SYSTEMS OF HESS-APPEL'ROT TYPE

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ABSTRACT. We construct higher-dimensional generalizations of the classical Hess-Appel'rot rigid body system. We give a Lax pair with a spectral parameter leading to an algebro-geometric integration of this new class of systems, which is closely related to the integration of the Lagrange bitop performed by us recently and uses Mumford relation for theta divisors of double unramified coverings. Based on the basic properties satisfied by such a class of systems related to bi-Poisson structure, quasi-homogeneity, and conditions on the Kowalevski exponents, we suggest an axiomatic approach leading to what we call the "class of systems of Hess-Appel'rot type".

Contents

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1. Introduction. Starting from the Kowalevski analysis

It is well known that Kowalevski, in her celebrated 1889 paper [28], starting with a careful analysis of the solutions of the Euler and the Lagrange case of rigid-body motion, formulated a problem *of describing the parameters* (A, B, C, x_0, y_0, z_0), *for which the Euler – Poisson equations have a general solution in a form of uniform functions only with moving poles as singularities.* Here, $I = \mathrm{diag}(A, B, C)$ represents the inertia operator, and $\chi = (x_0, y_0, z_0)$ is the centre of mass of the rigid body.

Then, in §1 of [28], some necessary conditions were formulated and a new case was discovered, now known as Kowalevski case, as a unique possible beside the cases of Euler and Lagrange. However, considering the situation where all momenta of inertia are different, Kowalevski came to a relation analogue to the following (see [24]):

$$x_0 \sqrt{A(B - C)} + y_0 \sqrt{B(C - A)} + z_0 \sqrt{C(A - B)} = 0,$$

and concluded that $x_0 = y_0 = z_0$, giving the Euler case.

But, it was Appel'rot who noticed in the beginning of 1890's, that the last relation admits one more case, not mentioned by Kowalevski:

$$x_0 \sqrt{A(B - C)} + z_0 \sqrt{C(A - B)} = 0, \quad y_0 = 0,$$

under the assumption $A > B > C$. Such systems were considered also by Hess, even before Appel'rot, in 1890. Such intriguing position corresponding to the overlook in the Kowalevski paper, made the Hess-Appel'rot systems very attractive for leading Russian mathematicians from the end of XIX century. After a few years, Nekrasov and Lyapunov managed to provide new arguments and they demonstrated that the Hess-Appel'rot systems didn't satisfy the condition investigated by Kowalevski, which means that conclusion of §1 of [28] *was correct.*

And, from that moment, the Hess-Appel'rot systems were basically left aside, even in modern times, when new methods of inverse problems, Lax representations, finite-zone integrations were applied to almost all known classical systems, until very recently.

A few years ago, we constructed a Lax representation for the Hess-Appel'rot system (see [15]).

Now, in this paper the first higher - dimensional generalizations of the Hess-Appel'rot systems are constructed. For each dimension $n > 3$, we give a family of such generalizations. We provide Lax representations for all new systems, generalizing the Lax pair from [15]. We show that the new systems are *isoholomorphic.* This class of systems was introduced and studied in [16], in connection with the Lagrange bitop.

Lax matrices of isoholomorphic systems have specific distributions of zero entries. Therefore standard integration techniques of [17], [1] cannot be applied directly. Its integration requires more detailed analysis of geometry of Prym varieties and it is based on Mumford's relation on theta - divisors of unramified double coverings.

In the present paper, in addition, we perform in detail the integration procedure in the first higher-dimensional case $n = 4$ of new Hess-Appel'rot type systems.

The L-operator, a quadratic polynomial in λ of the form $\lambda^2 C + \lambda M + \Gamma$, in the case $n = 4$, satisfies the condition

$$L_{12} = L_{21} = L_{34} = L_{43} = 0.$$

Such situation, explicitly excluded by Adler-van Moerbeke (see [1], Theorem 1) and implicitly by Dubrovin (see [17], Lemma 5 and Corollary) was studied for the first time in [16]. (A nice and natural cohomological interpretation of polynomial Lax equations has been studied in [26].)

Study of the spectral curve and the Baker-Akhiezer function for the four-dimensional Hess-Appel'rot systems shows that, similarly to [16], the dynamics of the system is related to a Prym variety Π. It is connected to the evolution of divisors of certain meromorphic differentials Ω_j^i. From the condition on zeroes of the Lax matrix, it follows that differentials

$$\Omega_2^1, \; \Omega_1^2, \; \Omega_4^3, \; \Omega_3^4$$

are *holomorphic* during the whole evolution. Compatibility of this requirement with dynamics is based on Mumford's relation (see [35], [16])

$$\Pi^- \subset \Theta,$$

where Π^- is a translation of a Prym variety Π, and Θ is the theta divisor.

The paper is organized as follows. In Section 2, the definition of the classical Hess-Appel'rot system is given and a few of its basic properties are listed such as the $L - A$ pair from [15] and the Zhukovskii geometric interpretation from [48, 31]. A construction of four-dimensional generalizations of the Hess-Appel'rot system is done in Section 3. In the same Section, a Lax representation is presented and the spectral curve calculated. The next Section contains generalizations of the Hess-Appel'rot systems to all dimensions higher than 4. In the cases $n > 4$ not only invariant relations exist, which are typical for the Hess-Appel'rot systems, but also values of some of the first integrals have to be fixed (and equal to zero). Thus, the systems we construct in the case $n > 4$ are also certain generalizations of the Goryachev-Chaplygin systems (see, for example, [24] for the definition). The Lax pairs are given in this section as well. In Section 5, a transformation of coordinates is performed for the four-dimensional Hess-Appel'rot systems, based on the decomposition $so(4) = so(3) \oplus so(3)$. In this manner, the integration of the four-dimensional Hess-Appel'rot systems reduces to integration of two coupled three-dimensional systems of Hess-Appel'rot type. Starting with Section 6, the algebro-geometric integration is performed. The principal observation is the relationship between the Baker-Akhiezer functions of the four-dimensional Hess-Appel'rot system and the Lagrange bitop. Then, in Sections 7 and 8, some most important facts from the algebro-geometric integration of the Lagrange bitop, done in [16], are reviewed. Analysis of a Prym variety Π is done and through the Mumford-Dalalyan theory, a connection between the algebro-geometric and the classical approach from Section 5 is explained. Differentials Ω_j^i are defined and the holomorphicity condition is derived. Therefore, the whole class of such systems is called **isoholomorphic** systems. The crucial point is application of *Mumford's relation on theta-divisors of unramified double coverings* to derive formulae in theta-functions for such systems. Their dynamics is realized on the odd part of the generalized Jacobian,

which is obtained by gluing of the infinite points of the spectral curve. In Section 9, additional equations, which differ the cases of the Lagrange bitop and higher-dimensional Hess-Appel'rot systems are derived. In the final part of the paper, the characteristic properties, common for the Hess-Appel'rot system and its higher-dimensional generalizations are studied. The most relevant ones are abstracted as the axioms of the **class of systems of Hess-Appel'rot type**. In this way, in Section 10, after analysis of relevant Poisson structures, the *Hamilton perturbation* and *bi-Poisson* axioms are formulated. These axioms give very simple and geometrically transparent description of the systems of Hess-Appel'rot type. Namely, suppose bi-Poisson structure $\{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$ is given, with a bihamiltonian system with the Hamiltonian H_0 corresponding to the first structure. Further, let $f_1,\dots,f_k$ be the commuting integrals of the system $(H_0,\{\cdot,\cdot\}_1)$, which are Casimirs for the second structure $\{\cdot,\cdot\}_2$. Then, the systems of Hess-Appel'rot type are Hamiltonian with respect to the first structure with a Hamiltonian

$$H = H_0 + \sum_{l=1}^{k} J_l b_l f_l,$$

where J_l are constants and b_l are certain functions on the phase space. The invariant relations are

$$f_l = 0, \quad l = 1,\dots,k.$$

Thus, the invariant manifolds are symplectic leaves of the second structure.

In Section 11, a Kowalevski analysis is performed. As a result, the *quasi-homogeneity* and the *arithmetic* axiom are formulated, providing characterization of Hess-Appel'rot systems in terms of arithmetic conditions on Kowalevski exponents. This gives strong constraints on the functions b_l in the above expressions.

In this way, the study of Hess-Appel'rot systems, in a sense, reaches its historical origins of Kowalevski, Appel'rot, Lyapunov and others, as briefly mentioned above.

Finally, based on these axioms we study three-dimensional Hess-Appel'rot systems and formulate conditions which determine uniquely the classical Hess-Appel'rot system among them. This confirms the reasonability of the chosen axioms. Classification of higher-dimensional Hess-Appel'rot systems looks like an interesting problem. We hope that detailed analysis of dynamical properties of systems of Hess-Appel'rot type will deserve sufficient attention.

2. Classical Hess-Appel'rot system.

The Euler-Poisson equations of the motion of a heavy rigid body in the moving frame are [24]:

$$\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{M} \times \Omega + \Gamma \times \chi, \\
\dot{\Gamma} &= \Gamma \times \Omega \\
\Omega &= \tilde{J}\mathbf{M}, \quad \tilde{J} = \mathrm{diag}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3),
\end{aligned}$$

(1)

where $\mathbf{M}$ is the kinetic momentum vector, Ω the angular velocity, $\tilde{J}$ a diagonal matrix, the inverse of inertia operator, Γ a unit vector fixed in the space and χ is the radius vector of the centre of masses.

It is well known ([24]) that equations (1) have three integrals of motion:

$$F_1 = \frac{1}{2}\langle \mathbf{M}, \Omega \rangle + \langle \Gamma, \chi \rangle$$

(2)
$$F_2 = \langle \mathbf{M}, \Gamma \rangle$$

$$F_3 = \langle \Gamma, \Gamma \rangle = 1.$$

Thus, for complete integrability, one integral more is necessary [24]. Let $\tilde{J}_1 < \tilde{J}_2 < \tilde{J}_3$ and $\chi = (x_0, y_0, z_0)$. Hess in [27] and Appel'rot in [4] found that if the inertia momenta and the radius vector of the centre of masses satisfy the conditions

$$y_0 = 0$$

(3)
$$x_0 \sqrt{\tilde{J}_3 - \tilde{J}_2} + z_0 \sqrt{\tilde{J}_2 - \tilde{J}_1} = 0,$$

then the surface

(4)
$$F_4 = M_1 x_0 + M_3 z_0 = 0$$

is invariant. Integration of such a system by classical techniques can be found in [24]. In [15], an L-A pair for the Hess-Appel'rot system is constructed:

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda \chi + \Omega, \quad C = \frac{1}{\tilde{J}_2}\chi,$$

where skew-symmetric matrices represent vectors denoted by the same letter. Also, basic steps in algebro-geometric integration procedure are given in [15].

The Zhukovskii geometric interpretation of the conditions (3) [48, 31]. Let us consider the ellipsoid

$$\frac{M_1^2}{\tilde{J}_1} + \frac{M_2^2}{\tilde{J}_2} + \frac{M_3^2}{\tilde{J}_3} = 1,$$

and the plane containing the middle axis and intersecting the ellipsoid at a circle. Denote by l the normal to the plane, which passes through the fixed point O. Then the condition (3) means that the centre of masses lies on the line l.

Having this interpretation in mind, we choose a basis of moving frame such that the third axis is l, the second one is directed along the middle axis of the ellipsoid, and the first one is chosen according to the orientation of the orthogonal frame. In this basis (see [13]), the particular integral (4) becomes

$$F_4 = M_3 = 0,$$

matrix $\tilde{J}$ obtains the form:

$$J = \begin{pmatrix} J_1 & 0 & J_{13} \\ 0 & J_1 & 0 \\ J_{13} & 0 & J_3 \end{pmatrix},$$

and $\chi = (0, 0, z_0)$. This will serve us as a motivation for a definition of the four-dimensional Hess-Appel'rot system.

3. Four-dimensional Hess-Appel'rot system.

The Euler-Poisson equations of motion of a heavy rigid body fixed at a point are Hamiltonian on the Lie algebra $e(3)$, which is the semi-direct product of Lie algebras R^3 and $so(3)$. Since R^3 is isomorphic to $so(3)$, there are two natural higher-dimensional generalizations of Euler-Poisson equations. One is to Lie algebra $e(n) = R^n \times so(n)$, and the second one, given by Ratiu in [37], is to the semi-direct product $so(n) \times so(n)$. The main result of this Section is a construction of an analogue of the Hess-Appel'rot system on $so(n) \times so(n)$.

Equations of a heavy n-dimensional rigid body on $so(n) \times so(n)$, introduced by Ratiu in [37], are:

$$(5) \qquad \begin{aligned} \dot{M} &= [M, \Omega] + [\Gamma, \chi] \\ \dot{\Gamma} &= [\Gamma, \Omega], \end{aligned}$$

where $M, \Omega, \Gamma, \chi \in so(n)$, and χ is a constant matrix. We will suppose that

$$\Omega = JM + MJ,$$

where J is a constant symmetric matrix. First, in this section, we consider equations (5) in dimension four. Motivated by the Zhukovskii geometric interpretation given at the end of the previous section, we start with the following definition

Definition 1. *The four-dimensional Hess-Appel'rot system is described by the equations* (5) *and satisfies the conditions:*
a)

$$(6) \qquad \Omega = MJ + JM, \quad J = \begin{pmatrix} J_1 & 0 & J_{13} & 0 \\ 0 & J_1 & 0 & J_{24} \\ J_{13} & 0 & J_3 & 0 \\ 0 & J_{24} & 0 & J_3 \end{pmatrix}$$

b)

$$\chi = \begin{pmatrix} 0 & \chi_{12} & 0 & 0 \\ -\chi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_{34} \\ 0 & 0 & -\chi_{34} & 0 \end{pmatrix}.$$

The invariant surfaces are determined in the next lemma.

Lemma 1. *For the four-dimensional Hess-Appel'rot system, the following relations take place:*

$$\begin{aligned} \dot{M}_{12} &= J_{13}(M_{13}M_{12} + M_{24}M_{34}) + J_{24}(M_{13}M_{34} + M_{12}M_{24}), \\ \dot{M}_{34} &= J_{13}(-M_{13}M_{34} - M_{12}M_{24}) + J_{24}(-M_{13}M_{12} - M_{24}M_{34}). \end{aligned}$$

In particular, if $M_{12} = M_{34} = 0$ hold at the initial moment, then the same relations are satisfied during the evolution in time.

