \quad x \in \mathbf{R} \ . \tag{2.33}$$

Evidently one has that

$$(2.34) \qquad w_X(x) \ = \ P(X > x), \qquad \frac{h_X(x)}{w_X(x)} \ = \ E\left[X - x \ \middle|\ X > x\right] \qquad x \in \mathbf{R} \ .$$

The beta function $\beta_X(\cdot)$ of X is then defined by

$$(2.35) \qquad \beta_X(x) \ = \ \frac{c_X(x)h_X(x)}{w_X(x)^2} \ = \ 1 + \frac{d}{dx}E\left[X - x \ \middle|\ X > x\right] \qquad x \in \mathbf{R} \ .$$

An important property of the beta function is that it is invariant under affine transformations. That is

$$(2.36) \qquad \beta_X(\lambda x + \mu) \ = \ \beta_{(X-\mu)/\lambda}(x) \ , \quad \lambda > 0, \ \mu, x \in \mathbf{R}.$$

One can also see that the function $h_X(\cdot)$ is *log concave* if and only if $\sup \beta_X(\cdot) \le 1$.

To understand the large time behavior of the CP model we first observe that the rate of coarsening equation (1.4) can be rewritten as

$$(2.37) \qquad \frac{d}{dt}\langle X_t\rangle \ = \ \beta_{X_t}(0) \ , \quad t > 0.$$

Furthermore, the beta function of the self-similar variable X_0 with pdf defined by (1.5) and parameter $\beta > 0$ is simply a constant $\beta_{X_0}(\cdot) \equiv \beta$. We have already shown that the time evolution of the CP equation (1.1) is given by the affine transformation (2.4). It is also relatively simple to establish that for a random variable X_0 corresponding to the initial data for (1.1), (1.2), then $\lim_{t\to\infty} F_{1/\Lambda_0}(0,t) = \|X_0\|_\infty$. Hence it follows from (2.35), (2.36) that if $\lim_{x\to\|X_0\|_\infty} \beta_{X_0}(x) = \beta$ for the initial data random variable X_0 of (1.1), (1.2), then the large time behavior of the CP model is determined by the self-similar solution (1.5) with parameter β.

We have already observed from (1.4) that the function $\Lambda_0(\cdot)$ in the CP model is increasing. If we assume that $\inf \beta_{X_0}(\cdot) > 0$, we can also see that $\lim_{t\to\infty} \Lambda_0(t) = \infty$. Hence in this case there exists a doubling time $T_{\rm double}$ for which $\Lambda_0(t) = 2\Lambda_0(0)$ when $t = T_{\rm double}$. Evidently $\inf \beta_{X_t}(\cdot) \ge \inf \beta_{X_0}(\cdot)$ and $\sup \beta_{X_t}(\cdot) \le \sup \beta_{X_0}(\cdot)$. The notion of doubling time can be a useful tool in obtaining an estimate on the rate of convergence of the solution of the CP model to a self-similar solution at large time.

We illustrate this by considering the CP model with Gaussian initial data. In particular we assume the initial data $c_0(\cdot)$ is given by the formula

$$(2.38) \qquad c_0(x) \ = \ K(L)\exp\left[-a(L)x - \{a(L)x\}^2/2L\right] \ ,$$

where $L \ge L_0 > 0$ and $K(L), a(L)$ are uniquely determined by the requirement that (1.2) holds and the function $\Lambda_0(\cdot)$ in (1.3) satisfies $\Lambda_0(0) = 1$. It is easy to see that the beta function for the initial data (2.38) is bounded above and below strictly larger than zero, uniformly in $L \ge L_0$. Hence from (2.37) there are constants $T_1, T_2 > 0$ depending only on L_0 such that $T_1 \le T_{\rm double} \le T_2$ for all $L \ge L_0$. It follows also from (2.2) that there are constants $\lambda_0, \lambda_1, \mu_0, \mu_1 > 0$ depending only on L_0 such that $F_{1/\Lambda_0}(x, T_{\rm double}) = \lambda(L)x + \mu(L), \ x \in \mathbf{R}$, where $0 < \lambda_0 \le \lambda(L) \le \lambda_1 < 1$ and $0 < \mu_0 \le \mu(L) \le \mu_1$ for $L \ge L_0$. Since F_{1/Λ_0} is a linear function, $c(\cdot, T_{\rm double})$ is also Gaussian. Rescaling so that the mean of X_t is now 1 at time $T_{\rm double}$, we see that $c(x, T_{\rm double})$ is given by the formula (2.38) with L replaced by $A(L)$, where

$$(2.39) \qquad A(L) \ = \ L[1 + \mu(L)a(L)/L]^2 = L + 2\mu(L)a(L) + \mu(L)^2a(L)^2/L \ .$$

Since we are assuming that $\Lambda_0(0) = 1$, there are constants $a_0, a_1 > 0$ depending only on L_0 such that $a_0 \le a(L) \le a_1$ for $L \ge L_0$. We conclude then from (2.39) that

$$L + \delta_0 \ \le A(L) \ \le L + \delta_1 \ , \quad \text{where } \delta_0, \delta_1 > 0 \text{ depend only on } L_0 \ . \tag{2.40}$$

It is easy to estimate from (2.40) the rate of convergence to the $\beta = 1$ self similar solution $c(x,t) = (1+t)^{-2}\exp[-x/(1+t)]$ for solutions to the CP model with Gaussian initial data. First we estimate the beta function of a Gaussian random variable.

Lemma 2.1. *Let $L > 0$ and Z_L be a positive random variable with pdf proportional to $e^{-z-z^2/2L}$, $z > 0$. Then for any $L_0 > 0$ there is a constant C depending only on L_0 such that if $L \ge L_0$ the beta function β_L for Z_L satisfies the inequality*

$$\left| \ \beta_L(z) - 1 + \frac{1}{L(1+z/L)^2} \ \right| \ \le \ \frac{C}{L^2(1+z/L)^4} \quad \text{for } z \ge 0. \tag{2.41}$$

Proof. We use the formula for the beta function $\beta(\cdot)$ of the pdf $c(\cdot)$ given by (2.35). Thus

(2.42)
$$\beta(z) \ = \ \frac{c(z)h(z)}{w(z)^2} \ , \quad w(z) \ = \ \int_0^\infty c(z+z') \ dz' \ , \quad h(z) \ = \ \int_0^\infty z'c(z+z') \ dz' \ .$$

Letting $c(z) = e^{-z-z^2/2L}$, we have that

(2.43)
$$w(z) \ = \ c(z)\int_0^\infty e^{-z'[1+z/L]-z'^2/2L} \ dz' \ , \quad h(z) \ = \ c(z)\int_0^\infty z'e^{-z'[1+z/L]-z'^2/2L} \ dz' \ .$$

It follows from (2.42), (2.43) on making a change of variable that

$$\beta_L(z) \ = \ \int_0^\infty xe^{-x-\delta x^2/2} \ dx \ \Big/ \left[\int_0^\infty e^{-x-\delta x^2/2} \ dx \ \right]^2 \ , \tag{2.44}$$

where $\delta = 1/L[1+z/L]^2$. It is easy to see that there is a universal constant K such that the RHS of (2.44) is bounded above by K for all $\delta > 0$. We also have by Taylor expansion in δ that $\beta_L(z) = 1 - \delta + O(\delta^2)$ if δ is small. The inequality (2.41) follows. □

Proposition 2.1. *Let $c_0(\cdot,\cdot)$ be the solution to the CP system (1.1), (1.2) with Gaussian initial data and $\Lambda_0(\cdot)$ be given by (1.3). Then there exists $t_0 > 2$ and constants $C_1, C_2 > 0$ such that*

$$1 - \frac{C_1}{\log t} \ \le \ \frac{d\Lambda_0(t)}{dt} \ \le \ 1 - \frac{C_2}{\log t} \quad \text{for} \quad t \ge t_0. \tag{2.45}$$

Proof. The initial data can be written in the form $c_0(x,0) = K_0\exp[-A_0(x+B_0)^2]$ where K_0, A_0, B_0 are constants with $K_0, A_0 > 0$. It follows from (2.2), (2.4) that for $t > 0$ one has $c_0(x,t) = K_t\exp[-A_t(x+B_t)^2]$, where $B_t = [B_0 + F_{1/\Lambda_0}(0,t)](A_0/A_t)^{1/2}$. Since $\lim_{t\to\infty} F_{1/\Lambda_0}(0,t) = \infty$, it follows that we may assume wlog that the initial data is of the form (2.38) and $\Lambda_0(0) = 1$. Evidently then $\beta_{X_0}(x) = \beta_L(a(L)x)$, $x \ge 0$, where β_L satisfies the inequality (2.41).

Assume now that the initial data for (1.1), (1.2) is given by (2.38) where $L = L_0 > 0$, and let L_t be the corresponding value of L determined by $c_0(\cdot,t)$. We have then from (2.40) and the discussion preceding it that for $N = 1, 2, ..,$ there exist times t_N such that

$$[2^N - 1]T_1 \le \ t_N \le [2^N - 1]T_2, \quad L_0 + N\delta_0 \le L_{t_N} \le L_0 + N\delta_1. \tag{2.46}$$

Since L_t is an increasing function of t, it follows from (2.41), (2.46) that $\beta_{X_t}(0)$ is bounded above and below as in (2.45). Now using the identity (2.37) we obtain the inequality (2.45). □

We wish next to compare the foregoing to the situation of the diffusive CP model with Gaussian initial data. From (4.15) the solution to (1.7) with initial data $c_0(\cdot)$ is given by

$$c_\varepsilon(x,t) \ = \ \int_0^\infty G_{\varepsilon,D}(x,y,0,t)c_0(y) \ dy, \quad x>0, \ t>0, \tag{2.47}$$

where $G_{\varepsilon,D}$ is the Dirichlet Green's function for the half space $\mathbf{R}^+$ defined by (4.14) with $A(s) = 1/\Lambda_\varepsilon(s)$. If we replace the Dirichlet Green's function $G_{\varepsilon,D}$ by the full space Green's function G_ε of (4.11) then the solution $c_\varepsilon(\cdot,t)$ is Gaussian for $t>0$ provided $c_\varepsilon(\cdot,0)$ is Gaussian, just as in the CP model. We shall see in §5 that it is legitimate to approximate $G_{\varepsilon,D}(x,y,0,t)$ by $G_\varepsilon(x,y,0,t)$ provided $x,y \ge M\varepsilon$ for some large constant M. Making the approximation $G_{\varepsilon,D} \simeq G_\varepsilon$ in (2.47), we obtain a formula similar to (2.39) for the length scale $A_\varepsilon(L)$ of the Gaussian at doubling time. It is given by

$$A_\varepsilon(L) \ = \ L[1+\mu_\varepsilon(L)a(L)/L]^2[1+\varepsilon a(L)^2\sigma_\varepsilon^2(L)\lambda_\varepsilon(L)^2/L]^{-1} \ . \tag{2.48}$$

As in (2.39) the functions $\lambda_\varepsilon(L), \mu_\varepsilon(L)$ are obtained from the coefficients of the linear function $F_{1/\Lambda_\varepsilon}$, when $t = T_{\varepsilon,\rm double}$, where $T_{\varepsilon,\rm double}$ denotes the doubling time for the diffusive model. The expression $\sigma_\varepsilon^2(L)$ is given by the formula for $\sigma_A^2(T)$ in (4.10) with $A(s) = 1/\Lambda_\varepsilon(s), \ s \le T$, and $T = T_{\varepsilon,\rm double}$. In §6 we shall study the $\varepsilon \to 0$ limit of the diffusive CP model. We prove that if the CP and diffusive CP models have the same initial data, then $\lim_{\varepsilon\to 0}\Lambda_\varepsilon(t) = \Lambda_0(t)$ uniformly in any finite interval $0 \le t \le T$. It follows that $\lim_{\varepsilon\to 0} A_\varepsilon(L) = A(L)$, where $A(L)$ is defined by (2.39).

We wish next to try to understand the evolution of the diffusive CP model when initial data is non-Gaussian. Let $w_\varepsilon(x,t), \ h_\varepsilon(x,t)$ be defined in terms of the solution $c_\varepsilon(\cdot,\cdot)$ to (1.7) by

$$w_\varepsilon(x,t) \ = \ \int_x^\infty c_\varepsilon(x',t) \ dx', \quad h_\varepsilon(x,t) \ = \ \int_x^\infty w_\varepsilon(x',t) \ dx' \ . \tag{2.49}$$

Then $w_\varepsilon(\cdot,t), h_\varepsilon(\cdot,t)$ are proportional to the functions (2.33) corresponding to the random variable X_t with pdf $c_\varepsilon(\cdot,t)/\int_0^\infty c_\varepsilon(x,t) \ dx$. Making the approximation $G_{\varepsilon,D} \simeq G_\varepsilon$, we see from (2.47), (2.49), (4.11) that

$$w_\varepsilon(x,t) \quad = \quad \exp\left[\int_0^t \frac{ds}{\Lambda_\varepsilon(s)}\right]\int_{-\infty}^\infty G_\varepsilon(x,y,0,t)w_\varepsilon(y,0) \ dy \ , \tag{2.50}$$

$$h_\varepsilon(x,t) \quad = \quad \exp\left[2\int_0^t \frac{ds}{\Lambda_\varepsilon(s)}\right]\int_{-\infty}^\infty G_\varepsilon(x,y,0,t)h_\varepsilon(y,0) \ dy \ . \tag{2.51}$$

Writing $h_\varepsilon(x,t) = \exp[-q_\varepsilon(x,t)]$, $x,t>0$, in (2.51), we see from (4.11) that the semi-classical approximation to $q_\varepsilon(x,t)$ is given by the formula
(2.52)

$$q_\varepsilon(x,t) \ = \ \frac{1}{2}\log 2\pi\varepsilon+\frac{1}{2}\log\sigma^2_{1/\Lambda_\varepsilon}(t)-2\int_0^t\frac{ds}{\Lambda_\varepsilon(s)}+\inf_y\left[\frac{\{x+m_{2,1/\Lambda_\varepsilon}(t)-m_{1,1/\Lambda_\varepsilon}(t)y\}^2}{2\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)}+q_\varepsilon(y,0)\right] .$$

Let us assume that $q_\varepsilon(\cdot,0)$ is given similarly to (2.38) by

$$q_\varepsilon(y,0) \ = \ \text{constant}+a(L)y+\{a(L)y\}^2/2L \ , \quad y>0. \tag{2.53}$$

The minimizer in (2.52) is then $y_{\min}(x,t)$ where
(2.54)

$$y_{\min}(x,t) \ = \ \frac{1}{1+\varepsilon a(L)^2\sigma^2_{1/\Lambda_\varepsilon}(t)/m_{1,1/\Lambda_\varepsilon}(t)^2L}\left[\frac{x+m_{2,1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)}-\frac{\varepsilon a(L)\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\right] .$$

If we substitute $y=y_{\min}(x,t)$ into (2.52) we obtain a quadratic formula for $q_\varepsilon(x,t)$ similar to (2.53). If $t=T_{\varepsilon,\text{double}}$ then L in (2.53) is replaced by $A_\varepsilon(L)$ as in (2.48).

More generally we can consider the case when $q_\varepsilon(\cdot,0)$ is convex so (2.52) is a convex optimization problem with a unique minimizer $y=y_{\min}(x,t)$. In that case it is easy to see that

$$\frac{\partial q_\varepsilon(x,t)}{\partial x} \ = \ \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)}\frac{\partial q_\varepsilon(y_{\min}(x,t),0)}{\partial y} \ , \tag{2.55}$$

$$\frac{\partial^2 q_\varepsilon(x,t)}{\partial x^2} \ = \ \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial^2 q_\varepsilon(y_{\min}(x,t),0)}{\partial y^2}\Bigg/\left[1+\frac{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\ \frac{\partial^2 q_\varepsilon(y_{\min}(x,t),0)}{\partial y^2}\right] .$$

It follows from (2.55) that if the inequality

$$\frac{\partial^2 q_\varepsilon(x,t)}{\partial x^2} \ \le \ \left[\frac{\partial q_\varepsilon(x,t)}{\partial x}\right]^2 \ , \quad x\ge 0, \tag{2.56}$$

holds at $t=0$ then it holds for all $t>0$. We define now the function β_ε : $[0,\infty)\times\mathbf{R}^+\to\mathbf{R}$ in terms of q_ε by the formula

$$\beta_\varepsilon(x,t) \ = \ 1-\frac{\partial^2 q_\varepsilon(x,t)}{\partial x^2}\Bigg/\left[\frac{\partial q_\varepsilon(x,t)}{\partial x}\right]^2 . \tag{2.57}$$

We can see from (2.35) that the function $h_\varepsilon(\cdot,t)=\exp[-q_\varepsilon(\cdot,t)]$ is proportional to $h_{X_t}(\cdot)$ for some random variable X_t if and only if $\beta_\varepsilon(\cdot,t)$ is non-negative. Hence by the remark after (2.56), if $q_\varepsilon(\cdot,0)$ corresponds to a random variable X_0, then $q_\varepsilon(\cdot,t)$ corresponds to a random variable X_t for all $t>0$. From (2.55), (2.57) we have that
(2.58)

$$1-\beta_\varepsilon(x,t) \ = \ [1-\beta_\varepsilon(y_{\min}(x,t),0)]\Bigg/\left[1+\frac{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\ \frac{\partial^2 q_\varepsilon(y_{\min}(x,t),0)}{\partial y^2}\right] .$$

It follows from (2.58) that if $\sup\beta_\varepsilon(\cdot,0)\le 1$ then $\sup\beta_\varepsilon(\cdot,t)\le 1$ for $t>0$. Furthermore, (2.58) also indicates that $\beta_\varepsilon(\cdot,t)$ should increase towards 1 as $t\to\infty$.

It is well known [9] that the solution q_ε to the optimization problem (2.52) satisfies a Hamilton-Jacobi PDE. We can easily see from (2.52), (2.55) that the

PDE is given by

$$\frac{\partial q_\varepsilon(x,t)}{\partial t}+\left[\frac{x}{\Lambda_\varepsilon(t)}-1\right]\frac{\partial q_\varepsilon(x,t)}{\partial x}+\frac{\varepsilon}{2}\left[\frac{\partial q_\varepsilon(x,t)}{\partial x}\right]^2+\frac{1}{\Lambda_\varepsilon(t)}-\frac{1}{2\sigma^2(t)} \;=\; 0 \; . \tag{2.59}$$

Differentiating (2.59) with respect to x and setting $v_\varepsilon(x,t)=\partial q_\varepsilon(x,t)/\partial x$, we see that $v_\varepsilon(x,t)$ is the solution to the inviscid Burgers' equation with linear drift,

$$\frac{\partial v_\varepsilon(x,t)}{\partial t}+\left[\frac{x}{\Lambda_\varepsilon(t)}-1+\varepsilon v_\varepsilon(x,t)\right]\frac{\partial v_\varepsilon(x,t)}{\partial x}+\frac{v_\varepsilon(x,t)}{\Lambda_\varepsilon(t)} \;=\; 0 \; . \tag{2.60}$$

If $q_\varepsilon(\cdot,t)$ corresponds to the random variable X_t, then $v_\varepsilon(x,t)=E[X_t-x \mid X_t > x]^{-1},\ x\ge 0$, and since $\Lambda_\varepsilon(t)=E[X_t]$ we have that

$$v_\varepsilon(0,t) \;=\; \frac{1}{\Lambda_\varepsilon(t)} \; . \tag{2.61}$$

The system (2.60), (2.61) is a model for the evolution of the pdf of a random variable X_t which is intermediate between the CP and diffusive CP models. To obtain the pdf of X_t from the function $v_\varepsilon(\cdot,t)$, we let $c_\varepsilon(\cdot,t)=c_{X_t}(\cdot),\ w_\varepsilon(\cdot,t)=w_{X_t}(\cdot),\ h_\varepsilon(\cdot,t)=h_{X_t}(\cdot)$ as in (2.33). Then $v_\varepsilon(x,t)=w_\varepsilon(x,t)/h_\varepsilon(x,t)$ and

$$\Gamma_\varepsilon(x,t) \;=\; v_\varepsilon(x,t)^2-\frac{\partial v_\varepsilon(x,t)}{\partial x} \;=\; \frac{c_\varepsilon(x,t)}{h_\varepsilon(x,t)} \; . \tag{2.62}$$

We also have that

$$v_\varepsilon(x,t) \;=\; -\frac{\partial}{\partial x}\log h_\varepsilon(x,t) \; , \quad \text{whence } h_\varepsilon(x,t) \;=A_\varepsilon(t)\exp\left[-\int_0^x v_\varepsilon(x',t)\,dx'\right] \; , \tag{2.63}$$

where $A_\varepsilon(\cdot)$ can be an arbitrary positive function. Evidently (2.62), (2.63) uniquely determine the pdf of X_t from the function $v_\varepsilon(\cdot,t)$.

We can do a more systematic derivation of the model (2.60), (2.61) by beginning with the solution c_ε to the diffusive CP model (1.7), (1.8). Setting $w_\varepsilon, h_\varepsilon$ to be given by (2.49), then we see on integration of (1.7) that w_ε is a solution to the PDE

$$\frac{\partial w_\varepsilon(x,t)}{\partial t}+\left[\frac{x}{\Lambda_\varepsilon(t)}-1\right]\frac{\partial w_\varepsilon(x,t)}{\partial x} \;=\; \frac{\varepsilon}{2}\frac{\partial^2 w_\varepsilon(x,t)}{\partial x^2} \; . \tag{2.64}$$

If we integrate (2.64) then we obtain a PDE for h_ε,

$$\frac{\partial h_\varepsilon(x,t)}{\partial t}+\left[\frac{x}{\Lambda_\varepsilon(t)}-1\right]\frac{\partial h_\varepsilon(x,t)}{\partial x}-\frac{h_\varepsilon(x,t)}{\Lambda_\varepsilon(t)} \;=\; \frac{\varepsilon}{2}\frac{\partial^2 h_\varepsilon(x,t)}{\partial x^2} \; . \tag{2.65}$$

Setting $h_\varepsilon(x,t)=\exp[-q_\varepsilon(x,t)]$, it follows from (2.65) that $q_\varepsilon(x,t)$ is a solution to the PDE

$$\frac{\partial q_\varepsilon(x,t)}{\partial t}+\left[\frac{x}{\Lambda_\varepsilon(t)}-1\right]\frac{\partial q_\varepsilon(x,t)}{\partial x}+\frac{\varepsilon}{2}\left[\frac{\partial q_\varepsilon(x,t)}{\partial x}\right]^2+\frac{1}{\Lambda_\varepsilon(t)} \;=\; \frac{\varepsilon}{2}\frac{\partial^2 q_\varepsilon(x,t)}{\partial x^2} \; . \tag{2.66}$$

If we differentiate (2.66) with respect to x we obtain a PDE for the function $v_\varepsilon(x,t)=\partial q_\varepsilon(x,t)/\partial x$, whence we have

$$\frac{\partial v_\varepsilon(x,t)}{\partial t}+\left[\frac{x}{\Lambda_\varepsilon(t)}-1+\varepsilon v_\varepsilon(x,t)\right]\frac{\partial v_\varepsilon(x,t)}{\partial x}+\frac{v_\varepsilon(x,t)}{\Lambda_\varepsilon(t)} \;=\; \frac{\varepsilon}{2}\frac{\partial^2 v_\varepsilon(x,t)}{\partial x^2} \; . \tag{2.67}$$

For $0<\nu\le 1$, we define the *viscous* CP model with viscosity ν as the solution to the PDE

(2.68)
$$\frac{\partial v_{\varepsilon,\nu}(x,t)}{\partial t}+\left[\frac{x}{\Lambda_{\varepsilon,\nu}(t)}-1+\varepsilon v_{\varepsilon,\nu}(x,t)\right]\frac{\partial v_{\varepsilon,\nu}(x,t)}{\partial x}+\frac{v_{\varepsilon,\nu}(x,t)}{\Lambda_{\varepsilon,\nu}(t)} \ = \ \frac{\varepsilon\nu}{2}\frac{\partial^2 v_{\varepsilon,\nu}(x,t)}{\partial x^2} \ , \ \ x,t>0,$$

with boundary condition

$$\frac{\partial v_{\varepsilon,\nu}(0,t)}{\partial x} \ = \ v_{\varepsilon,\nu}(0,t)^2 \ , \quad t>0, \tag{2.69}$$

and with the constraint

$$v_{\varepsilon,\nu}(0,t) \ = \ \frac{1}{\Lambda_{\varepsilon,\nu}(t)} \ , \quad t\ge 0. \tag{2.70}$$

Assuming that (2.68), (2.69) has a classical solution, we show that if the initial data for (2.68) corresponds to a random variable X_0, then $v_{\varepsilon,\nu}(\cdot,t)$ corresponds to a random variable X_t for $t>0$ in the sense that $v_{\varepsilon,\nu}(x,t)=E[X_t-x \mid X_t> x]^{-1}, \ x\ge 0$. To see this we define $\Gamma_{\varepsilon,\nu}$ similarly to Γ_ε in (2.62) but with v_ε replaced by $v_{\varepsilon,\nu}$ on the RHS. It follows from (2.68), (2.69) that $\Gamma_{\varepsilon,\nu}$ satisfies the PDE

$$\begin{aligned}\frac{\partial \Gamma_{\varepsilon,\nu}(x,t)}{\partial t}+\left[\frac{x}{\Lambda_{\varepsilon,\nu}(t)}-1+\varepsilon v_{\varepsilon,\nu}(x,t)\right]\frac{\partial \Gamma_{\varepsilon,\nu}(x,t)}{\partial x}+2\frac{\Gamma_{\varepsilon,\nu}(x,t)}{\Lambda_{\varepsilon}(t)} \\ = \ \frac{\varepsilon\nu}{2}\frac{\partial^2 \Gamma_{\varepsilon,\nu}(x,t)}{\partial x^2}+\varepsilon(1-\nu)\left(\frac{\partial v_{\varepsilon,\nu}(x,t)}{\partial x}\right)^2 \ ,\end{aligned} \tag{2.71}$$

with Dirichlet boundary condition $\Gamma_{\varepsilon,\nu}(0,t)=0, \ t>0$. Hence by the maximum principle [15], if $\Gamma_{\varepsilon,\nu}(\cdot,0)$ is non-negative then $\Gamma_{\varepsilon,\nu}(\cdot,t)$ is non-negative for $t>0$. We see from (2.62) that the non-negativity of $\Gamma_{\varepsilon,\nu}(\cdot,t)$ is equivalent to $v_{\varepsilon,\nu}(\cdot,t)$ corresponding to a random variable X_t. We have shown that for $0<\nu\le 1$ the viscous CP model corresponds to the evolution of a random variable $X_t, \ t\ge 0$. If $\nu=1$ the model is identical to the diffusive CP model (1.7), (1.8) with Dirichlet condition $c_\varepsilon(0,t)=0, \ t>0$.

