

Large deviations estimates for self-intersection local times for simple random walk in \mathbb{Z}^3 .

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Abstract

We obtain large deviations estimates for the self-intersection local times for a symmetric random walk in dimension 3. Also, we show that the main contribution to making the self-intersection large, in a time period of length n , comes from sites visited less than some power of $\log(n)$. This is opposite to the situation in dimensions larger or equal to 5. Finally, we present two applications of our estimates: (i) to moderate deviations estimates for the range of a random walk, and (ii) to moderate deviations for random walk in random sceneries.

Keywords and phrases: self-intersection local times, random walk, random sceneries.

AMS 2000 subject classification numbers: 60K35, 82C22, 60J25.

Running head: Large deviations for self-intersections in $d = 3$.

1 Introduction

In this paper, we focus on large deviations estimates for the self-intersection local times (SILT) for a simple random walk in dimension 3. Thus, P_x denotes the law of a nearest neighbors symmetric random walk $\{S_k, k \geq 0\}$ on \mathbb{Z}^d starting at site $x \in \mathbb{Z}^d$, and for any $y \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, the local time $l_n(y)$ is the number of visits of y up to time n . The SILT process is denoted $\{\Sigma_n^2, n \in \mathbb{N}\}$ with

$$\Sigma_n^2 = \sum_{x \in \mathbb{Z}^d} l_n^2(x). \quad (1.1)$$

This paper is a sequel to recent works [1] and [2] dealing with dimensions $d \geq 5$. There, the initial motivation came from establishing large deviations estimates for random walk in random sceneries: in [1], this problem was reduced to estimating the distribution of the size of the level sets of the local times. In other words, for large L and t , one needed to estimate

the probability of $\mathcal{A}_n(t, L) := \{|\{x : l_n(x) \sim t\}| > L\}$, where for a set Λ , we denote by $|\Lambda|$ its cardinal. A key tool in [1] was the following simple observation (see Lemma 2.1 of [1]): when $d \geq 3$, there is a constant κ_d such that for any subset Λ in \mathbb{Z}^d ,

$$P_0(l_n(\Lambda) > t) \leq \exp\left(-\kappa_d \frac{t}{|\Lambda|^{2/d}}\right), \quad \text{where } l_n(\Lambda) = \sum_{x \in \Lambda} l_n(x). \quad (1.2)$$

Then, in order to use (1.2), $\mathcal{A}_n(t, L)$ was partitioned as follows

$$\mathcal{A}_n(t, L) \subset \bigcup \{l_n(\Lambda) > t|\Lambda| : \Lambda \subset [-n, n]^d \ \& \ |\Lambda| = L\}.$$

Thus, the uniform estimate (1.2) yielded

$$P_0(\mathcal{A}_n(t, L)) \leq \mathcal{C}_n(L) \exp(-\kappa_d t L^{1-2/d}), \quad \text{with } \mathcal{C}_n(L) := |\{\Lambda \subset [-n, n]^d : |\Lambda| = L\}| \quad (1.3)$$

In [1], the combinatorial term $\mathcal{C}_n(L)$ in (1.3) had an innocuous rôle since $P_0(\mathcal{A}_n(t_n, L))$ was needed for a sequence $\{t_n\}$ so large that the trivial bound $L < n/t_n$ made $\mathcal{C}_n(L)$ negligible compared to $\exp(\kappa_d t_n L^{1-2/d})$. However, in [2], the combinatorial term ruined the naive bound (1.3). Thus, $\mathcal{A}_n(t, L)$ was first transformed into a question for the SILT

$$\mathcal{A}_n(t, L) \subset \left\{ \sum_{x \in \mathbb{Z}^d} 1_{\{l_n(x) \sim t\}} l_n^2(x) \geq Lt^2 \right\}. \quad (1.4)$$

Then the key estimate of [2] (Lemma 2.1) relied on bounding the self-intersection times of a given level set of the local times by the intersection times for two independent half-trajectories over a larger level set. This observation which twisted an idea of Le Gall [12] reads *morally* as

$$\left\{ \sum_{x: l_n(x) \sim t} l_n^2(x) \geq Lt^2 \right\} \longleftrightarrow \left\{ \sum_{x: l_{n/2}(x) \leq t} l_{n/2}(x) \tilde{l}_{n/2}(x) \geq Lt^2 \right\} \subset \left\{ \tilde{l}_{n/2}(\mathcal{D}) \geq Lt \right\}, \quad (1.5)$$

where $\mathcal{D} := \{l_{n/2}(x) \leq t\}$, and $\{\tilde{l}_n(x), x \in \mathbb{Z}^d\}$ is an independent copy of the local times with law \tilde{P}_0 . Thus, one reformulates the key tool (1.2) in order to get rid of the combinatorial factor as follows:

$$P_0 \otimes \tilde{P}_0 \left(\tilde{l}_{n/2}(\mathcal{D}) > z, |\mathcal{D}| < y \right) \leq \exp(-\kappa_d F(z, y)), \quad \text{with } F(z, y) = \frac{z}{y^{2/d}}. \quad (1.6)$$

Thus, we can evaluate $P_0(\tilde{l}_{n/2}(\mathcal{D}) \geq Lt)$ with the help of (1.6) as soon as a good bound on $|\mathcal{D}|$ obtains.

Note however, that in (1.5), the sum on the right hand side is over $\{x : l_{n/2}(x) \leq t\}$. This poses no trouble in $d \geq 5$, since the main contribution comes from *large* level sets. However, this approach fails in $d = 3$ and $d = 4$, and can at best bring a spurious logarithmic term as in the upper bound of (1.9). Besides, no indication can be extracted as to which level set gives a dominant contribution.

In this paper, the approach is somewhat opposite to that of [2]: we deal directly with level sets' distribution which in turn provides new estimates for the SILT process. The key idea is to transform any given level set of the local times of $\{S_0, S_1, \dots, S_{2n}\}$ into two sets:

- The sites that at least one of the *half* trajectories $\{S_n - S_{n-1}, \dots, S_n - S_0\}$, or $\{S_n - S_{n+1}, \dots, S_n - S_{2n}\}$ visits *nearly as often* as the whole trajectory.
- The sites that both trajectories visit *enough* times.

Then, we iterate this procedure chopping each trajectories near its midpoint from which stems two independent trajectories, and so forth until no piece of trajectories remains. This seemingly innocent strategy allows us to obtain some informations in dimension 3.

We show that the main contribution in making Σ_n^2 large comes from sites which are “not too often” visited. This is drastically different from the situation in $d \geq 5$, where only a few sites, where $l_n(x) \sim \sqrt{\Sigma_n^2}$, contributed to making Σ_n^2 large (see [2]). In dimension 4, it is still an open problem to understand which level sets give a dominant contribution to realize the large deviation $\{\Sigma_n^2 > ny\}$.

Proposition 1.1 *In dimension $d = 3$, there are positive constants \underline{c}, \bar{c} such that for y large enough*

$$\exp(-\underline{c}y^{2/3}n^{1/3}) \leq P_0 \left(\sum_{x \in \mathbb{Z}^d} l_n^2(x) > ny \right) \leq \exp(-\bar{c}y^{1/3}n^{1/3}). \quad (1.7)$$

Moreover, there is $\underline{\chi} > 0$ such that if $\bar{\mathcal{D}} := \{x : l_n(x) > \log(n)^{\underline{\chi}}\}$, then there is $\epsilon > 0$ such that

$$P_0 \left(\sum_{x \in \bar{\mathcal{D}}} l_n^2(x) > ny \right) \leq \exp(-\bar{c}n^{1/3} \log(n)^\epsilon). \quad (1.8)$$

Remark 1.2 It is a simple application of Lemma 2.1 of [2], and of our moment computations in Lemma 5.5 to obtain that in dimensions 3 and 4, for y large enough there are positive constants $\underline{c}, \bar{c}, \chi$ such that

$$\exp(-\underline{c}n^{1-2/d}) \leq P_0 \left(\sum_{x \in \mathbb{Z}^d} l_n^2(x) > ny \right) \leq \exp\left(-\bar{c} \frac{n^{1-2/d}}{\log(n)^\chi}\right). \quad (1.9)$$

However, the upper bound of (1.7) and most importantly (1.8) require a new treatment of the level sets.

A heuristic understanding of Proposition 1.1 comes from the following scenario realizing the lower bound in (1.9): we localize the walk in a ball $B(r_n)$ of radius r_n with $r_n^3 \sim n/y$. Indeed, assume that sites of $B(r_n)$ are visited uniformly: for $x \in B(r_n)$, $l_n(x) \sim n/r_n^3 \sim y$, and thus $\Sigma_n^2 \sim ny$. Now, the probability of staying in $B(r_n)$ a period of time n is larger than $\exp(-Cn/r_n^2)$ (for some $C > 0$), which yields the right exponent. However, we cannot say if, in the optimal strategy, the walk spends a fraction of its time outside $B(r_n)$, as expected by the result of van den Berg, Bolthausen & den Hollander[7] concerning the volume of the Wiener sausage, which is the continuous counterpart of the range of the walk $\mathcal{R}_n := \{x : l_n(x) > 0\}$. Indeed, a connection between the two problems (already noticed in [2]) is as follows:

$$\frac{n}{|\mathcal{R}_n|} \leq \frac{\sum_{x \in \mathbb{Z}^d} l_n^2(x)}{n} \implies P_0 \left(|\mathcal{R}_n| < \frac{n}{y} \right) \leq P_0 \left(\sum_{x \in \mathbb{Z}^d} l_n^2(x) > yn \right). \quad (1.10)$$

Note that in $d \geq 5$, the results of [2] show that the range does not shrink when realizing $\{\Sigma_n^2 > ny\}$, whereas in $d = 3$, the cost of the two deviations (i.e. small $|\mathcal{R}_n|$ and large Σ_n^2) correspond to the same speed $n^{1/3}$, and it would be interesting to know whether \mathcal{R}_n shrinks to produce $\{\Sigma_n^2 > ny\}$.

In dimension 2, large and moderate deviation principles are established for the SILT for Brownian motion in Bass & Chen [3], and for stable processes in Bass, Chen & Rosen[4]. Also, moderate deviations for the SILT and for the range of planar random walks were recently obtained by Bass, Chen & Rosen respectively in [5] and [6]. The approach of [3, 4, 5, 6] lies ultimately on the Donsker-Varadhan large deviation principle for the Brownian occupation measure [9], and might not be adequate when the dominant strategy to perform the large deviations is not a localization. Finally, for the $d = 1$ case, we refer the reader to Mansmann [13], and Chen & Li [8].

