

# Spectral singularities in $\mathcal{PT}$ -symmetric periodic finite-gap systems

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## Abstract

The origin of spectral singularities in finite-gap singly periodic  $\mathcal{PT}$ -symmetric quantum systems is investigated. We show that they emerge from a limit of band-edge states in a doubly periodic finite gap system when the imaginary period tends to infinity. In this limit, the energy gaps are contracted and disappear, every pair of band states of the same periodicity at the edges of a gap coalesces and transforms into a singlet state in the continuum. As a result, these spectral singularities turn out to be analogous to those in the non-periodic systems, where they appear as zero-width resonances. Under the change of topology from a non-compact into a compact one, spectral singularities in the class of periodic systems we study are transformed into exceptional points. The specific degeneration related to the presence of finite number of spectral singularities and exceptional points is shown to be coherently reflected by a hidden, bosonized nonlinear supersymmetry.

## 1 Introduction

The discovery of complex Hamiltonians with the combined space reflection and time reversal ( $\mathcal{PT}$ ) symmetry, which have a real spectrum [1], opened a new branch of quantum mechanics [2]. Recently, systems with  $\mathcal{PT}$ -symmetry have gained a lot of attention motivated by a possibility of its experimental observation in nature, particularly, in optical systems [3]. The ideas of  $\mathcal{PT}$ -symmetry have also been applied to different areas, including quantum field theory [4], gravitation [5] and relativistic quantum mechanics [6], among others.

An interesting peculiarity of non-Hermitian Hamiltonians is related with the presence of exceptional points [7] and spectral singularities [8] in their spectra. Exceptional points are particular states in the *discrete* spectrum of an operator where two eigenvectors of different energies coalesce to form a unique state. These states were studied in several contexts, see, e.g., [9, 10, 11, 12, 13, 14, 15] and references therein. Spectral singularities have a nature similar to that of exceptional points, but within the *continuous* spectrum of a non-Hermitian operator [16, 17, 18]. The energy values of the spectral singularities appear as poles in the resolvent of an operator, as zeros of a Wronskian of the Jost solutions, as well as divergences

in the scattering matrix. In 2009, Mostafazadeh showed that spectral singularities in *non-periodic* complex potentials appear as zero-width resonances [19]. Although the implications and applications of spectral singularities have been analyzed in several directions [20, 21, 22, 23, 24, 25, 26, 27, 28, 29], their meaning in complex *periodic* potentials remains, however, unknown.

In this article we study a certain class of  $\mathcal{PT}$ -symmetric complex potentials in which a finite number of spectral singularities does appear. Their origin is explained by analyzing a related, more general family of doubly periodic quantum models which belong to a class of finite-gap systems [30]. When the imaginary period (that can be treated as a hidden imaginary parameter of the potential) tends to infinity, energy gaps shrink and disappear, while the pairs of singlet band states of the same periodicity at the edges of each gap coalesce and produce singlet states inside the doubly degenerate continuum. These turn out to be the spectral singularities. This peculiar phenomenon is shown to be characteristic for complex potentials, there exists no analog for finite-gap real potentials. We show that the appearance of finite number of spectral singularities in the indicated class of non-Hermitian systems can be associated with a presence of a hidden, bosonized non-linear supersymmetry [31]. A compactification of the system, by imposing the appropriate periodicity condition for the wave functions, discretizes the spectrum and transforms spectral singularities into exceptional points. This provides a unified explanation for spectral singularities and exceptional points for certain class of related  $\mathcal{PT}$ -symmetric systems.

The paper is organized as follows. In section 2 we construct a family of periodic complex potentials with spectral singularities by applying Crum-Darboux transformations to a free particle. Section 3 is devoted to the description of spectral singularities and exceptional points in the spectra of the obtained related systems with non-compact and compact topologies, respectively. The origin of the same spectral singularities from a specific limit of finite-gap systems is explained in section 4. In section 5 we show that a hidden supersymmetry is associated with the presence of finite number of spectral singularities and exceptional points. Discussion is presented in section 6 .

## 2 Free particle and complex Darboux transformations

Let us consider a free particle on the real line <sup>1</sup>  $-\infty < x < \infty$ ,

$$H = -\frac{d^2}{dx^2}. \quad (2.1)$$

The spectrum of the system is continuous and is described by the states

$$\psi^\pm(x) = e^{\pm ikx}, \quad k \geq 0. \quad (2.2)$$

For  $k > 0$ , (2.2) are plane waves for doubly degenerate energy levels with  $E_k = k^2 > 0$ . A singlet state  $\psi = 1$  corresponds to  $E_0 = 0$  at the bottom of the spectrum. From the system (2.1) one can construct complex periodic Hamiltonians with finite number of spectral singularities by employing Crum-Darboux transformations. The procedure is analogous to

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<sup>1</sup>We work in the units  $\hbar = 2m = 1$ .

