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GLOBAL EXISTENCE OF SMALL EQUIVARIANT WAVE MAPS ON ROTATIONALLY SYMMETRIC MANIFOLDS

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Abstract. We introduce a class of rotationally invariant manifolds, which we call *admissible*, on which the wave flow satisfies smoothing and Strichartz estimates. We deduce the global existence of equivariant wave maps from admissible manifolds to general targets, for small initial data of critical regularity $H^{\frac{n}{2}}$. The class of admissible manifolds includes in particular asymptotically flat manifolds and perturbations of real hyperbolic spaces $\mathbb{H}^n$ for $n \geq 3$.

1. Introduction

Wave maps are functions $u : M^{1+n} \to N^\ell$ from a Lorentzian manifold (M^{1+n}, h) to a Riemannian manifold (N^ℓ, g), which are critical points for the functional on M^{1+n} with Lagrangian density $L(u) = Tr_h(u^*g)$, the trace with respect to the metric h of the pullback of the metric g through the map u. The space M^{1+n} is usually called the *base manifold* and N^ℓ the *target manifold*; both are assumed to be smooth, complete and without boundary. This notion extends to a Lorentzian setting the usual definition of harmonic maps between Riemannian manifolds. Wave maps arise in several different physical theories, and in particular they play an important role in general relativity.

When the base manifold is the flat Minkowski space $\mathbb{R} \times \mathbb{R}^n$, in local coordinates on the target, the Euler-Lagrange equations for $L(u)$ reduce to a system of derivative nonlinear wave equations

$$\Box u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0, \tag{1.1}$$

where Γ^a_{bc} are the Christoffel symbols on N^ℓ and we use implicit summation over repeated indices. The natural setting is then the Cauchy problem with data at $t = 0$

$$u(0, x) = u_0, \qquad u_t(0, x) = u_1. \tag{1.2}$$

The data are taken in suitable N^ℓ-valued Sobolev spaces

$$(u_0, u_1) \in H^s(\mathbb{R}^m, N^\ell) \times H^{s-1}(\mathbb{R}^m, TN^\ell) \tag{1.3}$$

which can be defined as follows, if N^ℓ is isometrically embedded in a euclidean $\mathbb{R}^{\ell'}$:

$$H^s(\mathbb{R}^m; N^\ell) := \{v \in H^s(\mathbb{R}^m; \mathbb{R}^{\ell'}),\ v(\mathbb{R}^m) \subseteq N^\ell\}. \tag{1.4}$$

Solutions belong to the space $C([0, T); H^s)$, with $T \leq \infty$. Starting with [18], [16] Problem (1.1), (1.2) has been studied extensively; see [37] and [14] for a review of the classical theory.

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Since equation (1.1) is invariant for the scaling $u(t,x) \mapsto u(\lambda t, \lambda x)$, the critical Sobolev space for the data corresponds to $s = \frac{n}{2}$. In dimension $n = 1$ energy conservation is sufficient to prove global well posedness, thus in the following we assume $n \geq 2$. Concerning local existence, the behaviour is rather clear; Problem (1.1)–(1.3) is

- locally well posed if $s > \frac{n}{2}$ (see [21], [23]). Note that classical energy estimates only allow to prove local existence for $s > \frac{n}{2} + 1$, and the sharp result requires bilinear methods which exploit the null structure of the nonlinearity.
- ill posed if $s < \frac{n}{2}$ (see [36], [12], [13]).

The problem of global existence with small data has been completely understood through the efforts of many authors during the last 20 years (see among the others [39], [47], [48], [22], [38], [25], [50]). The end result is that if the initial data belong to $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$, and their homogeneous $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$ norm is sufficiently small, then there exists a global solution, continuous with values in $H^{\frac{n}{2}}$, for general targets. Note that the solution also belongs to a suitable Strichartz space (more on this below), and uniqueness holds only under this additional constraint.

When the initial data are large, the geometry of the target manifold comes into play, and the problem presents additional difficulties; in particular, blow up in finite time may occur. For targets with positive curvature, when the dimension of the base space is $n \geq 3$, blow up examples with self similar structure were constructed already in [36], [39]. On the other hand, when the target is negatively curved, the available blow up examples require $n \geq 7$ [7].

The case $n = 2$ is especially interesting since the critical norm $\dot{H}^{\frac{n}{2}}$ coincides with the energy norm, which is conserved. The general conjecture is that large solutions may blow up for certain classes of targets with positive curvature, while they can be continued globally for geodesically convex targets. In this generality the conjecture remains open, and is being actively researched, but it has been confirmed in several cases and is supported by numerical evidence. Note however that for compact targets it was proved in [44], [43] that solutions are global as long as the energy of the initial data is below the energy of a minimal harmonic map (so that all solutions are global when such maps do not exist). When the target is the hyperbolic space $\mathbb{H}^2$, global existence was proved in [26] (see also [25]). In a rotationally symmetric setting, global existence for geodesically convex targets was obtained in [17], [9], [39], [46], while blow up solutions for the $\mathbb{S}^2$ target were constructed and analyzed in [24], [34], [31]. However, radially symmetric solutions into the 2-sphere never blow up [45]. See [49] for additional information and detailed references.

The more general case of a nonflat base manifold has received much less attention. If we restrict to maps defined on a product $\mathbb{R} \times M^n$, with M^n a Riemannian manifold, the wave map system in local coordinates (1.1) becomes

$$u^a_{tt} - \Delta_M u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0, \tag{1.5}$$

where Δ_M is the negative Laplace-Beltrami operator on M^n. To our knowledge, there are few results on (1.5). In [40] the stability of equivariant, stationary wave maps on $\mathbb{S}^2$ with values in $\mathbb{S}^2$ is proved, while [8] considers the local existence on Robertson-Walker spacetimes. More recently, in [28] global existence of small wave maps is proved in the case when $M^n = M^4$ is a four dimensional small perturbation of flat $\mathbb{R}^4$, and the stability of equivariant wave maps defined on $\mathbb{H}^2$ is studied in [29].

