

Distortion in Groups of Circle and Surface Diffeomorphisms

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1 Introduction

In his seminal article [18] S. Smale outlined a program for the investigation of the properties of generic smooth dynamical systems. He proposed as definition of the object of study the smooth action of a non-compact Lie group \mathcal{G} on a manifold M ; i.e., a smooth function

$$f : \mathcal{G} \times M \rightarrow M$$

satisfying $f(g_1, f(g_2, x)) = f(g_1 g_2, x)$ and $f(e, x) = x$ for all $x \in M$ and all $g_1, g_2 \in \mathcal{G}$, where e is the identity of \mathcal{G} . Equivalently one can consider the homomorphism

$$\phi : \mathcal{G} \rightarrow \text{Diff}(M)$$

from \mathcal{G} to the group of diffeomorphisms of M given by $\phi(g)(x) = f(g, x)$. The primary motivation, and by far the most studied case, has been that where \mathcal{G} is either the Lie group \mathbb{R} of real numbers or the discrete group \mathbb{Z} . As noted in the Introduction to this volume this study grew out of an interest in solution of differential equations where the group \mathbb{R} or \mathbb{Z} represents time (continuous or discrete).

In this article we will focus on the far less investigated case where \mathcal{G} is a subgroup of Lie group of dimension greater than one. The continuous and discrete cases when \mathcal{G} is \mathbb{R} or \mathbb{Z} share many characteristics with each other and

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it is often clear how to formulate (or even prove) an analogous result in one context based on a result in the other. Very similar techniques can be used in the two contexts. However, when we move to more complicated groups the difference between the actions of a connected Lie group and the actions of a discrete subgroup become much more pronounced. One must start with new techniques in the investigation of actions of a discrete subgroup of a Lie group.

As in the case of actions by \mathbb{R} and \mathbb{Z} one can impose additional structures on M , such as a volume form or symplectic form, and require that the group \mathcal{G} preserve them. For this article we consider manifolds of dimension two where the notion of volume form and symplectic form coincide. As it happens many of the results we will discuss are valid when a weaker structure, namely a Borel probability measure, is preserved.

The main object of this article is to provide some context for, and an exposition of, joint work of the author and Michael Handel which can be found in [8].

The ultimate aim is the study of the (non)-existence of actions of lattices in a large class of non-compact Lie groups on surfaces. A definitive analysis of the analogous question for actions on S^1 was carried out by É. Ghys in [9]. Our approach is topological and insofar as possible we try to isolate properties of a group which provide the tools necessary for our analysis. The two key properties we consider are almost simplicity of a group and the existence of a distortion element. Both are defined and described below.

We will be discussing groups of homeomorphisms and diffeomorphisms of the circle S^1 and of a compact surface S without boundary. We will denote the group of C^1 diffeomorphisms which preserve orientation by $\text{Diff}(X)$ where X is S^1 or S . Orientation preserving homeomorphisms will be denoted by $\text{Homeo}(X)$. If μ is a Borel probability measure on X then $\text{Diff}_\mu(X)$ and $\text{Homeo}_\mu(X)$ will denote the respective subgroups which preserve μ . Finally for a surface S we will denote by $\text{Diff}_\mu(S)_0$ the subgroup of $\text{Diff}_\mu(S)$ of elements isotopic to the identity.

An important motivating conjecture is the following.

Conjecture 1.1 (R. Zimmer [21]). *Any C^∞ volume preserving action of $SL(n, \mathbb{Z})$ on a compact manifold with dimension less than n , factors through an action of a finite group.*

This conjecture suggests a kind of exceptional rigidity of actions of $SL(n, \mathbb{Z})$ on manifolds of dimension less than n . The following result of D. Witte,

which is a special case of his results in [20], shows that in the case of $n = 3$ and actions on S^1 there is indeed a very strong rigidity.

Theorem 1.2 (D. Witte [20]). *Let \mathcal{G} be a finite index subgroup of $SL(n, \mathbb{Z})$ with $n \geq 3$. Any homomorphism*

$$\phi : \mathcal{G} \rightarrow \text{Homeo}(S^1)$$

has a finite image.

Proof. We first consider the case $n = 3$. If G has finite index in $SL(3, \mathbb{Z})$ then there is $k > 0$ such that

$$a_1 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}, \text{ and } a_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{pmatrix},$$

are all in \mathcal{G} . We will show that each of the a_i^k is in the kernel of ϕ . A result of Margulis (see Theorem 3.2 below) then implies that the kernel of ϕ has finite index. This result also implies that the case $n = 3$ is sufficient to prove the general result.

A straightforward computation shows that $[a_i, a_{i+1}] = e$ and $[a_{i-1}, a_{i+1}] = a_i^{\pm k}$, where the subscripts are taken modulo 6. Indeed $[a_{i-1}^m, a_{i+1}^n] = a_i^{\pm mnk}$.

Let $g_i = \phi(a_i)$. The group \mathcal{H} generated by g_1 and g_3 is nilpotent and contains g_2^k in its center. Since nilpotent groups are amenable there is an invariant measure for the group \mathcal{H} and hence the rotation number $\rho : \mathcal{H} \rightarrow \mathbb{R}/\mathbb{Z}$ is a homomorphism. Since g_2^k is a commutator, it follows that g_2^k has zero rotation number and hence it has a fixed point. A similar argument shows that for all i , g_i^k has a fixed point.

We will assume that one of the g_i^k , for definiteness say g_1^k , is not the identity and show this leads to a contradiction.

Let U_1 be any component of $S^1 \setminus \text{Fix}(g_1^k)$. Then we claim that there is a $U_2 \subset S^1$ which properly contains U_1 and such that U_2 is either a component of $S^1 \setminus \text{Fix}(g_6^k)$ or a component of $S^1 \setminus \text{Fix}(g_2^k)$. We postpone the proof of the claim and complete the proof.

