Hindawi Publishing Corporation
International Journal of Analysis
Volume 2013, Article ID 654659, 7 pages
http://dx.doi.org/10.1155/2013/654659

Research Article

Common Fixed Point Theorems for α-ψ-Contractive Type Mappings

Jalal Hassanzadeasl

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Correspondence should be addressed to Jalal Hassanzadeasl; jalal.hasanzadeh172@gmail.com

Received 8 November 2012; Revised 5 March 2013; Accepted 11 March 2013

Academic Editor: Leo G. Rebholz

Recently, Samet et al. (2012) introduced the notion of α-ψ-contractive type mappings. They established some fixed point theorems for these mappings in complete metric spaces. In this paper, we introduce the notion of a coupled α-ψ-contractive mapping and give a common fixed point result about the mapping. Also, we give a result of common fixed points of some coupled self-maps on complete metric spaces satisfying a contractive condition.

1. Introduction

We know fixed point theory has many applications and was extended by several authors from different views (see, e.g., [1–33]). Recently, Samet et al. introduced the notion of α-ψ-contractive type mappings [3]. Denote with Ψ the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ. It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$ [3]. Let (X, d) be a metric space, T a self map on X, $\psi \in \Psi$ and $\alpha : X\times X \to [0,\infty)$. Then, T is called a α-ψ-contractive mapping whenever $\alpha(x, y)d(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$. Also, we say that T is α-admissible whenever $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$ for all $x, y \in X$ [3]. Also, we say that X has the property (B_α) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \ge 1$ and $x_n \to x$, then $\alpha(x_n, x) \ge 1$ for all $n \ge 1$. Let (X, d) be a complete metric space and let T a α-admissible α-ψ-contractive mapping on X. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. If T is continuous or X has the property (B_α), then T has a fixed point (see [3]; Theorems 2.1 and 2.2). Finally, we say that X has the property (H_α) whenever for each $x, y \in X$ there exists $z \in X$ such that $\alpha(x, z) \ge 1$ and $\alpha(y, z) \ge 1$. If X has the property (H_α) in the Theorems 2.1 and 2.2, then X has a unique fixed point ([3]; Theorem 2.3). It is considerable that the results of Samet et al. generalize similar ordered results in the literature (see the results of the third section [3]). The aim of this paper is introducing the notion of generalized coupled α-ψ-contractive mappings and give a common fixed point result about the mappings.

Definition 1. Let $\mathcal{F}$ the family of functions $f : [0,\infty) \to \mathbb{R}$ satisfy:

(i) $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,+\infty)$;

(ii) f is continuous;

(iii) f is nondecreasing on $[0,+\infty)$;

(iv) $f(t_1 + t_2) \le f(t_1) + f(t_2)$ for all $t_1, t_2 \in (0,+\infty)$.

Definition 2. Let Ψ the family of functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfy

(ψ_1) ψ is nondecreasing;

(ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$, for all $t \in (0,+\infty)$.

These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for all $t > 0$.

Definition 3. Let (X, d) be a metric space, and let $T, S : X \to X$ with given coupled mappings. Let $\alpha : X \times X \to [0, +\infty)$, $f \in F$, $\psi \in \Psi$, and let

$$m(Ax, By) = \max\left\{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}\left[d(x, By) + d(y, Ax)\right]\right\}, \quad (1)$$

for all coupled mappings $A, B : X \to X$ and $x, y \in X$. One says that T, S are generalized coupled α-ψ-contractive mappings whenever

$$\alpha(x, y) f(d(Tx, Sy)) \leq \psi(f(m(Tx, Sy))),$$
$$\alpha(x, y) f(d(Sx, Ty)) \leq \psi(f(m(Sx, Ty))), \quad (2)$$

for all $x, y \in X$.

Definition 4. Let $T, S : X \to X$, and let $\alpha : X \times X \to [0, +\infty)$. One says that T, S are coupled α-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Sy) \geq 1 \quad \text{or} \quad \alpha(Sx, Ty) \geq 1 \quad (3)$$

for all $x, y \in X$.