We can think of the *inviscid* CP model (2.60), (2.61) as the limit of the viscous CP model (2.68), (2.69), (2.70) as the viscosity $\nu\to 0$. It is not clear however what happens to the boundary condition (2.69) in this limit. Unless the initial data $v_\varepsilon(\cdot,0)$ for (2.60) is increasing, the solution $v_\varepsilon(\cdot,t)$ develops discontinuities at some finite time [18]. For an entropy satisfying solution v_ε, discontinuities have the property that the solution *jumps down* across the discontinuity. Hence if $v_\varepsilon(\cdot,t)$ is discontinuous at the point z, then

$$\lim_{x\to z^-} v_\varepsilon(x,t) \ > \ \lim_{x\to z^+} v_\varepsilon(x,t) \ . \tag{2.72}$$

Observe now that for a random variable X, the function $x\to E[X-x \mid X>x]$ has discontinuities precisely at the atoms of X. In that case the function *jumps up* across the discontinuity. Since $v_\varepsilon(x,t)=E[X_t-x \mid X_t>x]^{-1}$ for some random variable X_t, it follows that at discontinuities of $v_\varepsilon(\cdot,t)$ the function jumps down. Thus discontinuities of $v_\varepsilon(\cdot,t)$ correspond to atoms of X_t, and the entropy condition for (2.60) is automatically satisfied.

We have already observed that the function $t\to\Lambda_0(t)$ in the CP model (1.1), (1.2) is increasing, and that the function $t\to\Lambda_\varepsilon(t)$ in the diffusive CP model (1.7), (1.8) is also increasing. To determine whether the function $t\to\Lambda_{\varepsilon,\nu}(t)$ in the viscous CP model (2.68)- (2.70) is increasing, we observe on setting $x = 0$ in (2.68) and using (2.70) that $v_{\varepsilon,\nu}(0,t)$ satisfies the equation

(2.73)
$$\frac{\partial v_{\varepsilon,\nu}(0,t)}{\partial t} + \{1-\varepsilon(1-\nu)v_{\varepsilon,\nu}(0,t)\}\,\Gamma_{\varepsilon,\nu}(0,t) + \varepsilon(1-\nu)v_{\varepsilon,\nu}(0,t)^3 + \frac{\varepsilon\nu}{2}\frac{\partial \Gamma_{\varepsilon,\nu}(0,t)}{\partial x} \ = \ 0.$$

We have already seen that $\Gamma_{\varepsilon,\nu}(\cdot,t)$ is a non-negative function, and from (2.69) it follows that $\Gamma_{\varepsilon,\nu}(0,t) = 0$ for $t > 0$. Hence $\partial\Gamma_{\varepsilon,\nu}(0,t)/\partial x \ge 0$ for $t > 0$. We conclude then from (2.73) that the function $t \to v_{\varepsilon,\nu}(0,t)$ is decreasing provided

$$v_{\varepsilon,\nu}(0,0) \ \le \ \frac{1}{\varepsilon(1-\nu)} \ . \tag{2.74}$$

Thus from (2.70) we see that if (2.74) holds, then the function $t \to \Lambda_{\varepsilon,\nu}(t)$ is increasing. Note that in the case of the diffusive CP model when $\nu = 1$ the condition (2.74) is redundant.

3. The Inviscid CP Model-Proof of Theorem 1.1

We shall restrict ourselves here to considering the solutions of (2.60), (2.61) when the initial data $v_\varepsilon(\cdot,0)$ is non-negative, increasing and also the function $\Gamma_\varepsilon(\cdot,0)$ of (2.62) is non-negative. The condition (2.74) becomes now $v_\varepsilon(0,t) \le \varepsilon^{-1}$, and assuming this holds also, we see that in this case (2.60) may be solved by the method of characteristics. To carry this out we set $\tilde{v}_\varepsilon(x,t) = m_{1,1/\Lambda_\varepsilon}(t)v_\varepsilon(x,t)$, where $m_{1,A}(\cdot)$ is defined by (2.2). Then (2.60) is equivalent to

$$\frac{\partial \tilde{v}_\varepsilon(x,t)}{\partial t} + \left[\frac{x}{\Lambda_\varepsilon(t)} - 1 + \varepsilon\frac{\tilde{v}_\varepsilon(x,t)}{m_{1,1/\Lambda_\varepsilon}t)}\right]\frac{\partial \tilde{v}_\varepsilon(x,t)}{\partial x} \ = \ 0 \ . \tag{3.1}$$

From (3.1) it follows that if $x(s),\ s \ge 0$, is a solution to the ODE

$$\frac{dx(s)}{ds} \ = \ \frac{x(s)}{\Lambda_\varepsilon(s)} - 1 + \varepsilon\frac{\tilde{v}_\varepsilon(x(s),s)}{m_{1,1/\Lambda_\varepsilon}(s)} \ , \tag{3.2}$$

and characteristics do not intersect, then $\tilde{v}_\varepsilon(x(t),t) \ = \ \tilde{v}_\varepsilon(x(0),0)$ for $t \ge 0$. We can therefore calculate the characteristics of (3.1) by setting $\tilde{v}_\varepsilon(x(s),s) = \tilde{v}_\varepsilon(x(0),0) = v_\varepsilon(x(0),0)$. We define the function $F_{\varepsilon,A}(x,t,v_0(\cdot))$ depending on $x,t \ge 0$ and increasing function $v_0 : [0,\infty) \to (0,\infty)$ by

$$z + \varepsilon\frac{\sigma_A^2(t)}{m_{1,A}(t)^2}v_0(z) \ = \ \frac{x + m_{2,A}(t)}{m_{1,A}(t)} \ = \ F_A(x,t) \ , \quad F_{\varepsilon,A}(x,t,v_0(\cdot)) \ = \ z, \tag{3.3}$$

where F_A is given by (2.2) and $\sigma_A^2(\cdot)$ by (4.10). Since $v_0(\cdot)$ is an increasing function there is a unique solution z to (3.3) for all $x \ge 0$ provided $v_0(0) \le \varepsilon^{-1}m_{1,A}(t)m_{2,A}(t)/\sigma_A^2(t)$. If this condition holds then the method of characteristics now yields the solution to (2.60) as

$$v_\varepsilon(x,t) \ = \ \frac{1}{m_{1,A}(t)}v_\varepsilon\left(F_{\varepsilon,A}(x,t,v_\varepsilon(\cdot,0)),0\right) \tag{3.4}$$

with $A = 1/\Lambda_\varepsilon$. It follows from (3.3), (3.4) that $v_\varepsilon(0,t) \ \le \ m_{2,A}(t)/\varepsilon\sigma_A^2(t) \le \varepsilon^{-1}$.

We wish to prove a global existence and uniqueness theorem for solutions of (2.60), (2.61). To describe our assumptions on the initial data $v_\varepsilon(\cdot,0)$ we shall consider functions $v_0 : [0,x_\infty) \to \mathbf{R}^+$ with the properties:

(3.5) The function $x \to v_0(x)$ is increasing on the interval $[0,x_\infty)$

and $v_0(0) > 0$. If $x_\infty < \infty$ then $\lim_{x\to x_\infty} v_0(x) = \infty$.

$$v_0(x_2) - v_0(x_1) \ \le \ \int_{x_1}^{x_2} v_0(x)^2 \ dx \quad \text{for } 0 \le x_1 < x_2 < x_\infty \ . \tag{3.6}$$

Note that (3.6) implies that $v_0(\cdot)$ is locally Lipschitz continuous in the interval $[0, x_\infty)$.

Lemma 3.1. *Assume the function $v_0(\cdot) = v_\varepsilon(\cdot, 0)$ satisfies (3.5), (3.6) and in addition that $(1+\delta_0)v_\varepsilon(0,0) < \varepsilon^{-1}$ for some $\delta_0 > 0$. Then there exists $\delta_1 > 0$ depending only on δ_0 such that there is a unique solution to (2.60), (2.61) for $0 \le t \le T = \delta_1/v_\varepsilon(0,0)$.*

Proof. Let $T, \delta_2 > 0$ and $\mathcal{E}$ be the space of continuous functions $V : [0,T] \to \mathbf{R}^+$ satisfying

$$V(0) \ = \ v_\varepsilon(0,0), \quad (1+\delta_2)^{-1} v_\varepsilon(0,0) \ \le \ V(t) \ \le (1+\delta_2) v_\varepsilon(0,0) \quad \text{for } 0 \le t \le T. \tag{3.7}$$

For $V \in \mathcal{E}$ we define a function $\mathcal{B}V(t), \ 0 \le t \le T$, by $\mathcal{B}V(t) = v_\varepsilon(0,t)$ where v_ε is the function (3.4) with $A(s) = V(s), \ 0 \le s \le T$. We shall show that if $T > 0$ is sufficiently small then $V \in \mathcal{E}$ implies $\mathcal{B}V \in \mathcal{E}$. To see this we first observe from (3.3) that $\mathcal{B}V(0) = v_\varepsilon(0,0)$. Next we note that for a function $v_0(\cdot)$ satisfying (3.5), (3.6), then

$$v_0(0) \ \le \ v_0(z) \le \frac{v_0(0)}{1 - zv_0(0)} \ , \quad \text{for } 0 \ \le z \ < 1/v_0(0) \ . \tag{3.8}$$

Since $V \in \mathcal{E}$, it follows from (3.7) that with $A(\cdot) = V(\cdot)$, then

$$t \exp\left[-(1+\delta_2)v_\varepsilon(0,0)t\right] \ \le \ \frac{m_{2,A}(t)}{m_{1,A}(t)} \ \le \ t \ , \quad 0 \le t \le T. \tag{3.9}$$

Similarly we have that

$$t \exp\left[-2(1+\delta_2)v_\varepsilon(0,0)t\right] \ \le \ \frac{\sigma_A^2(t)}{m_{1,A}(t)^2} \ \le \ t \ , \quad 0 \le t \le T. \tag{3.10}$$

From (3.9), (3.10) we have that for $\delta_1, \delta_2 > 0$ sufficiently small, depending only on δ_0, that

$$\varepsilon \frac{\sigma_A^2(t)}{m_{1,A}(t)^2} v_\varepsilon(0,0) \ \le \ \frac{m_{2,A}(t)}{m_{1,A}(t)} \quad \text{for } 0 \le t \le T. \tag{3.11}$$

Hence there is a unique solution $z(t) \le m_{2,A}(t)/m_{1,A}(t)$ to (3.3) with $x = 0$ provided $0 \le t \le T$. Since $\mathcal{B}V(t) \ = \ v_\varepsilon(z(t),0)/m_{1,A}(t)$, it follows from (3.8), (3.9) that on choosing $\delta_1 > 0$ sufficiently small, depending only on δ_2, that the function $\mathcal{B}V(t), \ 0 \le t \le T$, also satisfies (3.7).

Next we show that $\mathcal{B}$ is a contraction on the space $\mathcal{E}$ with metric $d(V_1, V_2) = \sup_{0 \le t \le T} |V_2(t) - V_1(t)|$ for $V_1, V_2 \in \mathcal{E}$. To see this let $z_1(t), \ z_2(t), \ 0 \le t \le T$, be the solutions to (3.3) with $x = 0$ corresponding to $V_1, V_1 \in \mathcal{E}$ respectively. Then from (3.4) we have

$$\mathcal{B}V_2(t) - \mathcal{B}V_1(t) \ = \ \frac{v_\varepsilon(z_2(t),0) - v_\varepsilon(z_1(t),0)}{m_{1,V_2}(t)} + \left[\frac{1}{m_{1,V_2}(t)} - \frac{1}{m_{1,V_1}(t)}\right] v_\varepsilon(z_1(t),0) \ . \tag{3.12}$$

The second term on the RHS of (3.12) is bounded as

$$\left| \left[\frac{1}{m_{1,V_2}(t)} - \frac{1}{m_{1,V_1}(t)} \right] v_\varepsilon(z_1(t),0) \right| \le (1+\delta_2) v_\varepsilon(0,0) \int_0^t |V_2(s)-V_1(s)| \, ds \ , \quad 0 \le t \le T. \tag{3.13}$$

We use (3.6) to bound the first term in (3.12). Thus we have that

$$\left| \frac{v_\varepsilon(z_2(t),0) - v_\varepsilon(z_1(t),0)}{m_{1,V_2}(t)} \right| \ \le \ (1+\delta_2)^2 v_\varepsilon(0,0)^2 |z_2(t)-z_1(t)| \ , \quad 0 \le t \le T. \tag{3.14}$$

From (3.3), (3.5) it follows that

$$|z_2(t)-z_1(t)| \ \le \ \left| \frac{m_{2,V_2}(t)}{m_{1,V_2}(t)} - \frac{m_{2,V_1}(t)}{m_{1,V_1}(t)} \right| \ , \quad 0 \le t \le T. \tag{3.15}$$

The RHS of (3.15) can be bounded similarly to (3.13), and so we obtain the inequality

$$|z_2(t)-z_1(t)| \ \le \ t \int_0^t |V_2(s)-V_1(s)| \, ds \ , \quad 0 \le t \le T. \tag{3.16}$$

It follows from (3.12)-(3.16) that

$$|\mathcal{B}V_2(t) - \mathcal{B}V_1(t)| \ \le \ 10\delta_1 \sup_{0 \le t \le T} |V_2(t)-V_1(t)| \ , \quad 0 \le t \le T = \delta_1 / v_\varepsilon(0,0), \tag{3.17}$$

provided $\delta_1 > 0$ is chosen sufficiently small depending only on δ_2. Evidently $\mathcal{B}$ is a contraction mapping on $\mathcal{E}$ and therefore has a unique fixed point if one also has $10\delta_1 < 1$. □

Lemma 3.2. *Let $v_\varepsilon(x,t)$, $x \ge 0$, $0 \le t \le T$ be the solution to (2.60), (2.61) constructed in Lemma 3.1. Then for any t satisfying $0 < t \le T$ the function $v_0(\cdot) = v_\varepsilon(\cdot,t)$ satisfies (3.5), (3.6) with $x_\infty = \infty$. In addition the function $t \to v_\varepsilon(0,t)$, $0 \le t \le T$, is continuous and decreasing.*

Proof. Since $\varepsilon > 0$ it follows from the fact that (3.5) holds for $v_0(\cdot) = v_\varepsilon(\cdot,0)$ that (3.3) has a unique solution $z < x_\infty$ for any $x > 0$. Hence $x_\infty = \infty$ if $t > 0$, and it is also clear that the function $x \to v_\varepsilon(x,t)$, $x \ge 0$, is increasing. We have therefore shown that (3.5) holds for $v_0(\cdot) = v_\varepsilon(\cdot,t)$ and $x_\infty = \infty$ if $0 < t \le T$.

Next we wish to show that (3.6) holds for $v_0(\cdot) = v_\varepsilon(\cdot,t)$ with $0 < t \le T$. To see this we observe from (3.4), (3.6) that for $0 \le x_1 \le x_2 < \infty$,

$$v_\varepsilon(x_2,t) - v_\varepsilon(x_1,t) \ = \ \frac{v_\varepsilon(z(x_2,t),0) - v_\varepsilon(z(x_1,t),0)}{m_{1,1/\Lambda_\varepsilon}(t)} \ \le \ \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)} \int_{z(x_1,t)}^{z(x_2,t)} v_\varepsilon(z,0)^2 \, dz \ , \tag{3.18}$$

where $z(x,t) = F_{\varepsilon,1/\Lambda_\varepsilon}(x,t,v_\varepsilon(\cdot,0))$. We see from (3.3) that $0 \le \partial z(x,t)/\partial x \le 1/m_{1,1/\Lambda_\varepsilon}(t)$, whence

$$\frac{1}{m_{1,1/\Lambda_\varepsilon}(t)} \int_{z(x_1,t)}^{z(x_2,t)} v_\varepsilon(z,0)^2 \, dz \ = \ m_{1,1/\Lambda_\varepsilon}(t) \int_{x_1}^{x_2} v_\varepsilon(x,t)^2 \frac{\partial z(x,t)}{\partial x} \, dx \ \le \ \int_{x_1}^{x_2} v_\varepsilon(x,t)^2 \, dx \ . \tag{3.19}$$

To show that the function $t \to v_\varepsilon(0,t)$ is continuous and decreasing we write $v_\varepsilon(0,t) = v_0(z(t))/m_{1,1/\Lambda_\varepsilon}(t)$ where $v_0(\cdot) = v_\varepsilon(\cdot,0)$ and $z(t)$ is the solution z to (3.3) with $x = 0$. We see from (3.3) that the function $t \to z(t)$ is Lipschitz continuous, whence the function $t \to v_\varepsilon(0,t)$ is continuous. If $z \to v_0(z)$ is differentiable at $z = z(t)$ then it follows from (2.2) that

$$\frac{\partial v_\varepsilon(0,t)}{\partial t} = \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)}\left[v_0'(z(t))\frac{dz(t)}{dt} - \frac{v_0(z(t))}{\Lambda_\varepsilon(t)}\right] . \tag{3.20}$$

Differentiating (3.3) with respect to t at $x = 0$ we obtain the equation

$$\left[1 + \varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}v_0'(z(t))\right]\frac{dz(t)}{dt} = \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)}\left[1 - \frac{\varepsilon v_0(z(t))}{m_{1,1/\Lambda_\varepsilon}(t)}\right] . \tag{3.21}$$

Hence (3.20), (3.21) imply that

$$\left[1 + \varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}v_0'(z(t))\right]m_{1,1/\Lambda_\varepsilon}(t)\frac{\partial v_\varepsilon(0,t)}{\partial t} = \tag{3.22}$$
$$\frac{v_0'(z(t))}{m_{1,1/\Lambda_\varepsilon}(t)} - \frac{v_0(z(t))}{\Lambda_\varepsilon(t)} - \frac{\varepsilon v_0'(z(t))v_0(z(t))}{m_{1,1/\Lambda_\varepsilon}(t)^2}\left[1 + \frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{\Lambda_\varepsilon(t)}\right] .$$

From (2.61), (3.4), (3.6) we have that

$$\frac{v_0'(z(t))}{m_{1,1/\Lambda_\varepsilon}(t)} \le \frac{v_0(z(t))^2}{m_{1,1/\Lambda_\varepsilon}(t)} = v_0(z(t))v_\varepsilon(0,t) = \frac{v_0(z(t))}{\Lambda_\varepsilon(t)} . \tag{3.23}$$

We conclude from (3.22), (3.23) that $\partial v_\varepsilon(0,t)/\partial t \le 0$. In the case when the function $z \to v_0(z)$ is not differentiable at $z = z(t)$, we can do an approximation argument to see that the function $s \to v_\varepsilon(0,s)$ is decreasing close to $s = t$. We have therefore shown that the function $t \to v_\varepsilon(0,t)$ is decreasing, whence $v_\varepsilon(0,t) \le v_\varepsilon(0,0) < \varepsilon^{-1}$ for $0 \le t \le T$. Since the RHS of (3.21) is the same as $[1 - \varepsilon v_\varepsilon(0,t)]/m_{1,1/\Lambda_\varepsilon}(t)$ this implies that the function $t \to z(t)$ is increasing. □

Proposition 3.1. *Assume the initial data for (2.60), (2.61) satisfies the conditions of Lemma 3.1. Then there exists a unique continuous solution $v_\varepsilon(x,t)$, $x, t \ge 0$, globally in time to (2.60), (2.61). The solution $v_\varepsilon(\cdot,t)$ satisfies (3.5), (3.6) for $t > 0$ with $x_\infty = \infty$, and the function $t \to v_\varepsilon(0,t)$ is decreasing. Furthermore there is a constant $C(\delta_0)$ depending only on δ_0 such that $\Lambda_\varepsilon(t) \le \Lambda_\varepsilon(0) + C(\delta_0)[\Lambda_\varepsilon(0) + t]$, $t \ge 0$.*

Proof. The global existence and uniqueness follows immediately from Lemma 3.1, 3.2 upon using the fact that the function $t \to v_\varepsilon(0,t)$ is decreasing. To get the upper bound on the function $\Lambda_\varepsilon(\cdot)$ we observe that Lemma 3.1 implies that with $T_0 = 0$,

$$\Lambda_\varepsilon(t) \le (1+\delta_2)\Lambda_\varepsilon(T_{k-1}) \text{ for } T_{k-1} \le t \le T_k, \quad T_k = T_{k-1} + \delta_1\Lambda_\varepsilon(T_{k-1}),\ k = 1, 2, ... \tag{3.24}$$

It follows from (3.24) that

$$\Lambda_\varepsilon(t) \le (1 + \delta_2)(T_k - T_{k-1})/\delta_1 \quad \text{for } T_{k-1} \le t \le T_k , \quad k = 1, 2, ... \tag{3.25}$$

We also have that

$$T_k - T_{k-1} \le \delta_1(1 + \delta_2)\Lambda_\varepsilon(T_{k-2}) \le (1 + \delta_2)T_{k-1} , \quad k = 2, 3, .. \tag{3.26}$$

From (3.25), (3.26) we conclude that $\Lambda_\varepsilon(t) \le (1+\delta_2)^2 t/\delta_1$ provided $t \ge T_1$, whence the result follows. □

The upper bound on the coarsening rate implied by Proposition 3.1 is independent of $\varepsilon > 0$ as $\varepsilon \to 0$. We can see from (3.22) that a lower bound on the rate of coarsening depends on ε. In fact if we choose $v_0(z) = 1/[\Lambda_\varepsilon(0) - z]$, $0 < z < \Lambda_\varepsilon(0)$, then $v_0'(z) = v_0(z)^2$, and so at $t = 0$ the RHS of (3.22) is zero if $\varepsilon = 0$. The random variable X_0 corresponding to this initial data is simply $X_0 \equiv$ constant, and it is easy to see that if $\varepsilon = 0$ then $X_t \equiv X_0$ for all $t > 0$. For $\varepsilon > 0$ however, we have the following:

Lemma 3.3. *Assume the initial data for (2.60), (2.61) satisfies the conditions of Lemma 3.1 and let $z(t)$ be as in Lemma 3.2. Then $\lim_{t\to\infty} z(t) = x_\infty$ and if $\varepsilon > 0$ one has $\lim_{t\to\infty} \Lambda_\varepsilon(t) = \infty$.*

Proof. Let $\lim_{t\to\infty} \Lambda_\varepsilon(t) = \Lambda_{\varepsilon,\infty}$ and assume first that $\Lambda_{\varepsilon,\infty} < \infty$. In that case $\lim_{t\to\infty} m_{1,\Lambda_\varepsilon}(t) = \infty$, and hence (2.61), (3.4) imply that $\lim_{t\to\infty} v_0(z(t)) = \infty$. We conclude from (3.5), (3.6) that if $\Lambda_{\varepsilon,\infty} < \infty$ then $\lim_{t\to\infty} z(t) = x_\infty$. We also have from (2.2) that

$$\limsup_{t\to\infty} \frac{m_{2,1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)} \ \le\ \Lambda_{\varepsilon,\infty}\ , \quad \liminf_{t\to\infty} \frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2} \ \ge\ \frac{\Lambda_\varepsilon(0)}{2}\ . \tag{3.27}$$

It follows now from (3.3), (3.27) that if $\varepsilon > 0$ then $\limsup_{t\to\infty} v_0(z(t)) \le 2\Lambda_{\varepsilon,\infty}/\varepsilon\Lambda_\varepsilon(0) < \infty$, which yields a contradiction.