the construction of reflectionless potentials with a finite number of bound states [32]. To define such a transformation, we introduce a complex operator

$$\mathcal{D}_{\alpha,\beta} = \frac{d}{dx} + \alpha \tan(x + i\delta) - \beta \cot(x + i\delta), \quad (2.3)$$

where  $\delta$  is a real parameter,  $0 < \delta < \pi/2$ , and construct a higher order differential operator

$$\mathcal{F}_{r,s} = \prod_{j=1}^r \mathcal{D}_{j,u}, \quad u = u(r, s; j) = \begin{cases} j - r + s & \text{if } j - r + s > 0, \\ 0 & \text{if } j - r + s \leq 0. \end{cases} \quad (2.4)$$

The upper index of the ordered product corresponds here to the first term on the left side while the lower index denotes the last term on the right side of the product. The parameters  $r$  and  $s$  take here integer values, and without any loss of generality we can assume that  $r > s$  (see next Section). Operator (2.4) intertwines the free particle Hamiltonian  $H$  with those of nontrivial systems described by Hamiltonians

$$H_{r,s} = -\frac{d^2}{dx^2} + \frac{r(r+1)}{\cos^2(x+i\delta)} + \frac{s(s+1)}{\sin^2(x+i\delta)}, \quad (2.5)$$

$$\mathcal{F}_{r,s} H = H_{r,s} \mathcal{F}_{r,s}, \quad (2.6)$$

where the free particle system  $H$  corresponds to the zero values of the parameters,  $H_{0,0} = H$ .

The nature of the continuous spectrum of  $H_{0,0}$  can be modified by changing the topology of the quantum problem. This is achieved by compactifying the coordinate,  $-\infty < x < +\infty \rightarrow 0 \leq x < 2\pi$ , via the introduction of the periodicity condition,

$$\psi^\pm(x + 2\pi) = \psi^\pm(x). \quad (2.7)$$

This condition is satisfied provided  $k \in \mathbb{Z}$  in (2.2), that transforms the energy spectrum of (2.1) into the discrete one,

$$E_\ell = \ell^2, \quad \ell = 0, \pm 1, \pm 2, \pm 3, \dots \quad (2.8)$$

In this case we have an infinite set of discrete doubly degenerate positive energy levels, while the ground state with  $\ell = 0$  ( $E_0 = 0$ ) is non-degenerate. Fig. 1 illustrates the spectrum in both cases, non-compact and compact ones.

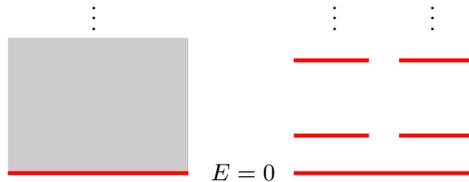


Figure 1: Spectrum of the free particle in the real line (left) and in the compactified case (right).

In the next section we discuss general properties of the Hamiltonian (2.5) and the description of spectral singularities, where the compactification scheme will be useful to understand

the relation with exceptional points in the case of periodic systems. It is worth to note that in the compactified case, the length of the space,  $2\pi$ , is twice the period of (2.5). This means that the compactification condition (2.7) comprises (unifies) both the periodic and anti-periodic Sturm-Liouville problems,  $\psi(0) = \pm\psi(\pi)$ ,  $\psi'(0) = \pm\psi'(\pi)$ , for the periodic non-Hermitian system (2.5). The same job is made, however, just by imposing the periodicity condition  $\psi(0) = \psi(2\pi)$ .

### 3 Spectral singularities of periodic $\mathcal{PT}$ -symmetric potentials

Periodic potentials of complex nature [33, 34, 35, 36, 37, 38, 39, 40] are of interest, particularly, in the context of optics and matter waves physics, see for example refs. in [41]. Hamiltonian (2.5) provides a good example of such a class of potentials, for which the properties of non-Hermitian systems can be analyzed using the advantage of exactly solvable systems.

The potential in (2.5) has a real period  $T = \pi$ , and can be treated as the complex version of the generalized trigonometric Pöschl-Teller potential. Although the potential is just a complex shift in the coordinate of the Hermitian case, its spectral properties are essentially different. Indeed, when  $\delta = 0$ , the potential in (2.5) is no longer complex, and has singularities located at  $x = l\pi$  and  $x = (l + \frac{1}{2})\pi$ , where  $l \in \mathbb{Z}$ . Consequently, the quantum interpretation in the real case is quite different in comparison with the complex one. The presence of singularities implies that the particle is confined in the region between the adjacent singularities, without tunneling through them. Therefore, one ought to impose the boundary conditions by requiring that the wave functions vanish at singular points. This results in an infinite number of bound states which correspond to non-degenerate energy levels. The picture is the same as in the case of the infinite square well potential. Moreover, both systems are related by the Crum-Darboux transformations (2.4) when  $\delta = 0$ , remembering that the infinite square well potential problem is no more than the free particle system subjected to the specific boundary conditions. In the Hermitian case, the quantum interpretation of a particle with the compactified coordinate requires also the necessary boundary conditions of vanishing of the wave functions at the singular points of the potentials.

For  $\delta \neq 0$  the situation radically changes. The singularities on the real line disappear, and the simultaneous action of the parity  $\mathcal{P}$ ,  $\mathcal{P}x\mathcal{P} = -x$ , and time reversal  $\mathcal{T}$ ,  $\mathcal{T}i\mathcal{T} = -i$ , operators leaves the Hamiltonian invariant. Before explaining the spectral properties of the family of systems (2.5), we review some general aspects of the Hamiltonian  $H_{r,s}$ .

