REED'S CONJECTURE ON SOME SPECIAL CLASSES OF GRAPHS

J.L. FOUQUET, J.M. VANHERPE

ABSTRACT. Reed conjectured that for any graph G, $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$, where $\chi(G)$, $\omega(G)$, and $\Delta(G)$ respectively denote the chromatic number, the clique number and the maximum degree of G. In this paper, we verify this conjecture for some special classes of graphs, in particular for subclasses of P_5-free graphs or *Chair*-free graphs.

October 30, 2012

1. Introduction

We consider here simple and undirected graphs. For terms which are not defined we refer to Bondy and Murty [2].

In 1998, Reed proposed the following Conjecture which gives, for any graph G, an upper bound of the chromatic number $\chi(G)$ in terms of the clique number $\omega(G)$ and the maximum degree $\Delta(G)$.

Conjecture 1 (Reed's Conjecture [8])**.** *For any graph G, $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$.*

In [1], Aravind et al. considered Conjecture 1 for some graph classes defined by forbidden configurations. In particular, when P_n, C_n and K_n respectively denote a chordless path, a chordless cycle and a complete graph on n vertices while *Chair*, *House*, *Bull*, *Dart* and *Kite* are the graphs depicted in Figure 1, Aravind et al. have shown that Conjecture 1 holds for :

- $(P_5, \overline{P_2 \cup P_3}, House, Dart)$-free graphs,
- $(P_5, Kite, Bull, (K_3 \cup K_1) + K_1)$-free graphs,
- (P_5, C_4)-free graphs,
- $(Chair, House, Bull, K_1 + C_4)$-free graphs,
- $(Chair, House, Bull, Dart)$-free graphs.

This paper proves that Reed's Conjecture holds for some classes of graphs. Our results extend those given in [1] on subclasses of P_5-free or *Chair*-free graphs.

2. Notations and preliminary results

2.1. **Odd hole expansions.** Given a graph H on n vertices $v_0 \dots v_{n-1}$ and a family of graphs $G_0 \dots G_{n-1}$, an *expansion* of H (or $H-$*expansion*), denoted $H(G_0 \dots G_{n-1})$ is obtained from H by replacing each vertex v_i of H with G_i for $i = 0 \dots n-1$ and joining a vertex x in G_i to a vertex y of G_j, $(i \neq j)$ if and only if v_i and v_j are adjacent in H. The graph G_i, $i = 0 \dots n-1$ is said to be the *component* of the expansion associated to v_i. For an expansion $H(G_0 \dots G_{n-1})$ of some graph H, we

1991 *Mathematics Subject Classification.* 035 C.
Key words and phrases. Vertex coloring, Chromatic number, Clique number, Maximum degree.

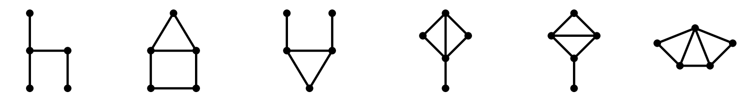

FIGURE 1. Configurations Chair, House, Bull, Dart,Kite

will assume in the following that the vertices of H are weighted with the chromatic number of their associated component while an edge of H is weighted with the sum of the weights of its endpoints.

When H is an odd hole, that is a chordless odd cycle of length at least 5, we shall say that $G = H(G_0 \dots G_{n-1})$ is an *odd hole expansion*.

Conjecture 1 was studied by Rabern [7].

Theorem 2. [7] *If $\overline{G}$ is disconnected then $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$.*

Moreover :

In [1] Aravind et al observed that the so-called *complete* expansion (every component of the expansion induces a complete graph) and *independent* expansion (every component of the expansion induces a stable) of an odd hole satisfy Conjecture 1. In [5] we have shown:

Theorem 3. [5] *Any expansion of a bipartite graph satisfies Conjecture 1.*

Theorem 4. [5] *Let $G = H(G_0 \dots G_{2k})$ be an expansion of an odd hole H of length $2k+1$ with $V(H) = \{v_0 \dots v_{2k}\}$and such that the edge v_0v_1 has maximum weigth in H. For $i = 0 \dots 2k$, let χ_i be the chromatic number of G_i. Let l be an index such that*

$$\chi_{l-1} + \chi_l + \chi_{l+1} = \underset{3 \leq i \leq 2k-1}{Min} \{\chi_{i-1} + \chi_i + \chi_{i+1}\}.$$