We now present two applications of our estimates on self-intersections. First, knowing that a random walk stays a time n in a ball $B(r_n)$ with $n/r_n^3 \gg 1$, we show that typically a proportion of the sites of $B(r_n)$ are visited about $n/|B(r_n)|$. Let $\sigma(r)$ be the first time the random walk exits the ball $B(r)$ of radius r . Also, we use the common notation $a_n = O(b_n)$ meaning that for some constant $A > 0$, $|a_n| \leq A|b_n|$.

Proposition 1.3 *Let $\{r_n\}$ be a sequence going to infinity with $r_n^3 = O(n)$. When ϵ_0 and δ_0 are small enough, we have*

$$\lim_{n \rightarrow \infty} P_0 \left(\left| \left\{ x : l_n(x) > \delta_0 \frac{n}{|B(r_n)|} \right\} \right| \geq \epsilon_0 |B(r_n)| \mid \sigma(r_n) > n \right) = 1. \quad (1.11)$$

Remark 1.4 Proposition 1.3 is based on the following estimate. For y large enough, the inequality (1.10) and the upper bound in (1.7) imply that there is a constant κ such that

$$P_0 \left(|\mathcal{R}_n| < \frac{n}{y} \right) \leq \exp(-\kappa y^{1/3} n^{1/3}). \quad (1.12)$$

This is weaker than the asymptotics of van de Berg, Bolthausen & den Hollander[7] for the volume of the Wiener sausage, and the proof is simpler. Also, to establish a lower bound similar to (1.12), note that the range is small if we localize the walk in a ball $B(r_n)$ with $|B(r_n)| = n/y$. Thus,

$$\{\sigma(r_n) > n\} \subset \{|\mathcal{R}_n| < \frac{n}{y}\} \implies P_0 \left(|\mathcal{R}_n| < \frac{n}{y} \right) \geq \exp\left(-C \frac{n}{(n/y)^{2/3}}\right) = e^{-Cy^{2/3}n^{1/3}}. \quad (1.13)$$

Secondly, we establish moderate deviations estimates for random walk in random sceneries (RWRS), following the approach of [2]. Thus, we consider a field $\{\eta(x), x \in \mathbb{Z}^d\}$ independent of the random walk $\{S_k, k \in \mathbb{N}\}$, and made up of centered i.i.d. with law denoted by P_η and tail decay

$$\lim_{t \rightarrow \infty} \frac{\log P_\eta(\eta(0) > t)}{t^\alpha} = -c, \quad \text{for a positive constant } c. \quad (1.14)$$

The random walk in random scenery is the process $X_n = \eta(S_0) + \dots + \eta(S_n)$. We present asymptotics for the probability, averaged over both randomness, that $\{X_n > n^\beta\}$ for $\beta > 1/2$ and $\alpha \geq 1$ in dimension 3. Our estimates are of the following type. For $\beta > 1/2$, and y large enough, there are two positive constants c_1, c_2 such that if $\mathbb{P} := P_0 \otimes P_\eta$

$$\exp(-c_1 n^\zeta) \leq \mathbb{P}(X_n > yn^\beta) \leq \exp(-c_2 n^\zeta). \quad (1.15)$$

Thus, the next result consists in characterizing the exponent ζ as a function of (α, β) .

Proposition 1.5 *Assume that dimension is 3.*

- In region I:= $\{(\alpha, \beta) : 1 \leq \alpha, 1/2 < \beta \leq 2/3\}$, we have $\zeta_I = 2\beta - 1$.
- In region II:= $\{(\alpha, \beta) : 1 \leq \alpha < 3/2, \beta > \frac{1+\alpha}{4-\alpha}\}$, we have $\zeta_{II} = \beta \frac{\alpha}{1+\alpha}$.
- In region III:= $\{(\alpha, \beta) : 1 \leq \alpha, 2/3 < \beta < \min(1, \frac{1+\alpha}{(4-\alpha)^+})\}$, we have $\zeta_{III} = \frac{4}{5}\beta - \frac{1}{5}$.

Remark 1.6 Compared with the situation in dimensions $d \geq 5$, we see that region III, which corresponds to localizing the walk, has expanded in $d = 3$. Note also that the lower bounds in regions I and II are already written in [2]. Also, we refer to [2] for a discussion of the behaviour of the walk and the environment leading to the exponent ζ in each region.

Note that region IV:= $\{(\alpha, \beta) : \alpha > 2/3, \beta \geq 1\}$ is treated in [11], where a large deviation principle is established. Also, a regime with $\alpha < 1$ is thoroughly studied in [10].

We prove Proposition 1.1 in Section 2, whose Subsection 2.1 is our main technical part. In Section 3, we establish Proposition 1.3, and the lower bound in Region III for Proposition 1.5. Finally, we have gathered in the Appendix a useful large deviation estimate and moments computations for intersection local times in $d = 3$ and $d = 4$.

2 Proof of Proposition 1.1

Note first that in order to obtain (1.8), we do not need to worry about the contribution of $\{x : l_n(x) \geq n^{1/3+\epsilon}\}$, for $\epsilon > 0$, since in dimension $d = 3$, $l_n(x)$ is bounded by a geometric variable and the upper bound of (1.8) follows easily for $P(\{x : l_n(x) > n^{1/3+\epsilon}\} \neq \emptyset)$. Also, we set for simplicity $n = 2^N$, and consider a subdivision $\{N^{\alpha_j}, j = 1, \dots, M_N\}$ of $[1, 2^{N(1/3+\epsilon)}]$, with $\alpha_j = (j-1)\alpha + \underline{\chi}$, for positive constants $\alpha, \underline{\chi}$ to be chosen later. Note also that M_N is of order $N/\log(N)$. We now form the level sets of the local times

$$\mathcal{L}_j = \{x : N^{\alpha_j} \leq l_{2^N}(x) < N^{\alpha_{j+1}}\} \quad \text{for } j > 0, \quad \text{and} \quad \mathcal{L}_0 = \{x : 1 \leq l_{2^N}(x) < N^{\underline{\chi}}\}. \quad (2.1)$$

Also, let $y_j = y/(2M_N)$ for $j > 0$, and $y_0 = y/2$ so that $y_0 + \dots + y_{M_N} = y$. We have the following decomposition

$$\begin{aligned} \left\{ \sum_{x \in \mathbb{Z}^d} l_{2^N}^2(x) > y2^N \right\} &\subset \bigcup_{j=0}^{M_N} \left\{ \sum_{\mathcal{L}_j} l_{2^N}^2(x) > y_j 2^N \right\} \cup \left\{ x : l_{2^N}(x) \geq 2^{N(\frac{1}{3}+\epsilon)} \right\} \\ &\subset \bigcup_{j=1}^{M_N} \left\{ |\mathcal{L}_j| > \frac{2^N y_j}{N^{2\alpha_{j+1}}} \right\} \cup \left\{ \sum_{\mathcal{L}_0} l_{2^N}^2(x) > y_0 2^N \right\} \cup \left\{ l_{2^N} \geq 2^{N(\frac{1}{3}+\epsilon)} \right\}. \end{aligned} \quad (2.2)$$

In Section 2.1, we deal with estimating the distribution of $|\mathcal{L}_j|$ for $j > 0$. In Section 2.2, we consider $\{\sum_{\mathcal{L}_0} l_{2^N}^2(x) > ny_0\}$. Finally, in Section 2.3 we repeat an argument of [2] to obtain the lower bound of Proposition 1.1.

2.1 Proof of (1.8)

We relabel our original trajectory as $\{S_k^{(0)}, k \in \mathbb{N}\}$ and its local time as $\{l_{k,1}^{(0)}, k \in \mathbb{N}\}$. We fix a time 2^N , and build from $\{S_0^{(0)}, \dots, S_{2^N}^{(0)}\}$ two independent trajectories running for times $k \in \{0, \dots, 2^{N-1}\}$

$$S_{k,1}^{(1)} = S_{2^{N-1}}^{(0)} - S_{2^{N-1}-k}^{(0)}, \quad \text{and} \quad S_{k,2}^{(1)} = S_{2^{N-1}}^{(0)} - S_{2^{N-1}+k}^{(0)}. \quad (2.3)$$

We denote by $\{l_{2^{N-1},i}^{(1)}(x), x \in \mathbb{Z}^d\}$ the local times of $\{S_{k,i}^{(1)}\}$ at time 2^{N-1} for $i = 1, 2$. Likewise, we proceed inductively, and consider at *generation* $l \leq N - 1$ two independent strands $\{S_{k,2i-1}^{(l)}, S_{k,2i}^{(l)}, k = 0, \dots, 2^{N-l}\}$ build from $\{S_{k,i}^{(l-1)}, k = 0, \dots, 2^{N-l+1}\}$ as in (2.3). Thus, for each generation $l < N$, we obtain a collection of 2^l independent local times $\{\{l_{2^{N-l},i}^{(l)}(x), x \in \mathbb{Z}^d\}, i = 1, \dots, 2^l\}$, associated with the trajectories $\{\{S_{k,i}^{(l)}, k = 0, \dots, 2^{N-l}\}, i = 1, \dots, 2^l\}$.

For any N and l , we define for $i = 1, \dots, 2^l$

$$\mathcal{D}_i^{(N, l)}(z) = \{x \in \mathbb{Z}^d : l_{2^{N-l},i}^{(l)}(x) > z\}, \quad (2.4)$$

and for $i = 1, \dots, 2^{l-1}$

$$\mathcal{C}_i^{(N, l)}(z) = \{x \in \mathbb{Z}^d : \min(l_{2^{N-l},2i-1}^{(l)}(x), l_{2^{N-l},2i}^{(l)}(x)) > z\}. \quad (2.5)$$

Step 1. We first show that if $\eta = \eta' + \eta''$, then for any $\delta \in]0, 1[$, and $l < N - 1$

$$\left\{ \sum_{i=1}^{2^l} |\mathcal{D}_i^{(N, l)}(z)| > \eta \right\} \subset \left\{ \sum_{i=1}^{2^{l+1}} |\mathcal{D}_i^{(N, l+1)}((1-\delta)z)| > \eta'' \right\} \cup \left\{ \sum_{i=1}^{2^l} |\mathcal{C}_i^{(N, l+1)}(\delta z)| > \eta' \right\}. \quad (2.6)$$

We first fix one strand $\{S_{k,i}^{(l)}, k = 0, \dots, 2^{N-l}\}$ at generation l . To lighten notations, we set $m = 2^{N-l-1}$. Then, on $\{l_{2m,i}^{(l)}(x) > z, S_{m,i}^{(l)} = \tilde{x}\}$ (for $\tilde{x} \in \mathbb{Z}^d$), we have

$$z < l_{2m,i}^{(l)}(x) \leq l_{m,2i-1}^{(l+1)}(\tilde{x} - x) + l_{m,2i}^{(l+1)}(\tilde{x} - x).$$