Assuming the claim suppose that U_2 is a component of $S^1 \setminus \text{Fix}(g_2^k)$ the other case being similar. Then again applying the claim, this time to g_2^k

we see there is U_3 which properly contains U_2 and must a component of $S^1 \setminus \text{Fix}(g_3^k)$ since otherwise U_1 would properly contain itself. But repeating this we obtain proper inclusions

$$U_1 \subset U_2 \dots U_5 \subset U_6 \subset U_1,$$

which is a contradiction. Hence $g_1^k = id$ which implies that $a_1^k \in \text{Ker}(\phi)$. A further application of the result of Margulis (Theorem 3.2 below) implies that $\text{Ker}(\phi)$ has finite index in \mathcal{G} and hence that $\phi(\mathcal{G})$ is finite.

To prove the claim we note that U_1 is an interval whose endpoints are fixed by g_1^k and we will first prove that it is impossible for these endpoints also to be fixed by g_6^k and g_2^k . This is because in this case we consider the action induced by the two homeomorphisms $\{g_6^k, g_2^k\}$ on the circle obtained by quotienting U_1 by g_1^k . These two circle homeomorphisms commute because $[g_6^k, g_2^k] = g_1^{\pm k^2}$ on \mathbb{R} so passing to the quotient where g_1 acts as the identity we obtain a trivial commutator. It is an easy exercise to see that if two degree one homeomorphisms of the circle, f and g , commute then any two lifts to the universal cover must also commute. (E.g. show that $[\tilde{f}, \tilde{g}]^n$ is uniformly bounded independent of n .) But this is impossible in our case because the universal cover is just U_1 and $[g_6^k, g_2^k] = g_1^{\pm k^2} \neq id$.

To finish the proof of the claim we note that if U_1 contains a point $b \in \text{Fix}(g_2^k)$ then $g_1^{nk}(b) \in \text{Fix}(g_2^k)$ for all n and hence

$$\lim_{n \rightarrow \infty} g_1^{nk}(b) \text{ and } \lim_{n \rightarrow -\infty} g_1^{nk}(b),$$

which are the two endpoints of U_1 must be fixed by g_2^k . A similar argument applies to g_6^k .

It follows that at least one of g_6^k and g_2^k has no fixed points in U_1 and does not fix both endpoints. I.e. there is U_2 as claimed. \square

It is natural to ask the analogous question for surfaces.

Example 1.3. *The group $SL(3, \mathbb{Z})$ acts smoothly on S^2 by projectivizing the standard action on \mathbb{R}^3 .*

Consider S^2 as the set of unit vectors in \mathbb{R}^3 . If $x \in S^2$ and $g \in SL(3, \mathbb{Z})$, we can define $\phi(g) : S^2 \rightarrow S^2$ by

$$\phi(g)(x) = \frac{gx}{|gx|}.$$

Question 1.4. *Can the group $SL(3, \mathbb{Z})$ act continuously or smoothly on a surface of genus at least one? Can the group $SL(4, \mathbb{Z})$ act continuously or smoothly on S^2 ?*

2 Distortion in Groups

A key concept in our analysis of groups of surface homeomorphisms is the following.

Definition 2.1. *An element g in a finitely generated group G is called distorted if it has infinite order and*

$$\liminf_{n \rightarrow \infty} \frac{|g^n|}{n} = 0,$$

where $|g|$ denotes the minimal word length of g in some set of generators. If \mathcal{G} is not finitely generated then g is distorted if it is distorted in some finitely generated subgroup.

It is not difficult to show that if \mathcal{G} is finitely generated then the property of being a distortion element is independent of the choice of generating set.

Example 2.2. *The subgroup G of $SL(2, \mathbb{R})$ generated by*

$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfies

$$A^{-1}BA = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = B^4 \text{ and } A^{-n}BA^n = B^{4^n}$$

so B is distorted.

Example 2.3. *The group of integer matrices of the form*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

is called the Heisenberg group.

If

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

then their commutator $f = [g, h] := g^{-1}h^{-1}gh$ is

$$f = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } f \text{ commutes with } g \text{ and } h.$$

This implies

$$[g^n, h^n] = f^{n^2}$$

so f is distorted.

Let ω denote Lebesgue measure on the torus \mathbb{T}^2 .

Example 2.4 (G. Mess [14]). *In the subgroup of $\text{Diff}_\omega(\mathbb{T}^2)$ generated by the automorphism given by*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and a translation $T(x) = x + w$ where $w \neq 0$ is parallel to the unstable manifold of A , the element T is distorted.

Proof. Let λ be the expanding eigenvalue of A . The element $h_n = A^n T A^{-n}$ satisfies $h_n(x) = x + \lambda^n w$ and $g_n = A^{-n} T A^n$ satisfies $g_n(x) = x + \lambda^{-n} w$. Hence $g_n h_n(x) = x + (\lambda^n + \lambda^{-n})w$. Since $\text{tr} A^n = \lambda^n + \lambda^{-n}$ is an integer we conclude

$$T^{\text{tr} A^n} = g_n h_n, \text{ so } |T^{\text{tr} A^n}| \leq 4n + 2.$$

But

$$\lim_{n \rightarrow \infty} \frac{n}{\text{tr} A^n} = 0,$$

so T is distorted. □

Question 2.5. *Is an irrational rotation of S^1 distorted in $\text{Diff}(S^1)$ or $\text{Homeo}(S^1)$? Is an irrational rotation of S^2 distorted in $\text{Diff}(S^2)$ or in the group of area preserving diffeomorphisms of S^2 ?*

Example 2.6 (D. Calegari [3]). *There is a C^0 action of the Heisenberg group on S^2 whose center is generated by an irrational rotation. Hence an irrational rotation of S^2 is distorted in $\text{Homeo}(S^2)$.*

Proof. Consider the homeomorphisms of \mathbb{R}^2 given by

$$G = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and a translation $H(x, y) = (x, y + 1)$. We compute $F = [G, H]$ to be a translation $F(x, y) = (x + 1, y)$. This defines an action of the Heisenberg group on \mathbb{R}^2 . Let C be the cylinder obtained by quotienting by the relation $(x, y) \sim (x + \alpha, y)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The quotient action is well defined. The two ends of C are fixed by every element of the action and hence if we compactify C to obtain S^2 by adding a point at each end, we obtain an action of the Heisenberg group on S^2 . \square

A theorem of Lubotzky, Mozes, and Raghunathan shows that there is a large class of non-uniform lattices which contain a distortion element.