Then

- *If $\chi_0 + \chi_1 \geq \chi_{l-1} + \chi_l + \chi_{l+1}$ then $\chi(G) = \chi_0 + \chi_1$*
- *else $\chi(G) = \chi_0 + \chi_1 + \lfloor \frac{\chi_{l-1}+\chi_l+\chi_{l+1}-\chi_0-\chi_1+1}{2} \rfloor$.*

Corollary 5. [5] *Conjecture 1 holds for an odd hole expansion when, in the conditions of Theorem 4, we have $\chi(A) = \omega(A)$ for $A \in \{G_0, G_1, G_l\}$.*

Theorem 6. [5] *If G is a C_5-expansion then G satisfies Conjecture 1.*

2.2. **Notations and definitions.**
Let $X \subseteq V(G)$, $N(X)$ will denote the set of vertices in $V(G)-X$ adjacent to at least one vertex in X while $G[X]$ will denote the subgraph of G induced by X. If $X = \{v\}$ we write $G - v$ instead of $G[V(G) - X]$. A vertex in $V(G) - X$ is said to be *partial* for X if it is adjacent to some (but not all) vertex of X. As usual, given a graph G, $\omega(G)$, $\chi(G)$ and $\Delta(G)$ denote respectively the maximum number of vertices in a clique of G, the chromatic number and the maximum degree. In addition, for a vertex $v \in V(G)$, $\omega(v)$ denotes the size of a maximum clique containing v, and $d(v)$ is the degree of v.

In [4], a *buoy* was defined as a special case of C_5-expansion, that is an expansion of the odd hole C_5. We extend here this notion to odd holes of length at least 5. We shall say that an induced subgraph of a graph G is an *buoy of length* $2k+1$, *$(k > 1)$*

whenever we can find a partition of its vertex set into $2k+1$ subsets (considered as organized in a cyclic order) such that any two consecutive sets in the list are joined by every possible edge, while no edges are allowed between two non consecutive sets, and such that these sets are maximal for these properties.

Observe that a buoy, as defined above is merely an odd hole expansion and that an buoy of length 5 is precisely as defined in [4]. Moreover, a buoy as well as its complement are connected graphs.

A graph G will be said a *minimal counter example to Conjecture 1* whenever $\chi(G) > \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$ and when Conjecture 1 holds for any subgraph of G.

2.3. Technical lemmas.

Lemma 7. *Let G be a minimal counter example to Conjecture 1 (if any). Then there are no two disjoint subsets $X \subseteq V(G)$ and $Y \subseteq V(G)$ such that $N(X) \subseteq N(Y)$ and $\chi(G[X]) \leq \chi(G[Y])$.*

Proof. Let $G^{'}$ be the subgraph obtained from G by deleting X. Since $G^{'}$ satisfies Conjecture 1 by hypothesis, we have $\chi(G^{'}) \leq \lceil \frac{\omega(G^{'})+\Delta(G^{'})+1}{2} \rceil$. We can then color the vertices of X by using the colors appearing in Y since $\chi(G[X]) \leq \chi(G[Y])$. Since $\omega(G) \geq \omega(G^{'})$ and $\Delta(G) \geq \Delta(G^{'})$, we have
$\chi(G) = \chi(G^{'}) \leq \lceil \frac{\omega(G^{'})+\Delta(G^{'})+1}{2} \rceil \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$, a contradiction. □

Lemma 8. *Let $G = H(G_0 \dots G_{n-1})$ be an expansion that is a minimal counter-example to Conjecture 1 (if any). Then each component G_i ($i \in \{0 \dots n-1\}$) is connected.*

Proof. Without loss of generality assume that the subgraph induced by G_0 is not connected. Let X and Y be two subset of $V(G_0)$ inducing a connected component and suppose that $\chi(G[X]) \leq \chi(G[Y])$. We get immediately a contradiction with Lemma 7 since it can be easily checked that $N(X) = N(Y)$. □

Lemma 9. *Let H be an induced subgraph of some graph G such that $\chi(H) = \chi(G)$. If $\chi(H) \leq \lceil \frac{\omega(H)+\Delta(H)+1}{2} \rceil$ then $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$.*

In [1] Aravind et al consider k-critical graphs in order to prove that every vertex in a minimum counter example to Conjecture 1 belongs to an odd hole.