The shift  $x \rightarrow x + \pi/2$  in the Hamiltonian (2.5) interchanges the parameters  $r$  and  $s$ ,  $H_{r,s}(x + \pi/2) = H_{s,r}(x)$ , without affecting the spectral characteristics of the system. Since the systems  $H_{r,s}$  and  $H_{s,r}$  can also be related by Crum-Darboux transformations constructed in terms of operators of the form (2.3), we say that they are self-isospectral [42].

This fact, in addition to the symmetries  $r \rightarrow -r - 1$  and  $s \rightarrow -s - 1$ , tells us that we can consider just non-negative integer values of the parameters. The case when  $r = s = l$  is

a particular case which is presented equivalently as

$$H_{l,l}(x + \pi/4) = 4H_{l,0}(\chi) = 4 \left( -\frac{d^2}{d\chi^2} + \frac{l(l+1)}{\cos^2(\chi + 2i\delta)} \right), \quad (3.1)$$

where  $\chi = 2x$ . Hence, the spectrum of the Hamiltonian (2.5) with  $r = s$  is given by rescaling of that for the case with  $s = 0$ , and then it is indeed sufficient to consider the cases  $r > s$ .

Eq. (2.6) represents the intertwining relation between the free Hamiltonian  $H_{0,0}$  and  $H_{r,s}$ . The eigenfunctions of the latter are given by application of the operator (2.4) to the plane waves (2.2),

$$\psi_{r,s}^{\pm k} = \mathcal{F}_{r,s} e^{\pm ikx}. \quad (3.2)$$

The spectrum of  $H_{r,s}$  is composed by a continuum of doubly degenerate states with energies  $E_k = k^2$ , except for the  $r + 1$  states

$$\psi_{r,s}^n = \mathcal{F}_{r,s} e^{i[n+u(r,s;n)]x}, \quad n = 0, 1, \dots, r, \quad (3.3)$$

which are singlets. The discrete parameter  $u(r, s; n)$  is defined here in the same way as in equation (2.4). The energy values of the singlet states (3.3) are

$$E_{r,s;n} = (n + u(r, s; n))^2, \quad n = 0, 1, \dots, r. \quad (3.4)$$

Clearly, the singlet states of the form (3.3) correspond to the states in (3.2) taken for the particular values of  $k$ . Two states obtained by the application of  $\mathcal{F}_{r,s}$  to the left and right moving plane waves with these special values of  $k \neq 0$  coincide modulo a constant factor. At the same time, the application of the Crum-Darboux generator  $\mathcal{F}_{r,s}$  to the singlet state  $\psi_0$  of the free particle produces a nontrivial singlet ground state of the system  $H_{r,s}$ . We shall return to this point below in the discussion of spectral singularities in terms of the Wronskian of the solutions.

The picture can be understood alternatively by observing that the operator  $\mathcal{F}_{r,s}$  annihilates  $r$  states of the free particle, which are complex linear combinations of the plane waves. Namely,

$$\mathcal{F}_{r,s} \cos \gamma(x + i\delta) = 0, \quad \gamma = n + u(r, s; n) = \text{odd}, \quad (3.5)$$

$$\mathcal{F}_{r,s} \sin \gamma(x + i\delta) = 0, \quad \gamma = n + u(r, s; n) = \text{even}, \quad (3.6)$$

with  $n = 1, 2, \dots, r$  and  $u(r, s; n)$  as defined above. Here the resulting  $r$  singlet states different from zero in the above relations correspond to spectral singularities, they are located inside the continuous spectrum. An additional singlet ground state ( $E_0 = 0$ ) at the bottom of the spectrum has a different nature with respect to the spectral singularities. States of this kind, being a Crum-Darboux-transformed ground state of the free particle, also appear in the Hermitian and non-Hermitian reflectionless potentials [43, 44]. In the next section the difference between the non-zero and zero energy singlet states of the Hamiltonian (2.5) will be clarified by applying a specific limit to doubly-periodic finite-gap systems.

Performing a compactification in the coordinate as in the free particle case, the eigenfunctions are determined by the condition (2.7) imposed on the states (3.2),

$$\psi_{r,s}^{\pm m} = \mathcal{F}_{r,s} e^{\pm imx}, \quad m = 0, 1, 2, \dots \quad (3.7)$$

The solutions (3.7) form an infinite discrete set of wave eigenfunctions for doubly degenerate energy levels, except the states (3.3), which are singlets. Again, we have  $r + 1$  singlet states, one of which is the ground state of zero energy while the rest are exceptional points.

This example, provided by the compactified  $\mathcal{PT}$ -symmetric systems, reveals a subtlety related to the definition of spectral singularities and exceptional points in a periodic case. In such class of the systems, they both have the same origin since the spectral singularities transform into exceptional points just by changing the topology of the quantum problem.