Thus, we have either of the two following possibilities on $\{l_{2m,i}^{(l)}(x) > z, S_{m,i}^{(l)} = \tilde{x}\}$: for $0 < \delta < 1$

(i) $\max(l_{m,2i-1}^{(l+1)}(\tilde{x} - x), l_{m,2i}^{(l+1)}(\tilde{x} - x)) > (1 - \delta)z.$

(ii) $\min(l_{m,2i-1}^{(l+1)}(\tilde{x} - x), l_{m,2i}^{(l+1)}(\tilde{x} - x)) > \delta z.$

Thus, by partitioning over $\{S_{m,i}^{(l)} = \tilde{x}, \tilde{x} \in \mathbb{Z}^d\}$, we obtain

$$\{x : l_{2m,i}^{(l)}(x) > z\} \subset \bigcup_{\tilde{x} \in \mathbb{Z}^d} \{S_{m,i}^{(l)} = \tilde{x}\} \cap \left(\{x : \max(l_{m,2i-1}^{(l+1)}(\tilde{x}-x), l_{m,2i}^{(l+1)}(\tilde{x}-x)) > (1-\delta)z\} \cup \{x : \min(l_{m,2i-1}^{(l+1)}(\tilde{x}-x), l_{m,2i}^{(l+1)}(\tilde{x}-x)) > \delta z\} \right) \quad (2.7)$$

Thus, by taking the cardinal of each set, we obtain for $i = 1, \dots, 2^l$,

$$\begin{aligned} |\mathcal{D}_i^{(N, l)}(z)| &\leq \sum_{\tilde{x} \in \mathbb{Z}^d} 1_{\{S_{m,i}^{(l)} = \tilde{x}\}} \left(\sum_{j=2i-1, 2i} |\{x : l_{m,j}^{(l+1)}(\tilde{x}-x) > (1-\delta)z\}| \right. \\ &\quad \left. + |\{x : \min(l_{m,2i-1}^{(l+1)}(\tilde{x}-x), l_{m,2i}^{(l+1)}(\tilde{x}-x)) > \delta z\}| \right) \\ &\leq \sum_{j=2i-1, 2i} |\{x : l_{m,j}^{(l+1)}(x) > (1-\delta)z\}| + |\{x : \min(l_{m,2i-1}^{(l+1)}(x), l_{m,2i}^{(l+1)}(x)) > \delta z\}| \\ &\leq |\mathcal{D}_{2i-1}^{(N, l+1)}((1-\delta)z)| + |\mathcal{D}_{2i}^{(N, l+1)}((1-\delta)z)| + |\mathcal{C}_i^{(N, l+1)}(\delta z)|. \end{aligned} \quad (2.8)$$

Thus, (2.6) follows at once.

Step 2. We show now that if we partition the size of the level-set η into $\eta' + \eta''$, then

$$\left\{ \sum_{i=1}^{2^l} |\mathcal{D}_i^{(N, l)}(z)| > \eta' + \eta'' \right\} \subset \left\{ \sum_{i=1}^{2^{l+1}} |\mathcal{D}_i^{(N, l+1)}((1-\delta)z)| > \eta'' \right\} \cup \mathcal{A} \quad (2.9)$$

with,

$$\mathcal{A} := \left\{ \sum_{i=1}^{2^l} l_{2^{N-l-1}, 2i}^{(l+1)} \left(\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z) \right) \geq \delta z \eta' \right\}.$$

Indeed, note that $\mathcal{C}_i^{(N, l+1)}(\delta z)$ are the sites of $\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z)$ where $l_{2^{N-l-1}, 2i}^{(l+1)} \geq \delta z$. Thus,

$$l_{2^{N-l-1}, 2i}^{(l+1)} \left(\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z) \right) \geq \delta z |\mathcal{C}_i^{(N, l+1)}(\delta z)|, \quad (2.10)$$

so that

$$\sum_{i=1}^{2^l} l_{2^{N-l-1}, 2i}^{(l+1)} \left(\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z) \right) > \delta z \sum_{i=1}^{2^l} |\mathcal{C}_i^{(N, l+1)}(\delta z)|, \quad (2.11)$$

and we deduce Step 2 from (2.6) and (2.11).

Step 3 We partition further (2.9) to get rid of the event that one of the $\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z)$ in \mathcal{A} be too large. Thus, for an arbitrary positive constant a to be chosen later,

$$\mathcal{A} \subset \left(\mathcal{A} \bigcap_{i=1}^{2^l} \left\{ |\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z)| \leq \frac{\eta}{\delta^a} \right\} \right) \bigcup_{i=1}^{2^l} \left\{ |\mathcal{D}_{2i-1}^{(N, l+1)}(\delta z)| > \frac{\eta}{\delta^a} \right\}. \quad (2.12)$$

Now, we denote

$$A_i^N(z, \eta) = P \left(\sum_{i=1}^{2^l} |\mathcal{D}_i^{(N, l)}(z)| > \eta \right), \quad (2.13)$$

and,

$$B_l^N(\eta; (z, w)) = P \left(\sum_{i=1}^{2^{l-1}} l_{2^{N-l}, 2^i}^{(l)} \left(\mathcal{D}_{2^{i-1}}^{(N, l)}(z) \right) > \eta; \quad \forall i = 1, \dots, 2^{l-1}, \quad |\mathcal{D}_{2^{i-1}}^{(N, l)}(z)| < w \right). \quad (2.14)$$

We consider a decomposition of η into $N - 1$ positive numbers $\eta_1, \dots, \eta_{N-1}$, and we denote $\bar{\eta}_i = \eta_{i+1} + \dots + \eta_{N-1}$. Now, at generation $l < N - 1$, we apply Step 1 and Step 2 and (2.12) with $\eta = \bar{\eta}_l$, $\eta' = \eta_l$ and $\eta'' = \bar{\eta}_{l+1}$. If we further take averages on both sides of (2.12), we obtain

$$A_l^N(z, \bar{\eta}_l) \leq A_{l+1}^N((1 - \delta)z, \bar{\eta}_{l+1}) + 2^l A_0^{N-l-1}(\delta z, \frac{\eta}{\delta a}) + B_{l+1}^N(\delta z \eta_{l+1}; (\delta z, \frac{\eta}{\delta a})). \quad (2.15)$$

Now, we define $\Theta(z, \eta) = (\delta z, \frac{\eta}{\delta a})$, $\Gamma(z, \eta) = ((1 - \delta)z, \eta)$ and for each $l \leq N$, $m_l(z, \eta) = (\frac{\delta}{l} z \eta; \Theta(z, \eta))$. By iterating (2.15) for the term A_{l+1}^N (until $l = N - 1$ since $A_k^k = 0$), and choosing $\eta_i = \eta / (N - 1)$ for $i = 1, \dots, N - 1$, we obtain

$$A_0^N(z, \eta) \leq \sum_{l=1}^{N-1} \{2^{l-1} A_0^{N-l} \circ \Theta + B_l^N \circ m_{N-l}\} \circ \Gamma^{l-1}(z, \eta). \quad (2.16)$$

On the right hand side of (2.16), we have desirable B -terms, and A_0^k -terms which we get rid off by iterating (2.16). Note that the action of iterates of Γ on (z, η) will be innocuous as we choose later δ very small; however, the action of Θ must be traced carefully. Thus, in (2.15), we say that in the A -terms of the right hand side, Θ acts once. Also, a given A -term, say A_0^{N-l} has argument $\Theta \circ \Gamma^{l-1}(z, \eta) = (\delta(1 - \delta)^{l-1} z, \frac{\eta}{\delta a})$, and in the induction, we need to decompose $\frac{\eta}{\delta a}$ into $N - l - 1$ equal parts so as to obtain

$$A_0^{N-l} \Theta \circ \Gamma^{l-1}(z, \eta) \leq \sum_{l'=1}^{N-l-1} \left\{ 2^{l'-1} A_0^{N-l-l'} \circ \Theta^2 + B_{l'}^{N-l} \circ m_{N-l-l'} \circ \Theta \right\} \circ \Gamma^{l-1+l'-1}(z, \eta).$$

We describe now in more details the B -terms we eventually obtain. In a *generic* B -term, let $\nu \geq 0$ be the number of times Θ has acted, and for $i = 1, \dots, \nu$, let l_i be the number of times Γ has acted between the $(i - 1)^{th}$ and i^{th} action of Θ , and let $l \geq 1$ be the number of times Γ acts after the ν -actions of Θ . We assume $1 \leq l_1 + \dots + l_\nu + l \leq N$. We set

$$k = l_1 + \dots + l_\nu, \quad \text{and} \quad k'' = k - \nu + l - 1.$$

For a single choice $(\nu, l_1, \dots, l_\nu, l)$, we have $2^{k-\nu}$ B -terms of the form B_l^{N-k} and with argument $m_{N-k} \circ \Theta^\nu \circ \Gamma^{k''}(z, \eta)$. Note that the total number of B -terms labelled B_l^{N-k} is the same as those labelled B_0^{N-k} , a number we call $c(k)$ which is easily seen from (2.16) to satisfy

$$c(k) \leq 2^0 c(k-1) + 2^1 c(k-2) + \dots + 2^{k-1} c(0), \quad \text{with} \quad c(0) = 1, \quad c(1) = 1, \quad c(2) = 3, \quad \text{etc...} \quad (2.17)$$

Now, an immediate induction shows that (2.17) imposes the bound $c(k) \leq 2^{2k}$. Thus, we obtain

$$A_0^N \leq \sum_{k,l: N \geq k+l} 2^{2k} \sup_{k'' \leq k+l} \sup_{\nu < k} B_l^{N-k} \circ m_{N-k} \circ \Theta^\nu \circ \Gamma^{k''} \quad (2.18)$$

We write in details the B -term in (2.18) for a choice of $(\nu', l_1, \dots, l_\nu, l)$, and $\nu = \nu' + 1$.

$$B_l^{N-k} \circ m_{N-k} \circ \Theta^{\nu'} \circ \Gamma^{k''}(z, \eta) = P \left(\sum_{i=1}^{2^{l-1}} l_{2^{N-k-l}, 2i}^{(l)}(D_{2i-1}) > \frac{\delta(1-\delta)^{k''}}{N-k} \frac{\delta^\nu z \eta}{\delta^{\nu a}}, \bigcap_{i=1}^{2^{l-1}} \mathcal{G}_i \right), \quad (2.19)$$

where, for $i = 1, \dots, 2^{l-1}$, we used the shorthand notations

$$D_{2i-1} := \mathcal{D}_{2i-1}^{(N-k, l)} \left((1-\delta)^{k''} \delta^\nu z \right), \quad \text{and} \quad \mathcal{G}_i := \left\{ |D_{2i-1}| < \frac{\eta}{\delta^{\nu a}} \right\}.$$

We take now $a = d/(d-2)$. To understand this choice, note that we deal in (2.19) with a sum of 2^{l-1} independent terms whose tail distribution is controlled by inequality (1.6). It will turn out, for the forthcoming choice of (z, η) , that the sum in (2.19) behaves similarly as one of its term. Now, if we were asking for the probability that

$$l_{2^{N-k-l}, 2i}^{(l)}(D_{2i-1}) > \frac{\delta^\nu z \eta}{\delta^{\nu a}}, \quad \text{with} \quad |D_{2i-1}| < \frac{\eta}{\delta^{\nu a}},$$

then, estimates (1.6) would give a bound $\exp(-\kappa_d F(\frac{\delta^\nu z \eta}{\delta^{\nu a}}, \frac{\eta}{\delta^{\nu a}}))$. Thus, Θ^ν will not ruin the use of estimate (1.6) if for the function F given in (1.6) we have that $F(\frac{\delta^\nu z \eta}{\delta^{\nu a}}, \frac{\eta}{\delta^{\nu a}})$ is independent of δ . This is what we achieve by choosing $a = d/(d-2)$.