A graph G is said to be *k-critical* if $\chi(G) = k$ and $\chi(G-v) < k$ for all $v \in V(G)$.

Theorem 10. [1] *If G is k-critical and $k > \frac{d(v)+\omega(v)+1}{2}$, for $v \in V(G)$, then v must belong to some odd hole in G.*

We can extend the result of [1] to minimal counter examples to Conjecture 1.

Lemma 11. *If G is a minimal counter example to Conjecture 1 then any vertex is contained in an odd hole.*

Proof. Since G is a minimal counter example to Conjecture 1, it follows that G is k-critical. Then, for every $v \in V(G)$, $k > \frac{d(v)+\omega(v)+1}{2}$, and hence by Theorem 10, v is part of some odd hole in G. □

3. On well-hooped graphs.

A hole in a graph G will be said *well-hooped*, if any vertex of G which is partial to C is connected to precisely three consecutive vertices of C or to precisely two vertices at distance two on C. The graph G itself will be said *well-hooped* when all odd holes of G are well-hooped.

Observe that the vertices of a well-hooped cycle C together the vertices which are partial to C induce a buoy which, by construction, is not distinguished from the outside.

Lemma 12 below comes from a result already stated in [4].

Lemma 12. *If G is a $(P_5, \overline{P_5})$-free graph then G is well-hooped.*

Proof. Let C be some odd hole in G and x be a vertex partial to C. Since G is P_5-free, C has length 5.

The neighbours of x in C are either two independant vertices or three consecutive vertices, otherwise the vertices of C together with x would contain an induced P_5 or $\overline{P_5}$, a contradiction. $\square$

Theorem 13. *Let G be a well-hooped graph. Any two distinct buoys are vertex disjoint or one is contained in the other.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two distinct buoys of G such that $V(\mathcal{B}_1) \cap V(\mathcal{B}_2) \neq \emptyset$, $V(\mathcal{B}_1) - V(\mathcal{B}_2) \neq \emptyset$ and $V(\mathcal{B}_2) - V(\mathcal{B}_1) \neq \emptyset$.

Observe that the vertices of $V(\mathcal{B}_1) - V(\mathcal{B}_2)$ as well as the vertices of $V(\mathcal{B}_2) - V(\mathcal{B}_1)$ are not partial with respect to $V(\mathcal{B}_1) \cap V(\mathcal{B}_2)$.

There is a vertex, say x, in $V(\mathcal{B}_1) - V(\mathcal{B}_2)$ that is connected to some vertex of $V(\mathcal{B}_1) \cap V(\mathcal{B}_2)$, otherwise $\mathcal{B}_1$ would be disconnected, a contradiction. By the definition of a$\not n$ buoy, x must be adjacent to all vertices of $V(\mathcal{B}_2)$. Consequently, there must be a vertex in $V(\mathcal{B}_2) - V(\mathcal{B}_1)$, say y, which is adjacent to x.

Let $z \in V(\mathcal{B}_1) - V(\mathcal{B}_2)$ be a vertex not connected to some vertex of $V(\mathcal{B}_1) \cap V(\mathcal{B}_2)$, then z cannot be connected to y since $y \in V(\mathcal{B}_2)$. But now, y is connected to x and not to z, another contradiction.