Definition 5. Let (X, d) be a compete metric space. For two subsets A, B of X, one marks $A \preceq B$, if for all $a \in A$, there exists $b \in B$ such that $a \preceq b$.

Definition 6. A partial metric on a nonempty set X is a function $\rho : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(ρ_1) $x = y \Leftrightarrow \rho(x, x) = \rho(x, y) = \rho(y, y)$;
(ρ_2) $\rho(x, x) \leq \rho(x, y)$;
(ρ_3) $\rho(x, y) = \rho(y, x)$;
(ρ_4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z)$.

A partial metric space is a pair (X, ρ) such that X is a nonempty set, and ρ is a partial metric on X. It is clear that if $\rho(x, y) = 0$, then from (ρ_1) and (ρ_2), $x = y$. But if $x = y$, $\rho(x, y)$ may not be 0. A basic example of a partial metric is the pair $(\mathbb{R}^+, \rho)$, where $\rho(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. If ρ is a partial metric on X, then the function $\rho^s : X \times X \to \mathbb{R}^+$ given by $\rho^s(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y)$ is a metric on X.

Example 7. Let $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the coupled mappings $T, S : X \to X$ by

$$Tx = \begin{cases} x + 1, & \text{if } x > 1, \\ x^2, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0, \end{cases}$$
$$Sx = \begin{cases} x - 1, & \text{if } x > 1, \\ x^3, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0. \end{cases} \quad (4)$$

We define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

If

$$\alpha(x, y) \geq 1 \Longrightarrow x, y \in [0, 1] \Longrightarrow x^2, y^3 \in [0, 1] \Longrightarrow \alpha(Tx, Sy) = \alpha(x^2, y^3) = 1. \quad (6)$$

Similarly, $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Ty) \geq 1$. This shows that T, S are coupled α-admissible.

Lemma 8. *Let (X, d) be a metric space. Suppose that $T, S : X \to X$ are generalized coupled α-ψ-contractive mappings. Then, $\mathcal{F}(T) = \mathcal{F}(S)$.*

Proof. We first show that any fixed point of T is also a fixed point of S and conversely. Define $\alpha(x, y) = 1$ for all $x, y \in X$. Since $\mathcal{F}(T) \neq \mathcal{F}(S)$, we may assume there exists $x^* \in X$ such that $x^* \in \mathcal{F}(T)$, but $x^* \notin \mathcal{F}(S)$. Since $d(x^*, Sx^*) > 0$, we have

$$\begin{aligned} m(Tx^*, Sx^*) &= \max\left\{d(x^*, x^*), d(x^*, Tx^*), d(x^*, Sx^*), \frac{1}{2}\left[d(x^*, Sx^*) + d(x^*, Tx^*)\right]\right\} \\ &= d(x^*, Sx^*), \\ f(d(x^*, Sx^*)) &= f(d(Tx^*, Sx^*)) \\ &\leq \alpha(x^*, x^*) f(d(Tx^*, Sx^*)) \\ &\leq \psi(f(m(Tx^*, Sx^*))) \\ &\leq \psi(f(d(x^*, Sx^*))) \\ &< f(d(x^*, Sx^*)). \end{aligned} \quad (7)$$

This contradiction establishes that $\mathcal{F}(T) \subseteq \mathcal{F}(S)$. A similar argument establishes the reverse containment, and therefore $\mathcal{F}(T) = \mathcal{F}(S)$. □

2. Main Results

Now, we are ready to state and prove our main results.