Next we assume that $\Lambda_{\varepsilon,\infty} = \infty$, which we have just shown always holds if $\varepsilon > 0$. The function $t \to z(t)$ is increasing, and let us suppose that $\lim_{t\to\infty} z(t) = z_\infty < x_\infty$. Then from (2.61), (3.4) we have that $\lim_{t\to\infty} m_{1,1/\Lambda_\varepsilon}(t) = \infty$. We use the fact that for any $T \ge 0$, there exists a constant K_T such that

$$\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2} \ \le\ \frac{m_{2,1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(T)m_{1,1/\Lambda_\varepsilon}(t)} + K_T, \quad \text{for } t \ge T. \tag{3.28}$$

From (3.3), (3.28) we obtain the inequality

$$\frac{m_{2,1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)} \ \le\ z_\infty + \varepsilon\left[\frac{m_{2,1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(T)m_{1,1/\Lambda_\varepsilon}(t)} + K_T\right] v_0(z_\infty) \quad \text{if } t \ge T. \tag{3.29}$$

Choosing T sufficiently large so that $m_{1,1/\Lambda_\varepsilon}(T) \ge 2\varepsilon$, we conclude from (3.28), (3.29) that there is a constant C_1 such that $\sigma^2_{1/\Lambda_\varepsilon}(t)/m_{1,1/\Lambda_\varepsilon}(t)^2 \le C_1$ for $t \ge T$. If we also choose T such that $m_{1,1/\Lambda_\varepsilon}(t) \ge \varepsilon v_0(z(t))/2$ for $t \ge T$ we have from (3.6), (3.21) that

$$\left[1 + C_1\varepsilon v_0(z_\infty)^2\right] \frac{dz(t)}{dt} \ \ge\ \frac{1}{2m_{1,1/\Lambda_\varepsilon}(t)} \quad \text{for } t \ge T. \tag{3.30}$$

Since the function $t \to z(t)$ is increasing and $\lim_{t\to\infty} z(t) = z_\infty < \infty$, it follows from (2.61), (3.4), (3.30) that there is a constant C_2 such that

$$\int_0^t \frac{ds}{\Lambda_\varepsilon(s)} \ \le\ v_0(z_\infty) \int_0^t \frac{ds}{m_{1,1/\Lambda_\varepsilon}(s)} \ \le C_2 \quad \text{for } t \ge 0. \tag{3.31}$$

However (3.31) implies that $\lim_{t\to\infty} m_{1,1/\Lambda_\varepsilon}(t) \le \exp[C_2]$ and so we have again a contradiction. We conclude that $\lim_{t\to\infty} z(t) = x_\infty$. □

Lemma 3.4. *Assume the initial data for (2.60), (2.61) satisfies the conditions of Lemma 3.1 with $x_\infty = \infty$, and that for $0 < \delta \le 1$, one has*

$$\limsup_{x\to\infty} \frac{v_\varepsilon\left(x+\delta/v_\varepsilon(x,0),0\right)}{v_\varepsilon(x,0)} \ \le\ 1+\gamma(\delta), \quad \text{where } \lim_{\delta\to 0}\frac{\gamma(\delta)}{\delta}=0. \tag{3.32}$$

Then $\lim_{t\to\infty}\Lambda_\varepsilon(t)/t = 1$ for any $\varepsilon \ge 0$.

Proof. The main point about the condition (3.32) is that it is invariant under the dynamics determined by (2.60), (2.61). It is easy to see this in the case $\varepsilon = 0$ since we have, on using the notation of (3.3), that

$$\frac{v_0\left(x+\delta/v_0(x,t),t\right)}{v_0(x,t)} \ =\ \frac{v_0\left(F_{1/\Lambda_0}(x,t)+\delta/v_0(F_{1/\Lambda_0}(x,t),0),0\right)}{v_0(F_{1/\Lambda_0}(x,t),0)} \ . \tag{3.33}$$

For $\varepsilon > 0$ we have

$$\frac{v_\varepsilon\left(x+\delta/v_\varepsilon(x,t),t\right)}{v_\varepsilon(x,t)} \ =\ \frac{v_\varepsilon\left(z(x+\delta/v_\varepsilon(x,t),t),0\right)}{v_\varepsilon(z(x,t),0)} \ . \tag{3.34}$$

Since the function $x \to v_\varepsilon(x,0)$ is increasing it follows from (3.3) that

$$z(x+\delta/v_\varepsilon(x,t),t) \ \le\ z(x,t)+\frac{\delta}{m_{1,1/\Lambda_\varepsilon}(t)v_\varepsilon(x,t)} \ . \tag{3.35}$$

We conclude now from (3.4), (3.34), (3.35) that

$$\frac{v_\varepsilon\left(x+\delta/v_\varepsilon(x,t),t\right)}{v_\varepsilon(x,t)} \ \le\ \frac{v_\varepsilon\left(z(x,t)+\delta/v_\varepsilon(z(x,t),0),0\right)}{v_\varepsilon(z(x,t),0)} \ . \tag{3.36}$$

From Lemma 3.3 and (3.36) there exists $T_0 \ge 0$ such that

$$\frac{v_\varepsilon\left(x+\delta/v_\varepsilon(x,t),t\right)}{v_\varepsilon(x,t)} \ \le\ 1+2\gamma(\delta) \quad \text{for } x \ge 0, \ t \ge T_0 \ . \tag{3.37}$$

We use (3.37) to estimate $\Lambda_\varepsilon(T_0+t)/\Lambda_\varepsilon(T_0)$ in the interval $0 \le t \le \delta/v_\varepsilon(0,T_0)$. Thus we have

$$\frac{\Lambda_\varepsilon(T_0)}{\Lambda_\varepsilon(T_0+t)} \ =\ \frac{v_\varepsilon(z(t),T_0)m_{1,1/\Lambda_\varepsilon}(T_0)}{v_\varepsilon(0,T_0)m_{1,1/\Lambda_\varepsilon}(T_0+t)} \ , \quad \text{where } 0 \le z(t) \le t. \tag{3.38}$$

We conclude from (3.37), (3.38) that

$$\frac{m_{1,1/\Lambda_\varepsilon}(T_0)}{\Lambda_\varepsilon(T_0)} \ \le\ \frac{dm_{1,1/\Lambda_\varepsilon}(T_0+t)}{dt} \ \le\ [1+2\gamma(\delta)]\frac{m_{1,1/\Lambda_\varepsilon}(T_0)}{\Lambda_\varepsilon(T_0)} \quad \text{for } 0 \le t \le \delta\Lambda_\varepsilon(T_0). \tag{3.39}$$

On integrating (3.39) we have

$$1+\frac{t}{\Lambda_\varepsilon(T_0)} \ \le\ \frac{m_{1,1/\Lambda_\varepsilon}(T_0+t)}{m_{1,1/\Lambda_\varepsilon}(T_0)} \ \le\ 1+\frac{[1+2\gamma(\delta)]t}{\Lambda_\varepsilon(T_0)} \quad \text{for } 0 \le t \le \delta\Lambda_\varepsilon(T_0). \tag{3.40}$$

Hence (3.38), (3.40) imply that

$$\frac{1}{1+2\gamma(\delta)}\left[1+\frac{t}{\Lambda_\varepsilon(T_0)}\right] \ \le\ \frac{\Lambda_\varepsilon(T_0+t)}{\Lambda_\varepsilon(T_0)} \ \le\ 1+\frac{[1+2\gamma(\delta)]t}{\Lambda_\varepsilon(T_0)} \quad \text{for } 0 \le t \le \delta\Lambda_\varepsilon(T_0). \tag{3.41}$$

We define now $T_1 > T_0$ as the minimum time $T_1 = T_0+t$ such that $\Lambda_\varepsilon(T_0+t) \ge (1+\delta/2)\Lambda_\varepsilon(T_0)$. The inequality (3.41) now yields bounds on $T_1 - T_0$ as

$$\frac{\delta/2}{1+2\gamma(\delta)} \ \le\ \frac{T_1-T_0}{\Lambda_\varepsilon(T_0)} \ \le\ [1+2\gamma(\delta)][1+\delta/2]-1 \ , \tag{3.42}$$

provided the RHS of (3.42) is less than δ. In view of (3.32) this will be the case if $\delta > 0$ is sufficiently small. We can iterate the inequality (3.42) by defining $T_k,\ k = 1, 2, ..,$, as the minimum time such that $\Lambda_\varepsilon(T_k) \ge (1+\delta/2)\Lambda_\varepsilon(T_{k-1})$. Thus we have that

$$\text{(3.43)} \quad \frac{\delta/2}{1+2\gamma(\delta)} \ \le\ \frac{T_k - T_{k-1}}{(1+\delta/2)^{k-1}\Lambda_\varepsilon(T_0)} \ \le\ [1+2\gamma(\delta)][1+\delta/2]-1\,, \quad k=1,2,...$$

On summing (3.43) over $k = 1, ..., N$ we conclude that
(3.44)

$$\left[1-\frac{1}{(1+\delta/2)^N}\right]\frac{1}{1+2\gamma(\delta)} \ \le\ \frac{T_N - T_0}{\Lambda_\varepsilon(T_N)} \ \le\ \left[1-\frac{1}{(1+\delta/2)^N}\right]\frac{[1+2\gamma(\delta)][1+\delta/2]-1}{\delta/2}\,.$$

It follows from (3.44) that
(3.45)

$$\frac{1}{(1+\delta/2)[1+2\gamma(\delta)]} \ \le\ \liminf_{t\to\infty}\frac{t}{\Lambda_\varepsilon(t)} \ \le\ \limsup_{t\to\infty}\frac{t}{\Lambda_\varepsilon(t)} \ \le\ (1+\delta/2)\frac{[1+2\gamma(\delta)][1+\delta/2]-1}{\delta/2}\,.$$

Now using the fact that $\lim_{\delta\to 0}\gamma(\delta)/\delta = 0$, we conclude from (3.45) that $\lim_{t\to\infty}\Lambda_\varepsilon(t)/t = 1$. □

Remark 1. *Theorem 5.4 of* [2] *implies in the case* $\varepsilon = 0$ *convergence to the exponential self similar solution for initial data* $v_\varepsilon(x,0),\ x \ge 0$, *which has the property* $\lim_{x\to\infty} v_\varepsilon(x,0)/x^\alpha = v_{\varepsilon,\infty}$ *with* $0 < v_{\varepsilon,\infty} < \infty$ *provided* $\alpha > -1$. *It is easy to see that if* $\alpha \ge 0$ *then such initial data satisfies the condition (3.32) of Lemma 3.4.*

In [1] *necessary and sufficient conditions -(5.18) and (5.19) of* [1]*- for convergence to the exponential self-similar solution are obtained in the case* $\varepsilon = 0$. *Note that (5.19) of* [1] *implies the condition (3.32) of Lemma 3.4.*

Next we obtain a rate of convergence theorem for $\lim_{t\to\infty}\Lambda_\varepsilon(t)/t$ which generalizes Proposition 2.1 to the system (2.60), (2.61). We assume that the function $x \to v_\varepsilon(x,0)$ is C^1 for large x, in which case the condition (3.32) becomes $\lim_{x\to\infty} v_\varepsilon(x,0)^{-2}\partial v_\varepsilon(x,0)/\partial x = 0$, or equivalently $\lim_{x\to\infty}\beta_{X_0}(x) = 1$ for the initial condition random variable X_0. More precisely, we have that if $v_\varepsilon(x,0)^{-2}\partial v_\varepsilon(x,0)/\partial x \le \eta$ for $x \ge x_\eta$ then

$$\text{(3.46)} \qquad \frac{1}{v_\varepsilon(x,0)} - \frac{1}{v_\varepsilon(x+\delta/v_\varepsilon(x,0),0)} \ \le\ \frac{\eta\delta}{v_\varepsilon(x,0)} \quad \text{for } x \ge x_\eta\ .$$

The inequality (3.46) implies (3.32) holds with $\gamma(\delta) \le \eta\delta/(1-\eta\delta)$. We conclude that (3.32) holds if $\lim_{x\to\infty} v_\varepsilon(x,0)^{-2}\partial v_\varepsilon(x,0)/\partial x = 0$.

The condition on the initial data to guarantee a logarithmic rate of convergence for $\Lambda_\varepsilon(t)/t$ is similar to (3.32). We require that there exists $\delta, \gamma(\delta), x_\delta > 0$ such that

$$\text{(3.47)} \quad \frac{v_\varepsilon(y,0)^2}{\partial v_\varepsilon(y,0)/\partial y} \ \ge\ \frac{v_\varepsilon(x,0)^2}{\partial v_\varepsilon(x,0)/\partial x} + \delta\gamma(\delta) \quad \text{for } y = x+\delta/v_\varepsilon(x,0),\ x \ge x_\delta\ .$$

Observe that if (3.47) holds for arbitrarily small $\delta > 0$ and $\liminf_{\delta\to 0} x_\delta = x_0$ then the function $x \to 1/v_\varepsilon(x,0)$ is convex for $x \ge x_0$. Furthermore if the function $x \to v_\varepsilon(x,0)$ is C^2, then on taking $\delta \to 0$ in (3.47) we obtain the second order differential inequality

$$\text{(3.48)} \qquad \frac{v_\varepsilon(x,0)\partial^2 v_\varepsilon(x,0)/\partial x^2}{[\partial v_\varepsilon(x,0)/\partial x]^2} \ \le\ 2-\gamma(0)\ .$$

Suppose now that (3.48) holds with $\gamma(0)=\eta>0$ for $x\ge x_0$. Then we have that

$$\frac{\partial}{\partial x}\frac{v_\varepsilon(x,0)^2}{\partial v_\varepsilon(x,0)/\partial x}\ge \eta v_\varepsilon(x,0)\quad \text{for } x\ge x_0\ . \tag{3.49}$$

On integrating (3.49) and using the fact that the function $x\to v_\varepsilon(x,0)$ is increasing, we conclude that (3.47) holds for all $\delta>0$ with $\gamma(\delta)=\eta$ and $x_\delta=x_0$.

It is easy to see that (3.48) is invariant under affine transformations. That is if the function $x\to v_\varepsilon(x,0)$ satisfies (3.48) for all $x>0$, then given any $\lambda,k>0$ so also does the function $x\to\lambda v_\varepsilon(\lambda x+k,0)$. We can solve the differential equation determined by equality in (3.48). The solution is given by the formula

$$v_\varepsilon(x,0)\ =\ a[1+\lambda x]^\alpha\ ,\quad \text{where } \alpha\ =\ 1/[\gamma(0)-1]\ . \tag{3.50}$$

Since we require $\gamma(0)>0$ it follows from (3.48) that α must satisfy either $\alpha>0$ or $\alpha<-1$. Note that the function $x\to 1/v_\varepsilon(x,0)$ of (3.50) is convex precisely for this range of α values.

Lemma 3.5. *Assume the initial data $x\to v_\varepsilon(x,0)$ for (2.60), (2.61) is C^1 increasing and that the function $x\to 1/v_\varepsilon(x,0)$ is convex for sufficiently large x. Assume further that there exists $\delta,\gamma(\delta),x_\delta>0$ such that (3.47) holds. Then there exists constants $C_0,t_0>0$ such that*

$$1-\frac{C_0}{\log t}\ \le\ \frac{d\Lambda_\varepsilon(t)}{dt}\ \le 1\quad \text{for } t\ge t_0\ . \tag{3.51}$$

Proof. Since the inequality (3.47) is invariant under affine transformations we see as in Lemma 3.4 that in the case $\varepsilon=0$ there exists $T_0>0$ such that if $t\ge T_0$ the function $x\to 1/v_\varepsilon(x,t)$ is convex for $x\ge 0$, and

$$\frac{v_\varepsilon(y,t)^2}{\partial v_\varepsilon(y,t)/\partial y}\ \ge\ \frac{v_\varepsilon(x,t)^2}{\partial v_\varepsilon(x,t)/\partial x}+\delta\gamma(\delta)\quad \text{for } y=x+\delta/v_\varepsilon(x,t),\ x\ge 0,\ t\ge T_0\ . \tag{3.52}$$

Next observe that since $\lim_{x\to\infty}v_\varepsilon(x,0)^{-2}\partial v_\varepsilon(x,0)/\partial x=0$, we may for any $\nu>0$ choose T_0 such that $v_\varepsilon(x,T_0)^{-2}\partial v_\varepsilon(x,T_0)/\partial x\le\nu$ for $x\ge 0$. It follows then from (3.46) that

$$\frac{v_\varepsilon(y,T_0)}{v_\varepsilon(0,T_0)}\ \le\ \frac{1}{1-\nu v_\varepsilon(0,T_0)y}\qquad \text{for } 0\le y<\frac{1}{\nu v_\varepsilon(0,T_0)}\ =\ \frac{\Lambda_\varepsilon(T_0)}{\nu}\ . \tag{3.53}$$

Hence as in (3.39) we see from (3.38), (3.53) that

$$\frac{dm_{1,1/\Lambda_\varepsilon}(T_0+t)}{dt}\ \le\ \frac{m_{1,1/\Lambda_\varepsilon}(T_0)}{[1-\nu v_\varepsilon(0,T_0)t]\Lambda_\varepsilon(T_0)}\quad \text{for } 0\le t<\frac{\Lambda_\varepsilon(T_0)}{\nu}\ . \tag{3.54}$$

Integrating (3.54) we conclude that for $0\le t<\Lambda_\varepsilon(T_0)/\nu$,

$$\frac{m_{1,1/\Lambda_\varepsilon}(T_0)}{m_{1,1/\Lambda_\varepsilon}(T_0+t)}\ \ge\ \left[1-\frac{1}{\nu}\log\left\{1-\frac{\nu t}{\Lambda_\varepsilon(T_0)}\right\}\right]^{-1}\ . \tag{3.55}$$

Using the inequality $-\log(1-z)\le 3z/2$ when $0\le z\le 1/3$, we conclude from (3.55) that

$$\frac{m_{1,1/\Lambda_\varepsilon}(T_0)}{m_{1,1/\Lambda_\varepsilon}(T_0+t)}\ \ge\ \left[1+\frac{3t}{2\Lambda_\varepsilon(T_0)}\right]^{-1}\quad \text{for } 0\le t\le\frac{\Lambda_\varepsilon(T_0)}{3\nu}\ . \tag{3.56}$$

Similarly to (3.41) we have from (3.56) that

$$\frac{\Lambda_\varepsilon(T_0+t)}{\Lambda_\varepsilon(T_0)} \ \le \ 1+\frac{3t}{2\Lambda_\varepsilon(T_0)} \quad \text{for } 0\le t\le \frac{\Lambda_\varepsilon(T_0)}{3\nu} \ . \tag{3.57}$$

In the case $\varepsilon=0$ the LHS of (3.56) is $dz(t)/dt$, so on integration we have that

$$z(t) \ \ge \ \frac{2\Lambda_\varepsilon(T_0)}{3}\log\left[1+\frac{3t}{2\Lambda_\varepsilon(T_0)}\right] \quad \text{for } 0\le t\le \frac{\Lambda_\varepsilon(T_0)}{3\nu} \ . \tag{3.58}$$

We choose now ν sufficiently small so that $2\log[1+1/2\nu]/3>\delta$ and let T_1 be the minimum T_0+t such that $z(t)\ge \delta/v_\varepsilon(0,T_0)$. Then we have that

$$T_1-T_0 \ \le \ \frac{\Lambda_\varepsilon(T_0)}{3\nu} \ , \quad \frac{v_\varepsilon\,(0,T_1)^2}{\partial v_\varepsilon(0,T_1)/\partial x} \ \ge \ \frac{v_\varepsilon\,(0,T_0)^2}{\partial v_\varepsilon(0,T_0)/\partial x}+\delta\gamma(\delta) \ . \tag{3.59}$$

Furthermore (3.57) implies that $\Lambda_\varepsilon(T_1)/\Lambda_\varepsilon(T_0)\le 1+1/2\nu$. We now iterate the foregoing to yield a sequence of times $T_k, \ k=1,2,..,$ with the properties that

$$T_k-T_{k-1} \ \le \ \frac{\Lambda_\varepsilon(T_{k-1})}{3\nu}, \quad \frac{\Lambda_\varepsilon(T_k)}{\Lambda_\varepsilon(T_{k-1})} \le 1+\frac{1}{2\nu}, \quad \frac{v_\varepsilon\,(0,T_k)^2}{\partial v_\varepsilon(0,T_k)/\partial x} \ \ge \ k\delta\gamma(\delta). \tag{3.60}$$

It follows from (3.60) that

$$T_N-T_0 \ \le \ \frac{2}{3}\left(1+\frac{1}{2\nu}\right)^N\Lambda_\varepsilon(T_0), \quad \frac{d\Lambda_\varepsilon(T_N)}{dt} \ge 1-\frac{1}{N\delta\gamma(\delta)}, \quad N=1,2,... \tag{3.61}$$

The inequality (3.61) implies the lower bound in (3.51) since the function $t\to d\Lambda_\varepsilon(t)/dt$ is increasing for $t\ge T_0$.

To deal with $\varepsilon>0$ we first assume that the function $x\to v_\varepsilon(x,0)$ is C^2 for $x>0$. Letting $z(x,t)$ be the solution to (3.3) we have that

$$\frac{\partial z(x,t)}{\partial x} \ = \ \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)}\left[1+\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial v_\varepsilon(z(x,t),0)}{\partial z}\right]^{-1} \ , \tag{3.62}$$

(3.63)

$$\frac{\partial^2 z(x,t)}{\partial x^2} \ = \ -\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial^2 v_\varepsilon(z(x,t),0)}{\partial z^2}\left[1+\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial v_\varepsilon(z(x,t),0)}{\partial z}\right]^{-1}\left(\frac{\partial z(x,t)}{\partial x}\right)^2 \ .$$

We also have that

$$\frac{\partial v_\varepsilon(x,t)}{\partial x} \ = \ \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)}\frac{\partial v_\varepsilon(z(x,t),0)}{\partial z}\frac{\partial z(x,t)}{\partial x}, \tag{3.64}$$

(3.65)

$$\frac{\partial^2 v_\varepsilon(x,t)}{\partial x^2} \ = \ \frac{1}{m_{1,1/\Lambda_\varepsilon}(t)}\left[\frac{\partial^2 v_\varepsilon(z(x,t),0)}{\partial z^2}\left(\frac{\partial z(x,t)}{\partial x}\right)^2+\frac{\partial v_\varepsilon(z(x,t),0)}{\partial z}\frac{\partial^2 z(x,t)}{\partial x^2}\right].$$

It follows from (3.62)-(3.65) that the ratio (3.48) for the function $x\to v_\varepsilon(x,t)$ is given by

(3.66)

$$\frac{v_\varepsilon(x,t)\partial^2 v_\varepsilon(x,t)/\partial x^2}{[\partial v_\varepsilon(x,t)/\partial x]^2} \ = \ \frac{v_\varepsilon(z(x,t),0)\partial^2 v_\varepsilon(z(x,t),0)/\partial z^2}{[\partial v_\varepsilon(z(x,t),0)/\partial z]^2}\left[1+\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial v_\varepsilon(z(x,t),0)}{\partial z}\right]^{-1} \ .$$

Hence if (3.48) holds for all sufficiently large x, then Lemma 3.3 implies that there exists $T_0>0$ such that the RHS of (3.66) is bounded above by 2 for $x\ge 0, t\ge T_0$. It follows that the function $x\to 1/v_\varepsilon(x,t)$ is convex for $x\ge 0$ provided $t\ge T_0$.

Observe that if $0<\gamma(0)\le 2$ in (3.48) then the RHS of (3.66) is bounded above by $2-\gamma(0)$. However if $\gamma(0)>2$ then we can only bound the RHS above by 0. It is easy to see from the example (3.50) that this is the best bound we can obtain. In fact if $\alpha>1$ in (3.50) then $\lim_{z\to\infty}\partial v_\varepsilon(z,0)/\partial z=\infty$, in which case the RHS of (3.66) converges to 0 as $x\to\infty$. We have shown that if (3.48) holds for all sufficiently large x then there exists $T_0>0$ such that

$$\frac{v_\varepsilon(x,t)\partial^2 v_\varepsilon(x,t)/\partial x^2}{[\partial v_\varepsilon(x,t)/\partial x]^2} \ \le\ \max[2-\gamma(0),0] \quad \text{for } x\ge 0,\ t\ge T_0\ . \tag{3.67}$$

In the case when we only assume that the function $x\to v_\varepsilon(x,0)$ is C^1 for $x>0$ we can make a more careful version of the argument of the previous paragraph. We have now from (3.62), (3.64) that

$$\frac{v_\varepsilon(x,t)^2}{\partial v_\varepsilon(x,t)/\partial x} \ =\ \frac{v_\varepsilon(z(x,t),0)^2}{\partial v_\varepsilon(z(x,t),0)/\partial z}+\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}v_\varepsilon(z(x,t),0)^2\ . \tag{3.68}$$

Since the function $x\to z(x,t)$ is increasing for $x\ge 0$, it follows that the second function on the RHS of (3.68) is increasing for $x\ge 0$. Since we are assuming that the function $z\to 1/v_\varepsilon(z,0)$ is convex for all large z, it follows that the first function on the RHS of (3.68) is also increasing for $x\ge 0$ provided $t\ge T_0$ and T_0 is sufficiently large. We conclude that the function $x\to 1/v_\varepsilon(x,t)$ is convex for $x\ge 0$ provided $t\ge T_0$.