Consequently, all vertices in $V(\mathcal{B}_1) - V(\mathcal{B}_2)$ are adjacent to all vertices in $V(\mathcal{B}_1) \cap \mathcal{B}_2)$, in other words $\overline{\mathcal{B}_1}$ is not connected, a final contradiction. $\square$

By Lemma 11, every vertex in a mimimal counter example to Conjecture 1 belongs to an odd hole, consequently :

Corollary 14. *Let G be a well-hooped graph which is a minimal counter example to Conjecture 1. There is a partition of the vertices of G in buoys.*

In [4] the following theorem was proved for the $(P_5, \overline{P_5})$-free graphs. This result can be easily extended to well-hooped graphs. We give here the proof for sake of completeness.

Theorem 15. *Let G be a well-hooped graph. If W is a minimum transversal of the odd cycles of G then $\omega(G[W]) \leq \omega(G) - 1$.*

Proof. For every vertex x of W, there exists an odd hole, denoted C_x, such that $W \cap V(C_x) = \{x\}$. Since W is a minimal transversal of the odd holes of G, we call C_x the private odd hole of x.

We have $\omega(G[W]) \leq \omega[G)$. Assume that $\omega(G(W]) = \omega(G)$ and let Q be a maximum clique of $G[W]$.

Let x be a vertex of Q such that the buoy which contains C_x, say $\mathcal{B}(C_x)$ is minimal among all buoys generated by private odd holes of vertices of Q, that is $\mathcal{B}(C_x)$ does not contain as a proper subset any other $\mathcal{B}(C_y)$ with $y \in Q$.

Assume that C_x has length $2k+1$ $(k > 1)$. We write $\mathcal{B}(C_x) = C_x(A_0, A_1 \ldots A_{2k})$ since $\mathcal{B}(C_x)$ is an odd hole expansion of length $2k+1$ and we suppose that $x \in A_0$.

If Q meets neither A_1 nor A_{2k} then $Q \subseteq A_0 \cup (N(\mathcal{B}(C_x)) - \mathcal{B}(C_x))$. Let y be a vertex of A_1, $\{y\} \cup Q$ is a clique of G, a contradiction.

We suppose now, without loss of generality, that Q meets A_1. Let $z \in Q \cap A_1$. By minimality of $\mathcal{B}(C_x)$ and by Theorem 13, $\mathcal{B}(C_x) \subseteq \mathcal{B}(C_z)$. Moreover, by the definition of a buoy, we have $\mathcal{B}(C_x) \subseteq \mathcal{B}(C_z)$.

We have $A_0 \subset W$ since every odd hole obtained from C_x by substituting another vertex of A_0 to x must intersect W. But C_z must instersect A_0 and $W \cap C_z \neq \{z\}$, a contradiction. $\square$

Using the Strong Perfect Graph Theorem [3], this result leads to

Theorem 16. *If G is a $(P_6, \overline{P_6})$-free well-hooped graph then $\chi(G) \leq \frac{\omega(G)(\omega(G)-1)}{2}$.*

Proof. Since G is $(P_6, \overline{P_6})$-free, the odd holes of G have length 5. If we remove a transversal W of the C_5's, we obtain a perfect graph . The perfection of $G - W$ implies that $\chi(G - W) \leq \omega(G)$ and by Theorem 15, $\omega(G[W]) \leq \omega(G) - 1$.

Applying recursively this observation we get $\chi(G) \leq \frac{\omega(G)(\omega(G)-1)}{2}$. $\square$

It follows from a result of King [6] that if G is a minimum counter-example to Conjecture 1 then $\omega(G) \leq \frac{2}{3}(\Delta(G)+1)$. Hence, if we restrict ourself to well-hooped graphs which are $(P_6, \overline{P_6})$-free, a minimum counter-example to this conjecture is such that $1 + \sqrt{\Delta(G) + 2} \leq \omega(G) \leq \frac{2}{3}(\Delta(G) + 1)$.

An *independent buoy* is a buoy such that any set of the associated partition is a stable set.

Theorem 17. *If G is a $(P_6, \overline{P_6})$-free well-hooped graph where each buoy of G is independent then G satisfies Conjecture 1.*

Proof. By Corollary 14, there is a partition of the vertex set into buoys.

Let W be a minimum transversal of the odd holes. We get immediately $\chi(G) \leq \chi(G - W) + \chi(G[W])$. Moreover, since $G - W$ and $G[W]$ does not contain any odd hole nor the complement of an odd hole, these graphs are perfect ([3]).