Theorem 9. *Let (X, d) be a complete metric space. Suppose that $T, S : X \to X$ are generalized coupled α-ψ-contractive mappings and satisfy the following conditions:*

(i) *T, S are coupled α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; or*
(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T or S is continuous.*

Then T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof. By Lemma 8, we have $\mathcal{F}(T) = \mathcal{F}(S)$. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 > 1$, then $x^* = x_{n_0}$ are a common fixed point for T, S. So, we can assume that $x_{2n} \neq Tx_{2n}$ and $x_{2n+1} \neq Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. Since T, S are coupled α-admissible, we have

$$\begin{aligned}\alpha(x_0, x_1) &= \alpha(x_0, Tx_0) \geq 1 \Longrightarrow \alpha(Tx_0, Sx_1)\\ &= \alpha(x_1, x_2) \geq 1 \Longrightarrow \alpha(Sx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.\end{aligned} \tag{8}$$

Inductively, we have

$$\begin{aligned}\alpha(x_{2n}, x_{2n+1}) &\geq 1,\\ \alpha(x_{2n+1}, x_{2n+2}) &\geq 1,\end{aligned} \tag{9}$$

for all $n \in \mathbb{N}_0$. We obtain

$$\begin{aligned}f(d(x_{2n+1}, x_{2n+2})) &= f(d(Tx_{2n}, Sx_{2n+1}))\\ &\leq \alpha(x_{2n}, x_{2n+1}) f(d(Tx_{2n}, Sx_{2n+1}))\\ &\leq \psi(f(m(Tx_{2n}, Sx_{2n+1}))).\end{aligned} \tag{10}$$

Now,

$$\begin{aligned}&m(Tx_{2n}, Sx_{2n+1})\\ &= \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}),\\ &\qquad \frac{1}{2}[d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})]\Big\}\\ &= \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),\\ &\qquad \frac{1}{2}[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]\Big\}\\ &\leq \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),\\ &\qquad \frac{1}{2}[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 0]\Big\}\\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}\end{aligned} \tag{11}$$

if

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2}). \tag{12}$$

So, in general,

$$\begin{aligned}f(d(x_{2n+1}, x_{2n+2})) &\leq \psi(f(d(x_{2n+1}, x_{2n+2})))\\ &< f(d(x_{2n+1}, x_{2n+2})),\end{aligned} \tag{13}$$

which is a contradiction since $d(x_{2n+1}, x_{2n+2}) > 0$. Thus,

$$f(d(x_{2n+1}, x_{2n+2})) \leq \psi(f(d(x_{2n}, x_{2n+1}))). \tag{14}$$

Similarly, if

$$f(d(x_{2n}, x_{2n+1})) \leq \psi(f(d(x_{2n-1}, x_{2n}))), \tag{15}$$

we have

$$f(d(x_n, x_{n+1})) \leq \psi(f(d(x_{n-1}, x_n))), \tag{16}$$

for all $n \in \mathbb{N}_0$. By induction, we get

$$\begin{aligned}f(d(x_n, x_{n+1})) &\leq \psi(\psi(f(d(x_{n-2}, x_{n-1}))))\\ &\leq \cdots \leq \psi^n(f(d(x_0, x_1))),\end{aligned} \tag{17}$$

for all $n \in \mathbb{N}_0$. Fix $\epsilon > 0$, and let $n(\epsilon) \in \mathbb{N}_0$ such that

$$\sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \epsilon. \tag{18}$$

Let $n, m \in \mathbb{N}_0$ with $m > n > n(\epsilon)$. Using the triangle inequality, we obtain

$$\begin{aligned}f(d(x_n, x_m)) &\leq \sum_{k=n}^{m-1} f(d(x_k, x_{k+1}))\\ &\leq \sum_{k=n}^{m-1} \psi^k(f(d(x_0, x_1)))\\ &\leq \sum_{n=n(\epsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \epsilon.\end{aligned} \tag{19}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d).

Since (X, d) is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. From the continuity of T, it follows that $x_{2n+1} = Tx_{2n} \to Tx^*$ as $n \to +\infty$, then $x^* = Tx^*$. Similarly if S is continuous, we have $x^* = Sx^*$. □

Corollary 10. *Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a generalized α-ψ-contractive mapping and satisfies the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; or*

(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ converges to the fixed point of T.