Step 4. We are now ready to evaluate the level sets distribution. Note that

$$\left\{ |\mathcal{L}_j| > \frac{2^N y_j}{N^{2\alpha_{j+1}}} \right\} \subset \left\{ |\mathcal{D}_1^{(N,0)}(N^{\alpha_j})| > \frac{2^N y_j}{N^{2\alpha_{j+1}}} \right\}. \quad (2.20)$$

We rewrite the B -term of (2.19) with $z = N^{\alpha_j}$ and $\eta = 2^N y_j / N^{2\alpha_{j+1}}$.

$$B_l^{N-k}(\cdot) = P \left(\sum_{i=1}^{2^{l-1}} X_i^{(l)} > x_N, \bigcap_{i=1}^{2^{l-1}} \mathcal{G}_i \right), \quad (2.21)$$

with

$$X_i^{(l)} = \delta^{\nu(a-1)} l_{2^{N-k-l}, 2i}^{(l)}(D_{2i-1}), \quad \text{with} \quad D_{2i-1} := \mathcal{D}_{2i-1}^{(N-k, l)} \left((1-\delta)^{k''} \delta^\nu N^{\alpha_j} \right), \quad (2.22)$$

and, as we chose $y_j = y/(2M_N)$ for $j > 0$,

$$x_N = \frac{\delta(1-\delta)^{k''}}{N-k} \frac{N^{\alpha_j}}{N^{2\alpha_{j+1}} 2M_N} 2^N y, \quad \text{and} \quad \mathcal{G}_i = \left\{ |D_{2i-1}| < \frac{2^N y}{N^{2\alpha_{j+1}} 2M_N \delta^{\nu a}} \right\}. \quad (2.23)$$

For B_l^{N-k} to be small, we need $2^{l-1} E[X_i^{(l)} 1_{\{\mathcal{G}_i\}}] < x_N/2$. Thus, we show in Lemma 5.5, that there is a constant C_0 such that

$$E[X_i^{(l)}] \leq C_0 \delta^{\nu(a-1)} 2^{\frac{2}{3}(N-k-l)} \exp \left(-\kappa_3 \frac{2}{3} \delta^\nu (1-\delta)^{k''} N^{\alpha_j} \right). \quad (2.24)$$

Now, to get rid of the term $(1 - \delta)^{k''}$ we take $\delta = 1/N$ (since $k'' \leq 2N$). Now, recall that when $d = 3$, then $a = 3$. Thus, we have $2^{l-1}E[X_i^{(l)}] < x_N/2$, if for some constant C_1

$$2^l 2^{\frac{2}{3}(N-k-l)} \delta^{2\nu} \exp(-c_0 \delta^\nu N^{\alpha_j}) \leq \frac{C_1 \delta}{N} \frac{2^N N^{\alpha_j} y}{N^{2\alpha_{j+1}} M_N}, \quad (2.25)$$

where we set $c_0 := \sup_N \left\{ \frac{2}{3} \kappa_3 (1 - \frac{1}{N})^{2N} \right\} > 0$. If we set $x = \delta^\nu N^{\alpha_j}$, then (2.25) holds as soon as

$$x^2 \exp(-c_0 x) \leq \frac{C_1 y}{N^3} \frac{N^{\alpha_j}}{N^{2(\alpha_{j+1} - \alpha_j)}} 2^{(N-l)\frac{1}{3}}. \quad (2.26)$$

Since $\alpha_{j+1} - \alpha_j = \alpha$ can be chosen arbitrarily small, (2.26) follows as soon as $\underline{\chi} > 3$. Thus, if we set $Y_i^{(l)} = X_i^{(l)} 1\{\mathcal{G}_i\}$ and $\bar{Y}_i^{(l)} = Y_i^{(l)} - E[Y_i^{(l)}]$, we have

$$P \left(\sum_{i=1}^{2^{l-1}} X_i^{(l)} > x_N, \quad \bigcap_{i=1}^{2^{l-1}} \mathcal{G}_i \right) \leq P \left(\sum_{i=1}^{2^{l-1}} \bar{Y}_i^{(l)} > \frac{x_N}{2} \right). \quad (2.27)$$

We have reached now a large deviation estimate for which Lemma 5.1 is devised. We first need tail estimates for $Y_i^{(l)}$.

Step 5: To obtain tail estimates, we rely on Lemma 1.2 of [1],

$$\begin{aligned} P \left(Y_i^{(l)} > u \right) &= P \left(X_i^{(l)} > u, \mathcal{G}_i \right) \leq E \left[\exp \left(-\frac{u \delta^{-\nu(a-1)}}{|\mathcal{D}_{2i-1}|^{2/d}} \right) 1\{\mathcal{G}_i\} \right] \\ &\leq \exp(-\xi_N u), \quad \text{with } \xi_N = \left(\frac{N^{2\alpha_{j+1}} M_N}{2^N y} \right)^{2/d}. \end{aligned} \quad (2.28)$$

We show now that for $\underline{\chi} \geq 2$ (and $\alpha < 1/2$), we have $2^{4/3l} \xi_N^2 E[(Y_i^{(l)})^2] \leq 1$. Using Lemma 5.5, there is a constant c

$$\begin{aligned} 2^{4/3l} \xi_N^2 E \left[(X_i^{(l)})^2 \right] &= 2^{4/3l} \left(\frac{N^{2\alpha_{j+1}} M_N}{2^N y} \right)^{4/d} \delta^{2\nu(a-1)} 2^{\frac{4}{3}(N-k-l)} \exp(-c \delta^\nu N^{\alpha_j}) \\ &\leq \left(\frac{N^{2(\alpha_{j+1} - \alpha_j)} M_N}{y} \right)^{4/3} \frac{1}{N^{(4-8/3)\alpha_j}} (\delta^\nu N^{\alpha_j})^4 \exp(-c \delta^\nu N^{\alpha_j}). \end{aligned} \quad (2.29)$$

The right hand side of (2.29) can be made smaller than 1 if $(2\alpha + 1)4/3 < (4 - 8/3)\alpha_j$, i.e. as $\underline{\chi} \geq 2$ and $\alpha < 1/2$. Thus, Lemma 5.1 with the choice $\gamma = 1/4$ yields

$$P \left(\sum_{i=1}^{2^{l-1}} \bar{Y}_i^{(l)} > \frac{x_N}{2} \right) \leq \exp \left(c_u \left(2^{4/3l} \xi_N^2 \gamma^2 E \left[(Y_i^{(l)})^2 \right] \right)^{1-\gamma} - \frac{\gamma \xi_N x_N}{4} \right). \quad (2.30)$$

Thus, we obtain that for some constant $c > 0$ and N large,

$$P \left(\sum_{i=1}^{2^{l-1}} \bar{X}_i^{(l)} > \frac{x_N}{2}, \quad \bigcap_{i=1}^{2^{l-1}} \mathcal{G}_i \right) \leq c \exp \left(-\frac{\xi_N x_N}{16} \right) \leq c \exp \left(-(2^N y)^{1/3} N^\zeta \right), \quad (2.31)$$

with

$$\zeta = \frac{\alpha_j}{3} - 2(\alpha_{j+1} - \alpha_j) - 2 - \frac{1}{3}. \quad (2.32)$$

Thus, $\zeta > 0$ as soon as $\underline{\chi} > 7$, and α small enough.

2.2 Proof of Upper Bound in (1.9)

Note that in dimension 3, we are left with showing that for $\mathcal{L}_0 := \{x : 0 < l_{2^N}(x) < N\underline{x}\}$, we have for $\bar{c} > 0$ and $y_0 = y/2$

$$P \left(\sum_{\mathcal{L}_0} l_{2^N}^2(x) > y_0 \right) \leq \exp(-\bar{c}y^{1/3}n^{1/3}).$$

The approach is close to the proof of Lemma 3.1 in [2]. However, in order to get rid of a logarithmic term, inherent in the proof in [2], additional work is needed. On the other hand, the proof we now present does not work in dimensions $d \geq 4$.