Let G^* be the simple graph obtained from G by shrinking each buoy of the partition of G and deleting multiple edges. It is an easy task to see that $2 \leq \omega(G) = \omega(G - W) = 2\omega(G[W]) = 2\omega(G^*)$. Hence we have $\chi(G) \leq \lceil \frac{3\omega(G)}{2} \rceil$.

Let v be a vertex contained in a maximum clique of G. Then $\Delta(G) \geq d(v) \geq 5(\omega(G) - 1) + 2$ and $\lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil \geq \lceil \frac{\omega(G)+5(\omega(G)-1)+2+1}{2} \rceil = 3\omega(G) - 1$.

We have thus $\chi(G) \leq \lceil \frac{\omega(G)+\Delta(G)+1}{2} \rceil$ as soon as $\lceil \frac{3\omega(G)}{2} \rceil \leq 3\omega(G) - 1$, a contradiction.

$\square$

An *full buoy* is a buoy such that any set of the associated partition is a clique. We have immediately by Corollary 5 that a full buoy satisfies Conjecture 1.

Theorem 18. *If G is a $\overline{P_6}$-free well-hooped graph where each buoy is full then $\chi(G) \leq \lceil \frac{3\omega(G)}{2} \rceil$.*

Proof. Since the buoys of G are full, a buoy cannot be contained into another, thus by Theorem 13 the buoys of G are pairwise disjoint. Let $(B_i)_{1 \leq i \leq k}$ be the set of buoys of G. Assume that the buoy B_i, $1 \leq i \leq k$, has length $2l_i + 1$, we write $B_i = C_{2l_i+1}(A^i_0, \dots A^i_{2l_i})$. Without loss of generality we can consider that $A^i_0 \cup A^i_1$ is a maximum clique of B_i and set $\omega_i = |A^i_0| + |A^i_1|$. Hence, we certainly have $|A^i_2| \leq \frac{\omega_i}{2}$ or $|A^i_{2l_i}| \leq \frac{\omega_i}{2}$. For $i \in \{1, \dots k\}$, let $W_i \in \{A^i_2, \dots A^i_{2l_i}\}$ be a set of minimum size and let $W = \cup_{i=1}^{k} W_i$.

Since $G - W$ and $G[W]$ do not contain any odd hole nor its complement (G is $\overline{P_6}$-free), theses graphs are perfect and $\chi(G) \leq \omega(G - W) + \omega(G([W])$. Without loss of generality we can write a maximum clique of $G([W])$ as the set $\cup_{i=1}^{q} W_i$ for some q. By Theorem 13 this maximum clique of $G([W])$ leads to a clique of $G - W$ which is $\cup_{i=1}^{q} A^i_0 \cup A^i_1$. Hence $\omega(G[W]) = \sum_{i=1}^{q} |W_i| \leq \sum_{i=1}^{q} \frac{\omega_i}{2} \leq \frac{\omega(G-W)}{2}$.

That is $\chi(G) \leq \frac{3\omega(G-W)}{2} \leq \frac{3\omega(G)}{2}$.

□

4. Applications

We do not know in general whether a well-hooped graph satisfies Conjecture 1. We are concerned here with various families of well-hooped graphs.

Theorem 19. *If G is a P_6-free well-hooped graph then G satisfies Conjecture 1 or G contains a subgraph isomorphic to a $P_4(C_5, C_5, C_5, C_5)$ and a subgraph isomorphic to $C_3(C_5, C_5, C_5)$.*

Proof.

Suppose that G is a P_6-free well-hooped graph being a minimal counter example to Conjecture 1. We can consider that G is connected. Since the graph is P_6-free, the odd holes of G have length 5.

By Corollary 14, there is a partition of the vertex set of G into buoys.

Let G^* be the graph obtained from G by shrinking each buoy of the above partition in a single vertex. Observe that G^* is C_5-free.

If G^* has only one vertex then G is a C_5 expansion and the result follows from Theorem 6.