Proof follows by direct calculations from equations (5), using (6).

Thus, in the four-dimensional Hess-Appel'rot case, there are two invariant relations

(7) $$M_{12} = 0, \quad M_{34} = 0.$$

Now we will give another definition of the four-dimensional Hess-Appel'rot conditions, starting from a basis where the matrix J is diagonal in.

Let $\tilde{J} = \mathrm{diag}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{J}_4)$.

Definition 1'. *The four-dimensional Hess-Appel'rot system is described by the equations (5) and satisfies the conditions:*
a)
$$\Omega = M\tilde{J} + \tilde{J}M, \quad \tilde{J} = \mathrm{diag}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{J}_4),$$

b)
$$\tilde{\chi} = \begin{pmatrix} 0 & \tilde{\chi}_{12} & 0 & \tilde{\chi}_{14} \\ -\tilde{\chi}_{12} & 0 & \tilde{\chi}_{23} & 0 \\ 0 & -\tilde{\chi}_{23} & 0 & \tilde{\chi}_{34} \\ -\tilde{\chi}_{14} & 0 & -\tilde{\chi}_{34} & 0 \end{pmatrix},$$

c)
$$\tilde{J}_3 - \tilde{J}_4 = \tilde{J}_2 - \tilde{J}_1,$$
$$\frac{\tilde{J}_3 - \tilde{J}_1}{\sqrt{1 + t_1^2}} = \frac{\tilde{J}_4 - \tilde{J}_2}{\sqrt{1 + t_2^2}}$$

where
$$t_1 := \frac{2(\tilde{\chi}_{14}\tilde{\chi}_{34} - \tilde{\chi}_{12}\tilde{\chi}_{23})}{\tilde{\chi}_{14}^2 - \tilde{\chi}_{34}^2 + \tilde{\chi}_{12}^2 - \tilde{\chi}_{23}^2},$$
$$t_2 := \frac{2(\tilde{\chi}_{14}\tilde{\chi}_{12} - \tilde{\chi}_{23}\tilde{\chi}_{34})}{-\tilde{\chi}_{14}^2 - \tilde{\chi}_{34}^2 + \tilde{\chi}_{12}^2 + \tilde{\chi}_{23}^2}.$$

Proposition 1. *There exists a bi-correspondence between sets of data from Definition 1 and Definition 1'.*

Proof. From $\tilde{J} = S^T J S$, where

$$S = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & \cos\varphi_1 & 0 & \sin\varphi_1 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & -\sin\varphi_1 & 0 & \cos\varphi_1 \end{pmatrix},$$

and $\varphi = \frac{1}{2}\arctan\frac{2J_{13}}{J_3 - J_1}$, $\varphi_1 = \frac{1}{2}\arctan\frac{2J_{24}}{J_3 - J_1}$, we have

$$\tilde{J} = \mathrm{diag}\left(\frac{J_1 + J_3 - A}{2}, \frac{J_1 + J_3 - A_1}{2}, \frac{J_1 + J_3 + A}{2}, \frac{J_1 + J_3 + A_1}{2}\right).$$

Here $A = \sqrt{(J_3 - J_1)^2 + 4J_{13}^2}$, $A_1 = \sqrt{(J_3 - J_1)^2 + 4J_{24}^2}$. The first part in the Definition 1'c)

$$\tilde{J}_3 - \tilde{J}_4 = \tilde{J}_2 - \tilde{J}_1,$$

follows from these relations.

From $\tilde{\chi} = S^T \chi S$, we have

$$\tilde{\chi} = S^T \chi S = \begin{pmatrix} 0 & \tilde{\chi}_{12} & 0 & \tilde{\chi}_{14} \\ -\tilde{\chi}_{12} & 0 & \tilde{\chi}_{23} & 0 \\ 0 & -\tilde{\chi}_{23} & 0 & \tilde{\chi}_{34} \\ -\tilde{\chi}_{14} & 0 & -\tilde{\chi}_{34} & 0 \end{pmatrix},$$

where

$$\tilde{\chi}_{12} = \chi_{12} \cos\varphi \cos\varphi_1 + \chi_{34} \sin\varphi \sin\varphi_1,$$
$$\tilde{\chi}_{14} = \chi_{12} \cos\varphi \sin\varphi_1 - \chi_{34} \sin\varphi \cos\varphi_1,$$
$$\tilde{\chi}_{23} = -\chi_{12} \sin\varphi \cos\varphi_1 + \chi_{34} \cos\varphi \sin\varphi_1,$$
$$\tilde{\chi}_{34} = \chi_{12} \sin\varphi \sin\varphi_1 + \chi_{34} \cos\varphi \cos\varphi_1.$$

From the last formulae, it follows:

$$(8) \qquad (\tilde{\chi}_{12}\sin\varphi + \tilde{\chi}_{23}\cos\varphi)\cos\varphi_1 + (\tilde{\chi}_{14}\sin\varphi - \tilde{\chi}_{34}\cos\varphi)\sin\varphi_1 = 0,$$

$$(9) \qquad (\tilde{\chi}_{12}\cos\varphi - \tilde{\chi}_{23}\sin\varphi)\sin\varphi_1 - (\tilde{\chi}_{14}\cos\varphi + \tilde{\chi}_{34}\sin\varphi)\cos\varphi_1 = 0.$$

From (8) and (9):

$$\tan 2\varphi = \frac{2(\tilde{\chi}_{14}\tilde{\chi}_{34} - \tilde{\chi}_{12}\tilde{\chi}_{23})}{\tilde{\chi}_{14}^2 - \tilde{\chi}_{34}^2 + \tilde{\chi}_{12}^2 - \tilde{\chi}_{23}^2} =: t_1,$$

$$\tan 2\varphi_1 = \frac{2(\tilde{\chi}_{14}\tilde{\chi}_{12} - \tilde{\chi}_{23}\tilde{\chi}_{34})}{-\tilde{\chi}_{14}^2 - \tilde{\chi}_{34}^2 + \tilde{\chi}_{12}^2 + \tilde{\chi}_{23}^2} =: t_2.$$

Thus, we get

$$(J_1 - J_3)^2 = \frac{(\tilde{J}_3 - \tilde{J}_1)^2}{1 + t_1^2},$$

$$(J_1 - J_3)^2 = \frac{(\tilde{J}_4 - \tilde{J}_2)^2}{1 + t_2^2}.$$

From the last formulae, we come to the last part of the Definition 1' c. This finishes the proof. □

Note. 1) In the case

$$J_{24} \neq 0, \chi_{34} = 0,$$

there is an additional relation

$$\tilde{\chi}_{12}\tilde{\chi}_{34} + \tilde{\chi}_{14}\tilde{\chi}_{23} = 0.$$

It follows from the system

$$\tilde{\chi}_{12}\sin\varphi + \tilde{\chi}_{23}\cos\varphi = 0,$$
$$\tilde{\chi}_{14}\sin\varphi - \tilde{\chi}_{34}\cos\varphi = 0,$$

as a consequence of (8, 9).

2) In the case

$$J_{24} = 0, \chi_{34} = 0,$$

additional relations are

$$\tilde{\chi}_{34} = \tilde{\chi}_{14} = 0,$$

and the second relation from the Definition 1' c) can be replaced by the relation

$$\tilde{\chi}_{12}\sqrt{\tilde{J}_2 - \tilde{J}_1} + \tilde{\chi}_{23}\sqrt{\tilde{J}_3 - \tilde{J}_2} = 0.$$

Notice the similarity of the last condition with the condition (3) for the classical three-dimensional case. (By ignoring the last coordinate one can recover the three-dimensional Hess-Appel'rot case.)

Theorem 1. *The four-dimensional Hess-Appel'rot system has the following Lax representation*

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \quad C = \frac{1}{J_1 + J_3}\chi.$$

Proof. Proof follows from

$$[C, \Omega] + [M, \chi] = \frac{1}{J_1 + J_3}\begin{pmatrix} 0 & 0 & D_{13} & 0 \\ 0 & 0 & 0 & D_{24} \\ -D_{13} & 0 & 0 & 0 \\ 0 & -D_{24} & 0 & 0 \end{pmatrix},$$

where

$$D_{13} = -\chi_{12}(J_{13}M_{12} + J_{24}M_{34}) + \chi_{34}(J_{13}M_{34} + J_{24}M_{12}),$$
$$D_{24} = -\chi_{12}(J_{13}M_{34} + J_{24}M_{12}) + \chi_{34}(J_{13}M_{12} + J_{24}M_{34}),$$

using relations (7). $\square$

One can calculate the spectral polynomial for the four-dimensional Hess-Appel'rot system:

$$p(\lambda, \mu) = \det(L(\lambda) - \mu \cdot 1) = \mu^4 + P(\lambda)\mu^2 + Q(\lambda)^2,$$

where

$$P(\lambda) = a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e$$
$$Q(\lambda) = f\lambda^4 + g\lambda^3 + h\lambda^2 + i\lambda + j$$

$a = C_{12}^2 + C_{34}^2,$

$b = 2C_{12}M_{12} + 2C_{34}M_{34}(= 0),$

$c = M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{24}^2 + M_{12}^2 + M_{34}^2 + 2C_{12}\Gamma_{12} + 2C_{34}\Gamma_{34},$

$d = 2\Gamma_{12}M_{12} + 2\Gamma_{13}M_{13} + 2\Gamma_{14}M_{14} + 2\Gamma_{23}M_{23} + 2\Gamma_{24}M_{24} + 2\Gamma_{34}M_{34}$

$e = \Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2 + \Gamma_{23}^2 + \Gamma_{24}^2 + \Gamma_{34}^2,$

$f = C_{12}C_{34}$

$g = C_{12}M_{34} + C_{34}M_{12}(= 0),$

$h = \Gamma_{34}C_{12} + \Gamma_{12}C_{34} + M_{12}M_{34} + M_{23}M_{14} - M_{13}M_{24},$

$i = M_{34}\Gamma_{12} + M_{12}\Gamma_{34} + M_{14}\Gamma_{23} + M_{23}\Gamma_{14} - \Gamma_{13}M_{24} - \Gamma_{24}M_{13},$

$j = \Gamma_{34}\Gamma_{12} + \Gamma_{23}\Gamma_{14} - \Gamma_{13}\Gamma_{24}.$

Let us consider standard Poisson structure on semidirect product $so(4) \times so(4)$. The functions d, e, i, j are Casimir functions (see [37]), c, h are first integrals, and $b = 0, g = 0$ are the invariant relations. General orbits of co-adjoint action are eight-dimensional, thus for complete integrability one needs four independent integrals in involution.

4. The n-dimensional Hess-Appel'rot systems.

In this Section, we introduce Hess-Appel'rot systems of arbitrary dimension.

Definition 2. *The n-dimensional Hess-Appel'rot system is described by the equations* (5), *and satisfies the conditions:*

a)

$$(10) \qquad \Omega = JM + MJ, \quad J = \begin{pmatrix} J_1 & 0 & J_{13} & 0 & 0 & \dots & 0 \\ 0 & J_1 & 0 & J_{24} & 0 & \dots & 0 \\ J_{13} & 0 & J_3 & 0 & 0 & \dots & 0 \\ 0 & J_{24} & 0 & J_3 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ . & . & . & . & . & \dots & . \\ . & . & . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & 0 & \dots & J_3 \end{pmatrix},$$

b)

$$\chi = \begin{pmatrix} 0 & \chi_{12} & 0 & \dots & 0 \\ -\chi_{12} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Direct calculations from (5) using (6) give the following lemma:

Lemma 2. *For the n-dimensional Hess-Appel'rot system, the following relations are satisfied:*

a)

$$\dot{M}_{12} = J_{13}(M_{12}M_{13} + M_{24}M_{34} + \sum_{p=5}^{n} M_{2p}M_{3p}) +$$

$$J_{24}(M_{12}M_{24} + M_{13}M_{34} - \sum_{p=5}^{n} M_{1p}M_{4p})$$

$$\dot{M}_{34} = -J_{13}(M_{13}M_{34} + M_{24}M_{12} + \sum_{p=5}^{n} M_{1p}M_{p4}) -$$

$$J_{24}(M_{13}M_{12} + M_{24}M_{34} + \sum_{p=5}^{n} M_{2p}M_{3p}),$$

$$\dot{M}_{3p} = -J_{13}(M_{13}M_{3p} + M_{2p}M_{12}) - J_{24}(M_{34}M_{2p} + M_{23}M_{4p}) +$$

$$M_{34}\Omega_{4p} - \Omega_{34}M_{4p} + \sum_{k=5}^{n}(M_{3k}\Omega_{kp} - \Omega_{3k}M_{4p}),\ \ p > 4,$$

$$\dot{M}_{4p} = J_{13}(-M_{14}M_{3p} + M_{1p}M_{34}) + J_{24}(M_{12}M_{1p} - M_{24}M_{4p}) -$$

$$M_{34}\Omega_{3p} + \Omega_{34}M_{3p} + \sum_{k=5}^{n}(M_{4k}\Omega_{kp} - \Omega_{4k}M_{4p}),\ \ p > 4,$$

b)

$$\dot{M}_{kl} = 0, \quad k,l > 4.$$

c) *The n-dimensional Hess-Appel'rot case has the following system of invariant relations*

(11) $$M_{12} = 0, \quad M_{lp} = 0, \quad l,p \geq 3.$$

By diagonalizing the matrix J, we come to another definition

Definition 2'. *The n-dimensional Hess-Appel'rot system is described by the equations* (5), *and satisfies the conditions*

a)

$$\Omega = \tilde{J}M + M\tilde{J}, \quad \tilde{J} = \operatorname{diag}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3, \tilde{J}_4, ..., \tilde{J}_4),$$

b)

$$\tilde{\chi} = \begin{pmatrix} 0 & \tilde{\chi}_{12} & 0 & \tilde{\chi}_{14} & \cdots & 0 \\ -\tilde{\chi}_{12} & 0 & \tilde{\chi}_{23} & 0 & \cdots & 0 \\ 0 & -\tilde{\chi}_{23} & 0 & \tilde{\chi}_{34} & \cdots & 0 \\ -\tilde{\chi}_{14} & 0 & -\tilde{\chi}_{34} & 0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ . & . & . & . & \cdots & . \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

c)

$$\tilde{J}_3 - \tilde{J}_4 = \tilde{J}_2 - \tilde{J}_1,$$

$$\frac{\tilde{J}_3 - \tilde{J}_1}{\sqrt{1 + t_1^2}} = \frac{\tilde{J}_4 - \tilde{J}_2}{\sqrt{1 + t_2^2}}$$

$$\tilde{\chi}_{12}\tilde{\chi}_{34} + \tilde{\chi}_{14}\tilde{\chi}_{23} = 0$$

where

$$t_1 := \frac{2(\tilde{\chi}_{14}\tilde{\chi}_{34} - \tilde{\chi}_{12}\tilde{\chi}_{23})}{\tilde{\chi}_{14}^2 - \tilde{\chi}_{34}^2 + \tilde{\chi}_{12}^2 - \tilde{\chi}_{23}^2},$$

$$t_2 := \frac{2(\tilde{\chi}_{14}\tilde{\chi}_{12} - \tilde{\chi}_{23}\tilde{\chi}_{34})}{-\tilde{\chi}_{14}^2 - \tilde{\chi}_{34}^2 + \tilde{\chi}_{12}^2 + \tilde{\chi}_{23}^2}.$$

As in the dimension four, there is an equivalence of the definitions.