We can also obtain an inequality (3.52) for a δ which is twice the δ which occurs in (3.47). To show this we consider two possibilities. In the first of these (3.52) follows from the monotonicity of the second function on the RHS of (3.68). We use the inequality

$$\begin{aligned}\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}v_\varepsilon\left(z\left(x+\frac{\eta}{v_\varepsilon(x,t)},t\right),0\right)^2 \ &\ge\ \varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}v_\varepsilon(z(x,t),0)^2+ \\ 2\varepsilon\eta\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\int_0^1 d\rho\,\frac{\partial v_\varepsilon}{\partial z}\left(z\left(x+\frac{\rho\eta}{v_\varepsilon(x,t)},t\right),0\right)&\left[1+\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial v_\varepsilon}{\partial z}\left(z\left(x+\frac{\rho\eta}{v_\varepsilon(x,t)},t\right),0\right)\right]^{-1},\end{aligned} \tag{3.69}$$

which follows from (3.4), (3.62), the monotonicity of the function $z\to v_\varepsilon(z,0)$, and Taylor's formula. Observe next from the convexity of the function $z\to 1/v_\varepsilon(z,0)$ that for $0<\rho'<\rho$,

$$\partial v_\varepsilon\left(z\left(x+\frac{\rho\eta}{v_\varepsilon(x,t)},t\right),0\right)\Big/\partial z \ \le\ \left[\frac{v_\varepsilon(x+\rho\eta/v_\varepsilon(x,t),t)}{v_\varepsilon(x+\rho'\eta/v_\varepsilon(x,t),t)}\right]^2\frac{\partial v_\varepsilon}{\partial z}\left(z\left(x+\frac{\rho'\eta}{v_\varepsilon(x,t)},t\right),0\right)\ . \tag{3.70}$$

Furthermore we have similarly to (3.53) that for $\nu>0$ there exists $T_0>0$ such that for $0<\rho'<\rho$ and $t\ge T_0$,

$$1\ \le\ \frac{v_\varepsilon(x+\rho\eta/v_\varepsilon(x,t),t)}{v_\varepsilon(x+\rho'\eta/v_\varepsilon(x,t),t)}\ \le\ \frac{1-\nu\rho'\eta}{1-\nu\rho\eta}\quad \text{provided } \rho<\frac{1}{\nu\eta}\ . \tag{3.71}$$

Now let us assume that $t\ge T_0$ and

$$\varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial v_\varepsilon}{\partial z}\left(z\left(x+\frac{\delta}{2v_\varepsilon(x,t)},t\right),0\right)\ \ge\ \frac{1}{4}\ . \tag{3.72}$$

Then (3.70), (3.71) imply upon setting $\eta = \delta/2$ in (3.69) and choosing ν less than some constant depending only on δ, that the second term on the RHS is bounded below by $\delta/6$ for $t \ge T_0$. This implies that (3.52) holds with $\gamma(\delta) = 1/6$.

Alternatively we assume that (3.72) does not hold. Then on choosing ν sufficiently small, depending only on δ, we see that (3.62), (3.70), (3.71) implies

$$z\left(x + \frac{2\delta}{v_\varepsilon(x,t)}, t\right) \ge \ z\left(x + \frac{\delta}{2v_\varepsilon(x,t)}, t\right) + \frac{11\delta}{10 v_\varepsilon(z(x,t),0)} \ . \tag{3.73}$$

Then we use the first term on the RHS of (3.68) and (3.47), (3.73) to establish (3.52) with 2δ in place of δ. We have proved then that there exists $T_0 > 0$ such that (3.52) holds (with δ replaced by 2δ).

We wish next to establish an inequality like (3.59) in the case $\varepsilon > 0$, in which case we need to examine the terms of (3.21) that depend on ε. Using the notation of (4.27), (4.28) the ε dependent coefficient on the LHS of (3.21) is given by

$$\varepsilon \frac{\sigma^2_{1/\Lambda_\varepsilon}(T_0, T_0+t)}{m_{1,1/\Lambda_\varepsilon}(T_0,T_0+t)^2} \frac{\partial v_\varepsilon(z(t),T_0)}{\partial z} \ = \ \varepsilon \frac{\sigma^2_{1/\Lambda_\varepsilon}(T_0,T_0+t)}{\Lambda_\varepsilon(T_0+t)^2}\left[v_\varepsilon(z(t),T_0)^{-2} \frac{\partial v_\varepsilon(z(t),T_0)}{\partial z}\right] \ . \tag{3.74}$$

The ε dependent coefficient on the RHS of (3.21) is given by

$$\varepsilon \frac{v_\varepsilon(z(t),T_0)}{m_{1,1/\Lambda_\varepsilon}(T_0,T_0+t)} \ = \ \frac{\varepsilon}{\Lambda_\varepsilon(T_0+t)} \ . \tag{3.75}$$

We choose now T_0 large enough so that $\varepsilon/\Lambda_\varepsilon(T_0) < 1/2$, whence (3.75) implies that the term in brackets on the RHS of (3.21) is at least $1/2$. We also have from (3.74) that

$$\varepsilon \frac{\sigma^2_{1/\Lambda_\varepsilon}(T_0,T_0+t)}{m_{1,1/\Lambda_\varepsilon}(T_0,T_0+t)^2} \frac{\partial v_\varepsilon(z(t),T_0)}{\partial z} \ \le \ \frac{\nu}{2} \frac{\sigma^2_{1/\Lambda_\varepsilon}(T_0,T_0+t)}{\Lambda_\varepsilon(T_0+t)} \ . \tag{3.76}$$

Now using (4.27), we conclude from (3.76) that for any $K > 0$,

$$\varepsilon \frac{\sigma^2_{1/\Lambda_\varepsilon}(T_0,T_0+t)}{m_{1,1/\Lambda_\varepsilon}(T_0,T_0+t)^2} \frac{\partial v_\varepsilon(z(t),T_0)}{\partial z} \ \le \ \frac{\nu K \exp(2K)}{2} \quad \text{for } 0 \le t \le K\Lambda_\varepsilon(T_0) \ . \tag{3.77}$$

It follows from (3.52), (3.62), (3.77) that there exists $T_1 > T_0$ such that (3.59) holds. Therefore we can define a sequence $T_k,\ k = 1, 2, ..,$ of times having the properties (3.60).

In order to estimate $d\Lambda_\varepsilon(T_{k-1}+t)/dt$ for $0 \le t \le T_k - T_{k-1}$, we need to examine the terms of (3.22) that depend on ε. Similarly to (3.77) we have from (2.61), (3.22) and the convexity of the function $x \to 1/v_\varepsilon(x, T_{k-1})$ that

$$\left[1 + \frac{\varepsilon \nu K \exp(2K)}{\Lambda_\varepsilon(T_{k-1})}\right] \frac{d\Lambda_\varepsilon(T_{k-1}+t)}{dt} \ \ge 1 - \frac{\partial v_\varepsilon(0,T_{k-1})/\partial x}{v_\varepsilon(0,T_{k-1})^2} \quad \text{for } 0 \le t \le T_k - T_{k-1} \ . \tag{3.78}$$

Noting from (3.41) that $\Lambda_\varepsilon(T_k)$ grows exponentially in k, we conclude from (3.60), (3.61) and (3.78) that the lower bound in (3.51) holds. To obtain the upper bound in (3.51) we use the identity

$$\frac{d\Lambda_\varepsilon(t)}{dt} \ = \ 1 - [1 - \varepsilon v_\varepsilon(0,t)] \frac{1}{v_\varepsilon(0,t)^2} \frac{\partial v_\varepsilon(0,t)}{\partial x} \ , \tag{3.79}$$

obtained from (2.60), (2.61). Evidently the RHS of (3.79) does not exceed 1. □

Lemma 3.6. *Assume $\varepsilon > 0$ and the initial data for (2.60), (2.61) satisfies the conditions of Lemma 3.1 with $x_\infty < \infty$. Then for any $t > 0$ the function $x \to v_\varepsilon(x,t)$ satisfies $\lim_{x\to\infty} v_\varepsilon(x,t)/x = v_\infty(t)$ for some $v_\infty(t) > 0$.*

Assume in addition that the initial data is C^1, the function $x \to 1/v_\varepsilon(x,0)$ is convex for x sufficiently close to x_∞, and $\liminf_{x\to x_\infty} \partial v_\varepsilon(x,0)/\partial x > 0$. Then for any $t > 0$ the function $x \to 1/v_\varepsilon(x,t)$ is convex for x sufficiently large, and the inequality (3.47) holds for all $\delta > 0$.

Proof. From (3.3), (3.4) we have that

$$\frac{x + m_{2,1/\Lambda_\varepsilon}(t) - m_{1,1/\Lambda_\varepsilon}(t)x_\infty}{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)} \ \le \ v_\varepsilon(x,t) \ \le \ \frac{x + m_{2,1/\Lambda_\varepsilon}(t)}{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)} \quad \text{for } x \ge 0. \tag{3.80}$$

Hence $\lim_{x\to\infty} v_\varepsilon(x,t)/x = 1/\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)$.

It is easy to see from our assumptions that (3.47) is satisfied if $v_\varepsilon(x,0)$ is C^2 for x sufficiently close to x_∞. In that case it follows from the convexity of the function $x \to 1/v_\varepsilon(x,0)$ close to x_∞ and (3.66) that there exists $\eta(t) > 0$ and $x_\eta(t)$ with

$$\frac{v_\varepsilon(x,t)\partial^2 v_\varepsilon(x,t)/\partial x^2}{[\partial v_\varepsilon(x,t)/\partial x]^2} \ \le \ 2\left[1 + \varepsilon\frac{\sigma^2_{1/\Lambda_\varepsilon}(t)}{m_{1,1/\Lambda_\varepsilon}(t)^2}\frac{\partial v_\varepsilon(z(x,t),0)}{\partial z}\right]^{-1} \le 2-\eta(t) \tag{3.81}$$

for $x > x_\eta(t)$. If we only assume the function $x \to v_\varepsilon(x,t)$ is C^1, then we use (3.68), whence (3.47) follows from (3.69). □

Proof of Theorem 1.1: Note the assumption that the function $x \to E[X_0 - x \mid X_0 > x]$ is decreasing implies that the initial data $v_\varepsilon(\cdot,0)$ for (2.60), (2.61) is continuous and increasing. Now $\lim_{t\to\infty}\langle X_t\rangle/t = 1$ follows from Lemma 3.4, the remark following it and Lemma 3.6. The inequality (1.13) follows from Lemma 3.5 and Lemma 3.6. □

4. Representations of Green's functions

Let $b : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ be a continuous function which satisfies the uniform Lipschitz condition

$$\sup\{|\partial b(y,t)/\partial y| \ : \ y,t \in \mathbf{R}\} \ \le \ A_\infty \tag{4.1}$$

for some constant A_∞. Then the terminal value problem

$$\frac{\partial u_\varepsilon(y,t)}{\partial t} + b(y,t)\frac{\partial u_\varepsilon(y,t)}{\partial y} + \frac{\varepsilon}{2}\frac{\partial^2 u_\varepsilon(y,t)}{\partial y^2} \ = \ 0, \quad y \in \mathbf{R}, \ t < T, \tag{4.2}$$

$$u_\varepsilon(y,T) \ = \ u_T(y), \quad y \in \mathbf{R}, \tag{4.3}$$

has a unique solution u_ε which has the representation

$$u_\varepsilon(y,t) \ = \ \int_{-\infty}^{\infty} G_\varepsilon(x,y,t,T)u_T(x)\,dx, \quad y \in \mathbf{R}, \ t < T, \tag{4.4}$$

where G_ε is the Green's function for the problem. The adjoint problem to (4.2), (4.3) is the initial value problem

$$\frac{\partial v_\varepsilon(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[b(x,t)v_\varepsilon(x,t)\right] \ = \ \frac{\varepsilon}{2}\frac{\partial^2 v_\varepsilon(x,t)}{\partial x^2}\ , \quad x \in \mathbf{R}, \ t > 0, \tag{4.5}$$

$$v_\varepsilon(x,0) \ = \ v_0(x), \quad y \in \mathbf{R}. \tag{4.6}$$

The solution to (4.5), (4.6) is given by the formula

$$v_\varepsilon(x,T) \ = \ \int_{-\infty}^{\infty} G_\varepsilon(x,y,0,T)v_0(y)\ dy, \quad x\in\mathbf{R},\ T>0. \tag{4.7}$$

For any $t<T$ let $Y_\varepsilon(s),\ s>t$, be the solution to the initial value problem for the SDE

$$dY_\varepsilon(s) \ = \ b(Y_\varepsilon(s),s)ds+\sqrt{\varepsilon}\ dB(s), \quad Y_\varepsilon(t)=y, \tag{4.8}$$

where $B(\cdot)$ is Brownian motion. Then $G_\varepsilon(\cdot,y,t,T)$ is the probability density for the random variable $Y_\varepsilon(T)$. In the case when the function $b(y,t)$ is linear in y it is easy to see that (4.8) can be explicitly solved. Thus let $A:\mathbf{R}\to\mathbf{R}$ be a continuous function and $b:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$ the function $b(y,t)=A(t)y-1$. The solution to (4.8) is then given by

$$Y_\varepsilon(s) \ = \ \exp\left[\int_t^s A(s')ds'\right]y-\int_t^s \exp\left[\int_{s'}^s A(s'')ds''\right]ds'+\sqrt{\varepsilon}\int_t^s \exp\left[\int_{s'}^s A(s'')ds''\right]dB(s')\ . \tag{4.9}$$

Hence the random variable $Y_\varepsilon(T)$ conditioned on $Y_\varepsilon(0)=y$ is Gaussian with mean $m_{1,A}(T)y-m_{2,A}(T)$ and variance $\varepsilon\sigma_A^2(T)$, where $m_{1,A},m_{2,A}$ are given by (2.2) and σ_A^2 by

$$\sigma_A^2(T)=\int_0^T \exp\left[2\int_s^T A(s')ds'\right]ds\ . \tag{4.10}$$

The Green's function $G_\varepsilon(x,y,0,T)$ is therefore explicitly given by the formula

$$G_\varepsilon(x,y,0,T)=\frac{1}{\sqrt{2\pi\varepsilon\sigma_A^2(T)}}\exp\left[-\frac{\{x+m_{2,A}(T)-m_{1,A}(T)y\}^2}{2\varepsilon\sigma_A^2(T)}\right]\ . \tag{4.11}$$

To obtain the formula (4.11) we have used the fact that the solution to the terminal value problem (4.2), (4.3) has a representation as an expectation value $u_\varepsilon(y,t)=E[u_0(Y_\varepsilon(T))\mid Y(t)=y\ ]$, where $Y_\varepsilon(\cdot)$ is the solution to the SDE (4.8). The initial value problem (4.5), (4.6) also has a representation as an expectation value in terms of the solution to the SDE

$$dX_\varepsilon(s) \ = \ b(X_\varepsilon(s),s)ds+\sqrt{\varepsilon}\ dB(s), \quad X_\varepsilon(T)=x,\ s<T. \tag{4.12}$$

run *backwards* in time. Thus in (4.12) $B(s),\ s<T$, is Brownian motion run backwards in time. The solution v_ε of (4.5), (4.6) has the representation

$$v_\varepsilon(x,T) \ = \ E\left[\exp\left\{-\int_0^T \frac{\partial b(X_\varepsilon(s),s)}{\partial x}\ ds\right\}v_0(X_\varepsilon(0))\ \middle|\ X_\varepsilon(T)=x\ \right]\ . \tag{4.13}$$

Next we consider the terminal value problem (4.2), (4.3) in the half space $y>0$ with Dirichlet boundary condition $u_\varepsilon(0,t)=0,\ t<T$. In that case the solution $u_\varepsilon(y,t)$ has the representation

$$u_\varepsilon(y,t) \ = \ \int_0^\infty G_{\varepsilon,D}(x,y,t,T)u_T(x)\ dx, \quad y>0,\ t<T, \tag{4.14}$$

in terms of the Dirichlet Green's function $G_{\varepsilon,D}$ for the half space. Similarly the solution to (4.5), (4.6) in the half space $x>0$ with Dirichlet condition $v_\varepsilon(0,t)=0,\ t>0$, has the representation

$$v_\varepsilon(x,T) \ = \ \int_0^\infty G_{\varepsilon,D}(x,y,0,T)v_0(y)\ dy, \quad x>0,\ T>0. \tag{4.15}$$

The function $G_{\varepsilon,D}(\cdot,y,t,T)$ is the probability density of the random variable $Y_\varepsilon(T)$ for solutions $Y_\varepsilon(s),\ s>t$, to (4.8) which have the property that $\inf_{t\le s\le T} Y_\varepsilon(s)>0$. No explicit formula for $G_{\varepsilon,D}(x,y,0,T)$ in the case of linear $b(y,t)=A(t)y-1$ is known except when $A(\cdot)\equiv 0$. In that case the method of images yields the formula (4.16)

$$G_{\varepsilon,D}(x,y,0,T)=\frac{1}{\sqrt{2\pi\varepsilon T}}\left\{\exp\left[-\frac{(x-y+T)^2}{2\varepsilon T}\right]-\exp\left[-\frac{2x}{\varepsilon}-\frac{(x+y-T)^2}{2\varepsilon T}\right]\right\} .$$

It follows from (4.11), (4.16) that

$$G_{\varepsilon,D}(x,y,0,T)/G_\varepsilon(x,y,0,T) \ = \ 1-\exp[-2xy/\varepsilon T] \ . \tag{4.17}$$

We may interpret the formula (4.17) in terms of conditional probability for solutions $Y_\varepsilon(s),\ s\ge 0$, of (4.8) with $b(\cdot,\cdot)\equiv -1$. Thus we have that

$$P(\inf_{0\le s\le T} Y_\varepsilon(s)>0 \mid Y_\varepsilon(0)=y,\ Y_\varepsilon(T)=x) \ = \ 1-\exp[-2xy/\varepsilon T] \ . \tag{4.18}$$

We wish to generalize (4.18) to the case of linear $b(y,t)=A(t)y-1$ in a way that is uniform as $\varepsilon\to 0$. To see what conditions on the function $A(\cdot)$ are needed we consider for $x,y\in\mathbf{R}, t<T$, the function $q(x,y,t)$ defined by the variational formula

$$q(x,y,t,T)=\min_{y(\cdot)}\left\{\frac{1}{2}\int_t^T\left[\frac{dy(s)}{ds}-b(y(s),s)\right]^2 ds \ \Big|\ y(t)=y,\ y(T)=x\right\}. \tag{4.19}$$

The Euler-Lagrange equation for the minimizing trajectory $y(\cdot)$ of (4.19) is

$$\frac{d}{ds}\left[\frac{dy(s)}{ds}-b(y(s),s)\right]+\frac{\partial b}{\partial y}(y(s),s)\left[\frac{dy(s)}{ds}-b(y(s),s)\right]=0, \quad t\le s\le T, \tag{4.20}$$

and we need to solve (4.20) for the function $y(\cdot)$ satisfying the boundary conditions $y(t)=y,\ y(T)=x$. In the case $b(y,t)=A(t)y-1$ equation (4.20) becomes

$$\left[-\frac{d^2}{ds^2}+A'(s)+A(s)^2\right]y(s) \ = \ A(s), \quad t\le s\le T. \tag{4.21}$$

It is easy to solve (4.21) with the given boundary conditions explicitly. In fact taking $t=0$ we see from (4.20) that

$$\frac{dy(s)}{ds}-b(y(s),s) \ = \ C(x,y,T)\exp\left[\int_s^T A(s')ds'\right] \ , \quad 0\le s\le T, \tag{4.22}$$

where the constant $C(x,y,T)$ is given by the formula

$$C(x,y,T) \ = \ [x+m_{2,A}(T)-m_{1,A}(T)y]/\sigma_A^2(T) \ , \tag{4.23}$$

with $m_{1,A}(T), m_{2,A}(T)$ as in (2.2) and $\sigma_A^2(T)$ as in (4.10). It follows from (4.11), (4.19), (4.22), (4.23) that the Green's function $G_\varepsilon(x,y,0,T)$ is given by the formula

$$G_\varepsilon(x,y,0,T)=\frac{1}{\sqrt{2\pi\varepsilon\sigma_A^2(T)}}\exp\left[-q(x,y,0,T)/\varepsilon\right] \ . \tag{4.24}$$

The minimizing trajectory $y(\cdot)$ for (4.19) has probabilistic significance as well as the function $q(x,y,t,T)$. One can easily see that for solutions $Y_\varepsilon(s),\ 0\le s\le T$, of (4.8) the random variable $Y_\varepsilon(s)$ conditioned on $Y_\varepsilon(0)=y,\ Y_\varepsilon(T)=x$, is Gaussian with mean and variance given by

$$E[Y_\varepsilon(s) \mid Y_\varepsilon(0)=y,\ Y_\varepsilon(T)=x] \ = \ y(s), \quad 0\le s\le T, \tag{4.25}$$

(4.26) $$\mathrm{Var}[Y_\varepsilon(s) \mid Y_\varepsilon(0)=y,\ Y_\varepsilon(T)=x] \ = \ \varepsilon\sigma_A^2(0,s)\sigma_A^2(s,T)/\sigma_A^2(T)\ ,$$

where the function $\sigma_A^2(s,t)$ is defined by

(4.27) $$\sigma_A^2(s,t) \ = \ \int_s^t \exp\left[2\int_{s'}^t A(s'')ds''\right] ds' \quad \text{for } s\le t.$$

Let $m_{1,A}(s,t),\ m_{2,A}(s,t)$ be defined by

(4.28) $$m_{1,A}(s,t)=\exp\left[\int_s^t A(s')ds'\right], \quad m_{2,A}(s,t)=\int_s^t \exp\left[\int_{s'}^t A(s'')ds''\right]ds' \quad \text{for } s\le t.$$

The minimizing trajectory $y(\cdot)$ for the variational problem (4.19) is explicitly given by the formula

(4.29) $$\begin{aligned}\sigma_A^2(T)y(s) \ &= \ xm_{1,A}(s,T)\sigma_A^2(0,s)+ym_{1,A}(0,s)\sigma_A^2(s,T) \\ &+ \ m_{1,A}(s,T)m_{2,A}(s,T)\sigma_A^2(0,s)-m_{2,A}(0,s)\sigma_A^2(s,T)\ .\end{aligned}$$

Now the process $Y_\varepsilon(s),\ 0\le s\le T$, conditioned on $Y_\varepsilon(0)=y,\ Y_\varepsilon(T)=x$, is in fact a Gaussian process with covariance independent of x,y,

(4.30) $$\mathrm{Covar}[Y_\varepsilon(s_1),Y_\varepsilon(s_2) \mid Y_\varepsilon(0)=y,\ Y_\varepsilon(T)=x] \ = \ \varepsilon\Gamma_A(s_1,s_2)\ , \quad 0\le s_1,s_2\le T,$$

where the symmetric function $\Gamma:[0,T]\times[0,T]\to\mathbf{R}$ is given by the formula

(4.31) $$\Gamma_A(s_1,s_2) \ = \ \frac{m_{1,A}(s_1,s_2)\sigma_A^2(0,s_1)\sigma_A^2(s_2,T)}{\sigma_A^2(T)}\ , \quad 0\le s_1\le s_2\le T.$$

The function Γ_A is the Dirichlet Green's function for the operator on the LHS of (4.21). Thus one has that

(4.32) $$\left[-\frac{d^2}{ds_1^2}+A'(s_1)+A(s_1)^2\right]\Gamma_A(s_1,s_2) \ = \ \delta(s_1-s_2), \quad 0<s_1,s_2<T,$$

and $\Gamma_A(0,s_2)=\Gamma_A(T,s_2)=0$ for all $0<s_2<T$.

We can obtain a representation of the conditioned process $Y_\varepsilon(\cdot)$ in terms of the white noise process, which is the derivative $dB(\cdot)$ of Brownian motion, by obtaining a factorization of Γ corresponding to the factorization

(4.33) $$-\frac{d^2}{ds^2}+A'(s)+A(s)^2 \ = \ \left[-\frac{d}{ds}-A(s)\right]\left[\frac{d}{ds}-A(s)\right]\ .$$

To do this we note that the boundary value problem

(4.34) $$\left[\frac{d}{ds}-A(s)\right]u(s) \ = \ v(s), \quad 0<s<T, \quad u(0)=u(T)=0,$$

has a solution if and only if the function $v:[0,T]\to\mathbf{R}$ satisfies the orthogonality condition

(4.35) $$\int_0^T \frac{v(s)}{m_{1,A}(s)}\ ds \ = \ 0.$$

Hence it follows from (4.33) that we can solve the boundary value problem

(4.36) $$\left[-\frac{d^2}{ds^2}+A'(s)+A(s)^2\right]u(s) \ = \ f(s), \quad 0<s<T, \quad u(0)=u(T)=0,$$

by first finding the solution $v : [0,T] \to \mathbf{R}$ to

$$\left[-\frac{d}{ds} - A(s) \right] v(s) \;=\; f(s), \quad 0 < s < T, \tag{4.37}$$

which satisfies the orthogonality condition (4.35). Then we solve the differential equation in (4.34) subject to the condition $u(0) = 0$.