Theorem 2.7 (Lubotzky-Mozes-Raghunathan [12]). *Suppose Γ is a non-uniform irreducible lattice in a semi-simple Lie group \mathcal{G} with \mathbb{R} -rank ≥ 2 . Suppose further that \mathcal{G} is connected, with finite center and no nontrivial compact factors. Then Γ has distortion elements, in fact, elements whose word length growth is at most logarithmic.*

3 Distortion in almost simple groups

Definition 3.1. *A group is called almost simple if every normal subgroup is finite or has finite index.*

As we saw in the proof of the theorem of Witte (Theorem 1.2), the fact that $SL(n, \mathbb{Z})$ is almost simple when $n \geq 3$ plays a crucial role. This will also be true for our analysis of surface diffeomorphisms.

Theorem 3.2 (Margulis [13]). *Assume Γ is an irreducible lattice in a semi-simple Lie group with \mathbb{R} -rank ≥ 2 , e.g. any finite index subgroup of $SL(n, \mathbb{Z})$ with $n \geq 3$. Then Γ is almost simple.*

The following observation is a very easy consequence of the fact that \mathbb{R} has no distortion elements and no elements of finite order. Nevertheless, it is a powerful tool in our investigations.

Proposition 3.3 ([8]). *If \mathcal{G} is a finitely generated almost simple group which contains a distortion element and $\mathcal{H} \subset \mathcal{G}$ is a normal subgroup, then the only homomorphism from \mathcal{H} to \mathbb{R} is the trivial one.*

Proof. Since \mathcal{G} is almost simple, \mathcal{H} is either finite or has finite index. Clearly the result is true if \mathcal{H} is finite, so we assume it has finite index. If u is a distortion element in \mathcal{G} then $v := u^k \in \mathcal{H}$ for some $k > 0$. Let \mathcal{D} be the smallest normal subgroup of \mathcal{G} containing v , i.e. the group generated by $\{g^{-1}vg \mid g \in \mathcal{G}\}$. Then \mathcal{D} is infinite and normal and hence has finite index in \mathcal{G} ; it is obviously contained in \mathcal{H} . Thus \mathcal{D} has finite index in \mathcal{H} . Since \mathbb{R} contains neither torsion nor distortion elements, v , and hence \mathcal{D} is in the kernel of ψ for every homomorphism $\psi : \mathcal{H} \rightarrow \mathbb{R}$. Since \mathcal{D} has finite index in \mathcal{H} we conclude that $\psi(\mathcal{H})$ is finite and hence trivial. \square

The last important ingredient we will need is the following result of Thurston, originally motivated by the study of foliations.

Theorem 3.4 (Thurston stability theorem [19]). *Let \mathcal{G} be a finitely generated group and M a connected manifold. Suppose*

$$\phi : \mathcal{G} \rightarrow \text{Diff}^1(M)$$

is a homomorphism and there is $x_0 \in M$ such that for all $g \in \phi(\mathcal{G})$

$$g(x_0) = x_0 \text{ and } Dg(x_0) = I.$$

Then either ϕ is trivial or there is a non-trivial homomorphism from \mathcal{G} to \mathbb{R} .

Proof. The proof we give is due to W. Schachermayer [17]. Let $\{g_i\}$ be a set of generators for $\phi(\mathcal{G})$. The proof is local so there is no loss of generality in assuming $M = \mathbb{R}^m$ and that $x_0 = 0$ is not in the interior of the points fixed by all of $\phi(\mathcal{G})$.

For $g \in \phi(\mathcal{G})$ let $\widehat{g}(x) = g(x) - x$, so $g(x) = x + \widehat{g}(x)$ and $D\widehat{g}(0) = 0$. We compute

$$\begin{aligned} \widehat{gh}(x) &= g(h(x)) - x \\ &= h(x) - x + g(h(x)) - h(x) \\ &= \widehat{h}(x) + \widehat{g}(h(x)) \\ &= \widehat{h}(x) + \widehat{g}(x + \widehat{h}(x)) \\ &= \widehat{g}(x) + \widehat{h}(x) + (\widehat{g}(x + \widehat{h}(x)) - \widehat{g}(x)). \end{aligned}$$

Hence we have shown that for all $g, h \in \mathcal{G}$ and for all $x \in \mathbb{R}^m$

$$\widehat{gh}(x) = \widehat{g}(x) + \widehat{h}(x) + (\widehat{g}(x + \widehat{h}(x)) - \widehat{g}(x)). \quad (1)$$

Choose a sequence $\{x_n\}$ in \mathbb{R}^m converging to 0 such that for some i we have $|\widehat{g}_i(x_n)| \neq 0$ for all n . This is possible since 0 is not in the interior of the points fixed by all of $\phi(\mathcal{G})$.

Let $M_n = \max\{|\widehat{g}_1(x_n)|, \dots, |\widehat{g}_k(x_n)|\}$. Passing to a subsequence we may assume that for each i the limit

$$L_i = \lim_{n \rightarrow \infty} \frac{\widehat{g}_i(x_n)}{M_n}$$

exists and that $\|L_i\| \leq 1$. For some i we have $\|L_i\| = 1$; for definiteness say for $i = 1$.

If g is an arbitrary element of \mathcal{G} such that the limit

$$L = \lim_{n \rightarrow \infty} \frac{\widehat{g}(x_n)}{M_n}$$

exists then for each i we will show that

$$\lim_{n \rightarrow \infty} \frac{\widehat{g_i g}(x_n)}{M_n} = L_i + L.$$

Indeed because of Equation (1) above it suffices to show

$$\lim_{n \rightarrow \infty} \frac{\widehat{g}_i(x_n + \widehat{g}(x_n)) - \widehat{g}_i(x_n)}{M_n} = 0. \quad (2)$$

By the mean value theorem

$$\lim_{n \rightarrow \infty} \left\| \frac{\widehat{g}_i(x_n + \widehat{g}(x_n)) - \widehat{g}_i(x_n)}{M_n} \right\| \leq \lim_{n \rightarrow \infty} \sup_{t \in [0,1]} \|D\widehat{g}_i(z_n(t))\| \left\| \frac{\widehat{g}(x_n)}{M_n} \right\|,$$

where $z_n(t) = x_n + t\widehat{g}(x_n)$. But

$$\lim_{n \rightarrow \infty} \frac{\widehat{g}(x_n)}{M_n} = L \text{ and } \lim_{n \rightarrow \infty} \sup_{t \in [0,1]} \|D\widehat{g}_i(z_n(t))\| = 0,$$

since $D\widehat{g}_i(0) = 0$ and hence Equation (2) is established.