Proposition 2. *There exists a bi-correspondence between sets of data from Definition 2 and Definition 2'.*

Proof follows the steps in Proposition 1.

Next theorem gives a Lax pair for the n-dimensional Hess-Appel'rot system.

Theorem 2. *The n-dimensional Hess-Appel'rot system has the following Lax pair*

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)],$$

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma, \quad A(\lambda) = \lambda\chi + \Omega, \quad C = \frac{1}{J_1 + J_3}\chi.$$

Proof. The statement follows from

$$[C,\Omega] + [M,\chi] = -\frac{\chi_{12}}{J_1 + J_3}.$$

$$. \begin{pmatrix} 0 & 0 & J_{13}M_{12} + J_{24}M_{34} & 0 & J_{24}M_{45} & ... & J_{24}M_{4n} \\ 0 & 0 & J_{13}M_{34} + J_{24}M_{12} & J_{13}M_{45} & ... & J_{13}M_{3n} \\ & 0 & 0 & 0 & 0 & ... & 0 \\ & & ... & & & & \\ & & & ... & & & \\ & & & & ... & & \\ & & & & & & 0 \end{pmatrix}$$

and relations (11). $\square$

Note. Let us note that invariant relations (11) exist in a more general case, with matrix J given by:

$$J := \begin{pmatrix} J_1 & 0 & J_{13} & J_{14} & ... & J_{1n} \\ 0 & J_1 & J_{23} & J_{24} & ... & J_{2n} \\ J_{13} & J_{23} & J_3 & 0 & ... & 0 \\ J_{14} & J_{24} & 0 & J_3 & ... & 0 \\ & & & & ... & \\ & & & & ... & \\ & & & & ... & \\ J_{1n} & J_{2n} & 0 & 0 & ... & J_3 \end{pmatrix}$$

But, using transformations $J \mapsto T^t J T$, where T is a block-diagonal matrix with 2×2 - block $A \in SO(2)$ and $(n-2) \times (n-2)$-block $B \in SO(n-2)$ on the diagonal, such a more general matrix J can be transformed to the case considered above.

Let us mention again that the Goryachev-Chaplygin system is a classical case integrable for the fixed level of a first integral. According to Lema 2b) and 2c) in

the n-dimensional Hess-Appel'rot system we also fix values of certain first integrals. But also, we have invariant relations which do not exist in the Goryachev-Chaplygin case.

5. The decomposition $so(4) = so(3) \oplus so(3)$ and integration of the four-dimensional Hess-Appel'rot system.

Starting from the well-known decomposition $so(4) = so(3) \oplus so(3)$, let us introduce

$$M_1 = \frac{1}{2}(M_+ + M_-) \qquad M_2 = \frac{1}{2}(M_+ - M_-),$$

(and similarly for Ω, Γ, χ), where M_+, M_- are vectors in R^3 defined with following correspondence between two three-dimensional vectors and four-dimensional antisymmetric matrices

$$(M_+, M_-) \rightarrow \begin{pmatrix} 0 & -M_+^3 & M_+^2 & -M_-^1 \\ M_+^3 & 0 & -M_+^1 & -M_-^2 \\ -M_+^2 & M_+^1 & 0 & -M_-^3 \\ M_-^1 & M_-^2 & M_-^3 & 0 \end{pmatrix}.$$

Then, equations of the motion become

(12)
$$\dot{M}_1 = 2(M_1 \times \Omega_1 + \Gamma_1 \times \chi_1) \qquad \dot{\Gamma}_1 = 2(\Gamma_1 \times \Omega_1)$$
$$\dot{M}_2 = 2(M_2 \times \Omega_2 + \Gamma_2 \times \chi_2) \qquad \dot{\Gamma}_2 = 2(\Gamma_2 \times \Omega_2),$$

and

$$\chi_1 = (0, 0, -\frac{1}{2}(\chi_{12} + \chi_{34})), \quad \chi_2 = (0, 0, -\frac{1}{2}(\chi_{12} - \chi_{34})).$$

Integrals of the motion are

(13)
$$\langle M_i, M_i \rangle + 2\frac{1}{J_1 + J_3}\langle \chi_i, \Gamma_i \rangle = h_i,$$
$$\langle \Gamma_i, \Gamma_i \rangle = 1, \qquad i = 1, 2,$$
$$\langle M_i, \Gamma_i \rangle = c_i,$$
$$\langle \chi_i, M_i \rangle = 0.$$

Connections between M and Ω are

$$\Omega_1 = ((J_1 + J_3)M_{(1)1} - (J_{13} - J_{24})M_{(2)3}, (J_1 + J_3)M_{(1)2},$$
$$(J_1 + J_3)M_{(1)3} + (J_1 - J_3)M_{(2)3} - (J_{13} + J_{24})M_{(2)1}),$$
$$\Omega_2 = ((J_1 + J_3)M_{(2)1} - (J_{13} + J_{24})M_{(1)3}, (J_1 + J_3)M_{(2)2},$$
$$(J_1 + J_3)M_{(2)3} + (J_1 - J_3)M_{(1)3} - (J_{13} - J_{24})M_{(1)1}),$$

where $M_{(i)j}$ is the j-th component of vector M_i. Using these expressions, equations

(12) can be rewritten in the following form:

$$\dot{M}_{(1)1} =2[(J_1 - J_3)M_{(1)2}M_{(2)3} - (J_{13} + J_{24})M_{(1)2}M_{(2)1} + \Gamma_{(1)2}\chi_{(1)3}],$$
$$\dot{M}_{(1)2} =2[-(J_1 - J_3)M_{(2)3}M_{(1)1} - (J_{13} - J_{24})M_{(1)3}M_{(2)3}+$$
$$(J_{13} + J_{24})M_{(1)1}M_{(2)1} - \Gamma_{(1)1}\chi_{(1)3}],$$
$$\dot{M}_{(1)3} =2(J_{13} - J_{24})M_{(1)2}M_{(2)3},$$
$$(14) \quad \dot{\Gamma}_{(1)1} =2[\Gamma_{(1)2}((J_1 + J_3)M_{(1)3} + (J_1 - J_3)M_{(2)3} - (J_{13} + J_{24})M_{(2)1})-$$
$$\Gamma_{(1)3}(J_1 + J_3)M_{(1)2}],$$
$$\dot{\Gamma}_{(1)2} =2[\Gamma_{(1)3}((J_1 + J_3)M_{(1)1} - (J_{13} - J_{24})M_{(2)3})-$$
$$\Gamma_{(1)1}((J_1 + J_3)M_{(1)3} + (J_1 - J_3)M_{(2)3} - (J_{13} + J_{24})M_{(2)1})],$$
$$\dot{\Gamma}_{(1)3} =2[\Gamma_{(1)1}(J_1 + J_3)M_{(1)2} - \Gamma_{(1)2}((J_1 + J_3)M_{(1)1} - (J_{13} - J_{24})M_{(2)3})],$$

and

$$\dot{M}_{(2)1} =2[(J_1 - J_3)M_{(2)2}M_{(1)3} - (J_{13} - J_{24})M_{(2)2}M_{(1)1} + \Gamma_{(2)2}\chi_{(2)3}],$$
$$\dot{M}_{(2)2} =2[-(J_1 - J_3)M_{(1)3}M_{(2)1} - (J_{13} + J_{24})M_{(2)3}M_{(1)3}+$$
$$(J_{13} - J_{24})M_{(2)1}M_{(1)1} - \Gamma_{(2)1}\chi_{(2)3}],$$
$$\dot{M}_{(2)3} =2(J_{13} + J_{24})M_{(2)2}M_{(1)3},$$
$$(15) \quad \dot{\Gamma}_{(2)1} =2[\Gamma_{(2)2}((J_1 + J_3)M_{(2)3} + (J_1 - J_3)M_{(1)3} - (J_{13} - J_{24})M_{(1)1})-$$
$$\Gamma_{(2)3}(J_1 + J_3)M_{(2)2}],$$
$$\dot{\Gamma}_{(2)2} =2[\Gamma_{(2)3}((J_1 + J_3)M_{(2)1} - (J_{13} + J_{24})M_{(1)3})-$$
$$\Gamma_{(2)1}((J_1 + J_3)M_{(2)3} + (J_1 - J_3)M_{(1)3} - (J_{13} - J_{24})M_{(1)1})],$$
$$\dot{\Gamma}_{(2)3} =2[\Gamma_{(2)1}(J_1 + J_3)M_{(2)2} - \Gamma_{(2)2}((J_1 + J_3)M_{(2)1} - (J_{13} + J_{24})M_{(1)3})].$$

From the equations (14) and (15), it follows that $M_{(1)3} = M_{(2)3} = 0$, giving two invariant relations introduced before.

Now, we are going to proceed the integration in a classical manner.

First, let us introduce coordinates K_i and l_i as follows:

$$M_{(i)1} = K_i \sin l_i, \qquad M_{(i)2} = K_i \cos l_i, \qquad i = 1, 2.$$

From the sixth equation of (14), using integrals (13), we have that

$$\dot{\Gamma}_{(1)3}^2 = 4(J_1 + J_3)^2 \left[(1 - \Gamma_{(1)3}^2)(h_1 - \frac{2}{J_1 + J_3}\chi_{(1)3}\Gamma_{(1)3}) - c_1^2\right] = P_3(\Gamma_{(1)3}).$$

Thus $\Gamma_{(1)3}$ can be solved by an elliptic quadrature. Also from the energy integral (the first one in (13)) we have that

$$K_1^2 = h_1 - \frac{2}{J_1 + J_3}\chi_{(1)3}\Gamma_{(1)3}.$$

Since $\tan l_1 = \frac{M_{(1)1}}{M_{(1)2}}$, using first two equations in (14), we have:

$$\dot{l}_1 = -2(J_{13} + J_{24})K_2 \sin l_2 + \frac{2\chi_{(1)3}c_1}{K_1^2}.$$

Also from the second and third integral in (13), we have that

$$K_1^2\Gamma_{(1)2}^2 - 2c_1 M_{(1)2}\Gamma_{(1)2} + c_1^2 - M_{(1)1}^2(1 - \Gamma_{(1)3}^2) = 0.$$

Similarly, from equations (15), we get:

$$\dot{\Gamma}_{(2)3}^2 = 4(J_1 + J_3)^2 \left[(1 - \Gamma_{(2)3}^2)(h_2 - \frac{2}{J_1 + J_3}\chi_{(2)3}\Gamma_{(2)3}) - c_2^2\right] = P_3(\Gamma_{(2)3}),$$

$$K_2^2 = h_2 - \frac{2}{J_1 + J_3}\chi_{(2)3}\Gamma_{(2)3},$$

$$\dot{l}_2 = -2(J_{13} - J_{24})K_1 \sin l_1 + \frac{2\chi_{(2)3}c_2}{K_2^2},$$

$$K_2^2\Gamma_{(2)2}^2 - 2c_2 M_{(2)2}\Gamma_{(2)2} + c_2^2 - M_{(2)1}^2(1 - \Gamma_{(2)3}^2) = 0.$$

From the previous considerations, we conclude that for complete integration of the four-dimensional Hess-Appel'rot system one need to solve a system of two differential equations (for l_1 and l_2) of the first order and to calculate two elliptic integrals, associated with elliptic curves E_1 and E_2 defined by

(16) $$E_i: \quad y^2 = P_i(x) = 8A_i x^3 - 4B_i x^2 - 8A_i x - 4C_i, \quad i = 1, 2$$

where

$$A_i = (J_1 + J_3)\chi_{(i)3}, \ B_i = (J_1 + J_3)^2 h_i, \ C_i = (J_1 + J_3)^2(c_i^2 - h_i).$$

This is a typical situation for the Hess-Appel'rot systems that additional integrations are required (see [36, 24, 15, 13]). Now we pass to the algebro-geometric integration.