Example 11. Let $X = \mathbb{R}$ endowed with the standard metric $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$. Define the coupled mappings $T, S : X \to X$ by

$$Tx = \begin{cases} 2x - \frac{3}{2}, & \text{if } x > 1, \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0, \end{cases}$$
$$Sx = \begin{cases} 2x - \frac{5}{3}, & \text{if } x > 1, \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0. \end{cases} \tag{20}$$

We define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

If $f(t) = t$ and $\psi(t) = (1/2)t$ for all $t \geq 0$, we have

$$\alpha(x,y) f(d(Tx,Sy)) = \begin{cases} f(d(Tx,Sy)), & \text{if } x, y \in [0,1], \\ 0, & \text{otherwise,} \end{cases}$$
$$\begin{aligned} \alpha(x,y) f(d(Tx,Sy)) &\leq f(d(Tx,Sy)) \\ &= \left|\frac{x}{2} - \frac{y}{2}\right| \leq \frac{1}{2}|x-y| \leq \psi(f(m(Tx,Sy))), \end{aligned} \tag{22}$$

for all $x, y \in X$. Thus, T, S are generalized coupled α-ψ-contractive mappings. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T1) = 1$. Obviously, T is continuous, and so it remains to show that T, S are coupled α-admissible. To do so, let $x, y \in X$ such that $\alpha(x,y) \geq 1$. This implies that $x, y \in [0,1]$ by the definition of α. We then have $Tx \in [0,1]$, $Sy \in [0,1]$ and $\alpha(Tx, Sy)) = 1$. Then, T, S are coupled α-admissible. Now, all the hypotheses of Theorem 9 are satisfied.

Consequently, T, S have common fixed points. In this example, 0 is at least one common fixed point of T and S.

Now, we omit the continuity hypothesis of T and S.

Theorem 12. *Let (X,d) be a complete metric space. Suppose that $T, S : X \to X$ are generalized coupled α-ψ-contractive mappings and satisfy the following conditions:*

(i) *T, S are coupled α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; or*

(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k; or*

(iii)* *if $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x, x_{n(k)}) \geq 1$ for all k.*

Then, T, S have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T, S.

Proof. Following the proof of Theorem 9, we know that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X,d). Then, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. From Theorem 9 and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{2n(k)}, x^*) \geq 1$ for all k.

Applying Theorem 9, for all k, we get that

$$\begin{aligned} f(d(x_{2n(k)+1}, Sx^*)) &= f(d(Tx_{2n(k)}, Sx^*)) \\ &\leq \alpha(x_{2n(k)}, x^*) d(Tx_{2n(k)}, Sx^*) \\ &\leq \psi(f(m(Tx_{2n(k)}, Sx^*))). \end{aligned} \tag{23}$$

On the other hand, we have

$$\begin{aligned} m&(Tx_{2n(k)}, Sx^*) \\ &= \max\Big\{d(x_{2n(k)}, x^*), d(x_{2n(k)}, Tx_{2n(k)}), \\ &\quad d(x^*, Sx^*), \\ &\quad \frac{1}{2}[d(x_{2n(k)}, Sx^*) + d(x^*, Tx_{2n(k)})]\Big\}. \end{aligned} \tag{24}$$

Letting $k \to +\infty$, in the above equality, we get that

$$\lim_{k \to \infty} m(Tx_{2n(k)}, Sx^*) = d(x^*, Sx^*). \tag{25}$$

Suppose that $d(x^*, Sx^*) > 0$. From (25), for k large enough, we have $m(Tx_{2n(k)}, Sx^*) > 0$, which implies that

$$\psi(f(m(Tx_{2n(k)}, Sx^*))) < f(m(Tx_{2n(k)}, Sx^*)). \tag{26}$$

Thus, from (23), we have $f(d(x_{2n(k)+1}, Sx^*)) < m(Tx_{2n(k)}, Sx^*)$. Letting $k \to \infty$ in the above inequality, and using (25), we obtain that $f(d(x^*, Sx^*)) < f(d(x^*, Sx^*))$ which is a contradiction. Thus, we have $d(x^*, Sx^*) = 0$; that is, $x^* = Sx^*$. Similarly, it can be shown that $x^* = Tx^*$. □

Corollary 13. *Let (X,d) be a complete metric space. Suppose that $T : X \to X$ is generalized α-ψ-contractive mapping and satisfies thefollowing conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; or*

(ii)* *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k; or*

(iii)* *if $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x, x_{n(k)}) \geq 1$ for all k.*

Then, T has a fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ converges to the fixed point of T.