We keep the notations of Section 2.1.

$$\sum_{x \in \mathcal{L}_0} l_n^2(x) \leq n + 1 + 2Z^{(0)}, \quad \text{with} \quad Z^{(0)} = \sum_{x \in \mathcal{L}_0} \sum_{0 \leq k < k' \leq 2^N} 1\{S_k^{(0)} = S_{k'}^{(0)} = x\}. \quad (2.33)$$

Now,

$$\begin{aligned} Z^{(0)} &\leq \sum_x 1\{l_{2^{N-1}}^{(0)}(x) \leq N\underline{x}\} \sum_{0 \leq k < k' \leq 2^{N-1}} 1\{S_k^{(0)} = S_{k'}^{(0)} = x\} \\ &\quad + \sum_x 1\{l_{2^N}^{(0)}(x) - l_{2^{N-1}}^{(0)}(x) \leq N\underline{x}\} \sum_{2^{N-1} \leq k < k' \leq 2^N} 1\{S_k^{(0)} = S_{k'}^{(0)} = x\} \\ &\quad + \sum_x 1\{l_{2^{N-1}}^{(0)}(x) \leq N\underline{x}\} \sum_{0 \leq k \leq 2^{N-1} \leq k' \leq 2^N} 1\{S_k^{(0)} = S_{k'}^{(0)} = x\} \\ &\leq Z_1^{(1)} + Z_2^{(1)} + J_1^{(1)}. \end{aligned} \quad (2.34)$$

where we have defined for $i = 1$ and $i = 2$

$$Z_i^{(1)} = \sum_x 1\{l_{2^{N-1},i}^{(1)}(x) \leq N\underline{x}\} \sum_{0 \leq k < k' \leq 2^{N-1}} 1\{S_{k,i}^{(1)} = S_{k',i}^{(1)} = x\},$$

and the intersection times of the two independent strands over $\{l_{2^{N-1},1}^{(1)}(x) \leq N\underline{x}\}$ is

$$J_1^{(1)} = \sum_x 1\{l_{2^{N-1},1}^{(1)}(x) \leq N\underline{x}\} l_{2^{N-1},1}^{(1)}(x) l_{2^{N-1},2}^{(1)}(x).$$

Iterating this procedure, we get

$$Z^{(0)} \leq \sum_{l=1}^{N-1} \sum_{k=1}^{2^{l-1}} J_k^{(l)}, \quad (2.35)$$

where for each $l \in \{1, \dots, N-1\}$, the random variables $\{J_k^{(l)}; 1 \leq k \leq 2^{l-1}\}$ are i.i.d., with

$$J_k^{(l)} = \sum_x 1\{l_{2^{N-l},2k-1}^{(l)}(x) \leq N\underline{x}\} l_{2^{N-l},2k-1}^{(l)}(x) l_{2^{N-l},2k}^{(l)}(x),$$

We now introduce a partition of $\{l_{2^{N-l}, 2k-1}^{(l)}(x) \leq N^\chi\}$ in terms of

$$\mathcal{D}_{k,i}^{(l)} = \{x : N^{\chi_i} \leq l_{2^{N-l}, 2k-1}^{(l)}(x) < N^{\chi_{i+1}}\}, \quad \text{with} \quad \chi_i = \frac{\chi(1+\delta)^i}{\log(N)}, \quad i = 0, \dots, M_N. \quad (2.36)$$

We choose χ such that $N^{\chi_0} = 1$, and $\delta < 1/3$. The reason for such choices will become clear later. Note that M_N is of order $\log(\log(N))$. Also, we introduce for $k = 1, \dots, 2^{l-1}$

$$J_{k,i}^{(l)} = \sum_x 1\{x \in \mathcal{D}_{k,i}^{(l)}\} l_{2^{N-l}, 2k-1}^{(l)}(x) l_{2^{N-l}, 2k}^{(l)}(x), \quad \text{and} \quad J_k^{(l)} = \sum_{i=0}^{M_N} J_{k,i}^{(l)}. \quad (2.37)$$

Finally, we need the self-intersections of the 2^l strands at generation l

$$Z_k^{(l)} = \sum_x \sum_{0 \leq m < m' \leq 2^{N-l}} 1\{l_{2^{N-l}, k}^{(l)}(x) \leq N^\chi\} 1\{S_{m,k}^{(l)} = S_{m',k}^{(l)} = x\}, \quad (2.38)$$

We bootstrap a little differently than in the proof of Lemma 2.1 of [2]. Thus, at each generation l , and for level-set index i , we introduce the good-sets

$$\forall k = 1, \dots, 2^{l-1}, \quad \forall i = 0, \dots, M_N, \quad \mathcal{G}_{k,i}^{(l)} = \{|\mathcal{D}_{k,i}^{(l)}| < \frac{4y_0 2^N}{N^{2\chi_i}}\}, \quad \text{and} \quad \mathcal{G}^{(l)} = \bigcap_{k,i} \mathcal{G}_{k,i}^{(l)}.$$

As in equation (35) of [2], we have

$$(\mathcal{G}^{(l)})^c \subset \{Z_1^{(l)} + \dots + Z_{2^l}^{(l)} > y_0 2^N\}. \quad (2.39)$$

It is important to note that contrary to (35) of [2], we have kept the threshold $y_0 2^N$. Thus,

$$\begin{aligned} P(Z^{(0)} > y_0 2^N) &\leq P\left(\sum_{l=1}^{N-1} \sum_{k=1}^{2^{l-1}} J_k^{(l)} > y_0 2^N\right) \\ &\leq P\left(\sum_{l,k} J_k^{(l)} > y_0 2^N, \bigcap_{l=1}^{N-1} \mathcal{G}^{(l)}\right) + \sum_{l=1}^N P((\mathcal{G}^{(l)})^c) \\ &\leq P\left(\sum_{l,k} J_k^{(l)} > y_0 2^N, \bigcap_{l=1}^{N-1} \mathcal{G}^{(l)}\right) + \sum_{l=1}^N P\left(\sum_{j=1}^{2^l} Z_j^{(l)} > y_0 2^N\right) \end{aligned} \quad (2.40)$$

Now, by writing self-intersection in terms of intersection of independent strands, and proceeding by induction, we obtain

$$\begin{aligned} P(Z^{(0)} > y_0 2^N) &\leq \sum_{L=1}^N 2^{L-1} P\left(\sum_{l=L}^{N-1} \sum_{k=1}^{2^{l-1}} J_k^{(l)} > y_0 2^N, \bigcap_{l=L}^N \mathcal{G}^{(l)}\right) + 2^{N-1} P\left(\sum_{k=1}^{2^{N-1}} Z_k^{(N-1)} > y_0 2^N\right) \\ &\leq \sum_{L=1}^N 2^{L-1} P\left(\sum_{l=L}^{N-1} \sum_{i=0}^{M_N} \sum_{k=1}^{2^{l-1}} J_{k,i}^{(l)} 1\{\mathcal{G}_{k,i}^{(l)}\} > y_0 2^N\right). \end{aligned} \quad (2.41)$$

The last term of the first line in (2.41) has vanished since $Z_k^{(N-1)} \leq 1$, and we choose $y_0 > 1/2$. Also, note that (2.41) is different from inequality (36) of [2] in having the sum over l inside the probability. Now, Lemma 5.5 of the Appendix allow us to center the $J_k^{(l)}$, since

$$E \left[\sum_{l=1}^N \sum_{k=1}^{2^{l-1}} J_k^{(l)} \right] \leq C_3 2^N.$$

Actually, we rather need to center $Y_{k,i}^{(l)} := J_{k,i}^{(l)} 1\{\mathcal{G}_{k,i}^{(l)}\}$. Thus, let $\bar{Y}_{k,i}^{(l)} = Y_{k,i}^{(l)} - E[Y_{k,i}^{(l)}]$, and if we set $\bar{y} < (y_0 - 1)/2 - C_3$, and choose y large enough so that $\bar{y} > y_0/2$,

$$P \left(\sum_{x \in \mathcal{L}_0} l_n^2(x) \geq 2^N y_0 \right) \leq \sum_{L=1}^N 2^{L-1} P \left(\sum_{l=L}^{N-1} \sum_{i=0}^{M_N} \sum_{k=1}^{2^{l-1}} \bar{Y}_{k,i}^{(l)} \geq \bar{y} 2^N \right).$$

Now, fix L and note that for any sequences $\{q_i, p_l^{(i)}, i = 0, \dots, M_N, l = L, \dots, N\}$ with

$$\sum_i q_i \leq 1, \quad \text{and} \quad \sum_l p_l^{(i)} \leq 1, \quad \text{for any } i = 0, \dots, M_N,$$

we have

$$P \left(\sum_{l=L}^{N-1} \sum_{i=0}^{M_N} \sum_{k=1}^{2^{l-1}} \bar{Y}_{k,i}^{(l)} \geq \bar{y} 2^N \right) \leq \sum_{l=L}^{N-1} \sum_{i=0}^{M_N} P \left(\sum_{k=1}^{2^{l-1}} \bar{Y}_{k,i}^{(l)} \geq p_l^{(i)} q_i \bar{y} 2^N \right) \quad (2.42)$$

In order to use Lemma 5.1, we need exponential estimates for the $\bar{Y}_{k,i}^{(l)}$. Note first that

$$J_{k,i}^{(l)} \leq N^{\chi_{i+1}} l_{2^{N-l}, 2k}^{(l)} \left(\mathcal{D}_{k,i}^{(l)} \right), \quad (2.43)$$

We use Lemma 1.2 of [1] to obtain

$$P \left(Y_{k,i}^{(l)} > u \right) = P \left(J_{k,i}^{(l)} > u, \mathcal{G}_{k,i}^{(l)} \right) \leq c_3 E \left[\exp \left(-\kappa_3 \frac{u}{N^{\chi_{i+1}} |\mathcal{D}_{k,i}^{(l)}|^{2/3}} \right) 1\{\mathcal{G}_{k,i}^{(l)}\} \right].$$

We have two bounds on $|\mathcal{D}_{k,i}^{(l)}|$: either we recall that, on $\mathcal{G}_{k,i}^{(l)}$, the volume is bounded by $2y2^N/N^{2\chi_i}$, or the trivial bound by the total time 2^{N-l} . Thus,

$$P \left(Y_{k,i}^{(l)} > u \right) \leq C_3 \exp \left(-\xi_i^{(l)} u \right) \quad \text{with any} \quad \xi_i^{(l)} \leq \frac{\kappa_3}{N^{\chi_{i+1}} 2^{\frac{2}{3}N}} \max \left(2^l, \left(\frac{N^{2\chi_i}}{2y} \right) \right)^{2/3}. \quad (2.44)$$

We define $\xi_N = 1/2^{\frac{2}{3}N}$, and for a fixed i , we choose for convenience (for a $\delta < 1/3$)

$$\frac{\xi_i^{(l)}}{\xi_N} = \frac{\kappa_3}{N^{\chi_{i+1}}} \begin{cases} \frac{N^{\frac{4}{3}\chi_i}}{(2y)^{2/3}} & \text{for } l \leq l_i^*, \\ 2^{\frac{1}{6}l} & \text{for } l > l_i^*, \end{cases} \quad (2.45)$$

where l_i^* is such that

$$2^{\frac{1}{6}l_i^*} = \frac{N^{\frac{4}{3}\chi_i}}{(2y)^{2/3}}, \quad \text{so that} \quad l_i^* = \left(\frac{8}{\log(2)} \chi(1+\delta)^i - \frac{2}{3} \log(2y) \right)^+.$$

We wish now to use Lemma 5.1, or rather Remark (5.2), with $\Gamma = \xi_i^{(l)}$ and $X_i = Y_{k,i}^{(l)}$. Thus, we first bound $\Gamma^2 E[X_i^2]$ using Lemma 5.7 and (2.43)

$$\begin{aligned} (\xi_i^{(l)})^2 E[(Y_{k,i}^{(l)})^2] &\leq (\xi_i^{(l)})^2 E[(J_{k,i}^{(l)})^2] \leq C \frac{2^{\frac{2}{6}l} N^{\frac{8}{3}\chi_i}}{(2y)^{2\frac{4}{3}N}} 2^{\frac{4}{3}(N-l)} e^{-\kappa_3 N \chi_i} \\ &\leq C \frac{(N \chi_i)^{8/3} e^{-\kappa_3 N \chi_i}}{(2y)^2} \frac{1}{2^l}. \end{aligned} \quad (2.46)$$