6. Algebro-geometric integration

Before analyzing spectral properties of the matrices $L(\lambda)$, we will change the coordinates in order to diagonalize the matrix C. In this new basis the matrices $L(\lambda)$ have the form $\tilde{L}(\lambda) = U^{-1}L(\lambda)U$, where

$$U = \begin{pmatrix} 0 & 0 & \frac{i\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{i\sqrt{2}}{2} \\ \frac{i\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{i\sqrt{2}}{2} & 0 & 0 \end{pmatrix}$$

After straightforward calculations, we have

$$\tilde{L}(\lambda) = \begin{pmatrix} -i\Delta_{34} & 0 & -\beta_3^* - i\beta_4^* & i\beta_3 - \beta_4 \\ 0 & i\Delta_{34} & -i\beta_3^* - \beta_4^* & -\beta_3 + i\beta_4 \\ \beta_3 - i\beta_4 & -i\beta_3 + \beta_4 & -i\Delta_{12} & 0 \\ i\beta_3^* + \beta_4^* & \beta_3^* + i\beta_4^* & 0 & i\Delta_{12} \end{pmatrix}$$

where

$$\Delta_{12} = \lambda^2 C_{12} + \lambda M_{12} + \Gamma_{12},$$
$$\Delta_{34} = \lambda^2 C_{34} + \lambda M_{34} + \Gamma_{34},$$

$$\beta_3 = x_3 + \lambda y_3, \qquad\qquad x_3 = \frac{1}{2}\left(\Gamma_{13} + i\Gamma_{23}\right),$$

(17)
$$\beta_4 = x_4 + \lambda y_4, \qquad\qquad x_4 = \frac{1}{2}\left(\Gamma_{14} + i\Gamma_{24}\right),$$

$$\beta_3^* = \bar{x}_3 + \lambda\bar{y}_3, \qquad\qquad y_3 = \frac{1}{2}\left(M_{13} + iM_{23}\right),$$

$$\beta_4^* = \bar{x}_4 + \lambda\bar{y}_4, \qquad\qquad y_4 = \frac{1}{2}\left(M_{14} + iM_{24}\right).$$

Matrix $L(\lambda)$ is of the same form as the Lax matrix for the Lagrange bitop [15, 16]. It is a quadratic polynomial in the spectral parameter λ with matrix coefficients. General theories describing the isospectral deformations for polynomials with matrix coefficients were developed by Dubrovin [17, 18] in the middle of 70's and by Adler and van Moerbeke [1] a few years later. Dubrovin's approach was based on the Baker-Akhiezer function. Both approaches were applied in rigid body problems (see [32, 1] respectively).

But, as it was shown in [16], none of these two theories can be directly applied in cases like this. Necessary modifications were suggested in [16], where a procedure of algebro-geometric integration was presented. It is based on some nontrivial facts from the theory of Prym varieties, such as the Mumford relation on theta divisors of unramified double coverings and the Mumford-Dalalyan theory (see [16, 35, 34, 14, 40, 5]).

Here, we are going to follow closely the procedure from [16], with necessary changes, calculations and comments.

As usual, we start with the spectral curve

$$\Gamma : \ det\left(\tilde{L}(\lambda) - \mu \cdot 1\right) = 0.$$

We have
(18)
$$\Gamma : \ \mu^4 + \mu^2\left(\Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3\beta_3^* + 4\beta_4\beta_4^*\right) + \left[\Delta_{12}\Delta_{34} + 2i(\beta_3^*\beta_4 - \beta_3\beta_4^*)\right]^2 = 0.$$

There is an involution

$$\sigma : (\lambda, \mu) \to (\lambda, -\mu)$$

of the curve Γ, which corresponds to the skew-symmetricity of the matrix $L(\lambda)$. Denote the factor-curve by $\Gamma_1 = \Gamma/\sigma$.

Lemma 3. *The curve Γ_1 is a smooth hyperelliptic curve of the genus $g(\Gamma_1) = 3$. The arithmetic genus of the curve Γ is $g_a(\Gamma) = 9$.*

Proof. The curve

$$\Gamma_1 : \ u^2 + P(\lambda)u + [Q(\lambda)]^2 = 0,$$

is hyperelliptic, and its equation in the canonical form is:

(19)
$$u_1^2 = \frac{[P(\lambda)]^2}{4} - [Q(\lambda)]^2,$$

where $u_1 = u + P(\lambda)/2$. Since $\frac{[P(\lambda)]^2}{4} - [Q(\lambda)]^2$ is a polynomial of the degree 8, the genus of the curve Γ_1 is $g(\Gamma_1) = 3$. Γ is a double covering of Γ_1 and the ramification divisor is of degree 8. According to the Riemann-Hurwitz formula, $g_a(\Gamma) = 9$. $\square$

Lemma 4. *In generic case the spectral curve Γ has four ordinary double points $S_i, i = 1, \ldots, 4$. The genus of its normalization $\tilde{\Gamma}$ is five.* $\square$

Proof. From the equations

$$\frac{\partial p(\lambda, \mu)}{\partial \lambda} = 0, \quad \frac{\partial p(\lambda, \mu)}{\partial \mu} = 0,$$

where $p(\lambda, \mu) = det\left(\tilde{L}(\lambda) - \mu \cdot 1\right) = \mu^4 + \mu^2 P(\lambda) + [Q(\lambda)]^2$, the double points are $S_k = (\lambda_k, 0), k = 1, \ldots, 4$, where λ_k are zeroes of $Q(\lambda)$. Thus, $g(\tilde{\Gamma}) = g_a(\Gamma) - 4 = 5$. $\square$

Lemma 5. *Singular points S_i of the curve Γ are fixed by σ. The involution σ exchanges the two branches of Γ at S_i.*

Proof. Fixed points of the σ are defined with $\mu = 0$, thus S_i are fixed. Since their projections on Γ_1 are smooth points, σ exchanges the branches of Γ, which are given by the equation:

$$\mu^2 = \frac{-P(\lambda) + \sqrt{P^2(\lambda) - 4Q^2(\lambda)}}{2}.$$

$\square$

We start with the well-known eigen-problem

$$(20) \qquad \left(\frac{\partial}{\partial t} + \tilde{A}(\lambda)\right)\psi_k = 0, \quad \tilde{L}(\lambda)\psi_k = \mu_k \psi_k,$$

where ψ_k are eigenvectors with eigenvalues μ_k. Then $\psi_k(t, \lambda)$ form a 4×4 matrix with components $\psi_k^i(t, \lambda)$. Denote by φ_i^k its inverse matrix. Let us introduce

$$g_j^i(t, (\lambda, \mu_k)) = \psi_k^i(t, \lambda) \cdot \varphi_j^k(t, \lambda)$$

(there is no summation on k) or, in other words $g(t) = \psi_k(t) \otimes \varphi(t)^k$.

Matrix g is of rank 1, and we have $\partial\psi/\partial t = -\tilde{A}\psi$, $\partial\varphi/\partial t = \varphi\tilde{A}$, $\partial g/\partial t = [g, \tilde{A}]$. We can consider vector-functions $\psi_k(t, \lambda) = \left(\psi_k^1(t, \lambda), ..., \psi_k^4(t, \lambda)\right)^T$ as one function $\psi(t, (\lambda, \mu)) = \left(\psi^1(t, (\lambda, \mu)), ..., \psi^4(t, (\lambda, \mu))\right)^T$ on Γ defined by $\psi^i(t, (\lambda, \mu_k)) = \psi_k^i(t, \lambda)$. Similarly, we define $\varphi(t, (\lambda, \mu))$. Relations for divisors of zeroes and poles of functions ψ^i i φ_i in the affine part of Γ are:

$$(21) \qquad \left(g_j^i\right)_a = d_j(t) + d^i(t) - D_r - D'_s,$$

where $d_j(t)$ is divisor of zeroes of ψ_j, divisor $d^i(t)$ is divisor of zeroes of φ^i, D_r is the ramification divisor over λ plane (see [17]), D'_s is some subdivisor of D_s divisor of singular points defined by (21). One can easily calculate $deg\, D_r = 16, deg D_s = 8$.

Matrix elements $g_j^i(t, (\lambda, \mu_k))$ are meromorphic functions on Γ. We need their asymptotics in neighbourhoods of points P_k, which cover the point $\lambda = \infty$. Let $\tilde{\psi}_k$ be the eigenvector of the matrix $\tilde{L}(\lambda)$ normalized in P_k by the condition $\tilde{\psi}_k^k = 1$, and let $\tilde{\varphi}_i^k$ be the inverse matrix for $\tilde{\psi}_k^i$. We will also use another decomposition of matrix elements of g: $g_j^i = \psi_k^i \varphi_j^k = \tilde{\psi}_k^i \tilde{\varphi}_j^k$. It is an immediate consequence of proportionality of the vectors ψ_k and $\tilde{\psi}_k$ (φ^k and $\tilde{\varphi}^k$).

Lemma 6.

a) *Matrix g has the following representation*

$$g = \frac{\mu^3 + a_1\mu^2 + a_2\mu + a_3}{\partial p(\lambda, \mu)/\partial \mu},$$

where $a_1 = L, a_2 = P \cdot 1 + L^2, a_3 = PL + L^3$.

b) *For the Lax matrix L and λ_i such that $Q(\lambda_i) = 0$, it holds $a_3 = 0$.*

The proof of the Lemma follows from [17] and straightforward calculation. From the part (a) one can see that g could have poles in singular points of the spectral curve. But, from (b) we have

Corollary 1. *The matrix g has no poles in singular points of the curve Γ.*

So, from now on, taking Corollary 1 into account, we will consider all functions in this section as functions on the normalization $\tilde{\Gamma}$ of the curve Γ.

Since the functions $\tilde{\psi}_k^i$ and $\tilde{\varphi}_j^k$ are meromorphic in neighbourhoods of points P_k, their asymptotics can be calculated by expanding $\tilde{\psi}_k$ as a power series in λ^{-1} in a neighbourhood of the point $\lambda = \infty$ around the vector e_k, where $e_k^i = \delta_k^i$. We get

$$(22) \qquad \begin{aligned} & \left(\tilde{C} + \frac{\tilde{M}}{\lambda} + \frac{\tilde{\Gamma}}{\lambda^2}\right)\left(e_i + \frac{u_i}{\lambda} + \frac{v_i}{\lambda^2} + \frac{w_i}{\lambda^3} + \dots\right) \\ & = \left(\tilde{C}_{ii} + \frac{b_i}{\lambda} + \frac{d_i}{\lambda^2} + \frac{h_i}{\lambda^3} + \dots\right)\left(e_i + \frac{u_i}{\lambda} + \frac{v_i}{\lambda^2} + \frac{w_i}{\lambda^3} + \dots\right), \end{aligned}$$

where matrices $\tilde{C}, \tilde{M}$ and $\tilde{\Gamma}$ are defined by $\tilde{L}(\lambda) = \lambda^2\tilde{C} + \lambda\tilde{M} + \tilde{\Gamma}$. Comparing the same powers of λ, from (22) we get

$$(u_i)_i = 0, \quad (v_i)_i = 0, \quad (w_i)_i = 0$$

$$(u_i)_j = \frac{\tilde{M}_{ji}}{\tilde{C}_{ii} - \tilde{C}_{jj}} \qquad j \neq i$$

$$(23) \quad (v_i)_j = \frac{1}{\tilde{C}_{ii} - \tilde{C}_{jj}}\left(\sum_{k\neq i}\frac{\tilde{M}_{jk}\tilde{M}_{ki}}{\tilde{C}_{ii} - \tilde{C}_{kk}} - \frac{\tilde{M}_{ii}\tilde{M}_{ji}}{\tilde{C}_{ii} - \tilde{C}_{jj}} + \tilde{\Gamma}_{ji}\right)$$

$$(w_i)_j = \frac{1}{C_i - C_j}\left[\sum_{k\neq i}\tilde{M}_{jk}(v_i)_k + \sum_{k\neq i}\tilde{\Gamma}_{jk}(u_i)_k - b_i(v_i)_j - d_i(u_i)_j\right]$$

$$b_i = \tilde{M}_{ii}, \quad d_i = \sum_{k\neq i}\frac{\tilde{M}_{ik}\tilde{M}_{ki}}{\tilde{C}_{ii} - \tilde{C}_{kk}} + \tilde{\Gamma}_{ii}, \quad h_i = \sum_{k\neq i}\tilde{M}_{ik}(v_i)_k + \sum_{k\neq i}\tilde{\Gamma}_{jk}(u_i)_k$$

So, the matrix $\tilde{\psi} = \{\tilde{\psi}_k^i\}$ in a neighbourhood of $\lambda = \infty$ has the form:

$$(24) \qquad\qquad \tilde{\psi} = 1 + \frac{u}{\lambda} + \frac{v}{\lambda^2} + \frac{w}{\lambda^3} + O\left(\frac{1}{\lambda^3}\right).$$

Denote by $\tilde{d}_j$ and $\tilde{d}^i$ the following divisors:

$$\begin{aligned} \tilde{d}_1 = d_1 + P_2, \quad \tilde{d}_2 = d_2 + P_1, \quad \tilde{d}_3 = d_3 + P_4, \quad \tilde{d}_4 = d_4 + P_3, \\ \tilde{d}^1 = d^1 + P_2, \quad \tilde{d}^2 = d^2 + P_1, \quad \tilde{d}^3 = d^3 + P_4, \quad \tilde{d}^4 = d^4 + P_3. \end{aligned}$$

Analyzing the behavior of matrix g around points P_k, as in [16], we get

Proposition 3.
a) *Divisors of matrix elements of g are*

$$(25) \qquad (g_j^i) = \tilde{d}^i + \tilde{d}_j - D_r + 2(P_1 + P_2 + P_3 + P_4) - P_i - P_j$$

b) *Divisors $\tilde{d}_i, \tilde{d}^j$ are of the same degree*

$$\deg \tilde{d}_i = \deg \tilde{d}^j = 5.$$

Let us denote by $\Phi(t, \lambda)$ the normalized fundamental solution of

$$\left(\frac{\partial}{\partial t} + \tilde{A}(\lambda) \right) \Phi(t, \lambda) = 0, \quad \Phi(\tau) = 1.$$

Then, if we introduce the Baker-Akhiezer functions

$$(26) \qquad \hat{\psi}^i(t, \tau, (\lambda, \mu_k)) = \sum_s \Phi_s^i(t, \lambda) h^s(\tau, (\lambda, \mu_k))$$

where h^s are eigen-vectors of $L(\lambda)$ normalized by the condition $\sum_s h^s(t, (\lambda, \mu_k)) = 1$, it follows that:

$$(27) \qquad \hat{\psi}^i(t, \tau, (\lambda, \mu_k)) = \sum_s \Phi_s^i(t, \lambda) \frac{\psi_k^s(\tau, \lambda)}{\sum_l \psi_k^l(\tau, \lambda)} = \frac{\psi_k^i(t, \lambda)}{\sum_l \psi_k^l(\tau, \lambda)}.$$

Proposition 4. *Functions $\hat{\psi}^i$ satisfy the following properties*
a) *In the affine part of $\tilde{\Gamma}$, the function $\hat{\psi}^i$ has 4 time dependent zeroes which belong to the divisor $d^i(t)$ defined by formula (21), and 8 time independent poles, i.e.*

$$\left(\hat{\psi}^i(t, \tau, (\lambda, \mu_k)) \right)_a = d^i(t) - \bar{\mathcal{D}}, \qquad \deg \bar{\mathcal{D}} = 8.$$

b) *In points P_k, functions $\hat{\psi}^i$ have essential singularities as follows:*

$$\hat{\psi}^i(t, \tau, (\lambda, \mu)) = exp\left[-(t - \tau)R_k - iF_k \right] \hat{\alpha}^i(t, \tau, (\lambda, \mu))$$

where R_k and F_k are:

$$R_1 = i\left(\frac{\chi_{34}}{z} \right), \ R_2 = -R_1, \ R_3 = i\left(\frac{\chi_{12}}{z} \right), \ R_4 = -R_3,$$

$$F_1 = \left(\int_\tau^t \Omega_{34} dt \right), \ F_2 = -F_1, \ F_3 = \left(\int_\tau^t \Omega_{12} dt \right), F_4 = -F_3$$

and $\hat{\alpha}^i$ are holomorphic in a neighbourhood of P_k,

$$\hat{\alpha}^i(\tau, \tau, (\lambda, \mu)) = h^i(\tau, (\lambda, \mu)), \quad \hat{\alpha}^i(t, \tau, P_k) = \delta_i^k + \tilde{v}_k^i(t)z + O(z^2),$$

with

$$(28) \qquad \tilde{v}_k^i = \frac{\tilde{M}_{ki}}{\tilde{C}_{ii} - \tilde{C}_{kk}}.$$

Proof repeats the demonstration of Proposition 5 in [16].

