Proof that the Hydrogen-antihydrogen Molecule is Unstable.

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In the framework of nonrelativistic quantum mechanics we derive a necessary condition for four Coulomb charges $(m_1^+, m_2^-, m_3^+, m_4^-)$, where all masses are assumed finite, to form the stable system. The obtained stability condition is physical and is expressed through the required minimal ratio of Jacobi masses. In particular this provides the rigorous proof that the hydrogen-antihydrogen molecule is unstable. This is the first result of this sort for four particles.

INTRODUCTION

Recent success in the production of trapped antihydrogen atoms [1] has renewed interest in the interaction of matter with antimatter and especially in the hydrogen-antihydrogen system (H-$\overline{\text{H}}$). It has long been conjectured that with pure Coulomb forces no bound state of hydrogen-antihydrogen exists. The numerical evidence supports this conjecture [2], yet there is a lack of rigorous proof as remarked by some authors [3], [4]. Our aim in this Letter is: (i) to supply such proof under assumption that only Coulomb forces act between the constituents, (ii) to provide insight into the screening effect within the system of four charged particles.

The instability of H-$\overline{\text{H}}$ is explained by the screening effect just like the instability of the muonic hydrogen ion (Hμ^-). In the system (Hμ^-) the heavy muon gets so tight to the nucleus that screens the positive charge and the electron "sees" a tightly bound neutral combination $p\mu^-$ and departs from it making the whole system unbound. In [5] we have proved that the screening effect in the system of three charged particles is expressed through some critical ratio of Jacobi masses. From the physical point of view there are two orbits in this system to consider, namely one orbit within the pair of particles (the pair that sets up the dissociation threshold) and the orbit of the third particle in the field of this pair with respect to the pair's center of mass. Inverse Jacobi masses are proportional to the Bohr radii of these orbits. If, say, the orbit of one negative particle is outdistanced then the attraction from the positive charge is screened by the other negative particle and the system becomes unbound.

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The system of four unit charges $(m_1^+, m_2^-, m_3^+, m_4^-)$ can be unstable only against dissociation into two neutral pairs. Indeed, if the lowest dissociation threshold would be dissociation into one particle and the bound cluster of three particles, then these two objects would have opposite charges and the long-tailed Coulomb attraction between them would guarantee the existence of a bound state below the dissociation threshold. (Just the same argument explains why atoms are stable). This suggests that we have to consider three orbits, two inner orbits of the neutral pairs and the third orbit of the relative motion of these pairs. The Jacobi masses for the neutral pairs are $\mu_x = m_1^+ m_2^- / (m_1^+ + m_2^-)$ and $\mu_y = m_3^+ m_4^- / (m_3^+ + m_4^-)$. The Jacobi mass corresponding to the relative motion of these two pairs is $\mu_R = (m_1^+ + m_2^-)(m_3^+ + m_4^-)/(m_1^+ + m_2^- + m_3^+ + m_4^-)$. Pay attention that we order the particles so that among two possible rearrangements into neutral pairs the lowest energy threshold corresponds to the dissociation into $(m_1^+ m_2^-) + (m_3^+ m_4^-)$ and the pairs are ordered so that $\mu_x \geq \mu_y$.

The Jacobi masses are in fact not independent, it is easy to check that $\mu_R \geq 4\mu_y$ always holds if the particles are ordered as above. Let us consider the screening effect within the system of four particles keeping in mind that the Bohr radii of the orbits are inverse proportional to the Jacobi masses. The first possibility is $\mu_y \ll \mu_R$ and $\mu_R \approx \mu_x$. This would mean that three particles form a heavy cluster and one particle is outdistanced from it. Here everything depends on whether this heavy three-body cluster has a bound state. If it does then the whole system is stable because the cluster and the particle have opposite charges. Hence the whole situation reduces to the question whether there is any screening in the three-body system. For $\mu_R \approx \mu_x$ there is no apparent screening as follows from [5], [6]. For example, Bressanini *et. al.* [2] have collected the convincing evidence that the system $(m_1^+, 1^-, m_3^+, 1^-)$ is stable for any m_1^+ and m_3^+. The three-body system $(m_1^+, 1^-, 1^-)$ is always stable and if $m_3^+ \ll 1$ then we run into the situation where $\mu_y \ll \mu_R$ and $\mu_R \approx \mu_x$ and still the whole system is stable. This means that $\mu_y \ll \mu_R$ is not sufficient for the screening effect to take over.