Example 14. Let $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the coupled mappings $T, S : X \to X$ by

$$Tx = \begin{cases} 2x - \dfrac{3}{4}, & \text{if } x > 1, \\ \dfrac{x}{2}, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x < 0, \end{cases}$$
$$Sx = \begin{cases} 2x - \dfrac{5}{6}, & \text{if } x > 1, \\ \dfrac{x}{2}, & \text{if } 0 \le x \le 1, \\ 0, & \text{if } x < 0. \end{cases} \tag{27}$$

We define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

If $f(t) = t$ and $\psi(t) = (1/2)t$ for all $t \ge 0$, we have

$$\alpha(x, y) f(d(Tx, Sy)) = \begin{cases} f(d(Tx, Sy)), & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise,} \end{cases}$$
$$\alpha(x, y) f(d(Tx, Sy)) \le f(d(Tx, Sy)) = \left|\frac{x}{2} - \frac{y}{2}\right| \le \frac{1}{2}|x - y| \le \psi(f(m(Tx, Sy))), \tag{29}$$

for all $x, y \in X$. Thus, T, S are generalized coupled α-ψ-contractive mappings. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. In fact, for $x_0 = 1$, we have $\alpha(1, T1) = 1$. Let $\{x_n\} \in X$, $x_n \to x$ and $\alpha(x_{n+1}, x_n) \ge 1$ for all $n \ge 1$. By the definition of α we have, $x_n, x_{n+1} \in [0, 1]$ for all n, so $x \in [0, 1]$ and $\alpha(x_n, x) \ge 1$. It remains to show that T, S are coupled α-admissible. In doing so, let $x, y \in X$ such that $\alpha(x, y) \ge 1$. This implies that $x, y \in [0, 1]$ by the definition of α. We have $Tx \in [0, 1]$, $Sy \in [0, 1]$, and $\alpha(Tx, Sy) = 1$. Then T, S are coupled α-admissible. Now, all the hypotheses of Theorem 12 are satisfied. Consequently, T, S have common fixed points. In this example, 0 is at least one common fixed point of T and S.

3. Fixed Point Theorems on Ordered Metric Space

Theorem 15. *Let $(X, \preceq, d)$ be a complete ordered metric space, $\psi \in \Psi$, $f \in \mathcal{F}$, and T, S be coupled mappings on X such that $f(d(Tx, Sy)) \npreceq \psi(f(m(Tx, Sy)))$ and $f(d(Sx, Ty)) \le \psi(f(m(Sx, Ty)))$ for all $x, y \in X$ with $x \preceq y$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $(Tx_0 \preceq x_0)$, and if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ or $(x_{n+1} \preceq x_n)$ for all $n \ge 1$ and $x_n \to x$, then $x_n \preceq x$ or $(x \preceq x_n)$ for all $n \ge 1$. If $x \preceq y$ implies $Tx \preceq Sy$ or $Sx \preceq Ty$ or $(Ty \preceq Sx$ or $Sy \preceq Tx)$, then T, S have common fixed points.*

Proof. Define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ whenever $x \preceq y$, and define $\alpha(x, y) = 0$ whenever $x \npreceq y$. It is easy to check that T, S are a coupled α-admissible and generalized coupled α-ψ-contractive mappings on X. Now, by using Theorem 9, T, S have common fixed points. □

Corollary 16. *Let $(X, \preceq, d)$ be a complete ordered metric space, $\psi \in \Psi$, $f \in \mathcal{F}$, and T a mapping on X such that $f(d(Tx, Ty)) \le \psi(f(m(Tx, Ty)))$ for all $x, y \in X$ with $x \preceq y$. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $(Tx_0 \preceq x_0)$ and if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ or $(x_{n+1} \preceq x_n)$ for all $n \ge 1$ and $x_n \to x$, then $x_n \preceq x$ or $(x \preceq x_n)$ for all $n \ge 1$. If $x \preceq y$ implies $Tx \preceq Ty$ or $(Ty \preceq Tx)$, then T has a fixed point.*