For $\gamma > 0$ small, we denote for convenience $\sigma_N = \gamma^2 \Gamma^2 E[X_i^2]$. By Lemma 5.1, we obtain for any $\gamma \in]0, 1[$

$$P \left(\sum_{k=1}^{2^{l-1}} \bar{Y}_{k,i}^{(l)} \geq p_l^{(i)} q_i y 2^N \right) \leq \exp \left(-\frac{\gamma \xi_i^{(l)}}{2} p_l^{(i)} q_i \bar{y} 2^N + c_u 2^l \max(\sigma_N, \sigma_N^{1-\gamma}) \right). \quad (2.47)$$

Assume now that we can choose $p_l^{(i)}$ and q_i such that for some constant c , and y large (but fixed as N tends to infinity)

$$\frac{\xi_i^{(l)}}{\xi_N} p_l^{(i)} q_i \geq \frac{c}{y^{2/3}}. \quad (2.48)$$

Then, (2.47) yields the upper bound in (1.9) if

$$c \gamma y^{1/3} 2^{\frac{1}{3}N} \geq 8 c_u 2^l \max(\sigma_N, \sigma_N^{1-\gamma}). \quad (2.49)$$

Note that from (2.46)

$$\sigma_N \leq \frac{C}{(2y)^2} \sup_{x \geq 0} (x^{8/3} e^{-\kappa_3 x}) \frac{1}{2^l},$$

so that (2.49) holds if $2^{\frac{1}{3}N} \gg 2^{\gamma l}$. Now, since $l \leq N$, (2.49) holds as soon as $\gamma < 1/3$ for N large enough.

Finally, we choose $p_l^{(i)}$ and q_i to fulfill (2.48). We set $\alpha := \frac{1}{2}(\frac{1}{3} - \delta)\chi$, and

$$q_i = q \left(\frac{N^{\chi_{i+1}}}{N^{\frac{4}{3}\chi_i}} \right)^{1/2} = q \exp(-\alpha(1+\delta)^i), \quad \text{with } q \text{ such that } \sum_{i=0}^{M_N} q_i \leq 1. \quad (2.50)$$

Note that it is possible to find such a q which depends on χ and δ . Now, fix i , and choose

$$\forall l \leq l_i^*, \quad p_l^{(i)} = p_i^* \exp(-\alpha(1+\delta)^i), \quad \text{whereas if } l > l_i^*, \quad p_l^{(i)} = \bar{p}_i \frac{N^{\chi_{i+1}} e^{\alpha(1+\delta)^i}}{2^{\frac{1}{6}l}}, \quad (2.51)$$

with two normalizing constants p_i^* and \bar{p}_i to be chosen later. Note that for $l > l_i^*$

$$\frac{N^{\chi_{i+1}} e^{\alpha(1+\delta)^i}}{2^{\frac{1}{6}l}} \leq \left(\frac{N^{\frac{4}{3}\chi_i} N^{\chi_{i+1}}}{2^{\frac{1}{6}l_i^*} 2^{\frac{1}{6}l}} \right)^{1/2} \leq \left(\frac{(2y)^{2/3} N^{(1+\delta)\chi_i}}{2^{\frac{1}{6}l}} \right)^{1/2}. \quad (2.52)$$

Note that from the definition of l_i^* , and the choice $\delta < 1/3$, we have for $l > l_i^*$

$$N^{(1+\delta)\chi_i} \leq N^{\frac{4}{3}\chi_i} = (2y)^{2/3} 2^{\frac{1}{6}l_i^*} \implies p_l^{(i)} \leq \bar{p}_i (2y)^{2/3} 2^{-(l-l_i^*)/12}. \quad (2.53)$$

To see that it is possible to choose p_i^* and \bar{p}_i such that for each i , $\sum_l p_l^{(i)} \leq 1$, note that

$$\begin{aligned} \sum_l p_l^{(i)} &\leq p_i^* l_i^* \exp(-\alpha(1+\delta)^i) + \bar{p}_i (2y)^{2/3} \sum_{l>l_i^*} 2^{-(l-l_i^*)/12} \\ &\leq p_i^* \left(\frac{8\chi}{\log(2)} (1+\delta)^i e^{-\alpha(1+\delta)^i} \right) + \bar{p}_i (2y)^{2/3} \sum_{l>0} \frac{1}{2^{l/12}} \\ &\leq p_i^* \left(\frac{8\chi}{\log(2)} \sup_{x>0} \{x e^{-\alpha x}\} \right) + \bar{p}_i \frac{(2y)^{2/3}}{2^{1/12} - 1}. \end{aligned} \quad (2.54)$$

It suffices now to choose p_i^* as a small constant (depending only on χ), and \bar{p}_i as a small constant times $1/(2y)^{2/3}$. It is easy now to check that (2.48) holds.

Remark 2.1 When dimension $d = 4$, the proof of Lemma 3.1 of [2], with Remark 5.6 to obtain centering of the $J_k^{(l)}$ variables, can be used to obtain the upper bound (1.9). Indeed, in [2] dimension $d \geq 5$ was used to obtain that the first two moments of the intersection times of two independent walks were finite. This is actually much too strong, and a close inspection of the proof of Lemma 3.1 of [2] shows us that we actually only need (5.13). We omit to repeat the proof since it is similar.

2.3 Proof of the Lower Bound in (1.9)

The proof proceed as in (66) of [2], by using the comparison $\Sigma_n^2 \geq n^2/|\mathcal{R}_n|$ where we denoted by \mathcal{R}_n the range of the walk. Since it is a few lines, we reproduced it for the ease of reading. Indeed, $\Sigma_n^2 \geq n^2/|\mathcal{R}_n|$ follows by Jensen's inequality

$$\left(\frac{1}{|\mathcal{R}_n|} \sum_{\mathcal{R}_n} l_n(x) \right)^2 \leq \frac{1}{|\mathcal{R}_n|} \sum_{\mathcal{R}_n} l_n(x)^2. \quad (2.55)$$

Now, if $\sigma(r)$ is the first time the walk exits a ball $B(r)$, we have

$$\{\sigma(r) > n\} \subset \{|\mathcal{R}_n| < |B(r)|\} \subset \left\{ \Sigma_n^2 > \frac{n^2}{|B(r)|} \right\}. \quad (2.56)$$

Thus, if we choose a radius r_n such that $|B(r_n)| = n/y$, then $\{\sigma(r_n) > n\} \subset \{\Sigma_n^2 > yn\}$. We recall now the classical estimate $\mathcal{P}_0(\sigma(r_n) \geq n) \geq \exp(-Cn/r_n^2)$, for some constant C , and this yields the lower bound in(1.9).

3 Application of Section 2 to lower bounds.

3.1 Proof of Proposition 1.3

We assume, for simplicity, that we can divide $[0, n]$ into k_n periods of length $|B(r_n)|$. Let $T_i = (i-1)|B(r_n)|$, and $\mathcal{R}_i := \{0, S_{T_i+1} - S_{T_i}, \dots, S_{T_{i+1}} - S_{T_i}\}$ for $i = 1, \dots, k_n$. Note that

$\{\mathcal{R}_i, i = 1, \dots, k_n\}$ are independent, and that for ϵ_0 small, inequality (1.12) yields

$$P(|\mathcal{R}_i| < 2\epsilon_0|B(r_n)|) \leq \exp\left(-\frac{\kappa}{(2\epsilon_0)^{1/3}}|B(r_n)|^{1/3}\right). \quad (3.1)$$

Now, we introduce independent Bernoulli variables $X_i = 1\{|\mathcal{R}_i| < 2\epsilon_0|B(r_n)|\}$ for $i = 1, \dots, k_n$. We rewrite (3.1) with a rate $I(\epsilon_0)$ large when ϵ_0 is small, such that

$$E[X_i] \leq \exp(-I(\epsilon_0)|B(r_n)|^{1/3}).$$

By Chebychev's inequality, there is a constant c depending on (δ_0/ϵ_0) , such that when $\delta_0 < \epsilon_0$ and large n ,

$$P\left(\frac{1}{k_n} \sum_{i=1}^{k_n} X_i > 1 - \frac{\delta_0}{\epsilon_0}\right) \leq \exp(-cI(\epsilon_0)|B(r_n)|^{1/3}k_n). \quad (3.2)$$

On the complementary event $\{\sum_i X_i \leq (1 - \delta_0/\epsilon_0)k_n\}$, and there are $\frac{\delta_0}{\epsilon_0}k_n$ periods, say the *good* periods, where $\{|\mathcal{R}_i| \geq 2\epsilon_0|B(r_n)|\}$. We show now that if there are enough good periods, then a fraction of the sites of $B(r_n)$ are visited a fraction of the time $n/|B(r_n)|$. In other words,

$$\begin{aligned} \{\sigma(r_n) > n\} \cap \{|\{i \leq k_n : |\mathcal{R}_i| \geq 2\epsilon_0|B(r_n)|\}| > \frac{\delta_0}{\epsilon_0}k_n\} \\ \subset \{|\{x : l_n(x) \geq \delta_0k_n\}| \geq \epsilon_0|B(r_n)|\}. \end{aligned} \quad (3.3)$$

We take an issue in the left hand event in (3.3), and by way of contradiction, we assume that more than $(1 - \epsilon_0)|B(r_n)|$ sites belong to $\mathcal{D} := \{x : l_n(x) < \delta_0k_n\}$. Since we suppose $|\mathcal{D}| \geq (1 - \epsilon_0)|B(r_n)|$, in each *good* period, where $|\mathcal{R}_i| > 2\epsilon_0|B(r_n)|$, there are at least $\epsilon_0|B(r_n)|$ sites of \mathcal{D} which are visited. Thus, \mathcal{D} receives a total of at least $\epsilon_0|B(r_n)|(\delta_0/\epsilon_0)k_n$ visits. Necessarily, one site of \mathcal{D} receives more than δ_0k_n visits, and this contradicts the definition of \mathcal{D} . Now, from (3.3) we obtain

$$\begin{aligned} P\left(\left\{|\left\{x : l_n(x) \geq \delta_0 \frac{n}{|B(r_n)|}\right\}| \geq \epsilon_0|B(r_n)|\right\} \cap \{\sigma(r_n) > n\}\right) \\ \geq P(\sigma(r_n) > n) - P\left(\frac{1}{k_n} \sum_{i=1}^{k_n} X_i > 1 - \frac{\delta_0}{\epsilon_0}\right). \end{aligned} \quad (3.4)$$

Note that by classical estimates $P(\sigma(r_n) > n) \geq 2c_1 \exp(-c_2 n/r_n^2)$ for two constants c_1, c_2 . Finally, the possibility of having $cI(\epsilon_0)$ large, by reducing ϵ_0 , in (3.2) allows us to conclude (1.11).