It follows that if we define $\Theta : \phi(\mathcal{G}) \rightarrow \mathbb{R}^m$ by

$$\Theta(g) = \lim_{n \rightarrow \infty} \frac{\widehat{g}(x_n)}{M_n}$$

this gives a well defined homomorphism from $\phi(\mathcal{G})$ to \mathbb{R}^m . \square

The following theorem is much weaker than known results on this topic, for example the theorem of Witte cited above or the definitive results of É. Ghys [9] on C^1 actions of lattices on S^1 . For those interested in circle actions the articles of Ghys, [9] and [10], are recommended. We present this “toy” theorem because its proof is simple and this is the proof which we are able to generalize to surfaces.

Theorem 3.5 (Toy Theorem). *Suppose \mathcal{G} is a finitely generated almost simple group and has a distortion element and suppose μ is a finite probability measure on S^1 . If*

$$\phi : \mathcal{G} \rightarrow \text{Diff}_\mu(S^1)$$

is a homomorphism then $\phi(\mathcal{G})$ is finite.

Proof. We give a sketch of the proof. The rotation number $\rho : \text{Diff}_\mu(S^1) \rightarrow \mathbb{R}/\mathbb{Z}$ is a homomorphism because the group preserves an invariant measure. If f is distorted then $\rho(f)$ has finite order in \mathbb{R}/\mathbb{Z} since there are no distortion elements in \mathbb{R}/\mathbb{Z} . Thus for some $n > 0$, $\rho(f^n) = 0$ and $\text{Fix}(f^n)$ is non-empty.

For any homeomorphism of S^1 leaving invariant a probability measure μ and having fixed points the support $\text{supp}(\mu)$ is a subset of the fixed point set. Hence $\text{supp}(\mu) \subset \text{Fix}(f^n)$.

Define $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \phi(g) \text{ pointwise fixes } \text{supp}(\mu)\}$. It is infinite, since $f^n \in \mathcal{G}_0$, and it is normal in \mathcal{G} . Hence it has finite index in \mathcal{G} . It follows that $\phi(\mathcal{G}_0)$ is trivial. This is because at a point $x \in \text{supp}(\mu)$ the homomorphism from \mathcal{G}_0 to the multiplicative group \mathbb{R}^+ given by $g \mapsto D\phi(g)_x$ must be trivial by Proposition 3.3 above. Hence we may use the Thurston stability theorem (and another application of Proposition 3.3) to conclude that $\phi(\mathcal{G}_0)$ is trivial. Since \mathcal{G}_0 has finite index in \mathcal{G} the result follows. \square

We proceed now to indicate how the proof of the “toy theorem” generalizes to the case of surfaces. The statement that $\text{supp}(\mu) \subset \text{Fix}(f^n)$ if $\text{Fix}(f^n)$ is non-empty, is trivial for the circle, but generally false for surfaces. Nevertheless, it was a key ingredient of the proof of the “toy theorem.” This apparent gap is filled by the following theorem from [8].

Theorem 3.6 ([8]). *Suppose that S is a closed oriented surface, that f is a distortion element in $\text{Diff}(S)_0$ and that μ is an f -invariant Borel probability measure.*

1. *If S has genus at least two then $\text{Per}(f) = \text{Fix}(f)$ and $\text{supp}(\mu) \subset \text{Fix}(f)$.*

2. If $S = T^2$ and $\text{Per}(f) \neq \emptyset$, then all points of $\text{Per}(f)$ have the same period, say n , and $\text{supp}(\mu) \subset \text{Fix}(f^n)$
3. If $S = S^2$ and if f^n has at least three fixed points for some smallest $n > 0$, then $\text{Per}(f) = \text{Fix}(f^n)$ and $\text{supp}(\mu) \subset \text{Fix}(f^n)$.

We can now nearly copy the proof of the ‘‘Toy Theorem’’ to obtain the following.

Theorem 3.7 ([8]). *Suppose S is a closed oriented surface of genus at least one and μ is a Borel probability measure on S with infinite support. Suppose \mathcal{G} is finitely generated, almost simple and has a distortion element. Then any homomorphism*

$$\phi : \mathcal{G} \rightarrow \text{Diff}_\mu(S)$$

has finite image.

Proof. We present only the case that S has genus greater than one. Define $\mathcal{G}_0 := \{g \in \mathcal{G} \mid \phi(g) \text{ pointwise fixes } \text{supp}(\mu)\}$. It is infinite, since by Theorem 3.6 the distortion element is in \mathcal{G}_0 , and it is normal in \mathcal{G} . Hence \mathcal{G}_0 has finite index in \mathcal{G} .

We wish to show that $\phi(\mathcal{G}_0)$ is trivial using the Thurston stability theorem. Let x be a point in the frontier of $\text{supp}(\mu)$ which is an accumulation point of $\text{supp}(\mu)$. There is then a unit tangent vector $v \in TM_x$ which is fixed by $D\phi(g)_x$ for all $g \in \mathcal{G}_0$. If we denote the unit sphere in the tangent space TM_x by S^1 then projectivization of $D\phi(g)_x$ gives an action of \mathcal{G}_0 on S^1 with global fixed point v . There is then a homomorphism from \mathcal{G}_0 to \mathbb{R}^+ given by mapping g to the derivative at v of the action of g on S^1 . This must be trivial by Proposition 3.3 above. Hence we may apply the Thurston stability theorem to the action of \mathcal{G}_0 on S^1 to conclude that it is trivial, i.e., that $D\phi(g)_x = I$ for all $g \in \mathcal{G}_0$. We may now apply the Thurston stability theorem to the action of \mathcal{G}_0 on S to conclude that $\phi(\mathcal{G}_0)$ is trivial. Since \mathcal{G}_0 has finite index in \mathcal{G} the result follows. \square

This result was previously known in the special case of symplectic diffeomorphisms by a result of L. Polterovich [16].