In the present paper we initiate the study of equivariant solutions of (1.5) on more general base manifolds M^n, $n \geq 3$. Our main result is the global existence of equivariant wave maps for small data in the critical norm, provided the base manifold belongs to a class of manifolds which we call *admissible*. The class of admissible manifolds is rather large, and includes in particular asymptotically flat manifolds and perturbations of real hyperbolic spaces; see some examples in Remark 1.3 below and a more detailed discussion in Section 5 at the end of the paper. The precise definition is the following:

Definition 1.1 (Admissible manifolds)**.** Let $n \geq 3$. We say that a smooth manifold M^n is *admissible* if its metric has the form $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$ and $h(r)$ satisfies:

(i) $\exists h_\infty \geq 0$ such that $H(r) := h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})'' = h_\infty + O(r^{-2})$ for $r \gg 1$.

(ii) $H^{(j)}(r) = O(r^{-1})$ and $(h^{-\frac{1}{2}})^{(j)} = O(r^{-\frac{1}{2}-j})$ for $r \gg 1$ and $1 \leq j \leq [\frac{n-1}{2}]$.

(iii) There exist $c, \delta_0 > 0$ such that for $r > 0$ we have $h(r) \geq cr$ while the function $P(r) = rH(r) - rh_\infty + \frac{1-\delta_0}{4r}$ satisfies the condition $P(r) \geq 0 \geq P'(r)$.

Note that (i) is a form of asymptotic convexity, while (iii) is effective essentially on a bounded region. Condition (ii), on the other hand, is weaker and excludes singularities of the metric at infinity. The parameter h_∞ can be understood as a measure of the curvature of the manifold at infinity; $h_\infty = 0$ means essentially that the manifold is asymptotically flat, while the case $h_\infty > 0$ includes examples with large asymptotic curvature, like the hyperbolic spaces.

Now assume both M^n and N^ℓ are rotationally symmetric manifolds, with global metrics

$$M^n: \quad dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}, \qquad N^\ell: \quad d\phi^2 + g(\phi)^2 d\chi^2_{\mathbb{S}^{\ell-1}} \tag{1.6}$$

where $d\omega^2_{\mathbb{S}^{n-1}}$ and $d\chi^2_{\mathbb{S}^{\ell-1}}$ are the standard metrics on the unit sphere. We recall the *equivariant ansatz* (see [37]): writing the map $u = (\phi, \chi)$ in coordinates on N^ℓ, the radial component $\phi = \phi(t, r)$ depends only on time and r, the radial coordinate on M^n, while the angular component $\chi = \chi(\omega)$ depends only on the angular coordinate ω on M^n. It follows that $\chi : \mathbb{S}^{n-1} \to \mathbb{S}^{\ell-1}$ must be a harmonic polynomial map of degree k, whose energy density is $k(k+n-2)$ for some integer $k \geq 1$. On the other hand $\phi(t, r)$ must satisfy the $\bar{\ell}$*-equivariant wave map equation*

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + \frac{\bar{\ell}}{h(r)^2} g(\phi)g'(\phi) = 0 \tag{1.7}$$

where $\bar{\ell} = k(k+n-2)$ and for which one considers the Cauchy problem with initial data

$$\phi(0, r) = \phi_0(r), \qquad \phi_t(0, r) = \phi_1(r). \tag{1.8}$$

When $h(r) = r$ the base space is the flat $\mathbb{R}^n$ and (1.7) reduces to the equation originally studied in [39].

In the following statement we use the notation $|D_M| = (-\Delta_M)^{\frac{1}{2}}$, where Δ_M is the Laplace-Beltrami operator on M^n. If $v : M^n \to N^\ell$ is an equivariant map of the form $v = (\phi(r), \chi(\omega))$ with $\chi : \mathbb{S}^{n-1} \to \mathbb{S}^{\ell-1}$ a fixed harmonic map, its Sobolev $H^s(M^n; N^\ell)$ norm can be equivalently expresssed as

$$\|v\|_{H^s(M^n;N^\ell)} \simeq \|\phi\|_{H^s} := \|(1-\Delta_M)^{\frac{s}{2}}\phi\|_{L^2(M^n)}.$$

We define also the weighted Sobolev space $H^s_q(w)$ of radial functions on M^n with norm

$$\|\phi\|_{H^s_q(w)} := \|w^{-1}(|x|)\phi(|x|)\|_{H^s_q(\mathbb{R}^{n+2k})}, \qquad w(r) := r^k \frac{r^{\frac{n-1}{2}}}{h(r)^{\frac{n-1}{2}}}.$$

and we choose the indices (p, q) as

$$p = \frac{4(m+1)}{m+3}, \qquad q = \frac{4m(m+1)}{2m^2 - m - 5}, \qquad m = n + 2k. \tag{1.9}$$

The notation $L^\infty H^s \cap CH^s$ denotes the space of continuous bounded functions from $\mathbb{R}$ to H^s, while $L^p H^s_q(w)$ is the space of functions $\phi(t,r)$ which are L^p in time with values in $H^s_q(w)$. Our main result is the following:

Theorem 1.2 (Global existence in the critical norm). *Let $n \geq 3$, $k \geq 1$, $\bar{\ell} = k(k+n-2)$ and p,q as in (1.9). Assume M^n and N^ℓ are two rotationally invariant manifolds with metrics given by (1.6), with M^n admissible, and let h_∞ be the limit of $h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})''$ as $r \to \infty$. Consider the Cauchy problem (1.7), (1.8).*

If $h_\infty > 0$ and $\|\phi_0\|_{H^{\frac{n}{2}}} + \|\phi_1\|_{H^{\frac{n}{2}-1}}$ is sufficiently small, the problem has a unique global solution $\phi(t,r) \in L^\infty H^{\frac{n}{2}} \cap CH^{\frac{n}{2}} \cap L^p H_q^{\frac{n-1}{2}}(w)$.