It is useful to look at the peculiarity of these states from the viewpoint of the Wronskian. For the physical wavefunctions (3.2) and (3.7) we have

$$W[\psi_{r,s}^{+k}, \psi_{r,s}^{-k}] = -2ik \prod_{n=1}^r (k^2 - E_{r,s;n}), \quad (3.8)$$

where for the case of the states (3.7), the  $k$  should be replaced by  $m$ . Being independent of  $x$ , the Wronskian vanishes at the energies of the spectral singularities (3.4) as well as at the lowest energy  $E = 0$  (which can be treated as a trivial zero of  $W$  corresponding to the case of the coinciding arguments  $\psi_{r,s}^{+0} = \psi_{r,s}^{-0}$ ). This reflects linear dependence of the pairs of states in (3.2) and (3.7). The second, linear independent solutions of the stationary Schrödinger equation for those energy values (including  $E = 0$ ) are not periodic functions (being not bounded in the non-compact topology case), and they do not belong to the physical spectrum of  $H_{r,s}$ . In ref. [38], Samsonov and Roy observed a similar phenomenon in the Wronskian for the Hamiltonian  $H_{1,0}(x + \zeta)$ , where  $\zeta$  is a complex parameter. In their analysis, however, they imposed the boundary conditions  $\psi(\pi) = \psi(-\pi) = 0$ , which eliminate the existence of spectral singularities in the spectrum, producing an infinite number of discrete singlet states; this can be compared with the Hermitian case  $\delta = 0$  we discussed at the beginning of the section. This provides a further example of the importance of the topology in Hermitian and non-Hermitian Hamiltonians.

The differences in the spectrum for the discussed family of periodic systems with compact and non-compact topologies, and the comparison with the free particle are illustrated by Fig. 2 for the cases  $r = 2, s = 0$  and  $r = 2, s = 1$ .

In this section we explained the appearance of spectral singularities in the spectrum of  $H_{r,s}$  by exploiting their relation with the free particle system by means of Crum-Darboux transformations. The non-degenerate nature of the spectral singularities can be understood taking into account a peculiar feature of the action of Crum-Darboux generators. The operator (2.4) annihilates  $r$  states in the spectrum of  $H_{0,0}$ , see the relations (3.5) and (3.6). Hence, one naturally may expect the presence of singlet states, (3.3), in the spectrum of the intertwined Hamiltonian  $H_{r,s}$ , whose energies coincide with the energy values of the annihilated states of the free Hamiltonian. In other words, we say that the operators (2.4) remove every time one state located at the level  $E_{r,s;n} \neq 0$  from a doublet in the free particle spectrum, generating a spectral singularity in the spectra of Hamiltonians (2.5) at the same energy  $E_{r,s;n} \neq 0$ .

In next sections we will see that the origin of the singlet states, in particular the spectral singularities, and the specific degeneracy in the spectrum of  $H_{r,s}$  have a remarkable interpretation from the point of view of finite-gap potentials and a hidden bosonized nonlinear supersymmetry.

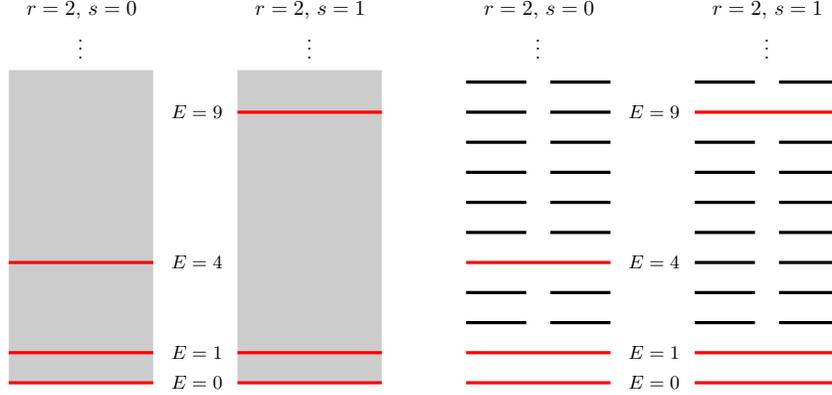


Figure 2: The spectra for  $r = 2, s = 0$  and  $r = 2, s = 1$  for the Hamiltonians with the non-compact (left) and compact (right) coordinate. The spectrum in the former case is a half-bounded infinite continuum, without spectral singularities above  $E = 4$  for  $r = 2, s = 0$  and  $E = 9$  for  $r = 2, s = 1$ , while in the latter case there is an infinite number of degenerate discrete states above those values corresponding to exceptional points.

## 4 Darboux-Treibich-Verdier potentials and the origin of spectral singularities

Generically, for a quantum periodic system free of singularities the spectrum is composed by an infinite number of bands and gaps. By the oscillation theorem [45], the number of nodes of the band-edge states within the period interval of the potential increases when energy increases. In the case of analytical potentials the width of the gaps decreases exponentially with increasing of the energy. There exists, however, an important class of the systems for which the number of bands and gaps is finite; the corresponding potentials are known as finite-gap. One example of a regular Hermitian finite-gap potential is provided by the family of associated Lamé potentials,

$$V_{r,s}^{AL}(x) = s(s+1)k^2\text{sn}^2x + r(r+1)k^2\text{sn}^2(x+K) \quad (4.1)$$

$$= s(s+1)k^2\text{sn}^2x + r(r+1)\frac{k^2\text{cn}^2x}{\text{dn}^2x}. \quad (4.2)$$