The result above also holds with $\text{supp}(\mu)$ finite if \mathcal{G} is a Kazhdan group (aka \mathcal{G} has property T). (see [11])

The fact that the hypotheses of Theorem 3.7 are satisfied by a large class of non-uniform lattices follows from the result of Lubotzky, Mozes, and

Raghunathan, Theorem 2.7, together with Theorem 3.2, the Margulis normal subgroup theorem.

An example illustrating Theorem 3.7 starts with an action on S^1 .

Example 3.8. Let \mathcal{G} be the subgroup of $PSL(2, \mathbb{Z}[\sqrt{2}])$ generated by

$$A = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

where $\lambda = \sqrt{2} + 1$. Note $\lambda^{-1} = \sqrt{2} - 1$.

These matrices satisfy

$$A^{-n}BA^n = \begin{pmatrix} 1 & \lambda^{2n} \\ 0 & 1 \end{pmatrix}$$

and

$$A^nBA^{-n} = \begin{pmatrix} 1 & \lambda^{-2n} \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that $m = \lambda^{2n} + \lambda^{-2n}$ is an integer. Hence

$$(A^{-n}BA^n)(A^nBA^{-n}) = \begin{pmatrix} 1 & \lambda^{2n} + \lambda^{-2n} \\ 0 & 1 \end{pmatrix} = B^m.$$

We have shown that $|B^m| \leq 4n + 2$ so

$$\liminf_{n \rightarrow \infty} \frac{|B^m|}{m} \leq \liminf_{n \rightarrow \infty} \frac{4n + 2}{\lambda^{2n}} = 0,$$

so B is distorted. The group \mathcal{G} acts naturally on \mathbb{RP}^1 (the lines through the origin in \mathbb{R}^2) which is diffeomorphic to S^1 . The element B has a single fixed point, the x -axis, and the only B invariant measure is supported on this point.

In example 1.6.K of [16] Polterovich considers the embedding $\psi : \mathcal{G} \rightarrow PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ where $\psi(g) = (g, \bar{g})$ with \bar{g} denoting the conjugate of g obtained by replacing an entry $a + b\sqrt{2}$ with $a - b\sqrt{2}$. He points out that the image of ψ is an irreducible non-uniform lattice in a Lie group of real rank 2. Of course $(B, \bar{B}) = (B, B)$ is a distortion element in $\psi(\mathcal{G})$ and in the product action of $PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$ on $T^2 = S^1 \times S^1$ it has only one fixed point (p, p) where p is the fixed point of B acting on S^1 . It is also

clear that the only (B, \bar{B}) invariant measure is supported on this point. It is easy to see that there are elements of $\psi(\mathcal{G})$ which do not fix this point, and hence there is no probability measure invariant under all of $\psi(\mathcal{G})$.

Under the stronger hypothesis that the group \mathcal{G} contains a subgroup isomorphic to the Heisenberg group we can remove the hypothesis that $\text{supp}(\mu)$ is infinite and allow the case that $S = S^2$.

Theorem 3.9 ([8]). *Suppose S is a closed oriented surface with Borel probability measure μ and \mathcal{G} is a finitely generated, almost simple group with a subgroup isomorphic to the Heisenberg group. Then any homomorphism*

$$\phi : \mathcal{G} \rightarrow \text{Diff}_\mu(S)$$

has finite image.

4 Parallels between $\text{Diff}(S^1)_0$ and $\text{Diff}_\mu(S)_0$

In general there seem to be strong parallels between results about $\text{Diff}(S^1)_0$ and $\text{Diff}_\mu(S)_0$. For example, Witte's theorem and our results above. There are several other examples which we now cite.

Theorem 4.1 (Hölder). *Suppose \mathcal{G} is a subgroup of $\text{Diff}(S^1)_0$ which acts freely (no non-trivial element has a fixed point). Then \mathcal{G} is abelian.*

See [5] for a proof. There is an analog of this result for dimension two. It is a corollary of the following celebrated result.

Theorem 4.2 (Arnold Conjecture: Conley-Zehnder). *Suppose ω is Lebesgue measure and*

$$f \in \text{Diff}_\omega(\mathbb{T}^2)_0$$

is in the commutator subgroup. Then f has (at least three) fixed points.

Corollary 4.3. *Suppose \mathcal{G} is a subgroup of $\text{Diff}_\omega(\mathbb{T}^2)_0$ which acts freely. Then \mathcal{G} is Abelian.*

Proof. If f is a commutator in \mathcal{G} . Then by the theorem of Conley and Zehnder it has a fixed point. Since \mathcal{G} acts freely only the identity element has fixed points. If all commutators of \mathcal{G} are the identity then \mathcal{G} is abelian. \square

Definition 4.4. A group \mathcal{N} is called nilpotent provided when we define

$$\mathcal{N}_0 = \mathcal{N}, \mathcal{N}_i = [\mathcal{N}, \mathcal{N}_{i-1}],$$

there is an $n \geq 1$ such that $\mathcal{N}_n = \{e\}$. Note if $n = 1$ it is Abelian.

Theorem 4.5 (Plante - Thurston [15]). *Let N be a nilpotent subgroup of $\text{Diff}^2(S^1)_0$. Then N must be Abelian.*

The result of Plante and Thurston requires the C^2 hypothesis as the following result shows.

Theorem 4.6 ([4]). *Every finitely-generated, torsion-free nilpotent group is isomorphic to a subgroup of $\text{Diff}^1(S^1)_0$.*

There is however an analogue of the Plante - Thurston Theorem for surface diffeomorphisms which preserve a measure.