If $h_\infty = 0$ and $\||D_M|^{\frac{1}{2}}\phi_0\|_{H^{\frac{n-1}{2}}} + \||D_M|^{-\frac{1}{2}}\phi_1\|_{H^{\frac{n-1}{2}}}$ is sufficiently small, the problem has a unique global solution $\phi(t,r)$ with $|D_M|^{\frac{1}{2}}\phi \in L^\infty H^{\frac{n-1}{2}} \cap CH^{\frac{n-1}{2}}$ and $\phi \in L^p H_q^{\frac{n-1}{2}}(w)$.

Remark 1.1 (Scattering). It is not difficult to prove that the solutions constructed in Theorem 1.2 scatter to solutions of the *linear* equivariant equation

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r = 0$$

in $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$ as $t \to \pm\infty$, by standard arguments; we omit the details.

Remark 1.2 (Local existence with large data). By a simple modification in the proof one can show that the small data assumption can be replaced by the weaker assumption that the linear part of the flow is sufficiently small. This in particular implies existence and uniqueness of a time local solution for large data in the same regularity class (see Remark 4.1 for a sketch of the proof).

Thus global existence of small equivariant wave maps on admissible manifolds holds in the critical space $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$, as in the case of a flat base manifold. The solution enjoys additional $L^p L^q$ integrability properties, determined by the Strichartz estimates used in the proof. This has the usual drawback that uniqueness holds only in a restricted space. *Unconditional uniqueness* in the critical space without additional restrictions was proved recently for general wave maps on Minkowski space in [30]. We conjecture that a similar result holds also in our situation; as a partial workaround, we prove that if the regularity of the initial data is increased by $\delta = \frac{1}{m+1}$ then uniqueness holds in the space $CH^{\frac{n}{2}+\frac{1}{m+1}}$:

Theorem 1.3 (Higher regularity and unconditional uniqueness). *Consider (1.7), (1.8) under the assumptions of Theorem 1.2, and let $0 \leq \delta < k$.*

If $h_\infty > 0$ and $\|\phi_0\|_{H^{\frac{n}{2}+\delta}} + \|\phi_1\|_{H^{\frac{n}{2}-1+\delta}}$ is sufficiently small, the problem has a unique global solution $\phi \in L^\infty H^{\frac{n}{2}+\delta} \cap CH^{\frac{n}{2}+\delta} \cap L^p H_q^{\frac{n-1}{2}+\delta}(w)$. Moreover, if $\delta \geq \frac{1}{m+1}$, this is the unique solution in $CH^{\frac{n}{2}+\delta}$.

If $h_\infty = 0$ and $\||D_M|^{\frac{1}{2}}\phi_0\|_{H^{\frac{n-1}{2}+\delta}(M)} + \||D_M|^{-\frac{1}{2}}\phi_1\|_{H^{\frac{n-1}{2}+\delta}(M)}$ is sufficiently small, Problem (1.7), (1.8) has a unique global solution ϕ with $|D_M|^{\frac{1}{2}}\phi \in L^\infty H^{\frac{n-1}{2}+\delta}(M) \cap CH^{\frac{n-1}{2}+\delta}(M)$ and $\phi \in L^p H_q^{\frac{n-1}{2}+\delta}(w)$. Moreover, if $\delta \geq \frac{1}{m+1}$, this is the unique solution with $|D_M|^{\frac{1}{2}}\phi \in CH^{\frac{n-1}{2}+\delta}$.

Remark 1.3 (Examples of admissible manifolds). In Section 5 we discuss at some length the admissibility assumption. In particular we prove that suitable perturbations of admissible manifolds are also admissible; this allows to substantially enlarge the list of explicit examples. The following manifolds are included in the class:

- The euclidean $\mathbb{R}^n$ and, more generally, rotationally invariant, asymptotically flat spaces of dimension $n \geq 3$,. The precise condition is the following:

the radial component of the metric has the form $h_\epsilon(r) = r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$

$$|\mu(r)| + r|\mu'(r)| + r^2|\mu''(r)| + r^3|\mu'''(r)| \leq \epsilon r \quad \text{for all} \quad r > 0$$

and

$$|\mu^{(j)}(r)| \lesssim r^{1-j} \quad \text{for} \quad r \gg 1, \quad j \leq [\tfrac{n-1}{2}] + 2.$$

- Real hyperbolic spaces $\mathbb{H}^n$ with $n \geq 3$, for which $h(r) = \sinh r$; more generally, rotationally invariant perturbations of $\mathbb{H}^n$ with a metric $h_\epsilon(r) = \sinh r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$

$$|\mu(r)| + |\mu'(r)| + |\mu''(r)| + |\mu'''(r)| \leq \epsilon \langle r\rangle^{-3} \sinh r \quad \text{for all} \quad r > 0$$

and

$$|\mu^{(j)}(r)| \lesssim r^{-1}e^r \quad \text{for} \quad r \gg 1, \quad j \leq [\frac{n-1}{2}] + 2.$$

- Some classes of rotationally invariant manifolds with a metric $h(r)$ of polynomial growth $h(r) \sim r^M$, wher M can be any $M \geq 1$.