4. Fixed Point Theorems on Metric Spaces Endowed with Partial Metric

If we substitute a partial metric ρ instead the metric d in Theorem 9, it is easy to check that a similar result holds for the partial metric space case as follows. We define

$$m^*(Ax, By) = \max\left\{\rho(x, y), \rho(x, Ax), \rho(y, By), \frac{1}{2}[\rho(x, By) + \rho(y, Ax)]\right\} \tag{30}$$

for all coupled mappings $A, B : X \to X$ and $x, y \in X$.

Theorem 17. *Let (X, ρ) be a complete partial metric space, $\alpha : X \times X \to [0, \infty)$ a function, $\psi \in \Psi$, $f \in \mathcal{F}$, and T, S self maps on X such that*

$$\alpha(x, y) f(\rho(Tx, Sy)) \le \psi(f(m^*(Tx, Sy))),$$
$$\alpha(x, y) f(\rho(Sx, Ty)) \le \psi(f(m^*(Sx, Ty))) \tag{31}$$

for all $x, y \in X$. Suppose that T, S are coupled α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ or $(\alpha(Tx_0, x_0) \ge 1)$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ or $(\alpha(x_{n+1}, x_n) \ge 1)$ for all n and $x_n \to x$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ or $(\alpha(x, x_{n(k)}) \ge 1)$ for all k. Then T, S have common fixed points.

Corollary 18. *Let (X, ρ) be a complete partial metric space, $\alpha : X \times X \to [0, \infty)$ a function, $\psi \in \Psi$, $f \in \mathcal{F}$, and T a self map on X such that*

$$\alpha(x, y) f(\rho(Tx, Ty)) \le \psi(f(m^*(Tx, Ty))) \tag{32}$$

for all $x, y \in X$. Suppose that T is α-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$ or $(\alpha(Tx_0, x_0) \ge 1)$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ or $(\alpha(x_{n+1}, x_n) \ge 1)$ for all n and $x_n \to x$ as $n \to +\infty$, then *there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *or* $(\alpha(x, x_{n(k)}) \geq 1)$ *for all* k. *Then* T *has a fixed point.*

Example 19. Let $X = [0, +\infty)$ endowed with the partial metric $\rho(x, y) = \max\{x, y\}$, for all $x, y \in X$. Define the mapping $T : X \to X$, by

$$Tx = \begin{cases} 2x - \dfrac{3}{2}, & \text{if } x > 1, \\ \dfrac{x^2}{1+x}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0. \end{cases} \tag{33}$$

We define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\begin{gathered} \alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0,1],\ x \leq y, \\ 0, & \text{otherwise}, \end{cases} \\ \alpha(x, y) \geq 1 \Longleftrightarrow x, y \in [0,1], \\ x \leq y \Longleftrightarrow Tx, Ty \in [0,1], \\ Tx \leq Ty \Longleftrightarrow \alpha(Tx, Ty) \geq 1 \end{gathered} \tag{34}$$

since T is α-admissible. If $\psi(t) = t^2/(1+t)$, we have

$$\begin{aligned} \alpha(x, y)\rho(Tx, Ty) &\leq \rho\left(\frac{x^2}{1+x}, \frac{y^2}{1+y}\right) \\ &= \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} \\ &= \frac{y^2}{1+y} \leq y \\ &= \rho\{x, y\}. \end{aligned} \tag{35}$$

Suppose that $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$; by definition of α, we have x_n, x_{n+1} and $x \in [0, 1]$, on the other hand $x_n \leq x_{n+1} \Rightarrow x_{n(k)} \leq x$; then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k. There exists $x_0 = 1$ such that $\alpha(x_0, Tx_0) = 1$. This show that all conditions of Corollary 18 are satisfied, and so T has a fixed point in X.

Acknowledgment

The author would like to thank Tehran Science and Research Branch, Islamic Azad University, for the financial support of this research, which is based on a research project contract.

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