3.2 Proof of the Lower Bound in Region III

We consider $\{X_n > n^\beta\}$. We fix $u = \frac{9}{5} - \frac{6}{5}\beta$, and $v = 1 - u$. Note that in Region III, u and v are positive, and $\zeta_{III} = 2\beta - 2v - u = 1 - \frac{2}{3}u$. We consider a sequence of radii with $|B(r_n)| = n^u$ and keep ϵ_0 and δ_0 of Proposition 1.3. Now, we set $\mathcal{G} := \{x : l_n(x) > \delta_0 n^v\}$, and use inequality (2.3) of Lemma 2.1 of [1], since we have assumed that the η 's are bell-shaped.

$$\mathbb{P}\left(\sum_{x \in \mathbb{Z}^d} \eta(x) l_n(x) > n^\beta\right) \geq \mathbb{P}\left(\sum_{\mathcal{G}} \eta(x) \delta_0 n^v > n^\beta\right)$$

$$\geq P(|\mathcal{G}| > \epsilon_0 n^u) P_\eta \left(\sum_{i=1}^{\epsilon_0 n^u} \eta_i > \frac{n^{\beta-v}}{\delta_0} \right), \quad (3.5)$$

where $\{\eta_j, j \in \mathbb{N}\}$ are i.i.d with the same law as $\eta(0)$. Note that the last probability estimate in (3.5) on the sum of η 's is on the moderate deviations regime, since (i) $\sqrt{n^u} \ll n^{\beta-v}$, and (ii) $n^u \gg n^{\beta-v}$. Indeed, (i) is equivalent to $\zeta_{III} > 0$ which holds, whereas (ii) is equivalent to $\beta < 1$. Now, in regime (i) and (ii), we have a gaussian lower bound

$$P_\eta \left(\sum_{i=1}^{\epsilon_0 n^u} \eta_i > \frac{n^{\beta-v}}{\delta_0} \right) \geq \exp \left(-c \frac{n^{2(\beta-v)}}{\delta_0^2 \epsilon_0 n^u} \right) = \exp \left(-\frac{c}{\delta_0^2 \epsilon_0} n^{\zeta_{III}} \right), \quad (3.6)$$

and Proposition 1.3 gives the same lower bound for $P(|\mathcal{G}| > \epsilon_0 n^u)$.

4 Upper bounds for deviations estimates for RWRS

We follow the approach of Section 4 of [2]. Thus, we partition the range of the RW into two domains $\bar{\mathcal{D}}_b = \{x \in \mathbb{Z}^d : l_n(x) \geq n^b\}$ and $\underline{\mathcal{D}}_b = \{x \in \mathbb{Z}^d : 0 < l_n(x) \leq n^b\}$, parametrized by a positive b .

According to Section 4 of [2], in each region of interest we choose $b = \beta - \zeta$, and it is sufficient to find constants C_1, C_2 such that for y large enough

$$P \left(\sum_{x \in \underline{\mathcal{D}}_b} l_n^2(x) > n^{\beta+b} y \right) \leq \exp(-C_1 n^\zeta) \quad (4.1)$$

and,

$$P \left(\sum_{x \in \bar{\mathcal{D}}_b} l_n^{\alpha^*}(x) \geq n^{\beta-b+\alpha^* b} y \right) \leq \exp(-C_2 n^\zeta), \quad \text{where } \alpha^* := \frac{\alpha}{\alpha - 1}. \quad (4.2)$$

Region I. We choose $\beta + b = 1$. Since, in Region I, $2\beta - 1 \leq 1/3$, (4.1) follows from the upper bound in (1.7). Finally, $b \geq 1/3$ implies that $P(\bar{\mathcal{D}}_b \neq \emptyset) \leq \exp(-Cn^{1/3})$, and (4.2) holds trivially.

Region II. We choose $b = \beta/(\alpha + 1)$. We consider two cases.

- First $\beta + b > 1$. The evaluation of $P(\Sigma_n^2 > n^{\beta+b} y)$ is straightforward from the proof of Lemma 2.1 of [2] supplied with the moment estimates of the Appendix. We omit to write this proof, since the argument is by now routine, and the result reads: for any $\epsilon > 0$

$$P \left(\sum_{x \in \underline{\mathcal{D}}_b} l_n^2(x) > n^{\beta+b} y \right) \leq \exp \left(-cn^{\beta+b-\frac{2}{3}-\epsilon} \right). \quad (4.3)$$

Now, we can find ϵ small enough so that in Region II, $\beta + b - \frac{2}{3} - \epsilon > \beta - b$, which is equivalent to $b > 1/3$. In region II, $b = \beta/(1 + \alpha) > 1/(4 - \alpha) \geq 1/3$. Thus, (4.1) holds.

- When $\beta + b = 1$ (and $\alpha = 1$), we have $\zeta_H = 1/3$. We can take $\epsilon = 0$ in (4.3), by (1.7).

In order to prove (4.2), we proceed along the same line as in [2], and rely on Proposition 3.2 of [2]. We omit to repeat the same computations.

Region III. We choose $5b = \beta + 1$. Note that $\beta + b > 1$, and with the help of (4.3), (4.1) follows as soon as $\beta + b - \frac{2}{3} > \beta - b$, which is equivalent to $\beta > \frac{2}{3}$.

We now prove (4.2). We consider two cases.

- $\alpha \geq d/2$. Condition (0), of Proposition 3.2 of [2], requires that $\beta - b \leq \frac{3}{2}b$ which is equivalent to $\beta \leq 1$. Condition (iii) of the same proposition requires that $\beta < 1$.
- $\alpha < d/2$. We need to check Conditions (i) and (ii) of Proposition 3.2 of [2]. Condition (i) imposes that

$$(\beta - b) \left(\frac{\alpha^*}{3/2} + 1 \right) < \beta - b + \alpha^*b \iff \beta < \frac{5}{2}b \iff \beta < 1. \quad (4.4)$$

(4.4) is satisfied in Region III. Condition (ii) requires

$$(\beta - b)\alpha^* < \beta - b + \alpha^*b \iff (\beta - b) \frac{\alpha^* - 1}{\alpha^*} < b \iff \frac{4}{5}\beta - \frac{1}{5} < \alpha \left(\frac{\beta}{5} + \frac{1}{5} \right) \iff \alpha > \frac{4\beta - 1}{\beta + 1}. \quad (4.5)$$

This last inequality holds in Region III.

5 Appendix

We have gathered in this section a handy large deviation estimate, as well as moments computations for variables related to self-intersection times in dimension 3 and 4.

5.1 On a large deviation estimate

Lemma 5.1 *Let $\{X, X_1, \dots, X_n\}$ be positive i.i.d. satisfying*

$$P(X > u) \leq C \exp(-u), \quad \text{with } C > 1. \quad (5.1)$$

We set $\bar{X}_i = X_i - E[X_i]$, and denote by $c_u = 3 + e^1 + C$. Then, for any $\gamma \in]0, 1[$, we have

$$P \left(\sum_{i=1}^n \bar{X}_i > x_n \right) \leq \exp \left(c_u n \max \left(\gamma^2 E[X^2], (\gamma^2 E[X^2])^{1-\gamma} \right) - \frac{\gamma x_n}{2} \right). \quad (5.2)$$

Remark 5.2 Lemma 5.1 will serve in regime where $x_n \sim n$. Estimate (5.2) allows us to take advantage of the smallness of $nE[X^2]/x_n$ to bypass the lack of Cramer's condition. Indeed,

assume for instance that instead of (5.1), we had for some $\Gamma > 0$ (that we think of as a small number which may depend on n) and for $0 < \gamma < 1$

$$P(X > u) \leq C \exp(-\Gamma u), \quad \text{and} \quad \max\left(\Gamma^2 E[X^2], (\Gamma^2 E[X^2])^{1-\gamma}\right) \leq \frac{\Gamma x_n}{4c_u n}, \quad (5.3)$$

then, the estimate (5.2) would read

$$P\left(\sum_{i=1}^n \bar{X}_i > x_n\right) \leq \exp\left(-\frac{\gamma \Gamma x_n}{4}\right). \quad (5.4)$$

Note that Lemma 1 of [4] does not achieve the same purpose, since even if $n\Gamma^2 E[X^2]$ were bounded, their proof would yield an estimate $P(\sum \bar{X}_i > x_n) \leq \exp(-c\Gamma x_n / \log(n))$.

Proof. Note that for any $\gamma \in]0, 1[$, we use (5.1) and Chebychev to obtain

$$P(X > u) \leq \left(\frac{E[X^2]}{u^2}\right)^{1-\gamma} C^\gamma e^{-\gamma u}. \quad (5.5)$$

Now, for any $0 < \lambda < 1$ we decompose $E[\exp(\lambda \bar{X})]$ as follows

$$\begin{aligned} E[\exp(\lambda \bar{X})] &= E\left[e^{\lambda \bar{X}} 1\{A\}\right] + E\left[e^{\lambda \bar{X}} 1\{A^c\}\right] \quad \text{with } A = \{\lambda X < 1\} \\ &\leq E\left[e^{\lambda \bar{X}} 1\{\lambda \bar{X} < 1\}\right] + E\left[e^{\lambda X} 1\{A^c\}\right] \\ &\leq E\left[e^{\lambda \bar{X}} 1\{\lambda \bar{X} < 1\}\right] + e^1 P(A^c) + \int_{1/\lambda}^{\infty} \lambda e^{\lambda u} P(X > u) du \\ &\leq 1 + \lambda E\left[\bar{X} 1\{\lambda \bar{X} < 1\}\right] + 2\lambda^2 E[X^2] + e^1 P(A^c) + \int_{1/\lambda}^{\infty} \lambda e^{\lambda u} P(X > u) du \\ &\leq 1 + \lambda E\left[|\bar{X}| 1\{\lambda \bar{X} \geq 1\}\right] + (e^1 + 2)\lambda^2 E[X^2] + \lambda \int_{1/\lambda}^{\infty} e^{\lambda u} P(X > u) du \end{aligned} \quad (5.6)$$

We have used that for $x \leq 1$, $e^x \leq 1 + x + 2x^2$ and that $E[\bar{X}] = 0$. Now, we choose $2\lambda = \gamma$ and (5.5) to estimate the last term in (5.6)

$$\begin{aligned} E[\exp(\lambda \bar{X})] &= 1 + (3 + e^1)\lambda^2 E[X^2] + \lambda C^\gamma \int_{1/\lambda}^{\infty} \left(\frac{E[X^2]}{u^2}\right)^{1-\gamma} e^{-\lambda u} du \\ &\leq 1 + (3 + e^1)\lambda^2 E[X^2] + C^\gamma (\lambda^2 E[X^2])^{1-\gamma} \\ &\leq \exp\left(c_u \max\left(\lambda^2 E[X^2], (\lambda^2 E[X^2])^{1-\gamma}\right)\right). \end{aligned} \quad (5.7)$$

The estimate (5.2) follows at once. ■

5.2 Moments computations

For notational convenience, we keep $n/2$ to denote the integer part of $n/2$.