Assume that G^* contains an induced path on four vertices $B_1B_2B_3B_4$. Since each buoy of G contains an induced C_5, this P_4 leads to a subgraph isomorphic to the expansion $P_4(C_5, C_5, C_5, C_5)$ as a subgraph of G.

If G^* is P_4-free and contains at least two vertices, it is well known (see Seinsche [9]) that its complement is not connected. Henceforth, $\overline{G}$ itself is not connected and G satisfies Conjecture 1 by Theorem 2.

Moreover, by Theorem 3 we can suppose that G is not bipartite. Consequently G^* contains a triangle, that means that G contains a subgraph isomorphic to $C_3(C_5, C_5, C_5)$. □

Theorem 19 above implies that any P_6-free well-hooped graph not containing some fixed subgraph of the expansion $P_4(C_5, C_5, C_5, C_5)$ nor some subgraph of the expansion of $C_3(C_5, C_5, C_5)$ satisfies Conjecture 1.

For example, Conjecture 1 holds for P_6-free well-hooped graphs of $\mathcal{G}$ with no induced K_6, since K_6 is a subgraph of $C_3(C_5, C_5, C_5)$.

Moreover, by this way we get shorter proofs of results given in [1].

Corollary 20. [1] *Any (C_4, P_5)-free graph satisfies Conjecture 1.*

Proof. Let G be a (C_4, P_5)-free graph. Since G is P_5-free, the odd holes of G have length 5. It is not difficult to check that a vertex partial to some odd hole of G, say C, is precisely connected to 3 consecutive vertices of C. By definition, a (C_4, P_5)-free graph is a P_6-free well-hooped graph. Since a $P_4(C_5, C_5, C_5, C_5)$ contains a C_4, the result follows from Theorem 19 □

Corollary 21. *Any $(P_5, \overline{P_5}, Dart)$-free graph satisfies Conjecture 1*

Proof. By Lemma 12, a $(P_5, \overline{P_5})$-free graph is well-hooped. Moreover, it is obviously a P_6-free graph. Since a $P_4(C_5, C_5, C_5, C_5)$ contains a $Dart$, the result follows from Theorem 19. □

Corollary 22. [1] *Any $(P_5, \overline{P_5}, Dart, \overline{P_2 \cup P_3})$-free graph satisfies Conjecture 1*

Corollary 23. *Any $(P_5, Kite)$-free graph satisfies Conjecture 1*

Proof. Let G be a $(P_5, Kite)$-free graph. Since G is P_5-free, the odd holes of G have length 5. It is not difficult to check that a vertex partial to some odd hole of G, say C, is precisely connected to 2 vertices at distance 2 on C. By definition G is well-hooped. Moreover G is P_6-free. Since a $P_4(C_5, C_5, C_5, C_5)$ contains a $Kite$, the result follows from Theorem 19 □

Corollary 24. [1] *Any $(P_5, Kite, Bull, (K_3 \cup K_1) + K_1)$-free graph satisfies Conjecture 1*

4.1. $(Chair, Bull)$-free graphs.

Lemma 25. *Let G be a (Chair, Bull)-free graph G and C_{2k+1} ($k > 1$) be an odd hole of G. Let x be a vertex of G partial to C_{2k+1}.*
One of the following holds :

(1) *x is adjacent to precisely 3 consecutive vertices of C_{2k+1},*
(2) *$k = 2$ and x is adjacent to precisely four vertices of C_{2k+1}.*

Proof. Let us write $C_{2k+1} = v_0 v_1 \dots v_{2k}$. Without loss of generality we can assume that x is adjacent to v_0 and not adjacent to v_{2k}.

The vertex x must have at least one neighbour in $\{v_1, v_2\}$ otherwise the set $\{x, v_{2k}, v_0, v_1, v_2\}$ would induce a $Chair$, a contradiction. If x is connected to v_1 and not to v_2, the set $\{x, v_{2k}, v_0, v_1, v_2\}$ would induce a $Bull$, a contradiction.