Theorem 4.7 ([8]). *Let \mathcal{N} be a nilpotent subgroup of $\text{Diff}_\mu^1(S)_0$ with μ a probability measure with $\text{supp}(\mu) = S$. If $S \neq S^2$ then \mathcal{N} is Abelian, if $S = S^2$ then \mathcal{N} is Abelian or has an index 2 Abelian subgroup.*

Proof. We sketch the proof in the case $\text{genus}(S) > 1$. Suppose

$$\mathcal{N} = \mathcal{N}_1 \supset \cdots \supset \mathcal{N}_m \supset \{1\}$$

is the lower central series of \mathcal{N} . then \mathcal{N}_m is in the center of \mathcal{N} . If $m > 1$ there is a non-trivial $f \in \mathcal{N}_m$ and elements g, h with $f = [g, h]$. No non-trivial element of $\text{Diff}^1(S)_0$ has finite order since S has genus > 1 . So g, h generate a Heisenberg group and f is distorted. Theorem 3.6 above says $\text{supp}(\mu) \subset \text{Fix}(f)$, but $\text{supp}(\mu) = S$ so $f = id$. This is a contradiction unless $m = 1$ and \mathcal{N} is abelian. \square

5 Detecting Non-Distortion

Given a diffeomorphism which we wish to prove is not distorted there are three properties, any one of which will give us the desired conclusion. In this section we will define these properties and show they are sufficient to establish non-distortion. These properties are

- exponential growth of length of a curve
- linear displacement in the universal cover
- positive *spread*

Definition 5.1. *If the surface S is provided with a Riemannian metric a smooth closed curve $\tau \subset S$ has a well defined length $l_S(\tau)$. Define the exponential growth rate by*

$$\text{egr}(f, \tau) = \liminf_{n \rightarrow \infty} \frac{\log(l_S(f^n(\tau)))}{n}.$$

This is easily seen to be independent of the choice of metric.

Proposition 5.2. *If G is a finitely generated subgroup of $\text{Diff}(S)_0$ and $f \in G$ is distorted in G then $\text{egr}(f, \tau) = 0$ for all closed curves τ .*

Proof. Choose generators g_1, \dots, g_j of G . There exists $C > 0$ such that $\|Dg_i\| < C$ for all i . Thus $l_S(g_i(\tau)) \leq Cl_S(\tau)$ for all τ and all i . It follows that

$$\liminf_{n \rightarrow \infty} \frac{\log(l_S(f^n(\tau)))}{n} \leq \liminf_{n \rightarrow \infty} \frac{\log(l_S(\tau)) + \log(C)|f^n|}{n} = 0.$$

□

Definition 5.3. *Assume that $f \in \text{Homeo}(S)_0$ and that $S \neq S^2$. A metric d on S lifts to an equivariant metric \tilde{d} on the universal cover \tilde{S} . We say that f has linear displacement if either of the following conditions hold.*

1. $S \neq T^2$, \tilde{f} is the identity lift and there exists $\tilde{x} \in \tilde{S} = H$ such that

$$\liminf_{n \rightarrow \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{x})}{n} > 0.$$

2. $S = T^2$ and there exist \tilde{f} and $\tilde{x}_1, \tilde{x}_2 \in \tilde{S} = \mathbb{R}^2$ such that

$$\liminf_{n \rightarrow \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}_1), \tilde{f}^n(\tilde{x}_2))}{n} > 0.$$

Proposition 5.4. *If G is a finitely generated subgroup of $\text{Homeo}(S)_0$ and $f \in G$ is distorted in G then f does not have linear displacement.*

Proof. We present only the case that S has genus > 1 . For the full result see [8]. In this case the identity lifts $\{\tilde{g} : g \in G\}$ form a subgroup \tilde{G} and \tilde{f} is a distortion element in \tilde{G} . Let d be the distance function of a Riemannian metric on S and let \tilde{d} be its lift to H . For generators g_1, \dots, g_j of G there exists $C > 0$ such that $\tilde{d}(\tilde{g}_i(\tilde{x}), \tilde{x}) < C$ for all $\tilde{x} \in H$ and all i . It follows that

$$\liminf_{n \rightarrow \infty} \frac{\tilde{d}(\tilde{f}^n(\tilde{x}), \tilde{x})}{n} \leq \liminf_{n \rightarrow \infty} C \frac{|f^n|}{n} = 0.$$

□

The final ingredient we use to detect non-distortion is *spread* which we now define. The following few paragraphs are taken almost verbatim from [8].

Suppose that $f \in \text{Diff}(S)_0$, that $\gamma \subset S$ is a smoothly embedded path with distinct endpoints in $\text{Fix}(f)$ and that β is a simple closed curve that crosses γ exactly once. We want to measure the relative rate at which points move ‘across γ in the direction of β ’.

Let A be the endpoint set of γ and let M be the surface with boundary obtained from S by blowing up both points of A . We now think of γ as a path in M and of β as a simple closed curve in M . Assume at first that $S \neq S^2$ and that M is equipped with a hyperbolic structure. We denote the universal covering space of S by H and the ideal points needed to compactify it by S_∞ . Choose non-disjoint extended lifts $\tilde{\beta} \subset H \cup S_\infty$ and $\tilde{\gamma} \subset H \cup S_\infty$ and let $T : H \cup S_\infty \rightarrow H \cup S_\infty$ be the covering translation corresponding to $\tilde{\beta}$, i.e. T^\pm are the endpoints of $\tilde{\beta}$. Denote $T^i(\tilde{\gamma})$ by $\tilde{\gamma}_i$. Each $\tilde{\gamma}_i$ is an embedded path in $H \cup S_\infty$ that intersects S_∞ exactly in its endpoints. Moreover, $\tilde{\gamma}_i$ separates $\tilde{\gamma}_{i-1}$ from $\tilde{\gamma}_{i+1}$.

An embedded smooth path $\alpha \subset S$ whose interior is disjoint from A can be thought of as a path in M . For each lift $\tilde{\alpha} \subset H \cup S_\infty$, there exist $a < b$ such that $\tilde{\alpha} \cap \tilde{\gamma}_i \neq \emptyset$ if and only if $a < i < b$. Define

$$\tilde{L}_{\tilde{\beta}, \tilde{\gamma}}(\tilde{\alpha}) = \max\{0, b - a - 2\}$$

and

$$L_{\beta,\gamma}(\alpha) = \max\{\tilde{L}_{\tilde{\beta},\tilde{\gamma}}(\tilde{\alpha})\}$$

as $\tilde{\alpha}$ varies over all lifts of α .

Suppose now that $S = S^2$ and hence that M is the closed annulus. In this case M is identified with $\mathbb{R} \times [0, 1]$, $T(x, y) = (x + 1, y)$ and $\tilde{\gamma}$ is an arc with endpoints in both components of $\partial\tilde{M}$. With these modifications, $L_{\beta,\gamma}(\alpha)$ is defined as in the $S \neq S^2$ case.