Lemma 5.3 *There is C_0 such that for $|x| > \sqrt{n}$ and $k < n/2$*

$$P_0(S_{n/2-k} = x) \leq C_0 P_0(S_{n-k} = x). \quad (5.8)$$

Remark 5.4 Note that this implies that for $|x| > \sqrt{n}$

$$\sum_{k=0}^n P_0(S_k = x) \leq (C_0 + 1) \sum_{k=n/2}^n P(S_k = x). \quad (5.9)$$

Proof. Since classical Gaussian estimates gives

$$\frac{C_1 e^{-|x|^2/2k}}{k^{d/2}} \leq P_0(S_k = x) \leq \frac{C_2 e^{-|x|^2/2k}}{k^{d/2}}, \quad (5.10)$$

(5.8) follows if there is a constant C , independent of $|x|$ and k such that

$$C \exp\left(\left(\frac{1}{n/2-k} - \frac{1}{n-k}\right) \frac{|x|^2}{2}\right) \geq \left(\frac{n-k}{n/2-k}\right)^{d/2}, \quad \text{for } |x|^2 > n, k < n/2. \quad (5.11)$$

Inequality (5.11) is equivalent to

$$C \exp\left(\frac{|x|^2}{2(n-k)} \left(\frac{n/2}{n/2-k}\right)\right) \geq \left(1 + \frac{n/2}{n/2-k}\right)^{d/2}. \quad (5.12)$$

Thus, since $|x|^2/(n-k) \geq n/(n/2) = 1/2$, it is enough to choose

$$C := \sup_{y>1} \left(\exp\left(-\frac{y}{4}\right) (1+y)^{d/2} \right).$$

We obtain 5.8 by choosing $C_0 = CC_2/C_1$. ■

We consider $\{\tilde{S}_n, n \in \mathbb{N}\}$ and independent copy of the random walk $\{S_n, n \in \mathbb{N}\}$, and denote by $\{\tilde{l}_n(x), x \in \mathbb{Z}^d\}$ its local times. Also, we denote $I_n = \sum_{\mathbb{Z}^d} l_n(x) \tilde{l}_n(x)$.

Lemma 5.5 *In dimension 3, there is a constant c_3 such that $E[I_n] \leq c_3 \sqrt{n}$. In dimension 4, there is a constant c_4 such that $E[I_n] \leq c_4 \log(n)$.*

Remark 5.6 Note that when $n = 2^N$, and $\{I_k^{(l)}, k = 1, \dots, 2^l\}$ are independent copies with the same distribution as $I_{2^{N-l}}$, we have both for $d = 3, 4$ constants C_3 and C_4 such that

$$E \left[I_k^{(l)} \right] \leq \begin{cases} C_3 \sqrt{2^{N-l}} & \text{for } d = 3 \\ C_4 (N-l) & \text{for } d \geq 4, \end{cases} \quad \text{and} \quad E \left[\sum_{l=1}^{N-1} \sum_{k=1}^{2^{l-1}} I_k^{(l)} \right] \leq \frac{C_d}{\sqrt{2}-1} 2^N. \quad (5.13)$$

Proof. If we denote by γ_d the probability of not returning to 0, i.e. $P_0(H_0 = \infty) = \gamma_d > 0$, then $E_0[l_\infty(0)] = 1/\gamma_d$, and

$$E[I_n] = \sum_{x \in \mathbb{Z}^d} (E_0[l_n(x)])^2 \leq \frac{1}{\gamma_d^2} \sum P_0(H_x \leq n)^2 \leq R_{n,1} + R_{n,2}, \quad (5.14)$$

with

$$\begin{aligned}
R_{n,1} &:= \frac{1}{\gamma_d^2} \sum_{|x| \leq \sqrt{n}} P_0(H_x \leq n)^2 \leq C \sum_{|x| \leq \sqrt{n}} \frac{1}{1 + |x|^{2(d-2)}} \\
&\leq C' \int_1^{\sqrt{n}} \frac{x^{d-1}}{x^{2(d-2)}} dx \leq C'' \begin{cases} \sqrt{n} & \text{for } d = 3 \\ \log(\sqrt{n}) & \text{for } d \geq 4. \end{cases} \quad (5.15)
\end{aligned}$$

Now, for $R_{n,2}$, we note that $P_0(H_x \leq n) \leq P_0(S_0 = x) + \dots + P_0(S_n = x)$, and use Lemma 5.3

$$R_{n,2} := \frac{1}{\gamma_d^2} \sum_{|x| > \sqrt{n}} P_0(H_x \leq n)^2 \leq \frac{(C_0 + 1)^2}{\gamma_d^2} \sum_{|x| > \sqrt{n}} \left(\sum_{k=n/2}^n P_0(S_k = x) \right)^2. \quad (5.16)$$

Now, note that from (5.10), there is C such that for $n \geq k \geq n/2$

$$P_0(S_k = x) \leq \frac{C e^{-x^2/(2n)}}{n^{d/2}}. \quad (5.17)$$

Thus,

$$\sum_{|x| > \sqrt{n}} \left(\sum_{k=n/2}^n P_0(S_k = x) \right)^2 \leq \sum_{|x| > \sqrt{n}} \frac{C^2 e^{-x^2/n}}{n^{d-2}}. \quad (5.18)$$

Finally, there is a constant C' such that

$$R_{n,2} \leq C' \int_{\sqrt{n}}^n \frac{e^{-x^2/n}}{n^{d-2}} x^{d-1} dx \leq C' n^{2-d/2} \int_1^\infty e^{-u^2} u^{d-1} du. \quad (5.19)$$

The result follows as we gather (5.16) and (5.19). \blacksquare

We denote now $\mathcal{D}_n(z) := \{x : l_n(x) > z\}$. The following Lemma estimates the first two moments of $\tilde{l}_n(\mathcal{D}_n(z))$.

Lemma 5.7 *There are positive constants $\kappa_3, \kappa_4, C_3, C_4$ such that*

$$E \otimes \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \leq C_d \begin{cases} n^{2/3} \exp(-\frac{2}{3}\kappa_3 z) & \text{for } d = 3, \\ \sqrt{n} \exp(-\frac{\kappa_4}{2} z) & \text{for } d \geq 4. \end{cases} \quad (5.20)$$

Moreover, we also have constants C'_3 and C'_4 such that

$$E \otimes \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z))^2 \right] \leq C'_d \begin{cases} n^{4/3} \exp(-\kappa_3 z) & \text{for } d = 3, \\ n \exp(-\kappa_4 z) & \text{for } d \geq 4. \end{cases} \quad (5.21)$$

Proof. We have seen in [1] that when $d \geq 3$, there is c_d independent of n and of the domain Λ such that

$$\sup_x E_x [l_n(\Lambda)] \leq c_d |\Lambda|^{2/d}. \quad (5.22)$$

Thus, using Holder's inequality

$$E \otimes \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \leq c_d E \left[|\mathcal{D}_n(z)|^{2/d} \right] \leq c_d (E [|\mathcal{D}_n(z)|])^{2/d}. \quad (5.23)$$

Now, since the expected number of visited sites at time n , is of order n , we have

$$|\mathcal{D}_n(z)| = \sum_{x \in \mathbb{Z}^d} 1\{l_n(x) > z\}, \quad \text{and} \quad \sup_x E_x [|\mathcal{D}_n(z)|] \leq c'_d n e^{-\kappa_d z}. \quad (5.24)$$

Thus,

$$E \otimes \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \leq c_d (c'_d n e^{-\kappa_d z})^{2/d}. \quad (5.25)$$

Inequality (5.20) follows at once. We now prove (5.21). First note that

$$\tilde{l}_n(\mathcal{D}_n(z))^2 = 2 \sum_{x, y \in \mathcal{D}_n(z)} \sum_{k < k' \leq n} 1\{\tilde{S}_k = x, \tilde{S}_{k'} = y\} + \tilde{l}_n(\mathcal{D}_n(z)). \quad (5.26)$$

Now, we average only over the walk $\{\tilde{S}_n\}$

$$\begin{aligned} \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z))^2 \right] &= 2 \sum_{x, y \in \mathcal{D}_n(z)} \sum_{k < k' \leq n} \tilde{P}_0(\tilde{S}_k = x) \tilde{P}_x(\tilde{S}_{k'-k} = y) + \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \\ &\leq 2 \sum_{x \in \mathcal{D}_n(z)} \sum_{k \leq n} \tilde{P}_0(\tilde{S}_k = x) \tilde{E}_x \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] + \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \\ &\leq 2 \left(\sup_x \tilde{E}_x \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \right)^2 + \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right]. \end{aligned} \quad (5.27)$$

From (5.22) we obtain

$$\left(\sup_x \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \right)^2 \leq C_d^2 |\mathcal{D}_n(z)|^{4/d}. \quad (5.28)$$

We average now with respect to the random walk $\{S_n\}$, (and use Jensen's inequality in $d = 3$)

$$E \left[\left(\sup_x \tilde{E} \left[\tilde{l}_n(\mathcal{D}_n(z)) \right] \right)^2 \right] \leq C_d^2 \begin{cases} (E [|\mathcal{D}_n(z)|^2])^{2/3} & \text{for } d = 3, \\ E [|\mathcal{D}_n(z)|] & \text{for } d \geq 4. \end{cases} \quad (5.29)$$

Finally, note that

$$|\mathcal{D}_n(z)|^2 \leq |\mathcal{D}_n(z)| + 2 \sum_{x \neq y} 1\{H_x < H_y \leq n, l_n(y) > z\}. \quad (5.30)$$

Taking the expectation in (5.30), we obtain

$$\begin{aligned} E [|\mathcal{D}_n(z)|^2] &\leq E [|\mathcal{D}_n(z)|] + 2 \sum_x E \left[1\{H_x < n\} \sum_y P_x(l_n(y) > z) \right] \\ &\leq \sup_x E_x [|\mathcal{D}_n(z)|] (1 + 2E_0[\{x : l_n(x) > 0\}]) \leq C n^2 e^{-\kappa_d z}. \end{aligned} \quad (5.31)$$

This concludes the proof. ■

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