If x is connected to v_2 but not to v_1, the vertex x must be adjacent v_{2k-1} or the vertices v_{2k-1}, v_{2k}, x, v_0 and v_1 would induce a $Chair$, a contradiction. Consequently, $k = 2$, otherwise the vertices v_2 and v_{2k-2} are distinct and independent and $\{v_{2k-2}, v_{2k-1}, v_{2k}, x, v_2\}$ induces a $Bull$ when x is adjacent to v_{2k-2} and a $Chair$ otherwise. But now the vertex x together with v_1, v_2, v_3 and v_4 would induce a $Bull$, a contradiction.

It follows that x is adjacent to v_1 and v_2.

If x has another neighbour on C_{2k+1}, say y, we have $y = v_{2k-1}$, otherwise the vertices x, y, v_0, v_1, v_{2k} iduce a $Bull$, a contradiction. Once again, we have $k = 2$, or the vertices v_2 and v_{2k-2} being distinct and independent, the set $\{v_{2k-2}, v_{2k-1}, v_{2k}, x, v_0, v_2\}$ would contain an induced $Chair$ when x and v_{2k-1} are not adjacent and an induced $Bull$ otherwise, a contradiction.

Hence, $k = 2$ and x is adjacent to precisely four vertices of the cycle C_5. □

Let us denote $\mathcal{F}$ the following set of graphs $\{House, Kite, Gem, C_5\}$ (see Figure 1).

Theorem 26. *If G is a $(Chair, Bull, F)$-free graph with $F \in \mathcal{F}$, then G satisfies Conjecture 1.*

Proof. Let G be a $(Chair, Bull, F)$-free graph. Assume that G is a minimal counter example to Conjecture 1. We can consider that G is connected. Since G is F-free, by Lemma 25, G is a well-hooped graph. Since G is $Chair$-free, it is not difficult to check that the buoys are full. By Corollary 14 there is a partition of the vertex set of G into buoys.

Let G^* be the graph obtained from G by shrinking each buoy of the above partition in a single vertex. Observe that G^* is odd hole free.

If G^* has only one vertex then G itself is a full odd hole expansion. By Corollary 5, Conjecture 1 holds for G.

In addition, G^* is P_4-free. As a matter of fact, since each vertex of G^* represents an odd hole, such a P_4 in G^* would represent a subgraph of G which is not $Chair$-free, a contradiction.

Consequently, if G^* contains at least two vertices, it is well known (see Seinsche [9]) that its complement is not connected. Henceforth, $\overline{G}$ itself is not connected and G satisfies Conjecture 1 by Theorem 2. □

Corollary 27. [1] *Any $(Chair, \overline{P_5}, Bull, K_1 + C_4)$-free graph satisfies Conjecture 1*

References

[1] N.R. Aravind, T. Karthick, and C.R. Subramanian. Bounding χ in terms of ω and Δ for some classes of graphs. *Discrete Mathematics*, 311:911–920, 2011.

[2] J.A. Bondy and U.S.R. Murty. *Graph Theory*, volume 244 of *Graduate Text in Mathematics*. Springer, 2008.

[3] M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas. The Strong Perfect Graph Theorem. *Annals of Math.*, 164:51–229, 2006.

[4] J.L. Fouquet, V. Giakoumakis, F. Maire, and H. Thuillier. On graphs without P_5 and $\overline{P_5}$. *Discrete Mathematics*, 146(33-44), 1995.

[5] J.L. Fouquet and J.M. Vanherpe. Reed's conjecture on hole expansions. Technical report, L.I.F.O., 2011.

[6] A.D. King. Hitting all maximum cliques with a stable set using lopsided incident transversals. *Journal of Graph Theory*, 2:111, 2010.

[7] L. Rabern. A note on Reed's conjecture. *SIAM Journal on Discrete Mathematics*, 22:820–827, 2008.

[8] B. Reed. ω, Δ and χ. *Journal of Graph Theory*, 27:177–212, 1998.

[9] D. Seinsche. On a property of the class of n-colorable graphs. *Journal of Combinatorial Theory*, Series B(16):191–193, 1974.

L.I.F.O., Faculté des Sciences, B.P. 6759, Université d'Orléans, 45067 Orléans Cedex 2, FR