Remark 1.4 (Strichartz estimates). The crucial tools in Theorems 1.2, 1.3 are smoothing and Strichartz estimates for wave equations defined on admissible manifolds, which are proved in Section 3. To our knowledge, Strichartz estimates on curved backgrounds were essentially known only for asymptotically flat manifolds, see e.g. [42], [32], [3] [35], among the others. For the case of manifolds with a nonvanishing curvature at infinity, such estimates are known in the special case of real hyperbolic spaces, see [1], [2].

Remark 1.5 (The case $n = 2$). Our proof does not cover the case $n = 2$. Indeed, the main obstruction is the smoothing estimate of Theorem 3.3, which fails when $n = 2$, and is the crucial step in the proof of Strichartz estimates. It is indeed possible to prove a smoothing estimate also in the low dimensional case, but this requires a substantial modification in the argument (in particular, it is necessary to use time dependent Morawetz multipliers). We plan to address this problem in a further work.

The plan of the paper is the following. In Section 2 we transform the equivariant wave map equation in an equation with potential defined on $\mathbb{R}^m$, with $m = n + 2k \geq 5$. Since we need Strichartz estimates at the level of $H^{\frac{n}{2}}$ regularity, we develop some tools to commute derivatives with the flow, and the lemmas to this purpose are proved in the same section. In Section 3 we prove smoothing and Strichartz estimates for the transformed linear equation; they hold under suitable assumptions on the potential, which translate into the definition of admissible manifold. Section 4 is devoted to the proof of global existence with small data for a radial nonlinear wave equation with critical nonlinearity depending also on the radial variable. Finally, in Section 5 we apply the result to the original wave map equation and we discuss the definition of admissible manifold in detail.

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2. Reduction to a perturbed problem and equivalence of norms

The component $g(s)$ in the metric of the target manifold is the restriction of an odd, smooth function on $\mathbb{R}$. Thus we can write

$$\bar{\ell}g(s)g'(s) = \bar{\ell}s + s^3\Gamma(s)$$

with $\Gamma(s)$ smooth. Applying the change of variable

$$\phi(t,r)=\psi(t,r)\cdot w(r), \qquad w(r):=\frac{r^{k+\frac{n-1}{2}}}{h(r)^{\frac{n-1}{2}}},$$

equation (1.7) reduces to

$$\psi_{tt}-\psi_{rr}-\frac{m-1}{r}\psi_r+V(r)\psi+\frac{r^{m-1}}{h(r)^{n+1}}\psi^3\Gamma\left(\frac{r^{\frac{m-1}{2}}}{h^{\frac{n-1}{2}}}\psi\right)=0 \tag{2.1}$$

where $m=2k+n$ and

$$V(r)=\frac{n-1}{2}\left[\frac{h''}{h}+\frac{n-3}{2}\left(\frac{h'^2}{h^2}-\frac{1}{r^2}\right)\right]+k(k+n-2)\left(\frac{1}{h^2}-\frac{1}{r^2}\right).$$

Note that the function $h(r)$ can be extended to a smooth odd function on $\mathbb{R}$ and satisfies $h(0)=h''(0)=0$, $h'(0)=1$. As a consequence, the potential $V(r)$ is smooth at the origin, and has a critical decay $\sim r^{-2}$ in general. Our main goal will be now to prove Strichartz estimates for the transformed equation (2.1), and this will be the object of the next section. Note that we require estimates at the level of the $H^{\frac{n}{2}}$ norm, thus we need to be able to commute $\frac{n}{2}$ derivatives with functions of the operator $-\Delta+V$; the rest of the section is devoted to the necessary tools for this step.

In following, $c(x)$ is a measurable real valued function on $\mathbb{R}^m$ and the operator

$$L=-\Delta+c(x)$$

is a selfadjoint Schrödinger operator on $L^2(\mathbb{R}^m)$. We shall make different sets of assumptions on the potential $c(x)$, but in all cases they imply that $-\Delta+c(x)$ has a unique selfadjoint extension by well known results. This fact will be tacitly used, and in particular we shall use the spectral calculus associated to the operator $-\Delta+c(x)$ without further notice.

The first result is contained in [5] but we include a short proof for completeness:

Lemma 2.1. *Let $m\geq 3$ and assume*

$$\tfrac{C}{|x|^2}\geq c(x)\geq -\tfrac{(m-2)^2-\delta}{4|x|^2} \tag{2.2}$$

for some $C,\delta>0$. Then $-\Delta+c$ is a positive operator, and for all $|s|\leq 1$ we have the equivalences

$$\|(-\Delta+c)^{\frac{s}{2}}u\|_{L^2}\simeq\|u\|_{\dot H^s}, \qquad \|(1-\Delta+c)^{\frac{s}{2}}u\|_{L^2}\simeq\|u\|_{H^s}. \tag{2.3}$$

Proof. By Hardy's inequality we have $\|\nabla u\|_{L^2}^2\gtrsim(Lu,u)\gtrsim\|\nabla u\|_{L^2}^2$ and this implies the case $s=1$. The case $s=-1$ is obtained by duality, and the remaining cases follow by interpolation. The proof of the second equivalence is almost identical. □

When the potential is smoother, we have a more general result for higher order nonhomogeneous norms. The following estimate is not sharp but sufficient for our purposes. We use the notation $\lceil s\rceil$ for the least integer $\geq s$.