The solution to (4.35), (4.37) is given by an expression

$$v(s) \;=\; K^* f(s) \;=\; \int_0^T k(s', s) f(s') \, ds' \ , \quad 0 \le s \le T, \tag{4.38}$$

where the kernel $k : [0,T] \times [0,T] \to \mathbf{R}$ is defined by

$$k(s', s) \;=\; \frac{m_{1,A}(s,s')\sigma^2(s',T)}{\sigma_A^2(T)} \quad \text{if } s' > s, \tag{4.39}$$

$$k(s', s) \;=\; \frac{\sigma_A^2(s',T)}{m_1(s',s)\sigma_A^2(T)} - \frac{1}{m_{1,A}(s',s)} \quad \text{if } s' < s.$$

If $v : [0,T] \to \mathbf{R}$ satisfies the condition (4.35) then

$$u(s) \;=\; Kv(s) \;=\; \int_0^T k(s, s') v(s') \, ds' \ , \quad 0 \le s \le T, \tag{4.40}$$

is the solution to (4.34). It follows that the kernel Γ_A of (4.31) has the factorization $\Gamma_A = KK^*$, and so the conditioned process $Y_\varepsilon(\cdot)$ has the representation

$$Y_\varepsilon(s) \;=\; y(s) + \sqrt{\varepsilon} \int_0^T k(s, s') \, dB(s') \ , \quad 0 \le s \le T, \tag{4.41}$$

where $y(\cdot)$ is the function (4.29). In the case $A(\cdot) \equiv 0$ equation (4.41) yields the familiar representation

$$Y_\varepsilon(s) \;=\; \frac{s}{T} x + \left(1 - \frac{s}{T} \right) y + \sqrt{\varepsilon} \left[B(s) - \frac{s}{T} B(T) \right] \ , \quad 0 \le s \le T, \tag{4.42}$$

for the Brownian bridge process.

We can obtain an alternative representation of the conditioned process $Y_\varepsilon(\cdot)$ in terms of Brownian motion by considering a stochastic control problem. Let $Y_\varepsilon(\cdot)$ be the solution to the stochastic differential equation

$$dY_\varepsilon(s) = \lambda_\varepsilon(\cdot, s) ds + \sqrt{\varepsilon} \, dB(s), \tag{4.43}$$

where $\lambda_\varepsilon(\cdot, s)$ is a non-anticipating function. We consider the problem of minimizing the cost function given by the formula

$$q_\varepsilon(x, y, t, T) = \min_{\lambda_\varepsilon} E \left[\frac{1}{2} \int_t^T \left[\lambda_\varepsilon(\cdot, s) - b(Y_\varepsilon(s), s) \right]^2 \ ds \ \middle| \ Y_\varepsilon(t) = y, \ Y_\varepsilon(T) = x \right]. \tag{4.44}$$

The minimum in (4.44) is to be taken over all non-anticipating $\lambda_\varepsilon(\cdot, s)$, $t \le s < T$, which have the property that the solutions of (4.43) with initial condition $Y_\varepsilon(t) = y$ satisfy the terminal condition $Y_\varepsilon(T) = x$ with probability 1. Formally the optimal controller λ^* for the problem is given by the expression

$$\lambda_\varepsilon(\cdot, s) = \lambda_\varepsilon^*(x, Y_\varepsilon(s), s) = b(Y_\varepsilon(s), s) - \frac{\partial q_\varepsilon}{\partial y} \ (x, Y_\varepsilon(s), s). \tag{4.45}$$

Evidently in the classical control case $\varepsilon = 0$ the solution to (4.43), (4.44) is the solution to the variational problem (4.19). If $b(y, t) = A(t)y - 1$ is a linear function of y then one expects as in the case of LQ problems that the difference between the cost functions for the classical and stochastic control problems is independent of y. Therefore from (4.11), (4.24) we expect that

(4.46)
$$\lambda_\varepsilon^*(x,y,t) \ = \ b(y,t) - \frac{\partial q(x,y,t,T)}{\partial y} \ = \ A(t)y - 1 - \frac{\partial}{\partial y}\frac{\{x+m_{2,A}(t,T)-m_{1,A}(t,T)y\}^2}{2\sigma_A^2(t,T)} \ .$$

It is easy to see that if we solve the SDE (4.43) with controller given by (4.46) and conditioned on $Y_\varepsilon(t)=y$ then $Y_\varepsilon(T)=x$ with probability 1 and in fact the process $Y_\varepsilon(s),\ t\le s\le T$, has the same distribution as the process $Y_\varepsilon(s),\ t\le s\le T$, satisfying the SDE (4.8) conditioned on $Y_\varepsilon(t)=y,\ Y_\varepsilon(T)=x$. Thus we have obtained the Markovian representation for the conditioned process of (4.8). Note however that the stochastic control problem with cost function (4.44) does not have a solution since the integral in (4.44) is logarithmically divergent at $s=T$ for the process (4.43) with optimal controller (4.46).

Solving (4.43) with drift (4.46) and $Y_\varepsilon(0)=y$, we see on taking $t=0$ that (4.41) holds with kernel $k:[0,T]\times[0,T]\to\mathbf{R}$ given by

$$k(s,s') \ = \ \frac{m_{1,A}(s',s)\sigma_A^2(s,T)}{\sigma_A^2(s',T)} \quad \text{if } s'<s, \qquad k(s,s') \ = \ 0 \quad \text{if } s'>s. \tag{4.47}$$

Observe that the kernel (4.47) corresponds to the Cholesky factorization $\Gamma_A=KK^*$ of the kernel Γ_A [3]. In the case $A(\cdot)\equiv 0$ equation (4.47) yields the Markovian representation

$$Y_\varepsilon(s) \ = \ \frac{s}{T}x + \left(1-\frac{s}{T}\right)y + \sqrt{\varepsilon}(T-s)\int_0^s \frac{dB(s')}{T-s'} \ , \quad 0\le s\le T, \tag{4.48}$$

for the Brownian bridge process.

We can also express the ratio (4.17) of Green's functions for the linear case $b(y,t)=A(t)y-1$ in terms of the solution to a PDE. Thus we assume $x>0$ and define

$$u(y,t) \ = \ P(\inf_{t\le s\le T} Y_\varepsilon(s)>0 \mid Y_\varepsilon(t)=y) \ , \quad y>0, t<T, \tag{4.49}$$

where $Y_\varepsilon(\cdot)$ is the solution to the SDE (4.43) with drift (4.46). Then $u(y,t)$ is the solution to the PDE

$$\frac{\partial u(y,t)}{\partial t} + \lambda_\varepsilon^*(x,y,t)\frac{\partial u(y,t)}{\partial y} + \frac{\varepsilon}{2}\frac{\partial^2 u(y,t)}{\partial y^2} \ = \ 0, \quad y>0, t<T, \tag{4.50}$$

with boundary and terminal conditions given by

$$u(0,t) \ = \ 0 \text{ for } t<T, \quad \lim_{t\to T} u(y,t) \ = \ 1 \text{ for } y>0. \tag{4.51}$$

In the case $A(\cdot)\equiv 0$ the PDE (4.50) becomes

$$\frac{\partial u(y,t)}{\partial t} + \left(\frac{x-y}{T-t}\right)\frac{\partial u(y,t)}{\partial y} + \frac{\varepsilon}{2}\frac{\partial^2 u(y,t)}{\partial y^2} \ = \ 0, \quad y>0, t<T. \tag{4.52}$$

Evidently the function u defined by

$$u(y,t) \ = \ 1 - \ \exp\left[-\frac{2xy}{\varepsilon(T-t)}\right] \ , \quad t<T, y>0, \tag{4.53}$$

is the solution to (4.51), (4.53). Observe that the RHS of (4.53) at $t=0$ is the same as the RHS of (4.17).

5. Estimates on the Dirichlet Green's function

In this section we shall obtain estimates on the ratio of the Dirichlet to the full space Green's function in the case of linear drift $b(y,t) = A(t)y - 1$. In particular we shall prove a limit theorem which generalizes the formula (4.17):

Proposition 5.1. *Assume $b(y,t) = A(t)y - 1$ where (4.1) holds and the function $A(\cdot)$ is non-negative. Then for $\lambda, y, T > 0$ the ratio of the Dirichlet to full space Green's function satisfies the limit*

$$\lim_{\varepsilon\to 0} \frac{G_{\varepsilon,D}(\lambda\varepsilon, y, 0, T)}{G_\varepsilon(\lambda\varepsilon, y, 0, T)} \ = \ 1 - \exp\left[-2\lambda\left\{1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right\}\ \right] \ , \tag{5.1}$$

where $m_{1,A}(T), m_{2,A}(T)$ are given by (2.2) and $\sigma_A^2(T)$ by (4.10).

Note that since we are assuming $A(\cdot)$ is non-negative in the statement of the proposition, it follows from (4.10) that $m_{2,A}(T)/\sigma_A^2(T) \le 1$. Hence the RHS of (5.1) always lies between 0 and 1. We can see why (5.1) holds from the representation (4.39), (4.41) for the conditioned process $Y_\varepsilon(s),\ 0 \le s \le T$. Thus we have that

$$Y_\varepsilon(s) \ = \ y(s) + \sqrt{\varepsilon}\left[\ \frac{m_{1,A}(s)\sigma_A^2(s,T)}{\sigma_A^2(T)}\int_0^T \ \frac{dB(s')}{m_{1,A}(s')} - m_{1,A}(s)\int_s^T \ \frac{dB(s')}{m_{1,A}(s')}\ \right] \ . \tag{5.2}$$

Since $\sigma_A^2(s,T) = O(T-s)$ the conditioned process $Y_\varepsilon(s)$ close to $s = T$ is approximately the same as

$$Y_\varepsilon(s) \ = \ \lambda\varepsilon - y'(T)(T-s) - \sqrt{\varepsilon}\int_s^T \ dB(s') \ . \tag{5.3}$$

Observe now from (4.29) that

$$-\,y'(T) \ = \ O(\varepsilon) + 1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)} \ . \tag{5.4}$$

Hence for s close to T the process $Y_\varepsilon(s),\ s < T$, is approximately Brownian motion with a constant drift. Thus let $Z_\varepsilon(t),\ t > 0$, be the solution to the initial value problem for the SDE

$$dZ_\varepsilon(t) \ = \ \mu dt + \sqrt{\varepsilon}\ dB(t), \quad Z_\varepsilon(0) = \lambda\varepsilon \ , \tag{5.5}$$

where we assume the drift μ is positive. Then from (5.3), (5.4) we see that $Y_\varepsilon(T-t) \simeq Z_\varepsilon(t)$ if μ is given by the formula

$$\mu \ = \ 1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)} \ . \tag{5.6}$$

Observe now that $P(\inf_{t>0} Z_\varepsilon(t) < 0) = e^{-2\lambda\mu}$, whence the RHS of (5.1) is simply $P(\inf_{t>0} Z_\varepsilon(t) > 0)$ when μ is given by (5.6). Since the time for which $Z_\varepsilon(t)$ is likely to become negative is $t \simeq O(\varepsilon)$ the approximations above are justified and so we obtain (5.1).

Proof of Proposition 5.1. Let $Y_\varepsilon(s),\ 0 \le s \le T$, be given by (5.2) where $y(T) = \lambda\varepsilon$. Then we have that for $0 < a\varepsilon \le T$,

$$P\left(\inf_{0\le s\le T} Y_\varepsilon(s) > 0\right) \ \le \ P\left(\inf_{0\le t\le a\varepsilon} Y_\varepsilon(T-t) > 0\right) \ = \ P\left(\inf_{0<t<a\varepsilon}[Z_\varepsilon(t) + \tilde{Z}_\varepsilon(t)] > 0\right) \ , \tag{5.7}$$

where $Z_\varepsilon(\cdot)$ is the solution to (5.5) with μ given by (5.6) and $\tilde{Z}_\varepsilon(\cdot)$ is given by the formula

$$\tilde{Z}_\varepsilon(t) \ = \ y(T-t)-y(T)+y'(T)t+ \tag{5.8}$$

$$\sqrt{\varepsilon}\left[\ \frac{m_{1,A}(T-t)\sigma_A^2(T-t,T)}{\sigma_A^2(T)}\int_0^T \ \frac{dB(s')}{m_{1,A}(s')}+\int_{T-t}^T\left[1-\frac{m_{1,A}(T-t)}{m_{1,A}(s')}\right] \ dB(s') \ \right] \ .$$

We use the inequality
(5.9)

$$P\left(\inf_{0<t<a\varepsilon}[Z_\varepsilon(t)+\tilde{Z}_\varepsilon(t)]>0\right) \ \le \ P\left(\inf_{0<t<a\varepsilon}Z_\varepsilon(t)>-b\lambda\varepsilon\right)+P\left(\sup_{0<t<a\varepsilon}\tilde{Z}_\varepsilon(t)>b\lambda\varepsilon\right) \ ,$$

which holds for any $a,b>0$ satisfying $a\varepsilon\le T$.

To estimate the first term on the RHS of (5.9) we observe by the method of images that

$$P\left(\inf_{0<t<a\varepsilon}Z_\varepsilon(t)<-b\lambda\varepsilon\right) \ = \tag{5.10}$$

$$e^{-2\mu(1+b)\lambda}\frac{1}{\sqrt{2\pi}}\int_{[(1+b)\lambda-\mu a]/\sqrt{a}}^\infty e^{-z^2/2} \ dz \ + \ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-[(1+b)\lambda+\mu a]/\sqrt{a}} e^{-z^2/2} \ dz \ .$$

To estimate the second term we write $\tilde{Z}_\varepsilon(t)$ in (5.8) as a sum of three quantities. The first of these is bounded as

$$\sup_{0\le t\le a\varepsilon}|y(T-t)-y(T)+y'(T)t| \ \le \ C[\lambda\varepsilon+y+1]a^2\varepsilon^2, \quad 0<a\varepsilon\le T, \tag{5.11}$$

for a constant C depending only on A_∞,T. The second is bounded as
(5.12)

$$\sup_{0\le t\le a\varepsilon}\left|\sqrt{\varepsilon} \ \frac{m_{1,A}(T-t)\sigma_A^2(T-t,T)}{\sigma_A^2(T)}\int_0^T \ \frac{dB(s')}{m_{1,A}(s')}\right| \ \le \ Ca\varepsilon^{3/2}\left|\int_0^T \ \frac{dB(s')}{m_{1,A}(s')}\right| \ ,$$

where C depends only on A_∞,T. Finally the third quantity is bounded as

(5.13)

$$\sup_{0\le t\le a\varepsilon}\left| \ \int_{T-t}^T\left[1-\frac{m_{1,A}(T-t)}{m_{1,A}(s')}\right] \ dB(s') \ \right| \ \le \ \sup_{0\le t\le a\varepsilon}\left| \ \int_{T-t}^T\left[1-\frac{m_{1,A}(T)}{m_{1,A}(s')}\right] \ dB(s') \ \right|$$

$$+\,Ca\varepsilon\sup_{0\le t\le a\varepsilon}\left|\int_{T-t}^T \ \frac{dB(s')}{m_{1,A}(s')}\right| \ , \quad \text{where } C \text{ depends only on } A_\infty,T.$$

We can estimate probabilities for the terms on the RHS of (5.12), (5.13) by using Martingale properties. Thus if $g:(-\infty,T)\to\mathbf{R}$ is a continuous function we define $X(t), \ t\ge 0$, by

$$X(t) \ = \ \int_{T-t}^T g(s) \ dB(s) \ . \tag{5.14}$$

Then for $\theta\in\mathbf{R}$
(5.15)

$$X_\theta(t) \ = \ \exp\left[\theta X(t)-\frac{\theta^2}{2}\int_{T-t}^T ds \ g(s)^2 \ \right] \quad \text{is a Martingale and } E[X_\theta(t)]=1.$$

Using the inequality

$$P\left(\ |X(0)| > M \ \right) \ \le \ 2\exp\left[-\theta M + \frac{\theta^2}{2}\int_0^T ds \ g(s)^2 \right] \quad \text{for } M,\theta > 0, \tag{5.16}$$

and optimizing the RHS of (5.16) with respect to $\theta > 0$ we conclude that

$$P\left(a\varepsilon^{3/2}\left| \int_0^T \frac{dB(s')}{m_{1,A}(s')}\right| > b\lambda\varepsilon/4 \right) \ \le \ 2\exp\left[-Cb^2\lambda^2/a^2\varepsilon \right] \ , \tag{5.17}$$

where the constant $C > 0$ depends only on A_∞, T. We use Doob's inequality to estimate probabilities for the terms on the RHS of (5.13). Thus we have for $\theta > 0$ that

$$\begin{aligned} P\left(\sup_{0\le t\le t_0} X(t) > M \right) \ &\le \ P\left(\sup_{0\le t\le t_0} X_\theta(t) > \exp\left[\theta M - \frac{\theta^2}{2}\int_{T-t_0}^T ds \ g(s)^2 \right] \right) \\ &\le \ \exp\left[-\theta M + \frac{\theta^2}{2}\int_{T-t_0}^T ds \ g(s)^2 \right] \ . \end{aligned} \tag{5.18}$$

Optimizing the term on the RHS of (5.18) with respect to $\theta > 0$ we conclude that

$$P\left(\sup_{0\le t\le t_0} |X(t)| > M \right) \ \le \ 2\exp\left[-M^2 \Big/ 2\int_{T-t_0}^T ds \ g(s)^2 \right] \ . \tag{5.19}$$

Hence we have from (5.19) for the first term on the RHS of (5.13) that

$$P\left(\sup_{0\le t\le a\varepsilon}\left| \int_{T-t}^T \left[1 - \frac{m_{1,A}(T)}{m_{1,A}(s')} \right] dB(s') \right| \ > \ b\lambda\varepsilon/4 \right) \ \le \ 2\exp\left[-Cb^2\lambda^2/a^3\varepsilon \right] \ , \tag{5.20}$$

where the constant $C > 0$ depends only on A_∞, T. Similarly we have that if C_1 depends only on A_∞, T then

$$P\left(C_1 a\varepsilon \sup_{0\le t\le a\varepsilon}\left| \int_{T-t}^T \frac{dB(s')}{m_{1,A}(s')} \right| \ > \ b\lambda\varepsilon/4 \right) \ \le \ 2\exp\left[-C_2 b^2\lambda^2/a^3\varepsilon \right] \ , \tag{5.21}$$

where the constant $C_2 > 0$ also depends only on A_∞, T.

We choose now $a = \varepsilon^{-\alpha}, b = \varepsilon^{\beta}$ for some $\alpha, \beta > 0$. Since $\mu > 0$ it follows from (5.10) that the first term on the RHS of (5.9) converges to $1 - e^{-2\lambda\mu}$ as $\varepsilon \to 0$. We also see from the estimates of the previous paragraph that the second term on the RHS of (5.9) converges to 0 as $\varepsilon \to 0$ provided $3\alpha + 2\beta < 1$. We have therefore shown that $\limsup_{\varepsilon\to 0} P\left(\inf_{0\le s\le T} Y_\varepsilon(s) > 0 \right)$ is bounded above by the RHS of (5.1).

To obtain the corresponding lower bound we use the inequality

$$P\left(\inf_{0\le s\le T} Y_\varepsilon(s) > 0 \right) \ \ge \ P\left(\inf_{T-a\varepsilon\le s\le T} Y_\varepsilon(s) > 0 \right) - P\left(\inf_{0\le s\le T-a\varepsilon} Y_\varepsilon(s) < 0 \right) \ . \tag{5.22}$$

Next we use the inequality similar to (5.9) that

$$P\left(\inf_{T-a\varepsilon\le s\le T} Y_\varepsilon(s) > 0 \right) \ \ge \ P\left(\inf_{0<t<a\varepsilon} Z_\varepsilon(t) > b\lambda\varepsilon \right) - P\left(\inf_{0<t<a\varepsilon} \tilde{Z}_\varepsilon(t) < -b\lambda\varepsilon \right) \ . \tag{5.23}$$

Arguing as previously we see from (5.23) on choosing $a = \varepsilon^{-\alpha}, b = \varepsilon^{\beta}$ with $3\alpha+2\beta < 1$ that $\liminf_{\varepsilon\to 0} P\left(\inf_{T-\varepsilon^{1-\alpha}\le s\le T} Y_\varepsilon(s) > 0\right)$ is bounded below by the RHS of (5.1). Next we need to obtain a bound on the second term on the RHS of (5.22) when $a = \varepsilon^{-\alpha}$ which vanishes as $\varepsilon \to 0$. Since $A(\cdot)$ is non-negative there is a positive constant C depending only on A_∞, T such that the function $y(\cdot)$ of (4.29) satisfies an inequality $y(s) \ge C(T-s)y$ for $0 \le s \le T$. Hence there is a positive constant c depending only on A_∞, T such that

$$(5.24)\quad P\left(\inf_{0\le s\le T-\varepsilon^{1-\alpha}} Y_\varepsilon(s) < 0\right) \ \le\ P\left(\left|\int_0^T \frac{dB(s')}{m_{1,A}(s')}\right| > \frac{cy}{\sqrt{\varepsilon}}\right) + P\left(\sup_{\varepsilon^{1-\alpha}\le t\le T}\left|\frac{1}{t}\int_{T-t}^T \frac{dB(s')}{m_{1,A}(s')}\right| > \frac{cy}{\sqrt{\varepsilon}}\right) .$$

We can bound the first term on the RHS of (5.24) similarly to (5.17). We bound the second term by using the inequality

$$(5.25)\quad P\left(\sup_{\varepsilon^{1-\alpha}\le t\le T}|X(t)| > cy/\sqrt{\varepsilon}\right) \ \le\ \sum_{k\ge 1} P\left(\sup_{k\varepsilon^{1-\alpha}\le t\le (k+1)\varepsilon^{1-\alpha}}|X(t)| > cy/\sqrt{\varepsilon}\right) .$$

From (5.19) we see that for $k \ge 1$,

$$(5.26)\quad P\left(\sup_{k\varepsilon^{1-\alpha}\le t\le (k+1)\varepsilon^{1-\alpha}}\left|\frac{1}{t}\int_{T-t}^T \frac{dB(s')}{m_{1,A}(s')}\right| > cy/\sqrt{\varepsilon}\right) \ \le\ \exp\left[-\frac{c_1 k y^2}{\varepsilon^\alpha}\right] ,$$

where $c_1 > 0$ depends only on A_∞, T. We conclude that the second term on the RHS of (5.22) converges when $a = \varepsilon^{-\alpha}$ with $\alpha > 0$ to zero as $\varepsilon \to 0$. Hence $\liminf_{\varepsilon\to 0} P\left(\inf_{0\le s\le T} Y_\varepsilon(s) > 0\right)$ is bounded below by the RHS of (5.1). □

Next we wish to obtain estimates on the LHS of (5.1) which are uniform as $\lambda \to 0$.