Let us denote by $\hat{\psi}^i_{LB}$ the Baker-Akhiezer function for the Lagrange bitop from [16] with analytical properties as in Proposition 4 a) above and with asymptotics given by:

at points P_k, functions $\hat{\psi}^i_{LB}$ have essential singularities as follows:

$$\hat{\psi}^i_{LB}(t,\tau,(\lambda,\mu)) = exp\left[-(t-\tau)R_k\right]\hat{\alpha}^i_{LB}(t,\tau,(\lambda,\mu)),$$

where R_k are given with

$$R_1 = i\left(\frac{\chi_{34}}{z}\right),\ R_2 = -R_1,\ R_3 = i\left(\frac{\chi_{12}}{z}\right),\ R_4 = -R_3,$$

and $\hat{\alpha}^i_{LB}$ are holomorphic in a neighbourhood of P_k,

$$\hat{\alpha}^i_{LB}(\tau,\tau,(\lambda,\mu)) = h^i_{LB}(\tau,(\lambda,\mu)), \quad \hat{\alpha}^i_{LB}(t,\tau,P_k) = \delta^k_i + \tilde{v}^i_k(t)z + O(z^2).$$

From the Proposition 4 and from Proposition 5 of [16], we have

Corollary 2. *A relationship between the data of generalized Hess-Appel'rot problem and the Lagrange bitop are given by:*
a)

$$\hat{\psi}^k_{HA} := \hat{\psi}^k = \exp(iF_k)\hat{\psi}^k_{LB}, \quad k = 1,\ldots,4;$$

b)

(29) $$v^k_{jHA} := \tilde{v}^k_j = \exp(i(F_k + F_j))v^k_j, \quad k,j = 1,\ldots,4.$$

(v^k_j we will also denote as v^k_{jLB}.)

7. A Prym variety

Let us recall that $d^j(t)$ is divisor defined in (21).

Lemma 7. *On the Jacobian* $\mathrm{Jac}(\tilde{\Gamma})$ *the following relation takes place :*

$$\mathcal{A}(d^j(t) + \sigma d^j(t)) = \mathcal{A}(d^j(\tau) + \sigma d^j(\tau))$$

where $\mathcal{A}$ *is the Abel map from the curve* $\tilde{\Gamma}$ *to* $\mathrm{Jac}(\tilde{\Gamma})$, *and* σ *is involution on* $\tilde{\Gamma}$.

The proof is the same as the one of the corresponding Lemma in [16].

From the previous Lemma, we see that vectors $\mathcal{A}(d^i(t))$ belong to some translation of a Prym variety $\Pi = Prym(\tilde{\Gamma}|\Gamma_1)$. More details concerning Prym varieties one can find in [41, 40, 21, 9, 34, 35, 5, 8]. A natural question arises to compare two-dimensional tori Π and $E_1 \times E_2$, where elliptic curves E_i are defined by (16).

Together with the curve Γ_1, one can consider curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by the equations

(30) $$\mathcal{C}_1 : v^2 = \frac{P(\lambda)}{2} + Q(\lambda), \quad \mathcal{C}_2 : v^2 = \frac{P(\lambda)}{2} - Q(\lambda).$$

Lemma 8. *Curves E_i defined by* (16) *are Jacobians of curves $\mathcal{C}_i$ given by* (30).

Proof. Follows by a straightforward calculation.

$\square$

Since the curve Γ_1 is hyper-elliptic, in a study of the Prym variety Π the Mumford-Dalalyan theory can be applied (see [14, 34, 40]). Thus, the previous Lemma allows us to use the following Theorem from [16].

Theorem 3.

a) *The Prymian Π is isomorphic to the product of curves E_i:*

$$\Pi = Jac(\mathcal{C}_1) \times Jac(\mathcal{C}_2).$$

b) *The curve $\tilde{\Gamma}$ is the desingularization of $\Gamma_1 \times_{\mathbb{P}^1} \mathcal{C}_2$ and $\mathcal{C}_1 \times_{\mathbb{P}^1} \Gamma_1$.*

c) *The canonical polarization divisor Ξ of Π satisfies*

$$\Xi = E_1 \times \Theta_2 + \Theta_1 \times E_2,$$

where Θ_i is the theta-divisor of E_i.

Theorem 3 explains the connection between the curves E_1, E_2 and the Prym variety Π. Further analysis of properties of Prym varieties necessary for understanding the dynamics of the Lagrange bitop will be done in the next section.

8. Isoholomorphisity condition, Mumford's relation and solutions for v_{jLB}^k

We saw that integration of the four-dimensional Hess-Appel'rot system is partially reduced to solutions of the Lagrange bitop. Now, we are goint to give the explicit formulae for the Baker-Akhiezer function for the Lagrange bitop, obtained in [16]. According to Proposition 4, the Baker-Akhiezer function Ψ satisfies usual conditions of normalized (n=)4-point function on a curve of genus $g = 5$ with the divisor $\bar{\mathcal{D}}$ of degree $\deg \bar{\mathcal{D}} = g + n - 1 = 8$, see [19, 18]. By the general theory, it should determine the whole dynamics uniquely.

Let us consider the differentials $\Omega_j^i = g_{ij} d\lambda$, $i, j = 1, \ldots, 4$. In the case of general position it was proved by Dubrovin that Ω_j^i is a meromorphic differential having poles at P_i and P_j, with residues v_j^i and $-v_i^j$ respectively. But here we have

Proposition 5. [16] *Differentials Ω_2^1, Ω_1^2, Ω_4^3, Ω_3^4 are holomorphic during the whole evolution.*

The proof is based on the fact that from the conditions $L_2^1 = L_1^2 = L_4^3 = L_3^4 = 0$ it follows that

$$(31) \qquad v_2^1 = v_1^2 = v_4^3 = v_3^4 = 0.$$

(For more details see [16]). We can say that the condition $L_2^1 = L_1^2 = L_4^3 = L_3^4 = 0$ implies *isoholomorphicity*. Let us recall the general formulae for v from [18]:

$$(32) \qquad v_j^i = \frac{\lambda_i \theta(A(P_i) - A(P_j) + tU + z_0)}{\lambda_j \theta(tU + z_0)\epsilon(P_i, P_j)}, \quad i \neq j,$$

where $U = \sum x^{(k)} U^{(k)}$ is a certain linear combination of b-periods $U^{(i)}$ of differentials of the second kind $\Omega_{P_i}^{(1)}$, which have a pole of order two at P_i; λ_i are nonzero scalars, and

$$\epsilon(P_i, P_j) := \frac{\theta[\nu](A(P_i - P_j))}{(-\partial_{U^{(i)}} \theta[\nu](0))^{1/2}(-\partial_{U^{(j)}} \theta[\nu](0))^{1/2}}.$$

(Here ν is an arbitrary odd non-degenerate characteristic.) Thus, from (32) we get:

Holomorphicity of some of the differentials Ω_j^i implies that the theta divisor of the spectral curve contains some tori.

In a case when the spectral curve is a double unramified covering

$$\pi : \tilde{\Gamma} \to \Gamma_1;$$

with $g(\Gamma_1) = g$, $g(\tilde{\Gamma}) = 2g - 1$, as we have here (assuming that $\tilde{\Gamma}$ is the normalization of the spectral curve Γ), it is really satisfied that the theta divisor contains a torus, see [35]. Let us denote by Π^- the set

$$\Pi^- = \left\{ L \in Pic^{2g-2}\tilde{\Gamma} \,|\, NmL = K_{\Gamma_1}, h^0(L) \text{ is odd} \right\},$$

where K_{Γ_1} is the canonical class of the curve Γ_1 and $Nm : Pic\tilde{\Gamma} \to Pic\Gamma_1$ is the norm map, see [35, 40] for details. For us, it is crucial that Π^- is a translate of the Prym variety Π and that Mumford's relation ([35]) holds:

$$(33) \qquad\qquad\qquad\qquad \Pi^- \subset \Theta_{\tilde{\Gamma}}.$$

Let us denote

$$(34) \qquad\qquad U = i(\chi_{34} U^{(1)} - \chi_{34} U^{(2)} + \chi_{12} U^{(3)} - \chi_{12} U^{(4)}),$$

where $U^{(i)}$ is the vector of $\tilde{b}$-periods of the differential of the second kind $\Omega_{P_i}^{(1)}$, which is normalized by the condition that $\tilde{a}$-periods are zero. We suppose here that the cycles $\tilde{a}, \tilde{b}$ on the curve $\tilde{\Gamma}$ and a, b on Γ_1 are chosen to correspond to the involution σ and the projection π, see [5, 40]:

$$\pi(\tilde{a}_0) = a_0; \quad \pi(\tilde{b}_0) = 2b_0, \quad \sigma(\tilde{a}_k) = \tilde{a}_{k+2}, \quad k = 1, 2.$$

The basis of normalized holomorphic differentials $[u_0, \ldots, u_5]$ on $\tilde{\Gamma}$ and $[v_0, v_1, v_2]$ on Γ_1 are chosen such that

$$\pi^*(v_0) = u_0, \quad \pi^*(v_i) = u_i + \sigma(u_i) = u_i + u_{i+2}, \quad i = 1, 2.$$

Now we have

Theorem 4. [16]
a) *If vector z_0 in (32) corresponds to the translation of the Prym variety Π to Π^-, and vector U is defined by (34) then conditions (31) are satisfied.*

b) *The explicit formula for z_0 is*

$$(35) \qquad z_0 = \frac{1}{2}(\hat{\tau}_{00}, \hat{\tau}_{01}, \hat{\tau}_{02}, \hat{\tau}_{01}, \hat{\tau}_{02}), \quad \hat{\tau}_{0i} = \int_{\tilde{b}_0} u_i, \quad i = 0, 1, 2.$$

Formulae for scalars λ_i from (32) will be given later in this section.

The evolution on the Jacobian of the spectral curve $\mathrm{Jac}(\tilde{\Gamma})$ gives a possibility to reconstruct the evolution of Lax matrix $L(\lambda)$ only up to a conjugation by diagonal matrices. To overcome this problem, we are going to consider, together with Dubrovin, a generalized Jacobian, obtained by gluing together the infinite points. Those points are P_1, P_2, P_3, P_4 and the corresponding Jacobian will be denoted by $\mathrm{Jac}(\tilde{\Gamma}|\{P_1, P_2, P_3, P_4\})$.

The generalized Jacobian can be understood as a set of classes of relative equivalence among the divisors on $\tilde{\Gamma}$ of a certain degree. Two divisors of the same degree D_1 and D_2 are called *equivalent relative to points* P_1, P_2, P_3, P_4 if there exists a meromorphic function f on $\tilde{\Gamma}$ such that $(f) = D_1 - D_2$ and $f(P_1) = f(P_2) = f(P_3) = f(P_4)$.

The generalized Abel map is defined with

$$\tilde{A}(P) = (A(P), \lambda_1(P), ..., \lambda_4(P)), \quad \lambda_i(P) = \exp \int_{P_0}^{P} \Omega_{P_i Q_0}, i = 1, ..., 4,$$

where A is the standard Abel map. Here $\Omega_{P_i Q_0}$ denotes the normalized differential of the third kind, with poles at P_i and at an arbitrary fixed point Q_0.

Then the generalized Abel theorem (see [21]) can be formulated as

Lemma 9 (the generalized Abel theorem). *Divisors D_1 and D_2 are equivalent relative P_1, P_2, P_3, P_4 if and only if there exist integer-valued vectors N, M such that*

$$A(D_1) = A(D_2) + 2\pi N + BM,$$
$$\lambda_j(D_1) = c\lambda_j(D_2)\exp(M, A(D_2)), \ j = 1, ..., 4$$

where c is some constant and B is the period matrix of the curve $\tilde{\Gamma}$.

A generalized Jacobi inverse problem can be formulated as a question of finding, for given z, points $Q_1, \ldots, Q_8$ such that

$$\sum_{1}^{8} A(Q_i) - \sum_{2}^{4} A(P_i) = z + K,$$
$$\lambda_j = c \exp \sum_{s=1}^{8} \int_{P_0}^{Q_s} \Omega_{P_j Q_0} + \kappa_j, \ j = 1, ...4,$$

where K is the Riemann constant and constants κ_j depend on $\tilde{\Gamma}$, points P_1, P_2, P_3, P_4 and the choice of local parameters around them.