Lemma 2.2. *Let $s\geq 0$, $1<p<\infty$ and assume $c(x)$ has bounded derivatives up to the order $2\lceil s\rceil-2$. Then there exists constants K_0,C depending on s,p and on the potential $c(x)$ such that, for all $K\geq K_0$,*

$$C^{-1}\|(K-\Delta)^s u\|_{L^p}\leq\|(K-\Delta+c)^s u\|_{L^p}\leq C\|(K-\Delta)^s u\|_{L^p}. \tag{2.4}$$

Proof. In the course of the proof, the letter C denotes several constants which are independent of K. Consider first the case $s=k>0$ is an integer. We can write

$$(K-\Delta+c)^k u=(K-\Delta)^k u+\textstyle\sum C\cdot K^h\partial^\alpha(c^\ell)(\partial^\beta u), \tag{2.5}$$

where the sum extends over all multiindices α, β and integers h, ℓ such that

$$\frac{1}{2}(|\alpha| + |\beta|) + h + \ell = k, \qquad \ell \geq 1.$$

Note that $|\alpha| \leq 2k-2$, so that $\|\partial^\alpha(c^\ell)\|_{L^\infty} \leq C\|c\|^\ell_{W^{2k-2,\infty}}$. If we take

$$K_1 := \|c\|_{W^{2k-2,\infty}}$$

we can estimate the L^p-norm of the generic term of the sum in (2.5) as follows

$$CK^h\|\partial^\alpha(c^\ell)\partial^\beta u\|_{L^p} \leq CK^h K_1^\ell \|\partial^\beta u\|_{L^p} = C(K_1/K)^\ell K^{h+\ell}\|\partial^\beta u\|_{L^p}.$$

By the Mikhlin multiplier theorem and recalling that $|\beta| + 2h + 2\ell \leq 2k$, this can be estimated with

$$\leq C(K_1/K)^\ell \|(K-\Delta)^k u\|_{L^p}.$$

Thus if we take $K \gg K_1 = \|c\|_{W^{2k-2,\infty}}$ we obtain from (2.5) that

$$(K-\Delta+c)^k u = (K-\Delta)^k u + I \quad \text{where} \quad \|I\|_{L^p} \leq \epsilon\|(K-\Delta)^k u\|_{L^p}, \quad \epsilon < 1$$

and this cocnludes the proof in the case $s = k$.

If s is not an integer, we consider the analytic family of operators $T_z = (K-\Delta+c)^z(K-\Delta)^{-z}$ and we apply Stein interpolation. In this step, the L^p boundedness of the operators $(K-\Delta+c)^{iy}$ for $y \in \mathbb{R}$ follows e.g. from the general method of [6] since the heat kernel $e^{-t(K-\Delta+c)}$ satisfies an upper gaussian estimate (note that $c + K \geq 0$). □

From Mikhlin-Hörmander we know that

$$\|(1-\Delta)^s v\|_{L^p} \simeq \|(K-\Delta)^s v\|_{L^p} \tag{2.6}$$

for all $K > 0$ and $1 < p < \infty$, with a constant depending on K. A similar property holds for the operators $(K-\Delta+c)^s$, at least in the case $p = 2$ and if the operator is positive:

Lemma 2.3. *Let $m \geq 3$, $s \in \mathbb{R}$ and assume $c(x)$ satisfies (2.2). Then for all $K > 0$ we have the equivalence*

$$\|(K-\Delta+c)^s v\|_{L^2(\mathbb{R}^m)} \simeq \|(1-\Delta+c)^s v\|_{L^2(\mathbb{R}^m)}. \tag{2.7}$$

Proof. By complex interpolation (which does not require a gaussian estimate since we are in the elementary L^2 case) it is sufficient to prove the equivalence when s is a half integer; and since $(K-\Delta+c)^{\frac{1}{2}}$ and $(1-\Delta+c)^{\frac{1}{2}}$ commute, it is sufficient to prove it for $s = \frac{1}{2}$. But in this case the equivalence is obvious since

$$((K-\Delta+c)v, v) \simeq K\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2}$$

by Lemma 2.1. □

Lemma 2.4. *Let $m \geq 3$, $s \geq 0$, and assume $c(x)$ satisfies for some $C, \delta, C_0 > 0$ condition (2.2) and*

$$|\partial^\alpha c(x)| \leq C_0\langle x\rangle^{-1}, \qquad |\alpha| \leq \lfloor s \rfloor + 2. \tag{2.8}$$

Then, using the notations $|D_c| = (-\Delta+c)^{\frac{1}{2}}$ and $\langle D_c\rangle = (1-\Delta+c)^{\frac{1}{2}}$, the following equivalences hold:

$$\||D|^{\frac{1}{2}}\langle D\rangle^s u\|_{L^2} \simeq \||D_c|^{\frac{1}{2}}\langle D_c\rangle^s u\|_{L^2} \simeq \||D|^{\frac{1}{2}}\langle D_c\rangle^s u\|_{L^2} \tag{2.9}$$

and

$$\||D|^{-\frac{1}{2}}\langle D\rangle^s u\|_{L^2} \simeq \||D_c|^{-\frac{1}{2}}\langle D_c\rangle^s u\|_{L^2} \simeq \||D|^{-\frac{1}{2}}\langle D_c\rangle^s u\|_{L^2} \tag{2.10}$$

with implicit constants depending on $c(x)$ and s.

Proof. Note that it is sufficient to prove (2.9) with $(K-\Delta)^{\frac{1}{2}}$ and $(K-\Delta+c)^{\frac{1}{2}}$ in place of $\langle D\rangle$ and $\langle D_c\rangle$ respectively, with an arbitrarily large K, thanks to the equivalence (2.7). In the following we shall use the same notation for all values of K.