The potential is expressed in terms of the doubly periodic Jacobi elliptic functions  $^2\text{sn}(x, k)$ ,  $\text{cn}(x, k)$  and  $\text{dn}(x, k)$ . Hereafter we will not display the dependence on the modular parameter  $0 < k < 1$  in them. The potential (4.1) has a real,  $2K$ , and an imaginary,  $2iK'$ , periods, where  $K = K(k)$  is the elliptic complete integral of the first kind and  $K' = K(k')$ ,  $k' = \sqrt{1-k^2}$  [46]. The finite-gap nature of the potentials happens when both parameters  $r$  and  $s$  take integer values. Specifically, if we take  $r > s$ , the spectrum has exactly  $r$  gaps (the case  $r = s$  reduces to the  $s = 0$  case with a real period  $K$ , see [47]). This property is correlated with the fact that the potentials (4.1) satisfy the non-linear stationary equation of order  $r$  of the Korteweg de-Vries hierarchy.

<sup>2</sup>In (4.2), the properties of the Jacobi elliptic functions under the real half-period displacement were used, see [46].

When the modular parameter tends to the limit values, we obtain two systems with essentially different spectra. In the limit  $k \rightarrow 0$ , when the imaginary period turns into infinity, any system with a finite number of gaps is reduced to the free particle. The gaps between all the allowed bands disappear, and two states of the same periodicity and number of nodes at the edges of a gap transform into two different states of the same energy; these correspond to sine and cosine combinations of the plane-wave states of the free particle. In the other limit  $k \rightarrow 1$ , when the real period tends to infinity, the system is transformed into the hyperbolic reflectionless Pöschl-Teller potential, with finite number of bound states equals to the number of gaps. The valence bands shrink, and each pair of band-edge states of the same valence band coalesces forming a unique bound state. Both described situations happen in a generic case of Hermitian finite-gap potentials. In the case of complex potentials, the picture radically changes, and leads to the origin of spectral singularities.

A regular complex finite-gap potential can be obtained by a complex displacement of the coordinate in (4.1). Performing the shift in the half of the imaginary period plus for an imaginary constant  $i\delta$ ,  $0 < \delta < K'$ ,  $x \rightarrow x + iK' + i\delta$ , the potential becomes

$$V_{r,s}^{ALS}(x) \equiv V_{r,s}^{AL}(x + iK' + i\delta) = r(r+1) \frac{\operatorname{dn}^2(x + i\delta)}{\operatorname{cn}^2(x + i\delta)} + s(s+1) \frac{1}{\operatorname{sn}^2(x + i\delta)}. \quad (4.3)$$

This potential is  $\mathcal{PT}$ -symmetric and the Hamiltonian  $H_{r,s}^{ALS} = -\frac{d^2}{dx^2} + V_{r,s}^{ALS}$  is a doubly periodic generalization of (2.5). When  $\delta = 0$ , the sum of (4.1) and (4.3) gives rise to the well-known family of Darboux-Treibich-Verdier potentials [48],

$$V^{DTV} = V_{r',s'}^{AL}(x) + V_{r,s}^{ALS}(x). \quad (4.4)$$

The limit cases of the modular parameter give us the systems with a single, real or pure imaginary, period,

$$H_{r,s}^{ALS} \xrightarrow{k \rightarrow 0} -\frac{d^2}{dx^2} + \frac{r(r+1)}{\cos^2(x + i\delta)} + \frac{s(s+1)}{\sin^2(x + i\delta)}, \quad (4.5)$$

$$H_{r,s}^{ALS} \xrightarrow{k \rightarrow 1} -\frac{d^2}{dx^2} + \frac{s(s+1)}{\tanh^2(x + i\delta)} + r(r+1). \quad (4.6)$$

First we discuss the spectral properties of complex finite-gap potentials with the non-compact topology.

The limit  $k \rightarrow 1$  produces, in analogy with the Hermitian case, complex reflectionless Hamiltonians (with a single imaginary period) that have bound states in their spectra [44, 49].

Surprisingly, another limit gives us something not noticed before in the literature. When  $k \rightarrow 0$ , the gaps shrink and disappear, and the band edge states at the edges of the same energy gap coalesce into one singly periodic state producing a spectral singularity.

As an example of this peculiar situation, let us consider the case  $r = 2, s = 0$ , which corresponds to a 2-gap system. The band-edge states are given by

$$\psi_0 = 1 + k^2 + \sqrt{1 - k^2 + k^4} - \frac{3}{\operatorname{sn}^2(x + i\delta)}, \quad (4.7)$$

$$\psi_1 = \frac{\operatorname{cn}(x + i\delta)\operatorname{dn}(x + i\delta)}{\operatorname{sn}^2(x + i\delta)}, \quad \psi_2 = \frac{\operatorname{cn}(x + i\delta)}{\operatorname{sn}^2(x + i\delta)}, \quad \psi_3 = \frac{\operatorname{dn}(x + i\delta)}{\operatorname{sn}^2(x + i\delta)}, \quad (4.8)$$