We will denote by Q_s the points which belong to the divisor $\bar{\mathcal{D}}$ from Proposition 4, and by E the prime form from [21]. Then we have

Proposition 6. *Scalars λ_j from formula (32) are given with*

$$\lambda_j = \lambda_j^0 \exp \sum_{k \neq j} ix^{(k)} \gamma_j^k, \qquad \lambda_j^0 = c \exp \sum_{s=1}^{8} \int_{P_0}^{Q_s} \Omega_{P_j Q_0} + \kappa_j,$$

where $\vec{x} = (x^{(1)}, \dots, x^{(4)}) = t(\chi_{34}, -\chi_{34}, \chi_{12}, -\chi_{12})$ and

$$\gamma_i^j = \frac{d}{dk_j^{-1}} ln E(P_i, P)|_{P=P_j}.$$

(k_j^{-1} is a local parameter around P_j.)

To give formulae for the Baker-Akhiezer function, we need some notations. Let

$$\alpha^j(\vec{x}) = \exp[i \sum \tilde{\gamma}_m^j x^{(m)}] \frac{\theta(z_0)}{\theta(i \sum x^{(k)} U^{(k)} + z_0)},$$

where

$$\tilde{\gamma}_m^j = \int_{P_0}^{P_j} \Omega_{P_m}^{(1)}, \quad m \neq j,$$

and $\tilde{\gamma}_m^m$ is defined by the expansion

$$\int_{P_0}^{P} \Omega_{P_m}^{(1)} = -k_m + \tilde{\gamma}_m^m + O(k_m^{-1}), \quad P \to P_m.$$

Denote

$$\phi^j(\vec{x}, P) = \alpha^j(\vec{x}) \exp(-i \int_{P_0}^{P} \sum x^{(m)} \Omega_{P_m}^{(1)}) \frac{\theta(A(P) - A(P_j) - i \sum x^{(k)} U^{(k)} - z_0)}{\theta(A(P) - A(P_j) - z_0)}.$$

Finally we come to

Proposition 7. [16] *The Baker-Akhiezer function is given by*

$$\psi^j(\vec{x}, P) = \phi^j(\vec{x}, P) \frac{\lambda_j^0 \frac{\theta(A(P-P_j) - z_0)}{\epsilon(P, P_j)}}{\sum_{k=1}^{4} \lambda_k^0 \frac{\theta(A(P-P_k) - z_0)}{\epsilon(P, P_k)}}, \quad j = 1, \dots, 4,$$

where z_0 is given by (35).

9. The restrictively integrable part − equations for the functions $F_i, i = 1, \dots, 4$

Let us denote

$$\phi_1 := F_1 + F_3, \quad \phi_2 := F_1 - F_3;$$

and also

$$N_1 := M_{14} - M_{23}, \quad N_3 := M_{24} - M_{13},$$
$$N_2 := -M_{24} - M_{13}, \quad N_4 := M_{14} + M_{23}.$$

From (29) we have

$$\varphi_1 := arg(v_{3HA}^1) = \phi_1 + \alpha_1(t), \quad \varphi_2 := arg(v_{4HA}^1) = \phi_2 + \alpha_2(t),$$

where $\alpha_1(t) = arg(v_{3LB}^1)$ and $\alpha_2(t) = arg(v_{4LB}^1)$ are known function of time. Let us denote $u_i = \tan \varphi_i$.

Basic relationships among those quantities are given in the next proposition.

Proposition 8. *The following relations take place*
a)

$$(36) \qquad u_1 = \frac{N_1}{N_2}, \quad u_2 = -\frac{N_3}{N_4};$$

b)

$$(37) \qquad N_1 = -2|v_{3LB}^1(\tilde{C}_{11} - \tilde{C}_{33})|\sin\varphi_1, \quad N_4 = 2|v_{4LB}^1(\tilde{C}_{11} - \tilde{C}_{44})|\cos\varphi_2;$$

c)

$$(38) \qquad \dot{\phi}_1 = N_1(J_{24} + J_{13}), \quad \dot{\phi}_2 = -N_4(J_{24} - J_{13}).$$

Proof. a) follows from the formulae for $\tilde{L}$ (28) and (29). Part b) also uses Corollary 2 b, Proposition 4 b. Note that from the condition $\Omega = JM + MJ$ and the invariant relations we have:

$$\Omega_{12} = M_{14}J_{24} + M_{32}J_{13};$$
$$\Omega_{34} = M_{32}J_{24} + M_{14}J_{13}.$$

From the last relation and the definition of functions F_i from the Proposition 4, c) follows. $\square$

Using formulae (37), (38) we get

$$\dot{\phi}_1 = -2(J_{24} + J_{13})|(\tilde{C}_{11} - \tilde{C}_{33})v_{3LB}^1|\sin(\phi_1 + \alpha_1(t))$$
$$\dot{\phi}_2 = -2(J_{24} - J_{13})|(\tilde{C}_{11} - \tilde{C}_{44})v_{4LB}^1|\cos(\phi_2 + \alpha_2(t))$$

10. Restrictive integrability in an abstract Poisson algebra settings. Bihamiltonian structures for the Lagrange bitop and n-dimensional Lagrange top

From the analysis given in this paper, it follows that the Hess-Appel'rot system and its generalizations can be understood as natural examples of the following, more abstract situation.

Suppose a Poisson manifold $(M^{2n}, \{\cdot, \cdot\})$ is given, together with $k+1$ functions $H, f_1, \ldots, f_k \in C^\infty(M)$, such that

(A1)

$$\{H, f_i\} = \sum_{j=1}^{k} a_{ij}f_j, \quad a_{ij} \in C^\infty(M), \quad i, j = 1, \ldots, k;$$

(A2)

$$\{f_i, f_j\} = 0, \quad i, j = 1, \ldots, k.$$

The Hamiltonian system (M^n, H) will be called *restrictively integrable*, if it satisfies the axioms (A1-A2).

A more general case can be obtained by replacing condition (A2) with

(A2')

$$\{f_i, f_j\} = \sum_{l=1}^{k} d_{ij}^l f_l, \quad d_{ij}^l = const, \quad i, j = 1, ..., k.$$

In this case, the algebra of invariant relations is a noncommutative Lie algebra.

Starting from the Hamiltonian system (M, H_0) with k integrals in involution $f_1, \ldots, f_k$, choosing functions $b_j \in C^\infty(M), \quad j = 1, \ldots, k$, one comes to a restrictively integrable system:

(HP) **Hamiltonian perturbation**

The system (M, H) where

$$H = H_0 + \sum_{j=1}^{k} b_j f_j,$$

will be called a Hamiltonian perturbation. It satisfies (A1) with

$$a_{ij} = \{b_j, f_i\}, \quad i, j = 1, \ldots, k.$$

Natural question is the converse one: when a restrictively integrable system is of the form (HP)?

Denote

$$c_{il}^j := \{a_{ij}, f_l\}, \quad i, j, l = 1 \ldots, k.$$

From the Jacobi identity, and involutivity of functions f_i we get compatibility conditions.

Proposition 9. *If a restrictively integrable system which satisfies the axioms* (A1-A2) *is of the form* (HP), *then*

$$c_{il}^j = c_{li}^j, \quad i, j, l = 1 \ldots, k.$$

If in Proposition 9 we replace axiom A2 with A2' then c_{il}^j should satisfy

$$c_{il}^j = c_{li}^j + \sum_m d_{il}^m a_{jm}.$$

A three-dimensional Lagrange top is defined by the Hamiltonian:

$$H_L = \frac{1}{2} \left(\frac{M_1^2 + M_2^2}{I_1} + \frac{M_3^2}{I_3} \right) + z_0 \Gamma_3,$$

according to the standard Poisson structure

$$\{M_i, M_j\}_1 = -\epsilon_{ijk} M_k, \ \{M_i, \Gamma_j\}_1 = -\epsilon_{ijk} \Gamma_k, \ \{\Gamma_i, \Gamma_j\} = 0$$

on the Lie algebra $e(3)$. It is also well-known that three-dimensional Lagrange top is Hamiltonian in another Poisson structure, compatible with first one. This structure is defined by:

$$\{\Gamma_i, \Gamma_j\}_2 = -\epsilon_{ijk} \Gamma_k, \ \{M_1, M_2\}_2 = 1,$$

and the corresponding Hamiltonian is:

$$\tilde{H}_L = (a-1)M_3 \left(\frac{1}{2}(M_1^2 + M_2^2) + \Gamma_3 \right) + M_1\Gamma_1 + M_2\Gamma_2 + M_3\Gamma_3$$

where $I_1 = 1, I_3 = a, z_0 = 1$.

Casimir functions in the second structure are $\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2$ and M_3.

Let us observe that the Hamiltonian for the three-dimensional Hess-Appel'rot case is a quadratic deformation of Hamiltonian H_L of the Lagrange top:

$$H_{HA} = H_L + J_{13}M_1M_3.$$

The function M_3, which gives the invariant relation for the Hess-Appel'rot case, is a Casimir function of the second Poisson structure.

Having this observation in mind, next we are going to prove that the Lagrange bitop and the n-dimensional Lagrange top are also bihamiltonian systems.

The standard Poisson structure on the semi-direct product $so(4) \times so(4)$ is:

$$\{M_{ij}, M_{jk}\}_1 = -M_{ik}, \quad \{M_{ij}, \Gamma_{jk}\}_1 = -\Gamma_{ik}, \quad \{\Gamma_{ij}, \Gamma_{kl}\}_1 = 0.$$

Now let us introduce a new Poisson structure as follows:

$$\begin{aligned}
&\{\Gamma_{ij}, \Gamma_{jk}\}_2 = -\Gamma_{ik}, \quad \{M_{ij}, \Gamma_{kl}\}_2 = 0, \\
\text{(39)} \quad &\{M_{13}, M_{23}\}_2 = -\chi_{12}, \quad \{M_{14}, M_{24}\}_2 = -\chi_{12}, \\
&\{M_{13}, M_{14}\}_2 = -\chi_{34}, \quad \{M_{23}, M_{24}\}_2 = -\chi_{34}
\end{aligned}$$

Casimir functions in this structure are M_{12}, M_{34}, $\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{14}^2 + \Gamma_{23}^2 + \Gamma_{24}^2 + \Gamma_{34}^2$, and $\Gamma_{12}\Gamma_{34} + \Gamma_{23}\Gamma_{14} - \Gamma_{13}\Gamma_{24}$.

Proposition 10. *The Poisson structure* (39) *is compatible with the standard one.*

Proof. Two Poisson structures, defined with antisymmetric matrices A and B, are compatible if their Shouten bracket, defined by:

$$[A, B]_{ijk} = \sum_s \left(\frac{\partial A^{ij}}{\partial x^s} B^{sk} + \frac{\partial B^{ij}}{\partial x^s} A^{sk} \right) + \text{cyclic for } i, j, k,$$

vanishes (see [24]). Proof follows by direct calculation. $\square$

In the metric $\Omega = JM + MJ$, where $J = \mathrm{diag}(J_1, J_1, J_3, J_3)$, Hamiltonian function of the Lagrange bitop in the standard Poisson structure is:

$$\begin{aligned}
H_{LB} = \frac{1}{2}(&2J_1M_{12}^2 + (J_1 + J_3)M_{13}^2 + (J_1 + J_3)M_{14}^2 + \\
&(J_1 + J_3)M_{23}^2 + (J_1 + J_3)M_{24}^2 + 2J_3M_{34}^2) + \chi_{12}\Gamma_{12} + \chi_{34}\Gamma_{34}.
\end{aligned}$$

Let us assume that $J_1 = a$, $J_3 = 1 - a$.

Proposition 11. *The Lagrange bitop defined in the first Poisson structure by the Hamiltonian H_{LB} is a Hamiltonian system in the second Poisson structure (39) with the Hamiltonian:*

$$\tilde{H}_{LB} =$$

$$\frac{(2a-1)(\chi_{12}M_{12} + \chi_{34}M_{34})}{\chi_{12}^2 - \chi_{34}^2} \left(\frac{M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{24}^2}{2} + \chi_{12}\Gamma_{12} + \chi_{34}\Gamma_{34} \right)$$

$$+ \frac{(1-2a)(\chi_{12}M_{34} + \chi_{34}M_{12})}{\chi_{12}^2 - \chi_{34}^2}(M_{23}M_{14} - M_{13}M_{24} + \chi_{12}\Gamma_{34} + \chi_{34}\Gamma_{12}) +$$

$$M_{12}\Gamma_{12} + M_{13}\Gamma_{13} + M_{14}\Gamma_{14} + M_{23}\Gamma_{23} + M_{24}\Gamma_{24} + M_{34}\Gamma_{34}.$$

The situation with four-dimensional Hess-Appel'rot case is similar to the three-dimensional case: the Hamiltonian for the four-dimensional Hess-Appel'rot system in the first structure is again a quadratic deformation of H_{LB}:

$$H_{HA} = H_{LB} + J_{13}(-M_{12}M_{23} + M_{14}M_{34}) + J_{24}(M_{12}M_{14} - M_{23}M_{34})$$

Functions M_{12} and M_{34}, giving invariant relations for the four-dimensional Hess-Appel'rot system, are also Casimir functions for the second Poisson structure (39).

Putting $\chi_{34} = 0$, and assuming $\chi_{12} = 1$ in (39) and in expression for H_{LB}, we get the bihamiltonian structure for the four-dimensional Lagrange top introduced by Ratiu in [37].