The claim is that for a fixed $s\geq 0$ and $z=\frac{1}{2}$ the operators

$$|D|^z\langle D\rangle^s\langle D_c\rangle^{-s}|D_c|^{-z},\qquad |D_c|^z\langle D_c\rangle^s\langle D\rangle^{-s}|D|^{-z}$$

are bounded on L^2. By interpolation it is sufficient to prove the case $z=1$. Thus, if we define the analytic families of operators

$$T_z=|D|\langle D\rangle^z\langle D_c\rangle^{-z}|D_c|^{-1},\qquad S_z=|D_c|\langle D_c\rangle^z\langle D\rangle^{-z}|D|^{-1}$$

we have to prove that T_z,S_z are L^2 bounded when $z=s$. Again by interpolation, it is sufficient to prove that T_z and S_z are bounded when $z=2k$ is an even integer. Thus we are reduced to the estimate

$$\||D|(K-\Delta)^k v\|_{L^2}\simeq\||D_c|(K-\Delta+c)^k v\|_{L^2}.$$

Using Lemma 2.1 we can replace $|D_c|$ with $|D|$ at the r.h.s., and the claim is implied by

$$\|\nabla(K-\Delta)^k v\|_{L^2}\simeq\|\nabla(K-\Delta+c)^k v\|_{L^2}. \tag{2.11}$$

We prove (2.11) by induction on k, using the equivalence

$$\|(K-\Delta)^k v\|_{L^2}\simeq K^k\|v\|_{L^2}+\|v\|_{\dot H^{2k}}\simeq \textstyle\sum_{j=0}^{2k}K^{k-j/2}\|v\|_{\dot H^j}$$

(implicit constants independent of K). By the induction step $k\to k+1$ we obtain

$$\begin{aligned}\|\nabla(K-\Delta+c)^{k+1}v\|_{L^2}\simeq&\|(K-\Delta)^k\nabla[(K-\Delta+c)v]\|_{L^2}=\\ =&\|(K-\Delta)^{k+1}\nabla v+(K-\Delta)^k\nabla(cv)\|_{L^2}.\end{aligned} \tag{2.12}$$

We have

$$\|(K-\Delta)^k\nabla(cv)\|_{L^2}\simeq K^k\|cv\|_{\dot H^1}+\|cv\|_{\dot H^{2k+1}}$$

and also, using assumption (2.8),

$$\|cv\|_{\dot H^j}\lesssim C_0\||x|^{-1}v\|_{L^2}+C_0\|\nabla v\|_{H^{j-1}}\simeq C_0\|\nabla v\|_{H^{j-1}}$$

by Hardy's inequality, so that

$$\|(K-\Delta)^k\nabla(cv)\|_{L^2}\lesssim K^kC_0\|\nabla v\|_{L^2}+C_0\|\nabla v\|_{H^{2k}}\lesssim\tfrac{C_0}{K}\|(K-\Delta)^{k+1}\nabla v\|_{L^2}.$$

Taking $K\gg C_0$, we see that we can absorb the last term in (2.12) and we obtain (2.11).

The proof of (2.10) is analogous. Instead of (2.11) we arrive at

$$\||D|^{-1}(K-\Delta)^k v\|_{L^2}\simeq\||D|^{-1}(K-\Delta+c)^k v\|_{L^2}.$$

As before, the induction step gives

$$\||D|^{-1}(K-\Delta+c)^{k+1}v\|_{L^2}\simeq\|(K-\Delta)^{k+1}|D|^{-1}v+(K-\Delta)^k|D|^{-1}(cv)\|_{L^2} \tag{2.13}$$

and we must absorb the last term by taking K sufficiently large. We can write

$$\|(K-\Delta)^k|D|^{-1}(cv)\|_{L^2}\simeq K^k\||D|^{-1}(cv)\|_{L^2}+\|cv\|_{\dot H^{2k-1}}$$

and this must be controlled by the main term which is

$$\|(K-\Delta)^{k+1}|D|^{-1}v\|_{L^2}\simeq K^{k+1}\||D|^{-1}v\|_{L^2}+\|v\|_{\dot H^{2k+1}}\simeq\textstyle\sum_{j=0}^{2k+2}K^{k+1-j/2}\|v\|_{\dot H^{j-1}}$$

Hence, using assumption (2.8), we have by Hardy's inequality

$$\|cv\|_{\dot H^{2k-1}}\lesssim C_0\|v\|_{H^{2k}}\lesssim C_0K^{-\frac{1}{2}}\|(K-\Delta)^{k+1}|D|^{-1}v\|_{L^2};$$

in a similar way we have, also by Hardy's inequality,

$$\||D|c(x)|D|^{-1}v\|_{L^2}\simeq\|\nabla(c(x)|D|^{-1}v)\|_{L^2}\lesssim C_0\|v\|_{L^2}$$

and this estimate by duality is equivalent to

$$\||D|^{-1}(cv)\|_{L^2} \lesssim C_0 \||D|^{-1}v\|_{L^2}.$$

Summing up, we obtain

$$\|(K-\Delta)^k |D|^{-1}(cv)\|_{L^2} \lesssim C_0(K^{-1}+K^{-\frac{1}{2}})\|(K-\Delta)^{k+1}|D|^{-1}v\|_{L^2}$$

and taking K sufficiently large in (2.13) we conclude the proof. □

We shall also need a lemma relating Sobolev norms on spaces with different dimension for radial functions, which extends Lemma 1.3 in [39].

Lemma 2.5. *Let $n \geq 2$, $k \geq 1$ be integers and $s \geq 0$. Then for all radial functions $v(r)$ we have the equivalence of norms*

$$\||x|^k v(|x|)\|_{\dot{H}^s(\mathbb{R}^n_x)} \simeq \|v(|y|)\|_{\dot{H}^s(\mathbb{R}^{n+2k}_y)} \tag{2.14}$$

with implicit constants depending only on s, n, k.