$$\psi_4 = 1 + k^2 - \sqrt{1 - k^2 + k^4} - \frac{3}{\operatorname{sn}^2(x + i\delta)}, \quad (4.9)$$

and their energies are

$$E_0 = 2 \left( 1 + k^2 - \sqrt{1 - k^2 + k^4} \right), \quad (4.10)$$

$$E_1 = 1 + k, \quad E_2 = 1 + 4k, \quad E_3 = 4 + k, \quad (4.11)$$

$$E_4 = 2 \left( 1 + k^2 + \sqrt{1 - k^2 + k^4} \right). \quad (4.12)$$

In the limit  $k = 1$ , the first valence band between  $E_0$  and  $E_1$  disappears, edge state  $\psi_0 = 1/\sinh^2(x + i\delta)$  coincides up to a multiplicative constant with  $\psi_1$ , and forms a bound state. The same happens with the second valence band between  $E_2$  and  $E_3$ , and for  $\psi_2 = \psi_3 = \cosh(x + i\delta)/\sinh^2(x + i\delta)$ . Quasi-periodic Bloch states of the conduction band are transformed into scattering states of the continuous spectrum, at the bottom of which is located the state  $\psi_4 = 1 - 3/\tanh^2(x + i\delta)$ . The described picture is typical for Hermitian finite-gap potentials in the real infinite period limit.

Instead, in the limit when the complex period tends to infinity,  $k = 0$ , the state  $\psi_0 = 2 - 3/\sin^2(x + i\delta)$  is located at the bottom of the continuum. The remaining pairs of band edge states,  $\psi_1 = \psi_2 = \cos(x + i\delta)/\sin^2(x + i\delta)$  and  $\psi_3 = -3\psi_4 = 1/\sin^2(x + i\delta)$ , coincide in the form of spectral singularities. In this limit, the quasi-periodic states inside the valence and conduction bands transform into periodic states, whose period depends on energy. In the case with the compact coordinate, the singlet states transform in the same way discussed above, while the condition (2.7) selects the periodic states of a fixed period  $2\pi$  to be the double period of the resulting potential.

Fig. 3 illustrates this example, showing how the energies of the band-edge states are transformed into corresponding singlets, particularly, into spectral singularities when  $k = 0$ . It is worth to note that periodic  $\mathcal{PT}$ -symmetric potentials of the Darboux-Treibich-Verdier family were treated before in ref. [50], but the existence of the spectral singularities in the  $k = 0$  limit was not noticed.

## 5 Hidden supersymmetry and spectral singularities

The Hamiltonians  $H_{r,s}$  display a finite number of spectral singularities which appear as singlet energy levels immersed into doubly degenerate continuous spectrum in the non-compact topology case. The same happens with exceptional points in the discrete spectrum of the compactified systems. These features, the presence of several singlet states and double degeneration of the rest of energy levels, are typical for a hidden, bosonized (due to the absence of the spin degrees of freedom) non-linear supersymmetry [31]. This kind of symmetry was observed earlier in the reflectionless systems with real [43] and complex [44] potentials, as well as in Hermitian periodic finite-gap [51, 47] systems. One can expect therefore that such a peculiar supersymmetry can also be associated with the spectral singularities and exceptional points in the described class of the  $\mathcal{PT}$ -symmetric systems.

Indeed, from the viewpoint of the limit of finite-gap potentials (4.3) discussed above, one knows that each potential  $V_{r,s}^{ALS}$  satisfies a corresponding stationary non-linear equation of the Korteweg-de Vries hierarchy. This fact implies the existence of a nontrivial integral of