In general, in arbitrary dimension n, the standard Poisson structure on $so(n) \times so(n)$ is given by:

$$\{M_{ij}, M_{jk}\}_1 = -M_{ik}, \quad \{M_{ij}, \Gamma_{jk}\}_1 = -\Gamma_{ik}, \quad \{\Gamma_{ij}, \Gamma_{kl}\}_1 = 0, \ i, j, k = 1, ..., n.$$

In the metric $\Omega = JM + MJ$, the n-dimensional Lagrange top is defined with a Hamiltonian

$$H_L = \frac{1}{2} \left(2J_1 M_{12}^2 + (J_1 + J_3) \sum_{p=3}^{n} (M_{1p}^2 + M_{2p}^2) + 2J_3 \sum_{3 \le p < q \le n} M_{pq}^2 \right) + \chi_{12}\Gamma_{12}$$

The number of nontrivial integrals of the motion is $\frac{n(n-1)}{2} - \left[\frac{n}{2} \right]$. Casimir functions are given by (see[37])

$$(40) \qquad\qquad tr(\Gamma^{2k}), \quad tr(M\Gamma^{2k+1}).$$

We use the notation $J_1 = a$, $J_3 = 1 - a$, $\chi_{12} = 1$. Let us introduce a new Poisson structure:

$$(41) \qquad \{\Gamma_{ij}, \Gamma_{jk}\}_2 = -\Gamma_{ik}, \ \{M_{ij}, M_{kl}\}_2 = 0 \ \{M_{1l}, M_{2l}\}_2 = -1, \ l = 3, ..., n.$$

The dimension of a symplectic leaf in this structure is

$$\frac{(n-2)(n-3)}{2} - \left[\frac{n-2}{2} \right] + 4(n-2),$$

hence there are $\frac{n^2 - 5n + 8}{2} + \left[\frac{n}{2} \right]$ Casimir functions:

$$M_{12}, M_{pq}, Tr(\Gamma^{2k}), \ \ 2 < p < q \le n, \quad k = 1, ..., \left[\frac{n}{2} \right].$$

Proposition 12. *The n-dimensional Lagrange top is a Hamiltonian system in the Poisson structure* (41), *compatible with the standard one. Its Hamiltonian is:*

$$\tilde{H}_L = (2a-1)M_{12}\left(\frac{1}{2}\sum_{p=3}^{n}(M_{1p}^2 + M_{2p}^2) + \Gamma_{12}\right) +$$

$$(1-2a)\sum_{3\leq p<q\leq n} M_{pq}(M_{1q}M_{2p} - M_{2q}M_{1p} + \Gamma_{pq}) + \sum_{1\leq p<q\leq n} M_{pq}\Gamma_{pq}$$

Similarly as in dimension 3 and 4, Hamiltonian for the Hess-Appel'rot system in arbitrary dimension n is a quadratic deformation of the Hamiltonian for the n-dimensional Lagrange top:

$$H_{HA} = H_L + \sum_{k=1}^{n}(J_{13}M_{1k}M_{3k} + J_{24}M_{2k}M_{4k}),$$

and functions $M_{12}, M_{pq}, p, q \geq 3$, which give the invariant relations (11), are Casimir functions for the Poisson structure (41).

We can summarize the discussion of this section by saying that constructed Hess-Appel'rot systems satisfy the following.

(BP) (bi-Poisson condition) *There exist a pair of compatible Poisson structures, such that the system is Hamiltonian with respect to the first structure, having the Hamiltonian of the form* (HP), *such that f_i are Casimir functions with respect to the second structure.*

The invariant relations define symplectic leaves with respect to the second structure, and the system is Hamiltonian with respect the first one.

11. Back to Kowalevski properties

As we mentioned in the introduction, from the first years of its history, the Hess-Appel'rot systems were closely related to Kowalevski's analysis. Investigating the systems constructed in the first part of this paper, we have noticed that they are certain perturbations of the form (HP) of integrable systems, n-dimensional Lagrange tops and the Lagrange bitop. We also observed that perturbing functions f_i, which give the invariant relations, are Casimir functions of the second Poisson structure and the integrable systems are bihamiltonian corresponding to that structure. Up to now there is no restriction on the choice of perturbing functions b_i in (HP). In order to define more precisely a class of systems which has the same typical dynamical and analytical properties as the classical and the n-dimensional Hess-Appel'rot case we need to study them in more details. Finally, after that we will be able to extract the basic ones leading to the constrains on the functions b_i. The correctness of our choice is illustrated by the Theorem 5 in Section 12. Using the axioms one can easily construct large number of new examples of systems of Hess-Appel'rot type (beside semidirect product $so(n) \times so(n)$ in a study of generalized rigid body systems, one can consider, for example, semidirect product of R^n and $so(n)$).

To get the right choice of axioms, we have to turn back to the Kowalevski analysis. First, we are going to introduce some general notions, see [29].

Suppose a system of ODEs of the form

(42)
$$\dot{z}_i = f_i(z_1, \ldots, z_n), \quad i = 1, \ldots, n,$$

is given and there exist positive integers $g_i, \quad i = 1, \ldots, n$, such that

$$f_i(a^{g_1} z_1, \ldots, a^{g_n} z_n) = a^{g_i+1} f_i(z_1, \ldots, z_n), \quad i = 1, \ldots, n.$$

Then the system (42) is *quasi-homogeneous* and numbers g_i are *exponents of quasi-homogeneity*. Then, for any complex solution $C = (C_1, \ldots, C_n)$ of the system of algebraic equations:

(43)
$$-g_i C_i = f_i(C_1, \ldots, C_n), \quad i = 1, \ldots, n,$$

one can define *the Kowalevski matrix* $K = K(C) = [K_j^i(C)]$:

$$K_j^i(C) = \frac{\partial f_i}{\partial z_j}(C) + g_i \delta_j^i.$$

Eigen-values of the Kowalevski matrix are called *the Kowalevski exponents*. This terminology was introduced in [45]. In last twenty years, heuristic and theoretical methods in application of Kowalevski matrix and Kowalevski exponents in study of integrability and nonintegrability have been actively developing, see for example [2, 3, 46, 47, 25, 29]. But the notion of Kowalevski matrix and Kowalevski exponents were introduced by Kowalevski herself in [28]. The criterion she used [28, p. 183, l. 15-22] to detect a system which is now known as the Kowalevski top, can be formulated in Yoshida terminology as:

Kowalevski condition (Kc). *The 6×6 Kowalevski matrix should have* five *different positive integer Kowalevski exponents.*

Now we return to the study of Hess-Appel'rot systems. The systems we have constructed *are quasi-homogeneous.* Exponents of each M variable are $g = 1$, and for any Γ they are equal to two. We are going now to calculate Kowalevski exponents for the Hess-Appel'rot systems.

Three-dimensional Hess-Appel'rot case. Let us denote $(M_1, M_2, M_3, \Gamma_1, \Gamma_2, \Gamma_3)$ by $(z_1, \ldots, z_6)$. Then the Euler-Poisson equations take the form (42) with

$$
\begin{aligned}
f_1 &= (J_3 - J_1)z_2 z_3 + J_{13} z_1 z_2 + z_5; \\
f_2 &= (J_3 - J_1)z_1 z_3 + J_{13}(z_3^2 - z_1^2) - z_4; \\
f_3 &= J_{13} z_2 z_3; \\
f_4 &= J_3 z_5 z_3 - J_1 z_2 z_6 + J_{13} z_1 z_5; \\
f_5 &= -J_3 z_3 z_4 + J_1 z_1 z_6 + J_{13}(z_3 z_6 - z_1 z_4); \\
f_6 &= J_3 z_2 z_4 - J_1 z_1 z_5 - J_{13} z_3 z_5;
\end{aligned}
$$

and $g_i = 1, \quad i = 1, 2, 3$ and $g_i = 2, \quad i = 4, 5, 6$. The invariant relation corresponds to the constraint $c_3 = 0$. So, we are looking for solutions $(c_1, c_2, 0, c_4, c_5, c_6)$ of the system of the form (43). One can easily get $c_4 = -J_{13}c_1^2 + c_2, c_5 = -c_1(1 + J_{13}c_2)$, $c_6 = -(c_1^2 + c_2^2)/2$. Then, for $c_1 \neq 0$, we get four possible solutions for (c_1, c_2)

divided into two pairs: $(\pm i/J_{13}, -1/J_{13})$ and $(\pm 2i/J_{13}, -2/J_{13})$. The Kowalevski exponents are

$$(-1, -2, 2, 4, 3, 3), \qquad (-1, 1, 3, 2, 2, 2),$$

respectively.

Thus, it can easily be seen that classical Hess-Appel'rot system doesn't satisfy exactly the Kowalevski condition (Kc), although it is quite close to.

Four-dimensional Hess-Appel'rot systems. In the four-dimensional case, there are 12 variables $(M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}, \Gamma_{12}, \Gamma_{13}, \Gamma_{14}, \Gamma_{23}, \Gamma_{24}, \Gamma_{34})$. Denote them by $(z_1, \ldots, z_{12})$ and

$$f_i = f_i(z_1, \ldots, z_{12}), \quad i = 1, \ldots, 12,$$

the corresponding right sides of the Euler-Poisson equations. Exponents of quasi-homogeneity are

$$g_i = 1, \quad i = 1, \ldots, 6, \qquad g_i = 2, \quad i = 7, \ldots, 12.$$

We are looking for solutions $(d_1, \ldots, d_{12})$ of a system of the form (43). The invariant relations correspond to constraints

$$d_1 = d_6 = 0.$$

From the relations on f_2, f_3, f_4, f_5 one can express $(d_8, d_9, d_{10}, d_{11})$ as functions of $(d_2, \ldots, d_5)$. Then, from the equation on f_7, f_{12} one gets d_7, d_{12} as functions of $(d_2, \ldots, d_5)$. Consider two possible cases separately.

Example 1. First consider the case where one of perturbing constants is equal to zero, say $J_{13} = 0$. The solution of the last system of four equations on four unknowns $(d_2, \ldots, d_5)$, leads to the solution:

$$d_2 = -d_5(= d_9 = d_{10}), \quad d_3 = d_4 = \sqrt{-1 - d_2^2}(= d_{11} = -d_8), \quad (d_7 = d_{12} = 1),$$

with arbitrary d_2. Computing the Kowalevski matrix, we get finally the Kowalevski exponents

$$\left(0, -1, 3, 4, 2, 1, 2, 1, 2 + 2\sqrt{J_{24}^2(1 + d_2^2)}, 2 - 2\sqrt{J_{24}^2(1 + d_2^2)},\right.$$
$$\left. 1 + 2\sqrt{J_{24}^2(1 + d_2^2)}, 1 - 2\sqrt{J_{24}^2(1 + d_2^2)}\right).$$

By analyzing correspondent eigen-vectors, we see that eight of them are tangent to the symplectic leaf, and other four are transversal to the leaf. Nonintegral Kowalevski exponents

$$\left(2 + 2\sqrt{J_{24}^2(1 + d_2^2)}, 2 - 2\sqrt{J_{24}^2(1 + d_2^2)}, 1 + 2\sqrt{J_{24}(1 + d_2^2)}, 1 - 2\sqrt{J_{24}(1 + d_2^2)}\right)$$

correspond to a half of tangential eigen-vectors.

Example 2. Now, suppose that both J_{13} and J_{24} are nonzero. To simplify the computations, assume $\chi_{12} = 1$, $\chi_{34} = 2$. If $d_2 = 0$, then there are three cases of nontrivial solutions of the system (43):

1) $d_2 = 0$, $d_3 = 0$, $d_4 = 0$, $d_5 = \pm\frac{i}{2}$;

2) $d_2 = 0$, $d_3 = \mp\frac{i}{4}$, $d_4 = \pm i/4$, $d_5 = 0$;

3) $d_2 = 0$, $d_3 = \pm\frac{i}{4}$, $d_4 = \pm\frac{i}{4}$, $d_5 = 0$.

Let us calculate Kowalevski exponents in the last case with $d_3 = i/4, d_4 = i/4$. We get first

$$d_7 = \frac{(J_1 + J_3)}{48}, d_8 = -\frac{i}{2}, d_9 = 0, d_{10} = 0, d_{11} = \frac{i}{2}, d_{12} = \frac{(J_1 + J_3)}{48},$$

and then the Kowalevski exponents

$$\left(0, \ -1, \ 3, \ 4, \ 1 + \frac{(J_{13} - J_{24})}{2}, \ 1 - \frac{(J_{13} - J_{24})}{2}, \right.$$
$$\left. 2 + \frac{(J_{13} - J_{24})}{2}, \ 2 - \frac{(J_{13} - J_{24})}{2}, \ 2, \ 1, \ 2, \ 1\right).$$

Suppose now that d_2 is arbitrary. Then there are two sets of solutions of (43) of the form

4)

$$d_2 = s, \ d_3 = \mp i\frac{\sqrt{1 + s^2(J_1 + J_3)^2}}{J_1 + J_3}, \ d_4 = \pm i\frac{\sqrt{1 + s^2(J_1 + J_3)^2}}{J_1 + J_3}, \ d_5 = s;$$

5)

$$d_2 = s, \ d_3 = \pm i\frac{\sqrt{1 + s^2(J_1 + J_3)^2}}{J_1 + J_3}, \ d_4 = \pm i\frac{\sqrt{1 + s^2(J_1 + J_3)^2}}{J_1 + J_3}, \ d_5 = -s.$$

In the case 4) we get further

$$d_7 = -\frac{1}{J_1 + J_3}, \ \ d_8 = i\frac{\sqrt{1 + s^2(J_1 + J_3)^2}}{J_1 + J_3}, \ \ d_9 = s,$$
$$d_{10} = -s, \ \ d_{11} = i\frac{\sqrt{1 + s^2(J_1 + J_3)^2}}{J_1 + J_3}, \ \ d_{12} = -\frac{1}{J_1 + J_3}.$$

The Kowalevski exponents are

$$(0, \ -1, \ 3, \ 4, \ 2, \ 2, \ 1, \ 1, \ 1 + A, \ 1 - A, \ 2 + A, \ 2 - A)$$

where

$$A = 2(J_{13} + J_{24})\sqrt{1 + s^2(J_1 + J_3)^2}.$$

Five-dimensional Hess-Appel'rot systems. In this case, there are 20 variables $(M_{12}, M_{13}, M_{14}, M_{15}, M_{23}, M_{24}, M_{25}, M_{34}, M_{35}, M_{45}, \Gamma_{12}, \Gamma_{13}, \Gamma_{14}, \Gamma_{15}, \Gamma_{23}, \Gamma_{24}, \Gamma_{25}, \Gamma_{34}, \Gamma_{35}, \Gamma_{45})$. As before, we denote them by $(z_1, \ldots, z_{20})$. Denoting also by

$$f_i = f_i(z_1, \ldots, z_{20}), \quad i = 1, \ldots, 20,$$

the corresponding right sides of the Euler-Poisson equations. Exponents of quasi-homogeneity are

$$g_i = 1, \quad i = 1, \ldots, 10, \qquad g_i = 2, \quad i = 11, \ldots, 20.$$

We are looking for solutions $(d_1, \ldots, d_{20})$ of a system of the form (43). The invariant relations correspond to constraints

$$d_1 = d_8 = d_9 = d_{10} = 0.$$

From the relations on $f_2 - f_7$ one can express $(d_{12}, \ldots, d_{17})$ as functions of $(d_2, \ldots, d_7)$. Then, from f_{11} one gets d_{11} as a function of $(d_2, \ldots, d_7)$, and after that, from (f_{18}, f_{19}, f_{20}) one gets d_{18}, d_{19}, d_{20} as functions of $(d_2, \ldots, d_7)$. The final step is solution of the system $(f_{12}, \ldots, f_{17})$ with the unknowns $(d_2, \ldots, d_7)$.