Lemma 5.1. *Assume the function $A(\cdot)$ is non-negative and that $0 < \lambda \le 1,\ 0 < \varepsilon \le T,\ y > 0$. Let $\Gamma : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be the function $\Gamma(a,b) = 1$ if $b > a^{-1/4}$ and otherwise $\Gamma(a,b) = a^{1/8}$. Then there is a constant C depending only on $A_\infty T$ such that*

$$(5.27)\quad \frac{G_{\varepsilon,D}(\lambda\varepsilon, y, 0, T)}{G_\varepsilon(\lambda\varepsilon, y, 0, T)} \ \le\ 1 - \exp\left[-2\lambda\left\{1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right\}\right] + C\lambda\Gamma\left(\frac{\varepsilon}{T}, \frac{y}{T}\right)\left[1 + \frac{y}{T}\right] .$$

Proof. We make the change of variable $s \leftrightarrow t$ in which

$$(5.28)\quad \frac{ds}{dt} \ =\ -\left[\frac{m_{1,A}(s)}{m_{1,A}(T)}\right]^2 , \quad s(0) \ =\ T.$$

Hence $s \simeq T - t$ if t is small and

$$(5.29)\quad m_{1,A}(T)\int_s^T \frac{dB(s')}{m_{1,A}(s')} \ =\ \int_0^t d\tilde{B}(t') \quad \text{where } \tilde{B}(\cdot) \text{ is a Brownian motion.}$$

Letting $s(\tilde{T})=0$, we see from (5.2), (5.29) that $Y_\varepsilon(s)=\tilde{Y}_\varepsilon(t)$ where

$$\tilde{Y}_\varepsilon(t) \ = \ \tilde{y}(t)+\sqrt{\varepsilon}\left[\frac{m_{1,A}(s)\sigma^2(s,T)}{m_{1,A}(T)\sigma^2(T)}\int_0^{\tilde{T}} d\tilde{B}(t') - \frac{m_{1,A}(s)}{m_{1,A}(T)}\int_0^t d\tilde{B}(t')\right] \ , \tag{5.30}$$

and $\tilde{y}(t)=y(s)$, where $y(\cdot)$ is the function (4.29). We consider any a for which $0<a\varepsilon\le\tilde{T}$ and observe as in (5.7) that if $M>0$ then

$$\begin{aligned} P\left(\inf_{0\le s\le T} Y_\varepsilon(s)>0\right) \ &\le \ P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_\varepsilon(t)>0\right) \\ &\le \ P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_\varepsilon(t)>0; \ \sup_{0\le t\le a\varepsilon}\left|\int_0^t d\tilde{B}(t')\right|\le M\right) \\ &\quad + P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_\varepsilon(t)>0; \ \sup_{0\le t\le a\varepsilon}\left|\int_0^t d\tilde{B}(t')\right|> M\right) \ . \end{aligned} \tag{5.31}$$

The first term on the RHS of (5.31) is bounded above by $P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_{0,\varepsilon}(t)>0\right)$ where $\tilde{Y}_{0,\varepsilon}(t)$ is given from (5.30) by the formula

$$\tilde{Y}_{0,\varepsilon}(t) \ = \ \tilde{y}(t)+\frac{C\sqrt{\varepsilon}Mt}{T}+\sqrt{\varepsilon}\,\frac{m_{1,A}(s)\sigma_A^2(s,T)}{m_{1,A}(T)\sigma_A^2(T)}\int_{a\varepsilon}^{\tilde{T}} d\tilde{B}(t')-\sqrt{\varepsilon}\frac{m_{1,A}(s)}{m_{1,A}(T)}\int_0^t d\tilde{B}(t')\,, \tag{5.32}$$

with C in (5.32) depending only on AT. To estimate the second term on the RHS of (5.31) we introduce the stopping time τ defined by

$$\tau=\inf\left\{ \ t<\tilde{T}: \ \left|\int_0^t d\tilde{B}(t')\right|>M\right\} \ . \tag{5.33}$$

Hence the second term is bounded above by $P\left(\inf_{0\le t\le\tau}\tilde{Y}_\varepsilon(t)>0; \ \tau<a\varepsilon \ \right)$. Observe now that for any $M_1>0$,

$$\begin{aligned} &P\left(\inf_{0\le t\le\tau}\tilde{Y}_\varepsilon(t)>0; \ \tau<a\varepsilon \ \right) \ = \\ &\sum_{n=1}^\infty P\left(\inf_{0\le t\le\tau}\tilde{Y}_\varepsilon(t)>0; \ \tau<a\varepsilon, \ (n-1)M_1\le\sup_{\tau\le t\le\tau+\tilde{T}}\left|\int_\tau^t d\tilde{B}(t')\right|<nM_1 \ \right) \ \le \\ &\sum_{n=1}^\infty P\left(\inf_{0\le t\le\tau}\tilde{Y}_{n,\varepsilon}(t)>0; \ \tau<a\varepsilon \ \right)P\left(\ (n-1)M_1\le\left|\sup_{\tau\le t\le\tau+\tilde{T}}\int_\tau^t d\tilde{B}(t') \ \right|<nM_1 \ \right) \\ &\ =\sum_{n=1}^\infty P\left(\inf_{0\le t\le\tau}\tilde{Y}_{n,\varepsilon}(t)>0; \ \tau<a\varepsilon \ \right)P\left(\ (n-1)M_1\le\left|\sup_{0\le t\le\tilde{T}}\int_0^t d\tilde{B}(t') \ \right|<nM_1 \ \right) \ , \end{aligned} \tag{5.34}$$

where $\tilde{Y}_{n,\varepsilon}$ is given by the formula

$$\tilde{Y}_{n,\varepsilon}(t) \ = \ \tilde{y}(t)+\frac{C\sqrt{\varepsilon}(M+nM_1)t}{T}-\sqrt{\varepsilon}\frac{m_{1,A}(s)}{m_{1,A}(T)}\int_0^t d\tilde{B}(t') \ , \tag{5.35}$$

and the constant C depends only on $A_\infty T$. Note that in (5.34) we are using the fact that the variables

$$\tau \ \text{and} \ \{\tilde{B}(t): \ 0<t\le\tau\} \quad \text{are independent of the variable} \ \left|\sup_{\tau\le t\le\tau+\tilde{T}}\int_\tau^t d\tilde{B}(t') \ \right| \ . \tag{5.36}$$

To estimate $P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_{0,\varepsilon}(t)>0\right)$ we compare $\tilde{Y}_{0,\varepsilon}(\cdot)$ to Brownian motion with constant drift as in (5.5). It follows from (5.32) that
(5.37)

$$P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_{0,\varepsilon}(t)>0\right)\ \le\ E\left[\ P\left(\inf_{0\le t\le a\varepsilon}Z_\varepsilon(t)>0\ \Big|\ \mu=\mu_{\rm rand},\ Z_\varepsilon(0)=\lambda\varepsilon[1+Ca\varepsilon/T]\ \right)\ \right]\ ,$$

where $\mu_{\rm rand}$ is the random variable
(5.38)

$$\mu_{\rm rand}\ =\ 1-\frac{m_{2,A}(T)}{\sigma_A^2(T)}+\frac{m_{1,A}(T)y}{\sigma_A^2(T)}+\frac{Ca\varepsilon}{T}\left[1+\frac{y}{T}\right]+\frac{C\lambda\varepsilon}{T}+\frac{C\sqrt{\varepsilon}}{T}\left[M+\left|\int_{a\varepsilon}^{\tilde{T}}d\tilde{B}(t')\right|\right]\ ,$$

and $C>0$ is a constant depending only on $A_\infty T$. To bound the RHS of (5.37) we use an identity similar to (5.10),

$$(5.39)\quad P\left(\inf_{0<t<a'\varepsilon}Z_\varepsilon(t)>0\ \Big|\ Z_\varepsilon(0)=\lambda'\varepsilon\ \right)\ =$$

$$\left\{1-e^{-2\mu\lambda'}\right\}\frac{1}{\sqrt{2\pi}}\int_{[\lambda'-\mu a']/\sqrt{a'}}^{\infty}e^{-z^2/2}\ dz\ +\ \frac{1}{\sqrt{2\pi}}\int_{[-\lambda'-\mu a']/\sqrt{a'}}^{[\lambda'-\mu a']/\sqrt{a'}}e^{-z^2/2}\ dz\ .$$

From (5.39) we obtain the upper bound

$$(5.40)\qquad P\left(\inf_{0<t<a'\varepsilon}Z_\varepsilon(t)>0\ \Big|\ Z_\varepsilon(0)=\lambda'\varepsilon\ \right)\ \le\ 1-e^{-2\mu\lambda'}+\frac{2\lambda'}{\sqrt{2\pi a'}}\ .$$

Using (5.40) we estimate the RHS of (5.37) when $a=\min\left[(T/\varepsilon)^\alpha,\ \tilde{T}/\varepsilon\right]$ for some α satisfying $0<\alpha<1$. In that case $\lambda'=\lambda[1+Ca\varepsilon/T]\le\lambda[1+C]$ for some constant C depending only on $A_\infty T$. Taking $M=C_1\sqrt{T}$ in (5.38) where C_1 depends only on $A_\infty T$ we conclude from (5.37), (5.40) that for $0<\lambda\le 1,\ 0<\varepsilon\le T$,

$$(5.41)\quad P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_{0,\varepsilon}(t)>0\right)\ \le$$

$$1-\exp\left[-2\lambda\left\{1-\frac{m_{2,A}(T)}{\sigma_A^2(T)}+\frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right\}\ \right]+C_2\lambda\left[\left(\frac{\varepsilon}{T}\right)^{1-\alpha}\left\{1+\frac{y}{T}\right\}+\left(\frac{\varepsilon}{T}\right)^{\alpha/2}\right]\ ,$$

where C_2 in (5.41) depends only on $A_\infty T$.

Next we estimate the probabilities on the RHS of (5.34). Evidently we have from (5.35) that

$$(5.42)\qquad \tilde{Y}_{n,\varepsilon}(\tau)\ =\ \tilde{y}(\tau)+\frac{C\sqrt{\varepsilon}(M+nM_1)\tau}{T}\pm M\sqrt{\varepsilon}\frac{m_{1,A}(s(\tau))}{m_{1,A}(T)}\ .$$

We choose $M_1=\sqrt{T}$ in (5.42) and $M=C_1\sqrt{T}$ for a constant C_1 depending only on $A_\infty T$ so that $M\sqrt{\varepsilon}/m_1(T)>2\varepsilon$. Since $Y_{n,\varepsilon}(\tau)>0$, it follows that if (5.42) holds with the $-$ sign then there is a constant $c>0$ depending only on $A_\infty T$ such that

$$(5.43)\qquad \tau>\ \tau_n\ =\ cT\sqrt{\frac{\varepsilon}{T}}\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \right]^{-1}\ .$$

Observe now from (5.43) that if $\alpha<1/2$ then $\tau_n>a\varepsilon$ provided

$$(5.44)\quad 1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \le\ 2c_1\left(\frac{T}{\varepsilon}\right)^{1/2-\alpha}\qquad \text{for } c_1>0 \text{ depending only on } A_\infty T.$$

Since $\tau < a\varepsilon$ it follows that (5.42) can hold with the minus sign only if

$$1+\frac{y}{T} \ \ge\ c_1\left(\frac{T}{\varepsilon}\right)^{1/2-\alpha} \quad \text{or } n \ \ge\ c_1\left(\frac{T}{\varepsilon}\right)^{1-\alpha} . \tag{5.45}$$

In the case when $\tau_n < a\varepsilon$ we see from (5.35) that there is a constant C depending only on $A_\infty T$ and

$$P\left(\inf_{0\le t\le\tau_n}\tilde{Y}_{n,\varepsilon}(t)>0\right) \ \le\ P\left(\inf_{0\le t\le\tau_n}Z_\varepsilon(t)>0 \ \Big|\ \mu=\mu_n,\ Z_\varepsilon(0)=\lambda\varepsilon[1+Ca\varepsilon/T]\right) , \tag{5.46}$$

where $Z_\varepsilon(\cdot)$ is the solution to the SDE (5.5). The drift μ_n is given by the formula

$$\mu_n \ = \ C\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \right] \quad \text{where } C \text{ depends only on } A_\infty T. \tag{5.47}$$

It follows then from (5.40), (5.46), (5.47) that

$$P\left(\inf_{0\le t\le\tau_n}\tilde{Y}_{n,\varepsilon}(t)>0\right) \ \le\ C_1\lambda\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}+\left(\frac{\varepsilon}{\tau_n}\right)^{1/2}\ \right] \ \le\ C_2\lambda\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \right] \tag{5.48}$$

for some constants C_1, C_2 depending only on $A_\infty T$. We conclude from (5.48) that

$$\begin{aligned} &\sum_{n\ge c(T/\varepsilon)^{1-\alpha}} P\left(\inf_{0\le t\le\tau_n}\tilde{Y}_{n,\varepsilon}(t)>0\right)P\left(\ (n-1)M_1\le\left|\sup_{0\le t\le\tilde{T}}\int_0^t d\tilde{B}(t')\ \right|<nM_1\right) \\ &\le\ C\lambda\sum_{n\ge c(T/\varepsilon)^{1-\alpha}}\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \right]e^{-n^2/2} \ \le\ C_1\lambda\left(1+\frac{y}{T}\right)\exp\left[\ -c_1\left(\frac{T}{\varepsilon}\right)^{2(1-\alpha)}\right] , \end{aligned} \tag{5.49}$$

where the constants C_1, c_1 depend only on $A_\infty T$.

We consider next the situation where (5.42) holds with the plus sign. One sees that

$$\begin{aligned} &P\left(\inf_{0\le t\le\tau}\tilde{Y}_{n,\varepsilon}(t)>0;\ \tau<a\varepsilon,\ \int_0^\tau d\tilde{B}(t')\ dt'=-M\ \right) \\ &\le\ P\left(\inf_{0\le t\le\tau}Z_\varepsilon(t)>0,\ \tau<a\varepsilon,\ \ Z_\varepsilon(\tau)\ge M\sqrt{\varepsilon}\ \Big|\ \mu=\mu_n,\ Z_\varepsilon(0)=\lambda\varepsilon[1+Ca\varepsilon/T]\right) , \end{aligned} \tag{5.50}$$

where μ_n is given by (5.47). Observe that the RHS of (5.50) is bounded by the probability that the diffusion $Z_\varepsilon(\cdot)$ started at $\lambda\varepsilon[1+O(\varepsilon^{1-\alpha})]$ exits the interval $[0, C_1T(\varepsilon/T)^{1/2}]$ through the rightmost boundary in time less than $T(\varepsilon/T)^{1-\alpha}$. This probability is bounded by $K(\varepsilon/T, n, y/T)\lambda$ for some function K which has the property that $\lim_{\varepsilon\to0}K(\varepsilon/T, n, y/T)=0$ provided $\alpha<1/2$. To find an expression for K we first choose C_1 depending only on $A_\infty T$ large enough so that $Z_\varepsilon(0) < C_1T(\varepsilon/T)^{1/2}/2$ for any ε satisfying $0<\varepsilon\le T$. It is easy to see that for $0<\lambda'<\Lambda'$,

$$P\left(0<Z_\varepsilon(t)<\Lambda'\varepsilon,\ t<\tau,\ Z_\varepsilon(\tau)=\Lambda'\varepsilon\ \Big|\ Z_\varepsilon(0)=\lambda'\varepsilon\ \right) \ = \ \frac{1-e^{-2\mu\lambda'}}{1-e^{-2\mu\Lambda'}} . \tag{5.51}$$

We apply (5.51) with $\Lambda'=C_1(T/\varepsilon)^{1/2},\ \mu=\mu_n$ and $\lambda'=\lambda[1+Ca\varepsilon/T]$, whence $\mu\Lambda'\ge c$ for some positive constant c depending only on $A_\infty T$. We conclude from

(5.51) that the function K satisfies the inequality

$$K(\varepsilon/T,n,y/T) \ \le \ C\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \right] \tag{5.52}$$

for some constant C depending only on $A_\infty T$.

To show that $\lim_{\varepsilon\to 0} K(\varepsilon/T,n,y/T)=0$ we assume $\lambda'<\Lambda'/2$ and use the inequality

$$P\left(\inf_{0<t<a\varepsilon} Z_\varepsilon(t)>0,\ \sup_{0<t<a\varepsilon} Z_\varepsilon(t)\ge \Lambda'\varepsilon \ \big|\ Z_\varepsilon(0)=\lambda'\varepsilon\ \right) \ \le \tag{5.53}$$
$$P\left(0<Z_\varepsilon(t)<\Lambda'\varepsilon/2,\ t<\tau,\ Z_\varepsilon(\tau)=\Lambda'\varepsilon/2\ \big|\ Z_\varepsilon(0)=\lambda'\varepsilon\ \right)\ P\left(\sup_{0<t<a\varepsilon} Z_\varepsilon(t)\ge\Lambda'\varepsilon\ \big|\ Z_\varepsilon(0)=\Lambda'\varepsilon/2\ \right) .$$

The second probability on the RHS of (5.53) can be bounded as

$$P\left(\sup_{0<t<a\varepsilon} Z_\varepsilon(t)\ge\Lambda'\varepsilon\ \big|\ Z_\varepsilon(0)=\Lambda'\varepsilon/2\ \right)\ \le\ C\exp\left[-\frac{\Lambda'^2}{32a}\right] \tag{5.54}$$

for some universal constant C provided $\mu a<\Lambda'/4$. Observe now from (5.47) that the condition $\mu_n a<\Lambda'/4$ is implied by (5.44). We conclude from (5.52), (5.54) that if (5.44) holds then

$$K(\varepsilon/T,n,y/T)\ \le\ C_2\left[1+\frac{y}{T}+n\sqrt{\frac{\varepsilon}{T}}\ \right]\exp\left[-c_2\left(\frac{T}{\varepsilon}\right)^{1-\alpha}\ \right] \tag{5.55}$$

for some positive constants C_2,c_2 depending only on $A_\infty T$. If (5.44) does not hold we can argue as before using (5.45), (5.52) to obtain an inequality similar to (5.49). The inequality (5.27) follows now from (5.41), (5.49), (5.52) on choosing $\alpha=1/4$. □

Lemma 5.2. *Assume the function $A(\cdot)$ is non-negative and that $0<\lambda\le 1,\ 0<\varepsilon\le T,\ y>0$. Then there are positive constants C,c depending only on $A_\infty T$ such that if $\gamma=c(T/\varepsilon)^{1/8}(y/T)\ge 5$ then*

$$\frac{G_{\varepsilon,D}(\lambda\varepsilon,y,0,T)}{G_\varepsilon(\lambda\varepsilon,y,0,T)}\ \ge\ [1+e^{-\gamma^2/4}]^{-2}\left(1-\exp\left[-\frac{2\lambda}{1+C(\varepsilon/T)^{1/8}}\left\{1-\frac{m_{2,A}(T)}{\sigma_A^2(T)}+\frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right\}\ \right]\ \right) . \tag{5.56}$$

Proof. We choose $a=\min\left[(T/\varepsilon)^\alpha,\tilde{T}/\varepsilon\right]$ with $0<\alpha<1$ as in Lemma 5.1 and observe from (4.29), (5.28) that there is a constant $c>0$ depending only on $A_\infty T$ such that $\tilde{y}(t)\ge cty/T$ for $0\le t\le\tilde{T}$. Hence there exists a constant $c_1>0$ depending only on $A_\infty T$ such that the process $\tilde{Y}_\varepsilon(\cdot)$ of (5.30) satisfies:

$$\tilde{Y}_\varepsilon(t)>0 \quad \text{for } a\varepsilon\le t\le\tilde{T} \ \text{ if for } k=1,2,.., \tag{5.57}$$
$$\left|\int_0^{a\varepsilon} d\tilde{B}(t')\right|\ <\ c_1\sqrt{T}\left(\frac{\varepsilon}{T}\right)^{1/2-\alpha}\frac{y}{T}\quad\text{and}\quad \sup_{a\varepsilon\le t\le(k+1)a\varepsilon}\left|\int_{a\varepsilon}^{t} d\tilde{B}(t')\right|\ \le\ c_1k\sqrt{T}\left(\frac{\varepsilon}{T}\right)^{1/2-\alpha}\frac{y}{T} .$$

It follows from (5.57) that

$$P\left(\inf_{0\le s\le T} Y_\varepsilon(s)>0\right)\ =\ P\left(\inf_{0\le t\le\tilde{T}}\tilde{Y}_\varepsilon(t)>0\right)\ \ge\ P\left(\inf_{0\le t\le a\varepsilon}\tilde{Y}_\varepsilon(t)>0\ ;\mathcal{E}\right) , \tag{5.58}$$

where $\mathcal{E}$ is the event defined by the second line of (5.57). It is easy to see from (5.30) that for $0 < t \le a\varepsilon$ on the event $\mathcal{E}$ there is a constant $C > 0$ depending only on $A_\infty T$ such that

$$\tilde{Y}_\varepsilon(t) > 0 \text{ if } \quad \tilde{Z}_\varepsilon(t) = \frac{\tilde{y}(t)}{1 + Ca\varepsilon/T} - Cc_1 t\frac{y}{T} - \sqrt{\varepsilon}\int_0^t d\tilde{B}(t') \ > 0, \tag{5.59}$$

where c_1 is the constant of (5.57). We conclude from (5.57), (5.59) that

$$P\left(\inf_{0\le s\le T} Y_\varepsilon(s) > 0\right) \ \ge \tag{5.60}$$
$$P\left(\ \tilde{Z}_\varepsilon(t) > 0, \ 0 < t \le a\varepsilon; \ \left|\int_0^{a\varepsilon} d\tilde{B}(t')\right| \ < \ c_1\sqrt{T}\left(\frac{\varepsilon}{T}\right)^{1/2-\alpha}\frac{y}{T} \ \right) \ P(\mathcal{E}) \ .$$

In order to bound $P(\mathcal{E})$ from below we consider for $\gamma > 0$ the event $\mathcal{E}_\gamma$ defined by

$$\left|\int_0^1 d\tilde{B}(t')\right| \ < \ \gamma \quad \text{and} \quad \sup_{1\le t\le (k+1)}\left|\int_1^t d\tilde{B}(t')\right| \ < \ k\gamma \quad \text{for } k = 1, 2, ... \tag{5.61}$$

Then we have that

$$P(\mathcal{E}) \ = \ P(\mathcal{E}_\gamma) \quad \text{where } \gamma \ = \ c_1\left(\frac{T}{\varepsilon}\right)^{\alpha/2}\left(\frac{y}{T}\right) \ . \tag{5.62}$$

Using the fact that

$$P\left(\sup_{1\le t\le (k+1)}\left|\int_1^t d\tilde{B}(t')\right| \ > \ k\gamma\right) \ \le \ 4e^{-k\gamma^2/2} \ , \tag{5.63}$$

we conclude that

$$P(\mathcal{E}_\gamma) \ \ge \ [1 + e^{-\gamma^2/4} \ ]^{-1} \quad \text{if } \gamma \ge 5. \tag{5.64}$$

We bound from below the first probability on the RHS of (5.60) by comparing it to the constant drift Brownian motion (5.5). To do this we use the inequality

$$\sigma_A^2(T)y(s) \ \ge \ xm_{1,A}(s,T)\sigma_A^2(0,s) + ym_{1,A}(0,s)\sigma_A^2(s,T) + [\sigma_A^2(0,s) - m_{2,A}(0,s)]\sigma_A^2(s,T) \ , \tag{5.65}$$

which follows from (4.29) and the assumption that $A(s) \ge 0, \ 0 \le s \le T$. Since the function $s \to \sigma_A^2(0,s) - m_{2,A}(0,s)$ is increasing we conclude from (5.28), (5.65) that there is a constant $C_1 > 0$ depending only on $A_\infty T$ such that for $0 \le t \le a\varepsilon$,

$$\tilde{y}(t) \ \ge \ \frac{\lambda\varepsilon + \mu_\varepsilon t}{1 + C_1 a\varepsilon/T} \quad \text{where } \mu_\varepsilon \ = \ \frac{m_{1,A}(T)y}{\sigma_A^2(T)} + \frac{\sigma_A^2(0, s(a\varepsilon)) - m_{2,A}(0, s(a\varepsilon))}{\sigma_A^2(T)} \ . \tag{5.66}$$

It follows from (5.59), (5.66) that for $0 \le t \le a\varepsilon$ there is a constant $C_2 > 0$ depending only on $A_\infty T$ such that

$$\tilde{Z}_\varepsilon(t) \ \ge \ Z_\varepsilon(t) \quad \text{with } Z_\varepsilon(0) = \frac{\lambda\varepsilon}{1 + C_2 a\varepsilon/T} \ , \ \mu \ = \ \frac{\mu_\varepsilon}{1 + C_2 a\varepsilon/T} - C_2 c_1\frac{y}{T} \ . \tag{5.67}$$

Hence the first probability on the RHS of (5.60) is bounded below by

$$P\left(\ Z_\varepsilon(t) > 0, \ 0 < t \le a\varepsilon; |Z_\varepsilon(a\varepsilon) - Z_\varepsilon(0) - a\varepsilon\mu| < \gamma\varepsilon\sqrt{a} \ \middle| \ Z_\varepsilon(0) = \frac{\lambda\varepsilon}{1 + C_2 a\varepsilon/T} \ \right) \ , \tag{5.68}$$

where γ is as in (5.62).