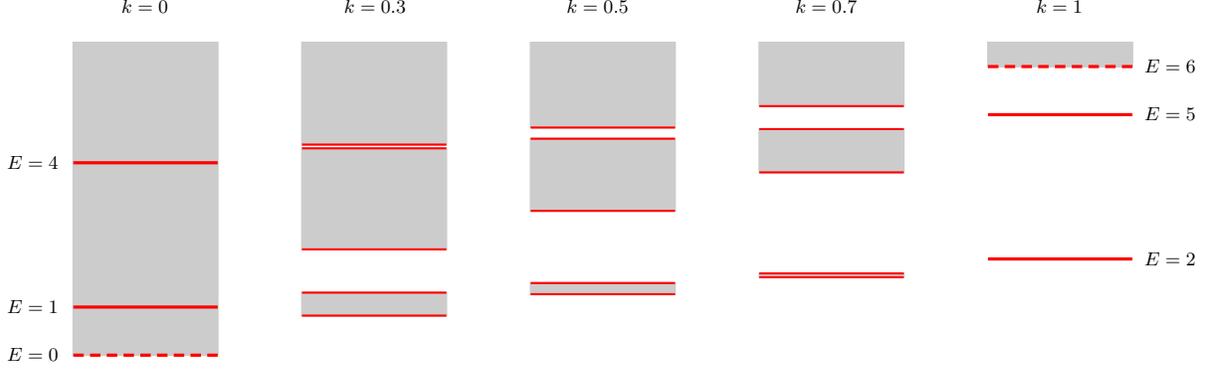


Figure 3: Spectra of the 2-gap system  $H_{2,0}^{ALS}$  are shown for the values of the modular parameter  $k = 0, 0.3, 0.5, 0.7, 1$ . In the limit when the imaginary period tends to infinity,  $k = 0$ , the system  $H_{2,0}$  is recovered. The spectral singularities (solid lines) appear at energies  $E = 1$  and  $E = 4$  when two band-edge states with the same periodicity, separated by a gap, coalesce, and the gap between them vanishes. In the other limit  $k = 1$ , a system with a pure imaginary periodicity is obtained; two bound states (solid lines) appear at energies  $E = 2$  and  $E = 5$ . The singlet state at the bottom of the continuous spectrum (dashed line) at energy  $E = 6$  ( $E = 0$ ) corresponds to the limit  $k = 1$  ( $k = 0$ ); it has a nature of the eigenfunction obtained by the application of the Crum-Darboux operator (2.4) to the free particle state  $\psi = 1$ . The system in the  $k = 1$  limit corresponds to the complexified Scarf II potential [44, 49].

motion  $\mathbb{A}_{2r+1}$  of differential order  $2r + 1$ , which underlies the non-linear nature of the hidden supersymmetry. Together with the Hamiltonian  $H_{r,s}$ , the operator  $\mathbb{A}_{2r+1}$  composes the Lax pair [30, 52],

$$[\mathbb{A}_{2r+1}, H_{r,s}] = 0, \quad -\mathbb{A}_{2r+1}^2 = P(H_{r,s}), \quad (5.1)$$

where  $P(H_{r,s})$  is an order  $2r + 1$  (spectral) polynomial in the Hamiltonian. Using the definition of the operator (2.3), one can identify the operator  $\mathbb{A}_{2r+1}$  as follows,

$$\mathbb{A}_{2r+1} = \mathcal{F}_{r,s} \frac{d}{dx} \mathcal{F}_{r,s}^T, \quad (5.2)$$

that is nothing else as a Crum-Darboux dressed free particle integral  $\frac{d}{dx}$  [53]. The transposition  $T$ , substituting here Hermitian conjugation in the case of a Hermitian Hamiltonian, is defined by inversion of the order of first order operator multipliers in Eq. (2.4) accompanied by the change  $\frac{d}{dx} \rightarrow -\frac{d}{dx}$ .

The remarkable property of the class of the systems described by  $H_{r,s}$  is related with the physical sense of the operator (5.2), where the  $\mathcal{PT}$ -symmetry plays a key role. In the Hermitian limit of the system when  $\delta = 0$ , the action of the operator  $\mathbb{A}_{2r+1}$  on physical states is ill-defined. As was noted in Section 3, when  $\delta$  vanishes, the potential in  $H_{r,s}$  becomes real, having singularities in the real line. As a result, the appropriate quantum treatment is such that the wave functions vanish at singular points, and an infinite number of the discrete singlet bound states appears. Though the higher order Lax operator is still commuting with Hermitian Hamiltonian, its action on physical (bound) states produces non-physical states,

$$\delta = 0 \quad \longrightarrow \quad \mathbb{A}_{2r+1} \psi_{\text{physical}} = \psi_{\text{non-physical}}. \quad (5.3)$$

Such a situation takes place also in the conformal mechanics model given by the inverse square potential  $n(n+1)x^{-2}$  [54], that corresponds to a rational limit (when both, real and imaginary periods tend to infinity) of finite-gap Hermitian periodic systems. However, for  $\delta \neq 0$  the  $\mathcal{PT}$ -symmetry provides the ‘‘cure’’ for the operator  $\mathbb{A}_{2r+1}$ : the states (3.2) and (3.7) are eigenstates of this operator,

$$\mathbb{A}_{2r+1}\psi_{r,s}^{\pm k} = \pm ik \prod_{n=1}^r (k^2 - E_{r,s;n})\psi_{r,s}^{\pm k}, \quad \mathbb{A}_{2r+1}\psi_{r,s}^{\pm m} = \pm im \prod_{n=1}^r (m^2 - E_{r,s;n})\psi_{r,s}^{\pm m}. \quad (5.4)$$

Particularly, the non-linear operator (5.2) annihilates all the singlet states. These are spectral singularities (exceptional points) and the state in the bottom of the spectrum in the systems with the non-compact (compact) topology. This can be seen also taking the square of  $\mathbb{A}_{2r+1}$ ,

$$-\mathbb{A}_{2r+1}^2 = H_{r,s} \prod_{n=1}^r (H_{r,s} - E_{r,s;n})^2, \quad (5.5)$$

where the roots of the operator-valued polynomial are the energies of the singlet states. In this sense we can say that the  $\mathcal{PT}$ -symmetry restores the physical meaning of  $\mathbb{A}_{2r+1}$ , which in the Hermitian case,  $\delta = 0$ , was broken [55].