Proof. The following pointwise equivalence is valid for all radial functions $v(r)$ and integers $N \geq 0$:

$$\sum_{|\alpha|=N} |D^\alpha v(|x|)| \simeq |\partial_r^N v(|x|)|, \tag{2.15}$$

where $\partial_r = \widehat{x} \cdot \nabla_x$ denotes the radial derivative and $\widehat{x} = \frac{x}{|x|}$; the implicit constants in (2.15) depend on N, n but not on v or x. We prove (2.15) by induction on N. For $N = 1$ it follows immediately from $\nabla v(|x|) = v'(|x|)\nabla|x| = v'(|x|)\widehat{x}$. Now assume the equivalence holds for some N; then we can write

$$\sum_{|\alpha|=N+1} |D^\alpha v| \simeq \sum_{\ell=1}^{n} \sum_{|\beta|=N} |D^\beta \partial_\ell v| \simeq \sum_{\ell=1}^{n} |\partial_r^N \partial_\ell v|.$$

Since $\partial_\ell v = \widehat{x}_\ell v'$ and $\partial_r \widehat{x}_\ell = 0$, we have $\partial_r \partial_\ell v = \partial_\ell \partial_r v$ and this implies

$$\simeq \sum_{\ell=1}^{n} |\partial_\ell \partial_r^N v| \simeq |\partial_r^{N+1} v|$$

which proves (2.15).

In order to prove (2.14), we note that the case of general k follows by repeated application of the case $k = 1$; moreover, if (2.14) is true for some $s = s_0$, by complex interpolation with the trivial case $s = 0$, it holds for all $0 \leq s \leq s_0$. In conclusion, it is sufficient to prove (2.14) when $k = 1$ and $s = N$ is an integer which we can assume large, say $N > n$. In this case we have, using (2.15),

$$\||x|v(|x|)\|^2_{\dot{H}^N(\mathbb{R}^n)} \simeq \|\partial_r^N(rv(r))\|^2_{L^2(\mathbb{R}^n)} \simeq \int_0^{+\infty} r^{n-1}|r\partial_r^N v + N\partial_r^{N-1} v|^2 dr$$

and to prove the claim it remains to check the equivalence

$$\int_0^{+\infty} r^{n-1}|r\partial_r^N v + N\partial_r^{N-1}v|^2 dr \simeq \int_0^{+\infty} r^{n+1}|\partial_r^N v|^2 dr. \tag{2.16}$$

One side of (2.16) follows by the Cauchy-Schwartz and then Hardy's inequality:

$$\int_0^{+\infty} r^{n+1}|r^{-1}\partial_r^{N-1}v|^2 dr \simeq \||x|^{-1}\partial_r^{N-1}v\|^2_{L^2(\mathbb{R}^{n+2})} \lesssim \|\nabla\partial_r^{N-1}v\|^2_{L^2(\mathbb{R}^{n+2})} \simeq \|\partial_r^N v\|^2_{L^2(\mathbb{R}^{n+2})}.$$

To prove the converse inequality, we expand the square at the left hand side of (2.16):

$$\int_0^{+\infty} r^{n+1}|\partial_r^N v|^2 dr + \int_0^{+\infty} r^{n-1}[N^2|\partial_r^{N-1}v|^2 + 2Nr\Re(\partial_r^{N-1}v\overline{\partial_r^N v})]dr,$$

then we integrate by parts the last term

$$2N\int_0^{+\infty} r^n \Re(\partial_r^{N-1}v\overline{\partial_r^N v})dr = N\int_0^{+\infty} r^n \partial_r|\partial_r^{N-1}v|^2 dr = -Nn\int_0^{+\infty} r^{n-1}|\partial_r^{N-1}v|^2 dr$$

and in conclusion we obtain that the left hand side of (2.16) is equal to

$$\int_0^{+\infty} r^{n+1}|\partial_r^N v|^2 dr + N(N-n)\int_0^{+\infty} r^{n-1}|\partial_r^{N-1} v|^2 dr \geq \int_0^{+\infty} r^{n+1}|\partial_r^N v|^2 dr$$

since $N > n$, which proves the claim. □

We finally consider a different type of equivalence, which is related to the change of variable

$$\phi(r) = w(r)\cdot\psi(r), \qquad w(r) := \frac{r^{\frac{m-1}{2}}}{h(r)^{\frac{n-1}{2}}} = w_0(r)\cdot r^{\frac{m-n}{2}}, \qquad w_0 := \left(\frac{r}{h}\right)^{\frac{n-1}{2}}. \tag{2.17}$$

Note that $w_0(0) = 1$. The Laplace-Beltrami operator Δ_M on M^n, with metric $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$, and the flat Laplacians Δ_n on $\mathbb{R}^n$ and Δ_m on $\mathbb{R}^m$ act on radial functions respectively as

$$\Delta_M\phi = \phi'' + (n-1)\frac{h'}{h}\phi', \qquad \Delta_n\psi = \psi'' + \frac{n-1}{r}\psi', \qquad \Delta_m\psi = \psi'' + \frac{m-1}{r}\psi'. \tag{2.18}$$

The operators Δ_M and Δ_n are connected by the formula

$$w_0^{-1}\Delta_M w_0 = \Delta_n - V_0(r), \qquad V_0 := \tfrac{n-1}{2}\left[\tfrac{h''}{h} + \tfrac{n-3}{2}\left(\tfrac{h'^2}{h^2} - \tfrac{1}{r^2}\right)\right]. \tag{2.19}$$