Example 3. We describe nonzero solutions of (43) under the assumption $d_5 = d_7 = 0$, $\chi_{12} = 1$. There are eight sets of solutions:

1)
$$d_2 = 0, \; d_3 = 0, \; d_4 = 0, \; d_6 = \pm \frac{2i}{J_1 + J_3};$$

2)
$$d_2 = \pm \frac{2i}{J_1 + J_3}, \; d_3 = 0, \; d_4 = 0, \; d_6 = 0;$$

3)
$$d_2 = \pm \frac{i}{J_1 + J_3}, \; d_3 = 0, \; d_4 = 0, \; d_6 = \pm \frac{i}{J_1 + J_3};$$

4)
$$d_2 = \pm \frac{2i}{J_{13}}, \; d_3 = 0, \; d_4 = 0, \; d_6 = -\frac{2}{J_{13}};$$

5)
$$d_2 = -\frac{1}{J_{13}}, \; d_3 = 0, \; d_4 = \pm \frac{i}{J_{13}}, \; d_6 = 0;$$

6)
$$d_2 = -\frac{4}{J_{13}}, \; d_3 = 0, \; d_4 = \pm \frac{4i}{J_{13}}, \; d_6 = \pm \frac{i}{J_1 + J_3};$$

7)
$$d_2 = \frac{1}{J_{13}}, \; d_3 = 0, \; d_4 = \pm \frac{i}{J_{13}}, \; d_6 = \pm \frac{2i}{J_1 + J_3};$$

8)
$$d_2 = -\frac{3}{J_{13}}, \; d_3 = \frac{i}{2J_{24}}, \; d_4 = \pm \frac{\sqrt{J_{13}^2 - 9J_{24}^2}}{2J_{13}J_{24}}, \; d_6 = 0.$$

We calculate the Kowalevski exponents in three representative cases: 1), 4) and 8).

In the case 1) only nonzero d_i, $i > 15$ are $d_{16} = 2/(J_1 + J_3)$, $d_{18} = -2i/(J_1 + J_3)$, and (under the assumption $J_3 = 1$, $J_{13} = 10$) the Kowalevski exponents are

$$\left(1 + A, \; 1 - A, \; A, \; -A, \; B, \; 1 - B, \; 1, \; -1, \; -1, \; -1, \; 3, \; 3, \; 3, \; 2, \; 2, \; 2, \; 2, \; 4, \; 4, \; 4 \right);$$

where $A = 2\frac{\sqrt{100-J_{24}^2}}{J_1+J_3}$ and $B = 2i\frac{J_{24}}{J_1+J_3}$.

In the case 2) $d_{11} = 2/(J_1 + J_3)$ and $d_{15} = 2i/(J_1 + J_3)$ are the only nonzero d_i, $i \geq 11$ and the Kowalevski exponents are

$$(-1,\ -1,\ -1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 1+A,\ 1-A,\ A,\ -A,\ B,\ 1-B),$$

where $A = \frac{4i\sqrt{21}}{J_3+1}$, $B = \frac{\sqrt{20i}}{J_3+1}$ and $J_1 = 1, J_{13} = 10, J_{24} = 4$.

In the case 4) we get that all d_i, $i \geq 11$ are zero except

$$d_{15} = \frac{2}{J_{13}}, \quad d_{17} = -\frac{2i}{J_{13}}.$$

The Kowalevski exponents are

$$(-2,\ 0,\ 0,\ 4,\ -1,\ 4,\ -1,\ 4,\ -1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3,\ 3,\ 3).$$

In the case 5) the Kowalevski exponents are

$$(-1,\ 0,\ 0,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 3,\ 3,\ 3).$$

In the case 8) we have the following nonzero d_i, $i \geq 11$:

$$d_{12} = \pm\frac{3i}{4J_{13}},\ d_{13} = \frac{1}{4J_{24}},\ d_{15} = \mp\frac{i\sqrt{J_{13}^2 - 9J_{24}^2}}{4J_{13}J_{24}},$$
$$d_{16} = \frac{3}{4J_{13}},\ d_{17} = \mp\frac{i}{4J_{24}},\ d_{18} = \mp\frac{\sqrt{J_{13}^2 - 9J_{24}^2}}{4J_{13}J_{24}}.$$

Corresponding Kowalevski exponents are

$$\left(0,\ -1,\ \frac{3}{2},\ 4,\ \frac{7}{2},\ \sqrt{2},\ -\sqrt{2},\ 1+\sqrt{2},\ 1-\sqrt{2},\ 1,\ \frac{1}{2},\ 3,\ \frac{5}{2},\ 1,\ \frac{1}{2},\ 3,\ \frac{5}{2},\ 2,\ 2,\ 2\right).$$

Six-dimensional Hess-Appel'rot systems. In this case, there are 30 variables $(M_{12}, \ldots, M_{56}, \Gamma_{12}, \ldots, \Gamma_{56})$. Denoting them by $(z_1, \ldots, z_{30})$, by

$$f_i = f_i(z_1, \ldots, z_{30}), \quad i = 1, \ldots, 30,$$

the corresponding right sides of the Euler-Poisson equations and exponents of quasi-homogeneity by

$$g_i = 1, \quad i = 1, \ldots, 15, \qquad g_i = 2, \quad i = 16, \ldots, 30,$$

we search to solutions $(d_1, \ldots, d_{30})$ of a system of the form (43). The invariant relations correspond to constraints

$$d_1 = d_{10} = d_{11} = d_{12} = d_{13} = d_{14} = d_{15} = 0.$$

The solution of the system follows the same lines as in the five-dimensional case.

Example 4. Under the following assumptions

$$J_1 = 1, \ J_3 = 3, \ \chi_{12} = 1, \ d_6 = \cdots = d_9 = 0,$$

we get six sets of solutions of the system (43):
1)

$$d_2 = 0, \ d_3 = \frac{i}{2}, \ d_4 = 0, \ d_5 = 0;$$

2)

$$d_2 = -\frac{3}{2J_{13}}, \ d_3 = \frac{i}{2J_{24}}, \ d_4 = 0, \ d_5 = \sqrt{\frac{J_{13}^2 - 9J_{24}^2}{J_{13}J_{24}}};$$

3)

$$d_2 = -\frac{2}{J_{13}}, \ d_3 = 0, \ d_4 = i\sqrt{\frac{s^2 J_{13}^2 + 4}{J_{13}}}, \ d_5 = s;$$

4)

$$d_2 = -\frac{1}{J_{13}}, \ d_3 = 0, \ d_4 = i\sqrt{\frac{s^2 J_{13}^2 + 1}{J_{13}}}, \ d_5 = s;$$

5)

$$d_2 = -\frac{3}{2J_{13}}, \ d_3 = \frac{i}{2J_{24}}, \ d_4 = \sqrt{\frac{J_{13}^2 - 9J_{24}^2}{J_{13}J_{24}}}, \ d_5 = 0;$$

6)

$$d_2 = -\frac{3}{2J_{13}}, \ d_3 = \frac{i}{2J_{24}}, \ d_4 = \sqrt{\frac{J_{13}^2 - 9J_{24}^2 - 4s^2 J_{24}^2 J_{13}^2}{J_{13}J_{24}}}, \ d_5 = s;$$

where s is an arbitrary parameter.

In the case 1) the only nonzero d are $d_{16} = \frac{1}{2}$ and $d_{21} = \frac{i}{2}$. The Kowalevski exponents are

$$(-1, \ -1, \ -1, \ -1, \ 1, \ 1, \ 1, \ 2, \ 2, \ 2, \ 2, \ 2, \ 2, \ 2, \ 3, \ 3, \ 3, \ 3,$$
$$4, \ 4, \ 4, \ 4, \ 1 + A, \ 1 - A, \ A, \ -A, \ 1 - iB, \ 1 - iB, \ iB, \ iB),$$

where $A = \frac{\sqrt{16 - J_{13}^2}}{2}$, $B = J_{13}/2$, $J_1 = 1, J_3 = 3$, $J_{24} = 4$.

Thus, using into account properties of Kowalevski exponents of algebraically-integrable Hamiltonian systems, we can conclude that for the systems we have constructed, functions b_i in the perturbation formula (HP) should satisfy two conditions:

(QH) (quasi-homogeneity) *The obtained system of Hamiltonian equations has to be quasi-homogeneous.*

In such a case, a Kowalevski matrix exists and we come to the last condition. Suppose the invariant relations correspond to equations $z_1 = 0, \ldots, z_k = 0$.

Denote by p number of Casimirs: $n = p + 2m$, where $2m$ is the dimension of a general symplectic leaf.

(ArA) (Arithmetic axiom) *For any nonzero solution $C = (0, ..., 0, c_{k+1}, ..., c_n)$ of the system (43), the Kowalevski matrix $K(C)$ has $n - p$ eigen-vectors tangent to the symplectic leaf and p transversal to it. Half of the Kowalevski exponents which correspond to tangential eigen-vectors and all of transversal ones are rational numbers. Irrational numbers among the second half of tangential Kowalevski exponents are divided into pairs such that the differences are integrally dependent.*

12. Description of three-dimensional systems of Hess-Appel'rot type

Now we would like to derive conditions which determine classical Hess-Appel'rot system among three-dimensional systems of Hess-Appel'rot type. More precisely, suppose a system is given by the Hamiltonian

$$(44) \qquad\qquad H_1 = H_0 + JbM_3,$$

where H_0 is the Hamiltonian of the Lagrange top corresponding to the first Poisson structure, M_3 is its integral and a Casimir for the second structure, J is a nonzero constant and b is a function, such that the axioms of the systems of Hess-Appel'rot are satisfied.

Analyzing the system (43) we come to the first, simple but very important, properties of such functions b. Denote by f the value of the function b at the point $(\hat{c}_1, \ldots, \hat{c}_6)$ of nonzero solution of (43).

Lemma 10. *For a nonzero solution $(\hat{c}_1, ..., \hat{c}_6)$ of (43) and the value $f = b(\hat{c}_1, ..., \hat{c}_6)$ it holds*
a) $\hat{c}_1^2 + \hat{c}_2^2 = 0$ *or* $f = 0$;
b) *if* $f \neq 0$ *and* $\hat{c}_1 = \pm i\hat{c}_2$ *then* $f = \mp i/J$ *or* $f = \mp 2i/J$.

The Lemma follows by straightforward calculation. It gives a possibility to reduce the analysis of functions b to analysis of their **germs**. By a *germ* of a function b, we mean $(f, f_1, \ldots, f_6)$, where f_i is the value of b_{z_i} calculated at points of solutions of system (43).

Lemma 10 leads to important simplifications in a study of Kowalevski matrices and their characteristic polynomials. We will denote by $K_1, \ldots, K_4$ Kowalevski matrices evaluated on germs, where

$$K_1 : (f = -i/J, \hat{c}_1 = i\hat{c}_2),$$
$$K_2 : (f = 2i/J, \hat{c}_1 = -i\hat{c}_2),$$
$$K_3 : (f = i/J, \hat{c}_1 = -i\hat{c}_2),$$
$$K_4 : (f = -2i/J, \hat{c}_1 = i\hat{c}_2),$$

and by $\mathrm{Pch}_i, \quad i = 1, \ldots, 4$ the corresponding characteristic polynomials, $\mathrm{Pch}_i(w) = \det(K_i - wId)$.

Proposition 13.
a) *Characteristic polynomials* Pch_i *have integer-valued coefficients.*
b) *These coefficients are J-independent.*
c) *The characteristic polynomial* Pch_1 *is of the form*

$$(45) \qquad \mathrm{Pch}_1(w) = w^6 + A_{15}w^5 + A_{14}w^4 + \cdots + A_{10},$$

where

(46) $A_{15} = (-9 - 2Jf_1\hat{c}_2), A_{10} = 12if_1J\hat{c}_2(-if_1J\hat{c}_2 + f_1f_2J^2\hat{c}_2^2 - iJ^2\hat{c}_2^2f_2^2 - i).$

The proof of Proposition 13 follows from Arithmetic axiom, Lemma 10 and straightforward calculations.

Proposition 14. *For $i = 1, \ldots, 4$ the following relation holds*

$$\mathrm{Pch}_i(-1) = 0.$$

Proposition 14 is a well-known property of Kowalevski matrices for authonomous systems, see [29]. For $i = 1$, using the notation

$$X := J\hat{c}_2f_2, \qquad Y := J\hat{c}_2f_1,$$

we get the following

Corollary. *For the first germ, the following relation holds*

(47) $$(2XY - 3iY + X - 3i)(-Y + X - 2) = 0.$$

In the same notations, from Proposition 13 we get

Lemma 11. *If $Y \neq 0$, then*

(48) $$(X - i)(Y - i(X + i)) = 0.$$

By systematical analysis of equations (45-48) finally we come to the following

Theorem 5. *The only non-zero polynomials b which give systems of Hess-Appel'rot type by relation* (44) *are of the form*

$$b(z_1, \ldots, z_6) = z_1 + kz_3.$$

All systems of Hess-Appel'rot type of the form (44) *are the classical Hess-Appel'rot systems.*

Example. One of possible solutions of the system (46, 47) is $X = i$, $Y = -3$. It leads to the function

(49) $$b(z_1, \ldots, z_6) = -3z_1 + iz_2.$$

Corresponding characteristic polynomial of the Kowalevski matrix K_1 is

$$\mathrm{Pch}_1(w) = w(w - 1)(w - 2)(w - 3)(w + 1)(w + 2).$$

However, the characteristic polynomial of the Kowalevski matrix K_3 is of the form

$$\mathrm{Pch}_3(w) = (w - 1)(w - 2)(w - 3)(w + 1)(2w^2 - 2w + 9).$$

Thus, the function b given by (49) only partially satisfies the Arithmetic axiom.

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