Let us consider the possibility when $\mu_R \ll \mu_x$, which would mean that the pair (m_1^+, m_2^-) has a very short inner orbit and other particles are outdistanced from it. In this case it is right to expect screening because other charged particles would "see" the tightly bound pair (m_1^+, m_2^-) as neutral and the system would fall apart. Our aim in this Letter is to present the rigorous and analytic proof of this screening effect, namely

$$\mu_R \leq 0.067\mu_x \implies \text{Instability}, \tag{1}$$

where under Instability we mean the absence of a bound state below the dissociation thresholds.

Eq. (1) manifests the screening effect for four particles. From Eq. (1) it easily follows that the hydrogen-antihydrogen molecule has no bound state and must decay into protonium and positronium. Muonic molecules $p\mu^- e^+ e^-$ and $\mu^+ \overline{p} e^+ e^-$ are unstable as well.

The proof of Eq. (1) is along the same line as in [5] (the basic idea of the proof is due to Thirring [7]). Before we proceed with the proof let us introduce the notations. Let $q_i, \mathbf{r}_i \in \mathbb{R}^3$ denote charges and position vectors of the particles $i = 1, 2, 3, 4$. We shall work in the system of units where $\hbar = 1$. We put $q_{1,3} = +1$, and $q_{2,4} = -1$, and the interactions between the particles are $V_{ik} = q_i q_k / |\mathbf{r}_i - \mathbf{r}_k|$ (remember how the particles are ordered). The stability problem with Coulomb interactions is invariant with respect to scaling all masses [6], so we can put $\mu_x = 2$. By the end we shall rescale the masses back.

Consider the system of four charged particles which is stable for $\mu_R < 3/8$ (this is weaker than in Eq. (1)). We separate the center of mass motion in the Jacobi frame [8] putting $\mathbf{x} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{y} = \mathbf{r}_4 - \mathbf{r}_3$, $\mathbf{R} = -a\mathbf{x} + \mathbf{r}_3 - \mathbf{r}_1 + b\mathbf{y}$, where $a = m_2/(m_1 + m_2)$ and $b = m_4/(m_3 + m_4)$ are the mass parameters invariant with respect to mass scaling. With Jacobi momenta defined as $\mathbf{p}_{x,y,R} = -i\nabla_{x,y,R}$ the Hamiltonian of the system takes the form

$$H = h_{12} + h_{34} + \frac{\mathbf{p}_R^2}{2\mu_R} + W, \tag{2}$$

where

$$W = V_{13} + V_{14} + V_{23} + V_{24}, \tag{3}$$

and $h_{12} = \mathbf{p}_x^2/4 - 1/x$, and $h_{34} = \mathbf{p}_y^2/(2\mu_y) - 1/y$ are the Hamiltonians of the pairs (1,2) and (3,4) (notation x is used instead of $|\mathbf{x}|$). The ground state wave function of h_{12} is $\phi_0 = \sqrt{8/\pi}\exp(-2x)$ so that $h_{12}\phi_0 = -\phi_0$. By the particle ordering the energy threshold corresponding to dissociation into two neutral pairs is $E_{th} = -1 - \mu_y/2$, which is the sum of the binding energies of the pairs (1,2) and (3,4). Following [5] we shall cut off the positive part of W by introducing $W_- \equiv (|W| - W)/2$ and $W_+ \equiv (|W| + W)/2$ which results in the decomposition $W = W_+ - W_-$, where $W_\pm \geq 0$. Instead of H we shall consider the Hamiltonian

$$\tilde{H} = h_{12} + h_{34} + \frac{\mathbf{p}_R^2}{2\mu_R} - W_-. \tag{4}$$

(The operator $\tilde{H}$ is self-adjoint on the same domain as H, see [5]). We shall assume that H is stable, *i.e.* H has a bound state Ψ with the energy $E < E_{th}$. Because $\tilde{H} \leq H$ we conclude that $\langle \Psi | \tilde{H} | \Psi \rangle < E_{th} \|\Psi\|^2$. Before we use this inequality let us introduce a projection operator P_0, which acts on any $f(\mathbf{x}, \mathbf{y}, \mathbf{R})$ as

$$P_0 f \equiv \phi_0(x) \int d\mathbf{x}' \phi_0(x') f(\mathbf{x}', \mathbf{y}, \mathbf{R}), \tag{5}$$

and put $\eta = P_0\Psi$ and $\xi = (1-P_0)\Psi$, where obviously $\eta \perp \xi$ and $\Psi = \eta + \xi$. We shall assume that $\|\xi\| \neq 0$ (later we shall get rid of this assumption), then we are free to choose such normalization of Ψ that $\|\xi\| = 1$. Now let us rewrite the inequality $\langle\Psi|\tilde{H}|\Psi\rangle < E_{th}\|\Psi\|^2$ decomposing Ψ into $\Psi = \eta + \xi$.