There is an equivalent definition of $L_{\beta,\gamma}(\alpha)$ that does not involve covers or blowing up. Namely, $L_{\beta,\gamma}(\alpha)$ is the maximum value k for which there exist subarcs $\gamma_0 \subset \gamma$ and $\alpha_0 \subset \alpha$ such that $\gamma_0\alpha_0$ is a closed path that is freely homotopic relative to A to β^k . We allow the possibility that γ and α share one or both endpoints. The finiteness of $L_{\beta,\gamma}(\alpha)$ follows from the smoothness of the arcs α and γ .

Definition 5.5. *Define the spread of α with respect to f, β and γ to be*

$$\sigma_{f,\beta,\gamma}(\alpha) = \liminf_{n \rightarrow \infty} \frac{L_{\beta,\gamma}(f^n \circ \alpha)}{n}.$$

Note that if γ' is another smoothly embedded arc that crosses β exactly once and that has the same endpoints as γ then $\sigma_{f,\beta,\gamma}(\alpha) = \sigma_{f,\beta,\gamma'}(\alpha)$ for all α . This follows from the fact that $\tilde{\gamma}'$ is contained in the region bounded by $\tilde{\gamma}_j$ and $\tilde{\gamma}_{j+J}$ for some j and J and hence $|L_{\beta,\gamma'}(\alpha) - L_{\beta,\gamma}(\alpha)| \leq 2J$ for all α .

Proposition 5.6. *If G is a finitely generated subgroup of $\text{Diff}(S)_0$ and $f \in G$ is distorted in G then $\sigma_{f,\beta,\gamma}(\alpha) = 0$ for all α, β, γ .*

This proposition is proved via three lemmas which we now state. For proofs see [8].

Lemma 5.7. *Suppose that $g \in \text{Diff}(S)$ and that η and η' are smoothly embedded geodesic arcs in S with length at most D . There exists a constant $C(g)$, independent of η and η' such that the absolute value of the algebraic intersection number of any subsegment of $g(\eta)$ with η' is less than $C(g)$.*

Let γ be a fixed oriented geodesic arc in S with length at most D , let $A = \{x, y\}$ be its endpoint set and let M be the surface with boundary obtained from $S \setminus A$ by blowing up x and y . For each ordered pair $\{x', y'\}$ of distinct points in S choose once and for all, an oriented geodesic arc $\eta = \eta(x', y')$ of length at most D that connects x' to y' and choose $h_\eta \in \text{Diff}(S)_0$ such that

$h_\eta(\gamma) = \eta$, $h_\eta(x) = x'$, $h_\eta(y) = y'$. There is no obstruction to doing this since both γ and η are contained in disks. If $x = x'$ and $y = y'$ we choose $\eta = \gamma$ and $h_\eta = id$.

Given $g \in \text{Diff}(S)$ and an ordered pair $\{x', y'\}$ of distinct points in S , let $\eta = \eta(x', y')$, $\eta' = \eta(g(x'), g(y'))$ and note that $g_{x', y'} := h_{\eta'}^{-1} \circ g \circ h_\eta$ pointwise fixes A . The following lemma asserts that although the pairs $\{x', y'\}$ vary over a non-compact space, the elements of $\{g_{x', y'}\}$ exhibit uniform behavior from the point of view of spread.

Lemma 5.8. *With notation as above, the following hold for all $g \in \text{Diff}(S)$.*

1. *There exists a constant $C(g)$ such that*

$$L_{\beta, \gamma}(g_{x', y'}(\gamma)) \leq C(g) \text{ for all } \beta \text{ and all } x', y'.$$

2. *There exists a constant $K(g)$ such that*

$$L_{\beta, \gamma}(g_{x', y'}(\alpha)) \leq L_{\beta, \gamma}(\alpha) + K(g) \text{ for all } \beta, \text{ all } \alpha \text{ and all } x', y'.$$

Lemma 5.9. *Suppose that $g_i \in \text{Diff}(S)_0$, $1 \leq i \leq k$, that f is in the group they generate and that $|f^n|$ is the word length of f^n in the generators $\{g_i\}$. Then there is a constant $C > 0$ such that*

$$L_{\beta, \gamma}(f^n(\alpha)) \leq L_{\beta, \gamma}(\alpha) + C|f^n|$$

for all α, β, γ and all $n > 0$.

Proof of Proposition 5.6 Since f is distorted in G

$$\liminf_{n \rightarrow \infty} \frac{|f^n|}{n} = 0.$$

According to the definition of spread and Lemma 5.9 we then have

$$\sigma_{f, \beta, \gamma}(\alpha) = \liminf_{n \rightarrow \infty} \frac{L_{\beta, \gamma}(f^n(\alpha))}{n} \leq \liminf_{n \rightarrow \infty} \frac{L_{\beta, \gamma}(\alpha) + C|f^n|}{n} = 0.$$

□

6 Sketch of Theorem 3.6

The following proposition is implicit in the paper of Atkinson [1]. This proof is taken from [6] but is essentially the same as an argument in [1].

Proposition 6.1. *Suppose $T : X \rightarrow X$ is an ergodic automorphism of a probability space (X, ν) and let $\phi : X \rightarrow \mathbb{R}$ be an integrable function with $\int \phi d\nu = 0$. Let $S(n, x) = \sum_{i=0}^{n-1} \phi(T^i(x))$. Then for any $\varepsilon > 0$ the set of x such that $|S(n, x)| < \varepsilon$ for infinitely many n is a full measure subset of X .*

Proof. Let A denote the set of x such that $|S(n, x)| < \varepsilon$ for only finitely many n . We will show the assumption $\mu(A) > 0$ leads to a contradiction. Suppose $\mu(A) > 0$ and let A_m denote the subset of A such that $|S(i, x)| < \varepsilon$ for m or fewer values of i . Then $A = \cup A_m$ and there is an $N > 0$ such that $\mu(A_N) > p$ for some $p > 0$.