To bound the probability in (5.68) we assume that the constant c_1 in (5.62) is small enough so that $\mu > 0$ and $\gamma < \mu\sqrt{a}$. Then similarly to (5.10), (5.39) we have that

$$(5.69)\quad P\left(\inf_{0<t<a\varepsilon} Z_\varepsilon(t) > 0;\ |Z_\varepsilon(a\varepsilon) - \lambda'\varepsilon - a\varepsilon\mu| < \gamma\varepsilon\sqrt{a}\ \Big|\ Z_\varepsilon(0) = \lambda'\varepsilon\right) =$$

$$\left\{1 - e^{-2\mu\lambda'}\right\}\frac{1}{\sqrt{2\pi}}\int_{2\lambda'/\sqrt{a}-\gamma}^{\gamma} e^{-z^2/2}\,dz \ + \ \frac{1}{\sqrt{2\pi}}\int_{-\gamma}^{2\lambda'/\sqrt{a}-\gamma} e^{-z^2/2}\,dz$$

$$-\, e^{-2\mu\lambda'}\frac{1}{\sqrt{2\pi}}\int_{\gamma}^{2\lambda'/\sqrt{a}+\gamma} e^{-z^2/2}\,dz \ \ge\ \left\{1 - e^{-2\mu\lambda'}\right\}\frac{1}{\sqrt{2\pi}}\int_{-\gamma}^{\gamma} e^{-z^2/2}\,dz\ .$$

We take $\lambda' = \lambda/[1 + C_2a\varepsilon/T]$ in (5.69) and choose $\alpha = 1/2$. Hence the drift μ of (5.67) satisfies the inequality

$$(5.70)\quad \mu \ \ge\ \frac{1}{1 + C_3\sqrt{\varepsilon/T}}\left\{1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right\} - C_3c_1\frac{y}{T} - C_3\left(\frac{\varepsilon}{T}\right)^{1/2}\ ,$$

for some constant $C_3 > 0$ depending only on $A_\infty T$. We choose now $c_1 = c(\varepsilon/T)^{1/8}$ where $c > 0$ depends only on $A_\infty T$. It is clear that if $\gamma \ge 5$ then the RHS of (5.70) is bounded below by

$$(5.71)\quad \frac{1}{1 + C_4(\varepsilon/T)^{1/8}}\left\{1 - \frac{m_{2,A}(T)}{\sigma_A^2(T)} + \frac{m_{1,A}(T)y}{\sigma_A^2(T)}\right\} \quad \text{where } C_4 \text{ depends only on } A_\infty T.$$

The inequality (5.56) follows now from (5.64), (5.69), (5.71). □

6. Convergence as $\varepsilon \to 0$ of solutions to the diffusive CP Model

Lemma 6.1. *Let $c_\varepsilon(x,t), \Lambda_\varepsilon(t),\ 0 < x,t < \infty$, be the solution to the diffusive CP system (1.7), (1.8) with non-negative initial data $c_0(x),\ 0 < x < \infty$, which is a locally integrable function satisfying*

$$(6.1)\qquad \int_0^\infty (1+x)c_0(x)\,dx < \infty, \qquad \int_0^\infty xc_0(x)\,dx \ = \ 1.$$

Then for any $T > 0$ there are positive constants C_1, C_2 depending only on T and $c_0(\cdot)$ such that

$$(6.2)\qquad C_1 \le \Lambda_\varepsilon(t) \le C_2 \quad \text{for } 0 < \varepsilon \le 1,\ 0 \le t \le T.$$

In addition the set of functions $\{\Lambda_\varepsilon : [0,T] \to \mathbf{R} :\ 0 < \varepsilon \le 1\}$ form an equicontinuous family. Denote by $c_0(x,t), \Lambda_0(t),\ 0 < x,t < \infty$, the solution to the CP system (1.1), (1.2) with $\varepsilon = 0$ and initial data $c_0(x),\ 0 < x < \infty$. Then for all $x,t \ge 0$

$$(6.3)\qquad \lim_{\varepsilon\to 0} w_\varepsilon(x,t) \ = \ w_0(x,t)\ ,$$

$$(6.4)\qquad \lim_{\varepsilon\to 0}\ \Lambda_\varepsilon(t) \ = \ \Lambda_0(t)\ ,$$

where w_ε is given in terms of c_ε by (2.49). The limit in (6.3), (6.4) is uniform for (x,t) in any finite rectangle $0 < x \le x_0,\ 0 < t \le T$.

Proof. It follows from (1.10) that $\Lambda_\varepsilon(t)$ is an increasing function of t, whence the lower bound in (6.2) follows. We first prove the upper bound for the CP model (1.1), (1.2) corresponding to $\varepsilon = 0$. We see from (1.2), (1.3) that

$$\int_{\Lambda_0(0)/2}^{\infty} xc_0(x)\ dx\ \ge \frac{1}{2}\int_0^{\infty} xc_0(x)\ dx\ =\ \frac{1}{2}\ . \tag{6.5}$$

Hence from (2.3) there is a positive constant $1/C_2$ depending only on $c_0(\cdot)$ such that $w_0(\Lambda_0(0)/2, 0) \ge 1/C_2$. It follows then from (2.2), (2.4) that $w_0(0,t) \ge 1/C_2$ for $0 \le t \le \Lambda_0(0)/2$, whence (1.2), (1.3) implies that $\Lambda_0(t) \le C_2$ for $0 \le t \le \Lambda_0(0)/2$. Furthermore we see from (2.3) that $\Lambda_0(t)$ is continuous in the interval $0 \le t \le \Lambda_0(0)/2$. Since $\Lambda_0(t)$ is an increasing function of t we can extend this argument in a finite number of steps to any interval $0 \le t \le T$. We have proven (6.2) in the case $\varepsilon = 0$.

To prove the upper bound in (6.2) for $0 < \varepsilon \le 1$ we use the representation

$$w_\varepsilon(x,t)\ =\ \int_0^\infty P\left(Y_\varepsilon(t) > x; \inf_{0\le s\le t} Y_\varepsilon(s) > 0\ \middle|\ Y_\varepsilon(0) = y\right) c_0(y)dy\ , \tag{6.6}$$

where $Y_\varepsilon(s)$ is the solution to the SDE (4.8) with $b(y,s) = y/\Lambda_\varepsilon(s) - 1$. Since $\Lambda_\varepsilon(s) \ge \Lambda_\varepsilon(0) = \Lambda_0(0)$ it follows from (4.9) that for any $\delta > 0, t > 0$ there is a positive constant p_1 depending only on $\delta, t, \Lambda_0(0)$ such that

$$P\left(\inf_{0\le s\le t}\{\ Y_\varepsilon(s) - E[Y_\varepsilon(s)]\ \} \ge -\delta\right)\ \ge\ p_1 \quad \text{for } 0 < \varepsilon \le 1. \tag{6.7}$$

We conclude from (6.7) by choosing δ appropriately that there is a positive constant p_2 depending only on $\Lambda_0(0)$ such that if $0 < \varepsilon \le 1$ then

$$P\left(Y_\varepsilon(t) > 0; \inf_{0\le s\le t} Y_\varepsilon(s) > 0\ \middle|\ Y_\varepsilon(0) = y\right)\ \ge\ p_2 \quad \text{for } t = \Lambda_0(0)/2,\ y \ge \Lambda_0(0)/2\ . \tag{6.8}$$

It follows now from (6.5), (6.6), (6.8) that there is a positive constant C_2 depending only on the initial data $c_0(\cdot)$ such that $w_\varepsilon(0,t) \ge 1/C_2$ for $0 < \varepsilon \le 1$ if $t = \Lambda_0(0)/2$. The upper bound in (6.2) for all T then follows as in the previous paragraph.

To prove that $\Lambda_\varepsilon(\cdot)$ is continuous we first note that for any fixed $t > 0$ the function $w_\varepsilon(x,t),\ x \ge 0$, is continuous by virtue of the representation (6.6), the fact that $\Lambda_\varepsilon(s) \ge \Lambda_0(0)$ for $0 \le s \le t$ and (4.11). The continuity is uniform for ε in the interval $0 < \varepsilon \le 1$ since $c_0(\cdot)$ is a locally integrable function. Next we observe from (4.9) that for $\Delta t > 0$ there exists $x(\Delta t)$ independent of ε in the interval $0 < \varepsilon \le 1$ such that $\lim_{\Delta t\to 0} x(\Delta t) = 0$ and

$$\begin{aligned} &P\left(Y_\varepsilon(t+\Delta t) > 0; \inf_{0\le s\le t+\Delta t} Y_\varepsilon(s) > 0\ \middle|\ Y_\varepsilon(0) = y\right)\ \ge \\ &[1-\Delta t]P\left(Y_\varepsilon(t) > x(\Delta t); \inf_{0\le s\le t} Y_\varepsilon(s) > 0\ \middle|\ Y_\varepsilon(0) = y\right) \quad \text{for } y \ge 0,\ 0 < \varepsilon \le 1. \end{aligned} \tag{6.9}$$

It follows from (6.6), (6.9) that $w_\varepsilon(0, t+\Delta t) \ge [1-\Delta t]w_\varepsilon(x(\Delta t), t)$ for $0 < \varepsilon \le 1$. Using the continuity of the function $w_\varepsilon(x,t),\ x \ge 0$, we conclude that $\lim_{\Delta t\to 0} w_\varepsilon(0, t+\Delta t) = w_\varepsilon(0,t)$ and the limit is uniform for $0 < \varepsilon \le 1$. Hence the function $\Lambda_\varepsilon(\cdot)$ is continuous, and in fact the family of functions $\Lambda_\varepsilon(\cdot),\ 0 < \varepsilon \le 1$, is equicontinuous.

To prove (6.3), (6.4) we first observe from the Ascoli-Arzela theorem that since the family of functions $\Lambda_\varepsilon(\cdot),\ 0 < \varepsilon \le 1$, is equicontinuous, the limit (6.4) holds uniformly on the interval $0 \le t \le T$ for a subsequence of $\varepsilon \to 0$. For such a sequence it follows from (2.2), (4.9), (6.6) that (6.3) holds with $w_0(x,t) = w_0(F_{1/\Lambda_0}(x,t), 0)$ and the conservation law (1.2) continues to hold for $\varepsilon = 0$. Hence the limits on the RHS of (6.3), (6.4) are the solution to the CP model (1.1), (1.2) and are therefore unique. Consequently (6.3), (6.4) hold for all $\varepsilon \to 0$. The uniformity of the limits follows by similar argument. □

To show that the coarsening rate (1.10) for the diffusive model (1.7), (1.8) converges as $\varepsilon \to 0$ to the coarsening rate (1.4) for the CP model (1.1), (1.2), it will be sufficient to prove the following:

Lemma 6.2. *Let $c_\varepsilon(x,t), \Lambda_\varepsilon(t),\ 0 < x, t < \infty$, and $c_0(x),\ 0 < x < \infty$, be as in Lemma 6.1 and satisfy (6.1). If $c_0(\cdot)$ is a continuous function then*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0,T)}{\partial x} \ = \ c_0(0,T) \quad \text{for any } T > 0. \tag{6.10}$$

Proof. We use the identity

$$\frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0,T)}{\partial x} \ = \ \lim_{\lambda \to 0} \frac{c_\varepsilon(\lambda\varepsilon, T)}{2\lambda} \tag{6.11}$$

and the representation for $c_\varepsilon(\lambda\varepsilon, T)$ from (4.15),

$$c_\varepsilon(\lambda\varepsilon, T) \ = \ \int_0^\infty G_{\varepsilon,D}(\lambda\varepsilon, y, 0, T) c_0(y)\ dy, \tag{6.12}$$

where $G_{\varepsilon,D}$ is the Dirichlet Green's function corresponding to the drift $b(y,t) = y/\Lambda_\varepsilon(t) - 1$. From (6.11), (6.12) and Lemma 5.1 we have that

$$\begin{gathered} \frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0,T)}{\partial x} \ \le \\ \int_0^\infty \left\{ 1 - \frac{m_{2,1/\Lambda_\varepsilon}(T)}{\sigma^2_{1/\Lambda_\varepsilon}(T)} + \frac{m_{1,1/\Lambda_\varepsilon}(T) y}{\sigma^2_{1/\Lambda_\varepsilon}(T)} + C\Gamma\left(\frac{\varepsilon}{T}, \frac{y}{T}\right) \left[1 + \frac{y}{T}\right] \right\} G_\varepsilon(0, y, 0, T) c_0(y)\ dy, \end{gathered} \tag{6.13}$$

where the constant C depends only on $T/\Lambda_0(0)$. We conclude from Lemma 6.1, (4.11) and (6.13) that

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0,T)}{\partial x} \ \le \ \frac{1}{m_{1,1/\Lambda_0}(T)} c_0 \left(\frac{m_{2,1/\Lambda_0}(T)}{m_{1,1/\Lambda_0}(T)} \right) \ , \tag{6.14}$$

provided the function $c_0(y),\ y > 0$, is continuous at $y = m_{2,1/\Lambda_0}(T)/m_{1,1/\Lambda_0}(T)$.

We can obtain a lower bound on the LHS of (6.10) by using Lemma 5.2. Thus we have that

$$\begin{gathered} \left[1 + C(\varepsilon/T)^{1/8}\right] \frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0,T)}{\partial x} \ \ge \\ \int_{5(\varepsilon/T)^{1/8} T/c}^\infty \left[1 + \exp\left(-c^2 (T/\varepsilon)^{1/4} (y^2/4T^2) \right) \right]^{-2} \ \times \\ \left\{ 1 - \frac{m_{2,1/\Lambda_\varepsilon}(T)}{\sigma^2_{1/\Lambda_\varepsilon}(T)} + \frac{m_{1,1/\Lambda_\varepsilon}(T) y}{\sigma^2_{1/\Lambda_\varepsilon}(T)} \right\} G_\varepsilon(0, y, 0, T) c_0(y)\ dy, \end{gathered} \tag{6.15}$$

where the constants $C, c > 0$ depend only on $T/\Lambda_0(0)$. We conclude from Lemma 6.1, (4.11) and (6.15) that

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon}{2} \frac{\partial c_\varepsilon(0,T)}{\partial x} \ \geq \ \frac{1}{m_{1,1/\Lambda_0}(T)} c_0 \left(\frac{m_{2,1/\Lambda_0}(T)}{m_{1,1/\Lambda_0}(T)} \right) , \tag{6.16}$$

provided the function $c_0(y),\ y > 0$, is continuous at $y = m_{2,1/\Lambda_0}(T)/m_{1,1/\Lambda_0}(T)$. Finally we observe that the RHS of (6.14), (6.16) is the same as $c_0(0,T)$. This follows by differentiating the function $w(x,t) = w_0(F_{1/\Lambda_0}(x,t), 0)$ with respect to x at $x = 0$, and using the formula (2.2) for the function F_{1/Λ_0}. □

7. Upper Bound on the Coarsening Rate of diffusive CP Model

In this section we prove Theorem 1.2. First we show that $\lim_{t\to\infty}\langle X_t \rangle = \infty$.

Lemma 7.1. *Let $c_\varepsilon(x,t), \Lambda_\varepsilon(t),\ 0 < x, t < \infty$, be the solution to (1.7), (1.8) with $\varepsilon > 0$ and non-negative initial data $c_0(x),\ 0 < x < \infty$, which is a locally integrable function satisfying (6.1). Then $\lim_{t\to\infty} \Lambda_\varepsilon(t) = \infty$.*

Proof. We have already noted that $\Lambda_\varepsilon(t)$ is an increasing function of t. It will therefore be sufficient to show that if for some finite Λ_∞ we have $\Lambda_\varepsilon(t) \leq \Lambda_\infty$ for all $t \geq 0$ then there is a contradiction. To see this we use the identity

$$\frac{d}{dt} \int_0^\infty x c_\varepsilon(x,t)\ dx \ = \ \frac{1}{\Lambda_\varepsilon(t)} \int_0^\infty x c_\varepsilon(x,t)\ dx - \int_0^\infty c_\varepsilon(x,t)\ dx , \tag{7.1}$$

which follows from (1.7). Using the conservation law (1.8) and (7.1) we see that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty x c_\varepsilon(x,t)\ dx \ &\geq \ \frac{1}{2\Lambda_\infty} \int_0^\infty x c_\varepsilon(x,t)\ dx - \int_0^{2\Lambda_\infty} c_\varepsilon(x,t)\ dx \\ = \ \frac{1}{2\Lambda_\infty} - \int_0^{2\Lambda_\infty} c_\varepsilon(x,t)\ dx \ &\geq \ \frac{1}{2\Lambda_\infty} - \int_0^{2\Lambda_\infty} dx \int_0^\infty dy\ G_\varepsilon(x,y,0,t) c_0(y) , \end{aligned} \tag{7.2}$$

where G_ε is the function (4.11) with $A(s) = 1/\Lambda_\varepsilon(s),\ s \geq 0$. Hence we conclude that

$$\frac{d}{dt} \int_0^\infty x c_\varepsilon(x,t)\ dx \ \geq \ \frac{1}{2\Lambda_\infty} - \frac{2\Lambda_\infty}{\Lambda_\varepsilon(0)\sqrt{2\pi\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)}} . \tag{7.3}$$

From (4.10) we see that $\sigma^2_{1/\Lambda_\varepsilon}(t) \geq t$ and hence (7.3) implies that

$$\lim_{t\to\infty} \int_0^\infty x c_\varepsilon(x,t)\ dx \ = \ \infty , \tag{7.4}$$

but this is a contradiction to the conservation law (1.8). □

We begin the proof of the inequality (1.14):

Lemma 7.2. *Suppose $c_0 : [0,\infty) \to \mathbf{R}^+$ satisfies (6.1) and $c_\varepsilon(x,t),\ x \geq 0, t > 0$ is the solution to (1.7), (1.8) with initial data $c_0(\cdot)$ and Dirichlet boundary condition $c_\varepsilon(0,t) = 0,\ t > 0$. Assume that $\Lambda_\varepsilon(0) = 1$ and that the function $h_\varepsilon(x,t)$ defined by (2.49) is log-concave in x at $t = 0$. Then there exist positive universal constants C, ε_0 with $0 < \varepsilon_0 \leq 1$ such that*

$$c_\varepsilon(\lambda\varepsilon, 1) \ \leq \ C\lambda c_\varepsilon(\varepsilon, 1) \quad \text{for } 0 < \varepsilon \leq \varepsilon_0,\ 0 < \lambda \leq 1. \tag{7.5}$$

Proof. Let X_0 be the positive random variable with pdf $c_0(x)/\int_0^\infty c_0(x')\,dx',\ x>0$. Then from (1.9) and the assumption $\Lambda_\varepsilon(0)=1$ we see that $\langle X_0\rangle=1$. Since the beta function (2.35) for X_0 is also bounded by 1 it follows from the Chebshev inequality and the identity (29) of [5] that for δ with $0<\delta<1$, there exists a constant $\nu(\delta)>0$ depending only on δ such that

$$P(X_0<\nu(\delta))+P(X_0>1/[1-\delta])\ \le 1-\delta/2. \tag{7.6}$$

Now recall that the Green's function (4.11) has the form (4.24) where the function $y\to q(x,y,0,t)$ takes its minimum at $y=F_A(x,t)$ where F_A is defined by (2.2). In the case of $A(\cdot)=1/\Lambda_\varepsilon(\cdot),\ x=\lambda\varepsilon$ and $t=1$, we see that $1-(1-\lambda\varepsilon)/e\le F_{1/\Lambda_\varepsilon}(\lambda\varepsilon,1)\le 1+\lambda\varepsilon$. This follows from the fact that the function $\Lambda_\varepsilon(\cdot)$ is increasing and $\Lambda_\varepsilon(0)=1$. We choose now $\delta,\varepsilon_0>0$ sufficiently small so that $F_{1/\Lambda_\varepsilon}(\lambda\varepsilon,1)-\nu(\delta)>1/(1-\delta)-F_{1/\Lambda_\varepsilon}(\lambda\varepsilon,1)>0$ for $0<\lambda\le 1,\ \varepsilon\le\varepsilon_0$. It follows then from Lemma 5.1 that

$$c_\varepsilon(\lambda\varepsilon,1)\ \le\ C_1\lambda\int_{\nu(\delta)}^{1/[1-\delta]}G_\varepsilon(\varepsilon,y,0,1)c_0(y)\,dy \qquad \text{for } 0<\varepsilon\le\varepsilon_0,\ 0<\lambda\le 1, \tag{7.7}$$

where $C_1>0$ depends only on δ,ε_0. Next we apply Lemma 5.2 to conclude that

$$\int_{\nu(\delta)}^{1/[1-\delta]}G_\varepsilon(\varepsilon,y,0,1)c_0(y)\,dy\ \le\ C_2\int_{\nu(\delta)}^{1/[1-\delta]}G_{\varepsilon,D}(\varepsilon,y,0,1)c_0(y)\,dy\ \le\ C_2c_\varepsilon(\varepsilon,1)\,, \tag{7.8}$$

for some constant $C_2>0$ depending only on δ,ε_0. Actually a strict application of Lemma 5.2 requires us to impose an additional restriction on $\varepsilon_0>0$ so that the condition $\gamma\ge 5$ of Lemma 5.2 holds. Now (7.5) follows from (7.7), (7.8). □

Lemma 7.3. *Suppose the initial data $c_0(\cdot)$ for (1.7), (1.8) satisfies the conditions of Lemma 7.2, the function $h_\varepsilon(x,t)$ is log-concave in x for all $t\ge 0$, and $0<\varepsilon\le\varepsilon_0$. Then there is a universal constant C such that $d\Lambda_\varepsilon(t)/dt\le C$ for $t\ge 1$.*

Proof. From (1.8), (1.10), (2.35), (2.49) and Lemma 7.2 we see there is a universal constant C_1 such that

$$\frac{d\Lambda_\varepsilon(1)}{dt}\ \le\ \frac{C_1c_\varepsilon(\varepsilon,1)h_\varepsilon(0,1)}{w_\varepsilon(0,1)^2}\ \le\ \frac{C_1\beta_{X_1}(\varepsilon)h_\varepsilon(0,1)}{h_\varepsilon(\varepsilon,1)}\ , \tag{7.9}$$

where X_t is the random variable with pdf proportional to $c_\varepsilon(\cdot,t)$. We can bound $h_\varepsilon(\varepsilon,1)$ below by a constant times $h_\varepsilon(0,1)$. To see this consider a positive random variable X and observe that

$$E[\ X-\langle X\rangle/2;\ X>\langle X\rangle/2\ ]\ \ge\ E[\ X-\langle X\rangle/2;\ X>3\langle X\rangle/4\ ]\ \ge \\ \frac{1}{3}E[\ X;\ X>3\langle X\rangle/4\ ]\ \ge\ \frac{1}{12}\langle X\rangle\ . \tag{7.10}$$

Since the function $\Lambda_\varepsilon(\cdot)$ is increasing, it follows that $\langle X_1\rangle\ge 1$. We conclude then from (7.10) that

$$h_\varepsilon(\varepsilon,1)\ =\ \int_\varepsilon^\infty(x-\varepsilon)c_\varepsilon(x,1)\,dx\ \ge\ \frac{1}{12}\int_0^\infty xc_\varepsilon(x,1)\,dx\ =\ \frac{1}{12}h_\varepsilon(0,1)\ , \tag{7.11}$$

provided $\varepsilon<1/2$. Since the log-concavity of $h_\varepsilon(\cdot,1)$ implies that $\beta_{X_1}(\varepsilon)\le 1$, we obtain from (7.9), (7.11) an upper bound on $d\Lambda_\varepsilon(t)/dt$ when $t=1$.

The upper bound for $t > 1$ now follows from the scaling property of (1.7), (1.8) mentioned in the discussion following the statement of Theorem 1.2. To see this we define a function $\tau(\lambda),\ \lambda \ge 1$, as the solution to the equation $\Lambda_\varepsilon(\lambda\tau(\lambda)) = \lambda$. Observe from (1.10) and the Hopf maximum principle [15] that the function $\Lambda_\varepsilon(\cdot)$ is strictly increasing. Hence $\tau(\lambda)$ is uniquely determined. Furthermore the function $\tau(\cdot)$ is continuous. Rescaling (1.7), (1.8) by λ, we conclude from the result of the previous paragraph that

$$\frac{d}{dt}\,\Lambda_\varepsilon(\lambda[\tau(\lambda)+t]) \ \le C\lambda \quad \text{at } t=1. \tag{7.12}$$

We have shown then that $d\Lambda_\varepsilon(t)/dt \le C$ at $t = \lambda[\tau(\lambda)+1]$. Since the function $\lambda \to \lambda\tau(\lambda)$ is monotonically increasing with range $[0,\infty)$ the result follows. □

To complete the proof of the inequality (1.14) we first observe that by Lemma 7.1 there exists $T_\varepsilon \ge 0$ such that $\varepsilon/\Lambda_\varepsilon(T_\varepsilon) \le \varepsilon_0$, where ε_0 is the universal constant of Lemma 7.2. We now rescale (1.7), (1.8) with $\lambda = \Lambda_\varepsilon(T_\varepsilon)$, which puts us into the situation of Lemma 7.3. The result follows on taking $T = T_\varepsilon + \Lambda_\varepsilon(T_\varepsilon)$, provided we have the log-concavity property of the function h_ε in the statement of Lemma 7.3. The assumption of Theorem 1.2, that the function $x \to E[X_0 - x \mid X_0 > x],\ 0 \le x < \|X_0\|_\infty$, decreases, is equivalent to the assumption that the function $h_\varepsilon(\cdot,0)$ is log-concave. Hence it remains to be shown that if $h_\varepsilon(\cdot,0)$ is log-concave, then $h_\varepsilon(\cdot,t)$ is also log-concave for all $t > 0$.

If we make the approximation (2.51) for h_ε, the log-concavity of $h_\varepsilon(\cdot,t)$ follows from the Prékopa-Leindler theorem (Theorem 6.4 of [21]). In our situation we follow the approach of Korevaar [10] and differentiate the PDE (2.65) which h_ε satisfies. Thus $v_\varepsilon(x,t) = -\frac{\partial}{\partial x}\log h_\varepsilon(x,t)$ is a solution of the PDE (2.67), whence $u_\varepsilon(x,t) = \partial v_\varepsilon(x,t)/\partial x$ is a solution to the PDE

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} + \left[\frac{x}{\Lambda_\varepsilon(t)} - 1 + \varepsilon v_\varepsilon(x,t)\right]\frac{\partial u_\varepsilon(x,t)}{\partial x} + \frac{2u_\varepsilon(x,t)}{\Lambda_\varepsilon(t)} + \varepsilon u_\varepsilon(x,t)^2 \ = \ \frac{\varepsilon}{2}\frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2}\ . \tag{7.13}$$

Observe now that

$$u_\varepsilon(x,t) \ = \ v_\varepsilon(x,t)^2[1 - c_\varepsilon(x,t)h_\varepsilon(x,t)/w_\varepsilon(x,t)^2]\ . \tag{7.14}$$

Since $\lim_{x\to 0} c_\varepsilon(x,t) = 0$ for $t > 0$ it follows from (7.14) that $\liminf_{x\to 0} u_\varepsilon(x,t) \ge 0$ for $t > 0$. If $h_\varepsilon(\cdot,0)$ is log concave then the initial data $u_\varepsilon(x,0),\ x > 0$, for (7.13) is also non-negative. We expect then from the maximum principle that $u_\varepsilon(x,t)$ is non-negative for all $x,t > 0$, and hence $h_\varepsilon(\cdot,t)$ is a log concave function for all $t > 0$.