$$\langle\eta|h_{34}|\eta\rangle + \langle\eta|\frac{\mathbf{p}_R^2}{2\mu_R} - W_-|\eta\rangle - \langle\eta|W_-|\xi\rangle - \langle\xi|W_-|\eta\rangle$$
$$+\langle\xi|h_{12}|\xi\rangle + \langle\xi|h_{34}|\xi\rangle - \langle\xi|\frac{\mathbf{p}_R^2}{2\mu_R} - W_-|\xi\rangle < -1 - (\mu_y/2)(\|\eta\|^2 + 1), \tag{6}$$

where we have used that the terms like $\langle\eta|\mathbf{p}_y^2|\xi\rangle$ cancel because P_0 commutes with the operators $\mathbf{p}_y^2, \mathbf{p}_R^2$ and $1/y$. Indeed, in this case for example $\langle\eta|\mathbf{p}_y^2|\xi\rangle = \langle\eta|P_0\mathbf{p}_y^2|\xi\rangle = \langle\eta|\mathbf{p}_y^2 P_0|\xi\rangle = 0$.

We are going to rewrite Eq. (6) using lower bounds for some of its terms. From the Hydrogen ground state and by the variational principle for the terms in Eq. (6) the following inequalities hold $\langle\eta|h_{34}|\eta\rangle \geq -(\mu_y/2)\|\eta\|^2$ and $\langle\xi|h_{34}|\xi\rangle \geq -\mu_y/2$. Introducing two non-negative constants $\alpha = \sqrt{\langle\eta|W_-|\eta\rangle}$ and $\beta = \sqrt{\langle\xi|W_-|\xi\rangle}$ we get by virtue of the Schwarz inequality $|\langle\xi|W_-|\eta\rangle| \leq \alpha\beta$. It remains to figure out the bound for the term $\langle\xi|h_{12}|\xi\rangle$. From the bound spectrum of the Hydrogen atom we have we have [7] $h_{12} \geq -P_0 - 1/4(1-P_0)$. (Indeed, P_0 projects on the ground state of h_{12} which has the energy -1 and the energy of all other excited states is greater or equal to $-1/4$ which is the energy of the second excited state). Hence for the first term in Eq. (6) we get the bound $\langle\xi|h_{12}|\xi\rangle \geq -1/4$. Substituting this into Eq. (6) leaves us with the main inequality

$$\langle\eta|\frac{\mathbf{p}_R^2}{2\mu_R} - W_-|\eta\rangle - 2\alpha\beta + \langle\xi|\frac{\mathbf{p}_R^2}{2\mu_R} - W_-|\xi\rangle < -\frac{3}{4}. \tag{7}$$

We shall focus on the third term on the lhs of Eq. (7).

First, let us prove that the inequality

$$\frac{\mathbf{p}_R^2}{2\mu_R} + AV_{14} + AV_{23} \geq -2A^2\mu_R \tag{8}$$

holds in the operator sense for any $A \geq 0$. Indeed, the interactions have the form $V_{14} = -1/|\mathbf{R} - \mathbf{z}_1|$ and $V_{23} = -1/|\mathbf{R} - \mathbf{z}_2|$, where the vectors $\mathbf{z}_1 = -a\mathbf{x} - (1-b)\mathbf{y}$ and $\mathbf{z}_2 = (1-a)\mathbf{x} + b\mathbf{y}$ play the role of parameters in Eq. (8). According to the result of Lieb and Simon [9] the energy of the Hamiltonian on the lhs of Eq. (8) is monotonically increasing with $|\mathbf{z}_1 - \mathbf{z}_2|$. Hence the minimum energy is attained when $\mathbf{z}_1 = \mathbf{z}_2 = 0$, which gives us Eq. (8). From Eq. (8) using the obvious inequality $-W_- \geq V_{14} + V_{23}$ we find that for any $A \geq 0$ and $\chi(\mathbf{x}, \mathbf{y}, \mathbf{R})$

$$\langle\chi|\frac{\mathbf{p}_R^2}{2\mu_R} - AW_-|\chi\rangle \geq -2A^2\mu_R\|\chi\|^2 \tag{9}$$

holds. With the help of Eq. (9) we obtain the following chain of inequalities

$$\langle\xi|\frac{\mathbf{p}_R^2}{2\mu_R} - W_-|\xi\rangle = \max_{\lambda\geq -1}\left[\langle\xi|\frac{\mathbf{p}_R^2}{2\mu_R} - (\lambda+1)W_-|\xi\rangle + \lambda\beta^2\right] \geq \tag{10}$$