The supersymmetric structure can be revealed by identifying the supercharges as follows,

$$Q_1 = i\mathbb{A}_{2r+1}, \quad Q_2 = i\Gamma Q_1, \quad (5.6)$$

where  $\Gamma$  is a  $\mathbb{Z}_2$ -grading operator,

$$\Gamma = \mathcal{P}e^{-2i\delta \frac{d}{dx}}, \quad [\Gamma, H_{r,s}] = 0, \quad \{\Gamma, Q_a\} = 0, \quad \Gamma^2 = 1. \quad (5.7)$$

Integral  $\Gamma$ , being  $\mathcal{PT}$ -symmetric,  $[\mathcal{PT}, \Gamma] = 0$ , produces a pure imaginary shifting of the coordinate for  $-2i\delta$  followed by the action of the parity operator. As the grading operator commutes with the  $\mathcal{PT}$  operator, from the definition of (5.2) and (2.4) it follows that the supercharges are  $\mathcal{PT}$ -symmetric operators,

$$[\mathcal{PT}, Q_a] = 0, \quad a = 1, 2. \quad (5.8)$$

Acting on the Hamiltonian eigenstates  $\psi_{r,s}^{+k}$  ( $\psi_{r,s}^{+m}$ ) and  $\psi_{r,s}^{-k}$  ( $\psi_{r,s}^{-m}$ ) given by Eqs. (3.2) and (3.7), operator  $\Gamma$  transforms them mutually,

$$\Gamma\psi_{r,s}^{\pm k} = (-1)^r e^{\pm 2\delta k} \psi_{r,s}^{\mp k}, \quad \Gamma\psi_{r,s}^{\pm m} = (-1)^r e^{\pm 2\delta m} \psi_{r,s}^{\mp m}. \quad (5.9)$$

On the other hand, all the singlet states are eigenstates of  $\Gamma$ . The corresponding  $N = 2$  nonlinear superalgebra generated by the Hamiltonian  $H_{r,s}$  and supercharges  $Q_a$  reads as

$$[Q_a, H_{r,s}] = 0, \quad \{Q_a, Q_b\} = 2\delta_{ab} H_{r,s} \prod_{n=1}^r (H_{r,s} - E_{r,s;n})^2. \quad (5.10)$$

As follows from (5.10), the nonlinear superalgebra detects all the singlet states in the spectrum; moreover, it distinguishes spectral singularities from the singlet ground state: unlike the energy level  $E = 0$ , all the spectral singularities appear as double roots of the polynomial in Hamiltonian. The same holds for exceptional points in the case of the compact topology.

## 6 Discussion

In this article we show from different points of view how spectral singularities appear in  $\mathcal{PT}$ -symmetric singly periodic finite-gap systems with non-compact topology. These states correspond to exceptional points when topology is changed for a compact one.

The examples discussed here test the effectiveness of Crum-Darboux transformations for non-Hermitian Hamiltonians. Applying them to the free particle Hamiltonian, we construct the systems which display singlet states inside the continuum. These states are known as spectral singularities; these are a specific feature of complex Hamiltonians. We note here that the models defined by (2.5) can be extended to a more generic family of singly periodic finite-gap systems by using complex Crum-Darboux transformations different from those in (2.4).

In ref. [44] it was shown that starting from the Hamiltonian  $H_{0,0}$ , it is possible to obtain complex reflectionless potentials by choosing non-physical states of the free system as a kernel of the Crum-Darboux operator. In a similar way, selecting physical solutions of the free Hamiltonian, displaced for a complex constant, as zero modes of the non-Hermitian Crum-Darboux operators, complex systems with spectral singularities can be constructed. Outside the scope of the present paper, an interesting approach for the comprehension of this kind of states would be that related to quasi-exact solvability, see refs. [47, 51, 15].

Non-Hermitian finite-gap potentials were analyzed from the point of view of the corresponding infinite period complex and real limits. We explain how the band-edge states coalesce and produce the spectral singularities when the complex period is infinite. In this picture the presence of spectral singularities is understood: it corresponds to a remarkable, peculiar feature of complex periodic potentials, with no analog in the Hermitian case. A more detailed investigation on the hidden supersymmetries and related properties of complex doubly periodic finite-gap systems deserves a separate, further investigation [56].

The existence of a finite number of spectral singularities leads to an additional feature of the systems discussed in this paper. Their non-degeneracy alongside with the doubly degenerate continuum are naturally explained by a hidden bosonized non-linear supersymmetry, whose structure also distinguishes spectral singularities from the singlet ground state. The hidden non-linear supersymmetry, which is related with the Lax pair of the KdV hierarchy, in the Hermitian limit, when  $\delta = 0$ , has a completely different nature. In such a limit the integral of motion is ill defined, producing non-physical states. In the sense we show that the pathologies of the hidden supersymmetry in the Hermitian case, originated from real singularities, can be circumvented by changing the Hermiticity property of the Hamiltonian for the  $\mathcal{PT}$ -symmetry.

It would be interesting to study the breaking of  $\mathcal{PT}$ -symmetry and the disappearance of spectral singularities by appropriate modification of the systems (2.5). In the same direction, the meaning of the investigated spectral singularities in the context of optics and matter waves could be also a relevant problem to investigate.

As a final remark, we note that it is interesting to apply the ideas of the present paper to study the models described by the first order Dirac-type Hamiltonians [57], particularly, to those related to the topologically nontrivial solutions in the Gross-Neveu model [58, 53], and to the physics of nanotubes [59].

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