The ergodic theorem applied to the characteristic function of A_N implies that for almost all x and all sufficiently large n (depending on x) we have

$$\frac{\text{card}(A_N \cap \{T^i(x) \mid 0 \leq i < n\})}{n} > p.$$

We now fix an $x \in A_N$ with this property. Let $B_n = \{i \mid 0 \leq i \leq n \text{ and } T^i(x) \in A_N\}$ and $r = \text{card}(B_n)$; then $r > np$. Any interval in \mathbb{R} of length ε which contains $S(i, x)$ for some $i \in B_n$ contains at most N values of $\{S(j, x) : j > i\}$. Hence any interval of length ε contains at most N elements of $\{S(i, x) \mid i \in B_n\}$. Consequently an interval containing the r numbers $\{S(i, x) \mid i \in B_n\}$ must have length at least $r\varepsilon/N$. Since $r > np$ this length is $> np\varepsilon/N$. Therefore

$$\sup_{0 \leq i \leq n} |S(i, x)| > \frac{np\varepsilon}{2N},$$

and hence by the ergodic theorem, for almost all $x \in A_N$

$$\left| \int \phi d\mu \right| = \lim_{n \rightarrow \infty} \frac{|S(n, x)|}{n} = \limsup_{n \rightarrow \infty} \frac{|S(n, x)|}{n} > \frac{p\varepsilon}{2N} > 0.$$

This contradicts the hypothesis so our result is proved. \square

Corollary 6.2. *Suppose $T : X \rightarrow X$ is an automorphism of a Borel probability space (X, μ) and $\phi : X \rightarrow \mathbb{R}$ is an integrable function. Let $S(n, x) =$*

$\sum_{i=0}^{n-1} \phi(T^i(x))$ and suppose $\mu(P) > 0$ where $P = \{x \mid \lim_{n \rightarrow \infty} S(n, x) = \infty\}$.
Let

$$\hat{\phi}(x) = \lim_{n \rightarrow \infty} \frac{S(n, x)}{n}.$$

Then $\int_P \hat{\phi} d\mu > 0$. In particular $\hat{\phi}(x) > 0$ for a set of positive μ -measure.

Proof. By the ergodic decomposition theorem there is a measure m on the space \mathcal{M} of all T invariant ergodic Borel measures on X with the property that for any μ integrable function $\psi : X \rightarrow \mathbb{R}$ we have $\int \psi d\mu = \int_{\mathcal{M}} I(\psi, \nu) dm$ where $\nu \in \mathcal{M}$ and $I(\psi, \nu) = \int \psi d\nu$.

The set P is T invariant. Replacing $\phi(x)$ with $\phi(x)\mathfrak{X}_P(x)$, where \mathfrak{X}_P is the characteristic function of P , we may assume that ϕ vanishes outside P . Then clearly $\hat{\phi}(x) \geq 0$ for all x for which it exists. Let \mathcal{M}_P denote $\{\nu \in \mathcal{M} \mid \nu(P) > 0\}$. If $\nu \in \mathcal{M}_P$ the fact that $\hat{\phi}(x) \geq 0$ and the ergodic theorem imply that $I(\phi, \nu) = \int \phi d\nu = \int \hat{\phi} d\nu \geq 0$. Also Proposition 6.1 implies that $\int \phi d\nu = 0$ is impossible so $I(\phi, \nu) > 0$. Then $\mu(P) = \int I(\mathfrak{X}_P, \nu) dm = \int \nu(P) dm = \int_{\mathcal{M}_P} \nu(P) dm$. This implies $m(\mathcal{M}_P) > 0$ since $\mu(P) > 0$.

Hence

$$\int \hat{\phi} d\mu = \int \phi d\mu = \int I(\phi, \nu) dm \geq \int_{\mathcal{M}_P} I(\phi, \nu) dm > 0$$

since $I(\phi, \nu) > 0$ for $\nu \in \mathcal{M}_P$ and $m(\mathcal{M}_P) > 0$. □

Outline of the proof of Theorem 3.6

We must show that if $f \in \text{Diff}_\mu(S)_0$ has infinite order and $\mu(S \setminus \text{Fix}(f)) > 0$ then f is not distorted. In light of the results of the previous section this will follow from the following proposition.

Proposition 6.3. *If $f \in \text{Diff}_\mu(S)_0$ has infinite order and $\mu(S \setminus \text{Fix}(f)) > 0$ then one of the following holds:*

1. *There exists a closed curve τ such that $\text{egr}(f, \tau) > 0$.*
2. *f has linear displacement.*
3. *After replacing f with some iterate $g = f^k$ and perhaps passing to a two-fold covering $g : S \rightarrow S$ is isotopic to the identity and there exist α, β, γ such that the spread $\sigma_{f, \beta, \gamma}(\alpha) > 0$.*

The idea of the proof of this proposition is to first ask if f is isotopic to the identity relative to $\text{Fix}(f)$. If not there is a finite set $P \subset \text{Fix}(f)$ such that f is not isotopic to the identity relative to P . We then consider the Thurston canonical form of f relative to P . If there is pseudo-Anosov component the property (1) holds. If there are no pseudo-Anosov components then there must be non-trivial Dehn twists in the Thurston canonical form. In this case it can be shown that either (2) or (3) holds. For details see [8]

We are left with the case that f is isotopic to the identity relative to $\text{Fix}(f)$. There are several subcases. It may be that S has negative Euler characteristic and the identity lift \tilde{f} has a point with non-zero rotation vector in which case (2) holds. It may be that $S = T^2$ and there is a lift \tilde{f} with a fixed point and a point with non-zero rotation vector in which case (2) again holds.

The remaining cases involve $M = S \setminus \text{Fix}(f)$. A result of Brown and Kister [2] implies that each component of M is invariant under f . If M has a component which is an annulus and which has positive measure then there is a positive measure set in the universal cover of this component which goes to infinity in one direction or the other. In this case Corollary 6.2, with ϕ the displacement by \tilde{f} in the covering space, implies there are points with non-zero rotation number. Since points on the boundary of the annulus have zero rotation number we can conclude that (3) holds.

The remaining case is that there is a component of M with positive measure and negative Euler characteristic (we allow infinitely many punctures). In this case it can be shown that there is a simple closed geodesic and a set of positive measure whose lift in the universal cover of this component tends asymptotically to an end of the simple closed geodesic. An argument similar to the annular case then shows that (3) holds.

More details can be found in [8] including the fact that these cases exhaust all possibilities.

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