$$\max_{\lambda\geq -1}\left[-2(\lambda+1)^2\mu_R + \lambda\beta^2\right] = \frac{\beta^4}{8\mu_R} - \beta^2, \tag{11}$$

where in Eq. (10) we have added and subtracted the term $\lambda\beta^2 = \lambda\langle\xi|W_-|\xi\rangle$. Substituting Eq. (10)–(11) into Eq. (7) leaves us with the inequality

$$\langle\eta|\frac{\mathbf{p}_R^2}{2\mu_R} - W_-|\eta\rangle + \frac{\beta^4}{8\mu_R} - \beta^2 - 2\alpha\beta < -\frac{3}{4}. \tag{12}$$

The following inequality always holds

$$\frac{\beta^4}{8\mu_R} - \beta^2 - 2\alpha\beta + \frac{3}{4} \geq -\left(\sqrt{\frac{3}{8\mu_R}} - 1\right)^{-1}\alpha^2. \tag{13}$$

To see that Eq. (13) is true it suffices to take all terms to the lhs and minimize over α. Substituting Eq. (13) into Eq. (12) and using $\alpha^2 = \langle\eta|W_-|\eta\rangle$ allows us to formulate the stability condition

$$\langle\eta|\frac{\mathbf{p}_R^2}{2\mu_R} - \left(1 + (\sqrt{3/(8\mu_R)} - 1)^{-1}\right)W_-|\eta\rangle < 0. \tag{14}$$

It remains to consider the case when $\|\xi\| = 0$. It is easy to see that in this case the substitution $\Psi = \eta$ into the inequality $\langle\Psi|\tilde{H}|\Psi\rangle < E_{th}\|\Psi\|^2$ leads to the condition even more stringent than Eq. (14).

It makes sense to introduce the effective potential $V_{eff}(\mathbf{y},\mathbf{R}) = \int d\mathbf{x}|\phi_0|^2 W_-$. The function η has the factorized form $\eta = \phi_0(\mathbf{x})f(\mathbf{y},\mathbf{R})$. From Eq. (14) we conclude that the system of four unit charges is unstable if for any fixed $\mathbf{y}$

$$\frac{\mathbf{p}_R^2}{2\mu_R} - \left(1 + (\sqrt{3/(8\mu_R)} - 1)^{-1}\right)V_{eff} \geq 0. \tag{15}$$

We shall make one simplification, which helps carrying out all calculations analytically. We have $W = W_1 + W_2$, where $W_1 = V_{14} + V_{24}$ and $W_2 = V_{13} + V_{23}$ and obviously $W_- \leq (W_1)_- + (W_2)_-$. Breaking the kinetic energy term in Eq. (15) we deduce that the system is unstable if both of the following inequalities are satisfied $\mathbf{p}_R^2 - 4\mu_R\left(1 + (\sqrt{3/(8\mu_R)} - 1)^{-1}\right)V_{eff}^{(i)} \geq 0$ for $i = 1, 2$, where $V_{eff}^{(i)} = \int d\mathbf{x}|\phi_0|^2(W_i)_-$. Using the explicit calculation from [5] we get $V_{eff}^{(1)} \leq (3/16)|\mathbf{R}+(1-b)\mathbf{y}|^{-2}$ and $V_{eff}^{(2)} \leq (3/16)|\mathbf{R} - b\mathbf{y}|^{-2}$. It is known [10] that $\mathbf{p}_R^2 - \lambda/R^2 \geq 0$ for $\lambda \leq 1/4$. Thus both inequalities are satisfied if $3\mu_R\left(1 + (\sqrt{3/(8\mu_R)} - 1)^{-1}\right) \leq 1$. Solving this inequality and rescaling the masses tells us that the system is unstable if $\mu_R \leq (13 - 2\sqrt{22})\mu_x/54$, which proves Eq. (1). One can improve the constant in Eq. (1) if one extracts everything from Eq. (15). We preferred to make the simplification by splitting W into two terms because this makes the whole derivation analytical. Let us also remark that Instability in Eq. (1) means that there is no bound state neither below nor *at* the threshold. Indeed, if we would have $H\Psi = E_{th}\Psi$ then, because one can choose $\Psi > 0$ in the ground state, we immediately get $\langle\Psi|\tilde{H}|\Psi\rangle < E_{th}\|\Psi\|^2$ which was the starting point of our analysis.

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