Although Korevaar's argument is simple in concept, the rigorous implementation requires that certain technical difficulties be overcome. Our first step towards rigorous implementation is to approximate an arbitrary non-negative random variable X satisfying $\langle X\rangle < \infty$, and having log-concave function $h_X(\cdot)$ as defined by (2.33), by random variables with some regularity. The approximating random variables Y have the properties:

$$Y \text{ is nonnegative with continuous pdf } c_Y(y),\ y \ge 0, \quad \text{and } c_Y(0) = 0. \tag{7.15}$$

There exists $K, a, L, y_0 > 0$ and $c_Y(y) = K\exp\left[-a(y-y_0) - \{a(y-y_0)\}^2/2L\right]$ for $y \ge y_0$.

The beta function (2.35) of Y satisfies $\beta_Y(y) < 1 \quad$ for all $y \ge 0$.

We assume that the function $\Lambda_\varepsilon : [0,\infty) \to \mathbf{R}^+$ of (1.7) is positive and continuous, and consider solutions c_ε to (1.7) with Dirichlet boundary condition $c_\varepsilon(0,t)=0,\ t>0$, and initial condition given by the integrable pdf $c_{X_0}(\cdot)$ of a non-negative random variable X_0 satisfying $\langle X_0\rangle<\infty$. We denote by X_t the random variable with pdf $c_\varepsilon(\cdot,t),\ t>0$.

Lemma 7.4. *Assume that the function $h_{X_0}(\cdot)$ for the initial data random variable X_0 of (1.7) is log-concave. Then there is a sequence of random variables $Y_0^k,\ k=1,2,..,$ satisfying (7.15) such that the functions $(x,t)\to h_{Y_t^k}(x),\ k=1,2,..,$ converge as $k\to\infty$, uniformly in any rectangle $\{(x,t)\ :\ 0\le x\le x_0,\ 0\le t\le T\}$, to the function $(x,t)\to h_{X_t}(x)$.*

Proof. We first assume that $\|X_0\|_\infty<\infty$, and define the beta function for Y_0^k in the interval $0\le x\le(1-2/k)\|X_0\|_\infty$ in terms of the beta function for X_0 as follows:

$$(7.16)\quad \beta_{Y_0^k}(x) \ = \ \left(1-\frac{1}{k}\right)\frac{k}{\|X_0\|_\infty}\int_0^x \beta_{X_0}(z)\ dz \quad \text{for } 0\le x\le \frac{\|X_0\|_\infty}{k}\ ,$$

$$\beta_{Y_0^k}(x) \ = \ \left(1-\frac{1}{k}\right)\frac{k}{\|X_0\|_\infty}\int_{x-\|X_0\|_\infty/k}^x \beta_{X_0}(z)\, dz \quad \text{for } \frac{\|X_0\|_\infty}{k}\le x\le\left(1-\frac{2}{k}\right)\|X_0\|_\infty\ .$$

It follows from (2.35), (7.16) that $c_{Y_0^k}(x)$ is continuous in the interval $0\le x\le(1-2/k)\|X_0\|_\infty$ and $c_{Y_0^k}(0)=0$. To continue the definition of $\beta_{Y_0^k}(\cdot)$, we choose $L_k>0$ sufficiently large so that the function β_{L_k} of Lemma 2.1 satisfies $\beta_L(0)\ge 1-1/k$ and $\beta_L(z)\le 1-1/2L(1+z/L)^2$ for $z\ge0,\ L\ge L_k$. We then define $\beta_{Y_0^k}(x)$ for $(1-2/k)\|X_0\|_\infty\le x\le(1-1/k)\|X_0\|_\infty$ by linear interpolation, taking the value $\beta_{Y_0^k}((1-2/k)\|X_0\|_\infty)$ at the left end of the interval and the value $\beta_{L_k}(0)$ at the right end. Finally we extend $c_{Y_0^k}(x)$ to $x\ge(1-1/k)\|X_0\|_\infty$ by setting it equal to the Gaussian in (7.15) with $y_0=(1-1/k)\|X_0\|_\infty$, choosing K so that $c_{Y_0^k}(\cdot)$ is continuous and a so that $E[Y_0^k-x\mid Y_0^k>x]=E[X_0-x\mid X_0>x]$ when $x=(1-1/k)\|X_0\|_\infty$. The random variable Y_0^k satisfies (7.15). In particular, since $h_{X_0}(\cdot)$ is log-concave it follows that $\beta_{Y_0^k}(x)\le1-1/k$ for $x\le(1-2/k)\|X_0\|_\infty$.

To construct the pdf $c_{Y_0^k}(\cdot)$ from the function $\beta_{Y_0^k}(\cdot)$ defined in the previous paragraph we first observe from (2.35) that

$$(7.17)\quad \frac{d}{dy}E[Y_0^k-y\mid Y_0^k>y] \ = \ \beta_{Y_0^k}(y)-1\ ,\quad E[Y_0^k-y^k\mid Y_0^k>y^k]=E[X_0-y^k\mid X_0>y^k]\ ,$$

where $y^k=(1-1/k)\|X_0\|_\infty$. The function $v^k(y)=E[Y_0^k-y\mid Y_0^k>y]^{-1},\ y\ge0$, is uniquely determined by (7.17). From (2.33), (2.34) we see that

$$(7.18)\qquad h_{Y_0^k}(x) \ = \ A^k\exp\left[-\int_0^x v^k(y)\ dy\right]\ ,\quad x\ge0,$$

for some constant A^k. If we define the function $f_k(\cdot)$ by

$$(7.19)\qquad f_k(x) \ = \ \beta_{Y_0^k}(x)v^k(x)^2\exp\left[-\int_0^x v^k(y)\ dy\right]\ ,\quad x\ge0,$$

then (2.35) implies that $c_{Y_0^k}(x)=A^kf_k(x),\ x\ge0$. Using the normalization condition for the probability measure $c_{Y_0^k}(\cdot)$, we conclude that the constant K in (7.15) is given by the formula

$$K = f_k(y^k)\Big/\int_0^\infty f_k(x)\ dx \ . \tag{7.20}$$

We show that the functions $w_{Y_0^k}(\cdot)$ converge as $k\to\infty$ to $w_{X_0}(\cdot)$, uniformly in $[0,\infty)$. To do this we use the identity

$$\int_x^\infty f_k(y)\ dy = v^k(x)\exp\left[-\int_0^x v^k(y)\ dy\right] \ , \quad x\ge 0, \tag{7.21}$$

obtained from (7.17), (7.19). From (7.17) we have that

$$E[Y_0^k - y \mid Y_0^k > y] = E[X_0 - y^k \mid X_0 > y^k] + \int_y^{y^k}\left[1-\beta_{Y_0^k}(y')\right]\ dy' \ , \quad 0\le y\le y^k \ . \tag{7.22}$$

It follows from (7.16), (7.22) that

$$\lim_{k\to\infty} E[Y_0^k - y \mid Y_0^k > y] = \int_y^{\|X_0\|_\infty}\left[1-\beta_{X_0}(y')\right]\ dy' \ , \quad 0\le y<\|X_0\|_\infty \ , \tag{7.23}$$

and the convergence is uniform in any interval $\{y\ :\ 0\le y<\|X_0\|_\infty(1-\delta)\}$ for which $\delta>0$. We conclude from (7.21), (7.23) upon setting $v^\infty(x) = E[X_0 - x \mid X > x]^{-1}$ that

$$\lim_{k\to\infty}\int_x^\infty f_k(y)\ dy = v^\infty(x)\exp\left[-\int_0^x v^\infty(y)\ dy\right] \ , \quad 0\le x<\|X_0\|_\infty \ , \tag{7.24}$$

and the convergence is uniform in any interval $\{x\ :\ 0\le x<\|X_0\|_\infty(1-\delta)\}$ for which $\delta>0$. Taking $x=0$ in (7.24) we have that $\lim_{k\to\infty} A_k = E[X_0]$. Hence (7.24) implies that

$$\lim_{k\to\infty}\int_x^\infty c_{Y_0^k}(y)\ dy = \int_x^\infty c_{X_0}(y)\ dy \ , \quad 0\le x<\|X_0\|_\infty \ , \tag{7.25}$$

and the convergence is uniform in any interval $\{x\ :\ 0\le x<\|X_0\|_\infty(1-\delta)\}$ for which $\delta>0$. In view of the integrability of $c_{X_0}(\cdot)$, we conclude that the convergence (7.25) is uniform for $0\le x<\infty$.

We can easily estimate $w_{Y_0^k}(x)$ for $x\ge\|X_0\|_\infty$ since the pdf of Y_0^k is Gaussian when $x\ge\|X_0\|_\infty$. To do this we define a function $g:[0,\infty)\to\mathbf{R}$ by

$$g(z) = E[Z \mid Z>z]^{-1} = e^{z^2/2}\int_z^\infty e^{-z'^2/2}\ dz' = \int_0^\infty e^{-z'z-z'^2/2}\ dz' \ , \tag{7.26}$$

where Z is the standard normal variable. Evidently we have from (7.26) that

$$g'(z) = -1 + z\int_0^\infty e^{-z'z-z'^2/2}\ dz' \ , \quad g(0) = \sqrt{\pi/2} \ . \tag{7.27}$$

We conclude from (7.26), (7.27) that $g(\cdot)$ is a decreasing function and $\lim_{z\to\infty} g(z) = 0$. We can estimate $g(z)$ for large z from the final integral on the RHS of (7.26) to obtain the inequality

$$0 < g(z) < \frac{1}{z}\left[1-\frac{1}{z^2}+\frac{3}{z^4}\right] \ , \tag{7.28}$$

whence it follows that

$$E[Z \mid Z>z] > z\left[1+\frac{1}{2z^2}\right] \quad \text{for } z>\sqrt{6}. \tag{7.29}$$

Since the final integral on the RHS of (7.26) is strictly less than $1/z$ for all $z > 0$, we conclude from (7.29) that there exists $\gamma_0 > 0$ and

$$E[Z \mid Z > z] - z \ \geq \ \min\{\gamma_0, 1/2z\} \ , \quad z \geq 0. \tag{7.30}$$

The random variable Y of (7.15) has for $y \geq y_0$ the pdf of a normal variable with mean $y_0 - L/a$ and variance L/a^2. We can estimate the value of a when $Y = Y_0^k$ by using the equality $E[Y_0^k - y^k \mid Y_0^k > y^k] = E[X_0 - y^k \mid X_0 > y^k] \leq \|X_0\|_\infty/k$. Observe now that

$$E[Y_0^k - y^k \mid Y_0^k > y^k] \ = \ \frac{\sqrt{L_k}}{a}\left(E[Z \mid Z > \sqrt{L_k}] - \sqrt{L_k}\right) \ . \tag{7.31}$$

Hence using the upper bound on the LHS of (7.31), we obtain from (7.30), (7.31) a lower bound

$$a \ \geq \ \frac{k\sqrt{L_k}}{\|X_0\|_\infty}\min\{\gamma_0, 1/2\sqrt{L_k}\} \tag{7.32}$$

for a. Since $\lim_{k\to\infty} L_k = \infty$, it follows from (7.32) that $\lim_{k\to\infty} a_k = \infty$, where a_k is the value of a in (7.15) when $Y = Y_0^k$.

We have from (7.15) that

$$w_{Y_0^k}(x) \ = \ \int_{x-y_k}^\infty K_k \exp[-a_k y - (a_k y)^2/2L_k] \ dy \quad \text{for } x \geq y_k \ . \tag{7.33}$$

Furthermore, from (7.25) it follows that for any $\eta > 0$ there exists an integer k_η such that $w_{Y_0^k}(y_k) < \eta$ for $k \geq k_\eta$. This implies a bound on K_k in (7.33) of the form $K_k \leq \eta a_k, \ k \geq k_\eta$. Hence from (7.33) it follows that $h_{Y_0^k}(y_k) \leq \eta/a_k$ and hence $\lim_{k\to\infty} h_{Y_0^k}(y_k) = 0$. We conclude from this and (7.25) that the functions $x \to h_{Y_0^k}(x), \ k = 1, 2, .., $ converge as $k \to \infty$, uniformly in any interval $\{x \ : \ 0 \leq x \leq x_0\}$, to the function $x \to h_{X_0}(x)$.

To prove that $h_{Y_t^k}(\cdot), \ k = 1, 2, ..,$ converges to $h_{X_t}(\cdot)$ for $t > 0$ we use the fact that the function w_ε defined by (2.49) is a solution to the PDE (2.64). Furthermore, the Dirichlet boundary condition $c_\varepsilon(0, t) = 0$ for (1.7) becomes a Neumann boundary condition $\partial w_\varepsilon(0, t)/\partial x = 0$ for (2.64). By the Hopf maximum principle [15] we then have that

$$\sup\left|w_{Y_t^k}(\cdot) - w_{X_t}(\cdot)\right| \ \leq \ \sup\left|w_{Y_0^k}(\cdot) - w_{X_0}(\cdot)\right| \ , \quad t \geq 0. \tag{7.34}$$

From (7.25), (7.34) we see that $w_{Y_t^k}(\cdot), \ k = 1, 2, ..,$ converges to $w_{X_t}(\cdot)$ for any $t \geq 0$. This implies convergence of $h_{Y_t^k}(\cdot), \ k = 1, 2, ..,$ to $h_{X_t}(\cdot)$ provided we can obtain a suitable uniform bound on $w_{Y_t^k}(x), \ k = 1, 2, ..,$ for large x. To carry this out we use the representation (6.6) for w_ε. Thus we have that

$$\begin{aligned} w_{Y_t^k}(x) \ &\leq \ \int_x^\infty dx' \int_0^\infty G_\varepsilon(x', y, 0, t) \ c_{Y_0^k}(y) \ dy \ = \\ &\int_x^\infty dx' \ G_\varepsilon(x', 0, 0, t) w_{Y_0^k}(0) + m_{1,1/\Lambda_\varepsilon}(t) \int_0^\infty G_\varepsilon(x, y, 0, t) \ w_{Y_0^k}(y) \ dy \ , \end{aligned} \tag{7.35}$$

where G_ε is given by (4.11) with $A(\cdot) = 1/\Lambda_\varepsilon(\cdot)$. Evidently (7.25) and (7.35) yield a uniform upper bound on $w_{Y_t^k}(x), \ k = 1, 2, ..,$ for large x, which decays rapidly as $x \to \infty$ to 0.

We have therefore proven the result for random variables X_0 which satisfy $\|X_0\|_\infty < \infty$. In the case when $\|X_0\|_\infty = \infty$ we proceed similarly, approximating X_0 with variables Y_0^k satisfying (7.15) by averaging the beta function of X_0 over intervals of length $1/k$ as in (7.16) for $0 \le x \le k,\ k = 1, 2, ...$ □

Lemma 7.5. *Assume that the function $h_{X_0}(\cdot)$ for the initial data random variable X_0 of (1.7) satisfies (7.15) and $T > 0$. Then there exists $x_T > 0$ such that $\beta_{X_t}(x) < 1$ for all $x \ge x_T,\ 0 \le t \le T$.*

Proof. We first obtain a lower bound on the ratio of the half line Dirichlet Green's function defined by (4.14) to the full line Green's function (4.11). Letting A_∞ be given by (4.1), we show that for any $\gamma > 0$, there are positive constants C_1, C_2 depending only on $\gamma,\ A_\infty T$ such that

$$\frac{G_{\varepsilon,D}(x,y,0,T)}{G_\varepsilon(x,y,0,T)} \ \ge\ 1 - \exp\left[-\frac{x^2}{C_1\varepsilon T}\right] \quad \text{for } y \ge \gamma x,\ x \ge C_2[T + \sqrt{\varepsilon T}]. \tag{7.36}$$

We see from (4.29), (4.41) that in order to establish (7.36) it is sufficient to show that there are constants C_1, C_2 depending only on $A_\infty T$ such that

$$P\left(\sup_{0\le s\le T}\left|\int_0^T k(s,s')\ dB(s')\right| > z\right) \ \le\ \exp\left[-\frac{z^2}{C_1 T}\right] \quad \text{for } z \ge C_2\sqrt{T}. \tag{7.37}$$

The inequality (7.37) follows from Doob's Martingale inequality as in the proof of Proposition 5.1.

We have now that

$$c_{X_t}(x) \ = \ \int_0^\infty G_{\varepsilon,D}(x,y,0,t)c_{X_0}(y)\ dy \ \le \\ \int_0^{y_0} G_\varepsilon(x,y,0,t)c_{X_0}(y)\ dy + \int_{y_0}^\infty G_\varepsilon(x,y,0,t)c_{X_0}(y)\ dy\ , \tag{7.38}$$

where $A(\cdot) = 1/\Lambda_\varepsilon(\cdot)$ in (4.11) and y_0 is given in (7.15). We can bound the first integral on the RHS of (7.38) using integration by parts to obtain the inequality

$$\int_0^{y_0} G_\varepsilon(x,y,0,t)c_{X_0}(y)\ dy \ \le\ G_\varepsilon(x,0,0,t)w_{X_0}(0) - m_{1,1/\Lambda_\varepsilon}(t)\frac{\partial}{\partial x}\int_0^{y_0} G_\varepsilon(x,y,0,t)w_{X_0}(y)\ dy\ . \tag{7.39}$$

Since $c_{X_0}(y)$ is Gaussian for $y \ge y_0$ as given in (7.15), the second integral on the RHS of (7.38) is bounded by a Gaussian,

$$\int_{y_0}^\infty G_\varepsilon(x,y,0,t)c_{X_0}(y)\ dy \ \le\ K\int_{-\infty}^\infty G_\varepsilon(x,y,0,t)\exp[-a(y-y_0) - \{a(y-y_0)\}^2/2L]\ dy \\ = \ K\exp[L/2]\left(\frac{L}{a^2\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t) + Lm_{1,1/\Lambda_\varepsilon}(t)^2}\right)^{1/2} \times \\ \exp\left[-\frac{\left(x + m_{2,1/\Lambda_\varepsilon}(t) - m_{1,1/\Lambda_\varepsilon}(t)y_0 + m_{1,1/\Lambda_\varepsilon}(t)L/a\right)^2}{2\left\{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t) + Lm_{1,1/\Lambda_\varepsilon}(t)^2/a^2\right\}}\right]\ . \tag{7.40}$$

We conclude from (4.11), (7.38)-(7.40) that there exist positive constants x_T, M_T such that for $0 < t \le T,\ x \ge x_T$,

(7.41)
$$c_{X_t}(x) \ \le\ \left[1+\exp[-x^2/M_T\right] K_t \exp\left[-\frac{(x+\bar{x}_t)^2}{2\left\{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)+Lm_{1,1/\Lambda_\varepsilon}(t)^2/a^2\right\}}\right] ,$$

where $K_t, \bar{x}_t$ are given by the formulas

$$(7.42)\qquad K_t \ =\ K\exp[L/2]\left(\frac{L}{a^2\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)+Lm_{1,1/\Lambda_\varepsilon}(t)^2}\right)^{1/2} ,$$
$$\bar{x}_t \ =\ m_{2,1/\Lambda_\varepsilon}(t)-m_{1,1/\Lambda_\varepsilon}(t)y_0+m_{1,1/\Lambda_\varepsilon}(t)L/a \ .$$

We can use (7.36) to obtain a lower bound on $c_{X_t}(\cdot)$. Thus we have that

$$(7.43)\qquad c_{X_t}(x) \ \ge\ \left\{1-\exp\left[-\frac{x^2}{C_1\varepsilon T}\right]\right\}\int_{\gamma x}^{\infty} G_\varepsilon(x,y,0,t)c_{X_0}(y)\ dy \ .$$

Assuming x sufficiently large so that $\gamma x \ge y_0$ then from (7.40) we see that

(7.44)
$$\int_{\gamma x}^{\infty} G_\varepsilon(x,y,0,t)c_{X_0}(y)\ dy \ =\ K_t\exp\left[-\frac{(x+\bar{x}_t)^2}{2\left\{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)+Lm_{1,1/\Lambda_\varepsilon}(t)^2/a^2\right\}}\right]$$
$$-\ K\int_{-\infty}^{\gamma x} G_\varepsilon(x,y,0,t)\exp[-a(y-y_0)-\{a(y-y_0)\}^2/2L]\ dy \ .$$

If $\gamma > 0$ is sufficiently small then the the second term on the RHS of (7.44) is much smaller than the first term. Hence we conclude that there exist positive constants x_T, M_T such that for $0 < t \le T,\ x \ge x_T$,

(7.45)
$$c_{X_t}(x) \ \ge\ \left[1-\exp[-x^2/M_T\right] K_t \exp\left[-\frac{(x+\bar{x}_t)^2}{2\left\{\varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)+Lm_{1,1/\Lambda_\varepsilon}(t)^2/a^2\right\}}\right] .$$

We can use (7.41), (7.45) to obtain an upper bound for $\beta_{X_t}(x)$ when $0 < t \le T,\ x \ge x_T$. In fact from the formula (2.35) we immediately conclude that

$$(7.46)\quad \beta_{X_t}(x) \ \le\ \left[1+\exp[-x^2/M_T\right]^2\left[1-\exp[-x^2/M_T\right]^{-2}\beta_Y(x) \ , \quad \text{for } x \ge x_T,$$

where $\beta_Y(\cdot)$ is the beta function for a Gaussian variable Y with mean m and variance σ^2 given by

$$(7.47)\qquad m \ =\ -\bar{x}_t \ , \quad \sigma^2 \ =\ \varepsilon\sigma^2_{1/\Lambda_\varepsilon}(t)+Lm_{1,1/\Lambda_\varepsilon}(t)^2/a^2 \ .$$

Now from Lemma 2.1 we see that if Y is Gaussian with mean m and variance σ^2, then there exists a universal constant C such that

$$(7.48)\qquad \beta_Y(x) \ \le\ 1-\frac{\sigma^2}{2(x-m)^2} \quad \text{for } x \ge C\sigma+m \ .$$

The result follows from (7.46), (7.48) upon choosing x_T sufficiently large. □

Remark 2. *The inequality (7.36) was easy to obtain from the explicit representation for the stochastic process* $Y_\varepsilon(s),\ 0 \le s \le T$, *of (4.8) conditioned on* $Y_\varepsilon(0) = y,\ Y_\varepsilon(T) = x$, *when the drift* $b(\cdot,\cdot)$ *is linear. It is much more difficult to obtain estimates on probabilities for the conditioned process in the case of non-linear* $b(\cdot,\cdot)$. *Theorem 1.2 of* [6] *proves a result for the cdf of* $Y_\varepsilon(t),\ 0 < t < T$, *in the case of* $b(\cdot,\cdot)$ *satisfying the uniform Lipschitz condition (4.1), which is analagous to (7.36).*

Proof of log-concavity of the function h_ε. We first assume that the initial condition random variable X_0 for (1.7) satisfies (7.15). Then by standard regularity theorems [8] for solutions to (1.7), the function u_ε of (7.13) is continuous on the closed set $\{(x,t)\ :\ x \ge 0, t \ge 0\}$. Furthermore Lemma 7.5 implies that for any $T > 0$ there exists $x_T > 0$ such that $u_\varepsilon(x,t) > 0$ for $0 \le t \le T,\ x \ge x_T$. In addition, (7.14) implies that $u_\varepsilon(0,t) > 0,\ 0 \le t \le T$, and (7.15) that $u_\varepsilon(x,0) > 0,\ x \ge 0$. Since u_ε is a classical solution to (7.13), the maximum principle [15] implies that $u_\varepsilon(x,t) > 0$ for $0 \le t \le T,\ 0 \le x \le x_T$. We have therefore proven that the function $h_\varepsilon(\cdot,t)$ is log-concave for $0 \le t \le T$ when the initial data random variable X_0 satisfies (7.15). The log-concavity of $h_\varepsilon(\cdot,t),\ t > 0$, for general log-concave initial data random variable X_0 now follows from Lemma 7.4. □

Remark 3. *The main difficulty in implementing Korevaar's argument is to show that the solution of the PDE is log-concave on the boundary of the region. We accomplished this here by taking advantage of the fact that the full line Green's function is Gaussian when the drift* $b(\cdot,\cdot)$ *for (4.2) is linear. In the case of non-linear* $b(\cdot,\cdot)$ *it is not possible to argue in this way. An alternative approach is to use Korevaar's observation* [10] *that a Dirichlet boundary condition implies log-concavity close to the boundary on account of the Hopf maximum principle* [15]. *Some log-concavity theorems for non-linear* $b(\cdot,\cdot)$ *are proved in the appendix of* [6] *using this method.*

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