Lemma 2.6. *Let $m > n \geq 3$, $s \geq 0$, M^n a smooth rotationally symmetric manifold with metric $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$, and let $w(r) = r^{\frac{m-1}{2}} h(r)^{\frac{1-n}{2}}$ and $V_0(r)$ as in (2.19). Assume that $V_0(|x|)$ has bounded derivatives on $\mathbb{R}^m$ up to the order $2\lceil s\rceil - 2$, that $\partial^\alpha V_0 = O(|x|^{-1})$ as $|x| \to \infty$ for $|\alpha| \leq [s]$, and that V_0 satisfies condition (2.2). Then for all smooth functions $\phi(r)$ on M^n which depend only on the radial coordinate, we have the equivalence*

$$\|\phi\|_{H^s(M^n)} \simeq \|w(|y|)^{-1}\phi(|y|)\|_{H^s(\mathbb{R}^m_y)} \tag{2.20}$$

Moreover, using the notations $|D_M| = (-\Delta_M)^{\frac12}$, $\langle D_M\rangle = (1-\Delta_M)^{\frac12}$, we have

$$\||D_M|^{\pm\frac12}\langle D_M\rangle^s\phi\|_{L^2(M^n)} \simeq \||D|^{\pm\frac12}\langle D\rangle^s[w(|y|)^{-1}\phi(|y|)]\|_{L^2(\mathbb{R}^m_y)} \tag{2.21}$$

Proof. From the first formula in (2.19) we obtain, for all $s, K \geq 0$,

$$w_0^{-1}(K-\Delta_M)^s w_0 = (K - \Delta_n + V_0(r))^s.$$

Since for radial functions and $p < \infty$

$$\|\psi\|^p_{L^p(\mathbb{R}^n)} = c_n\int_0^\infty |\psi(r)|^p r^{n-1}dr, \qquad \|\phi\|^p_{L^p(M)} = c_n\int_0^\infty |\phi(r)|^p h^{n-1}dr,$$

we get the identity

$$\|w^{\frac{2}{p}-1}(K-\Delta_M)^s\phi\|_{L^p(M)} = \|(r^k)^{\frac{2}{p}-1}(K-\Delta_n+V_0)^s w_0^{-1}\phi\|_{L^p(\mathbb{R}^n)}, \qquad k := \tfrac{m-n}{2}.$$

By repeating the proof of Lemma 2.2 with the weight $(r^k)^{\frac{2}{p}-1}$, we have

$$\|(r^k)^{\frac{2}{p}-1}(K-\Delta_n+V_0)^s w_0^{-1}\phi\|_{L^p(\mathbb{R}^n)} \simeq \|(r^k)^{\frac{2}{p}-1}(K-\Delta_n)^s w_0^{-1}\phi\|_{L^p(\mathbb{R}^n)}$$

provided K is large enough. Thus for $p = 2$ we have proved that

$$\|(K-\Delta_M)^s\phi\|_{L^2(M)} \simeq \|(K-\Delta_n)^s w_0^{-1}\phi\|_{L^2(\mathbb{R}^n)}$$

provided K is large enough. However, the two norms are equivalent to the H^{2s} norms on M^n and on $\mathbb{R}^n$ respectively (on M^n this follows easily from the spectral formula), thus we have proved that

$$\|\phi\|_{H^s(M^n)} \simeq \|w_0^{-1}\phi\|_{H^s(\mathbb{R}^n)}$$

for all $s \geq 0$ and all radial functions. In order to obtain (2.20) it is sufficient to prove the equivalence

$$\||x|^k v\|_{H^s(\mathbb{R}^n)} \simeq \|v\|_{H^s(\mathbb{R}^m)}$$

for all radial functions v, and this follows immediately from Lemma 2.5.

Proceeding in a similar way, and using (2.9), (2.10) (with $c = V_0$), we obtain

$$\||D_M|^{\pm\frac{1}{2}}\langle D_M\rangle^s\phi\|_{L^2(M^n)} \simeq \||D|^{\pm\frac{1}{2}}\langle D\rangle^s(w_0^{-1}\phi)\|_{L^2(\mathbb{R}^n)}$$

which is equivalent to

$$\simeq \||D|^{\pm\frac{1}{2}}(w_0^{-1}\phi)\|_{L^2(\mathbb{R}^n)} + \||D|^{s\pm\frac{1}{2}}(w_0^{-1}\phi)\|_{L^2(\mathbb{R}^n)}.$$

In order to conclude the proof of (2.21), it is now sufficient to apply to each term the equivalence for radial functions

$$\||D|^\sigma v\|_{L^2(\mathbb{R}^n)} \simeq \||D|^\sigma(r^{-k}v)\|_{L^2(\mathbb{R}^m)} \tag{2.22}$$

for $\sigma = s \pm \frac{1}{2}$ and $\sigma = \pm\frac{1}{2}$. When $\sigma \geq 0$, this follows from Lemma 2.5. However, (2.22) is valid also when $0 \geq \sigma \geq -1$; to check this fact we write (2.22) in the equivalent form

$$\||x|^{-k}(1-\Delta_n)^{\frac{\sigma}{2}}|x|^k w\|_{L^2(\mathbb{R}^m)} \simeq \|(1-\Delta_m)^{\frac{\sigma}{2}}w\|_{L^2(\mathbb{R}^m)};$$

since $|x|^{-k}(1-\Delta_n)|x|^k = (1-\Delta_m - \frac{\ell}{|x|^2})$ where $\ell = k(k+n-2)$, we see that (2.22) is also equivalent to

$$\|(1-\Delta_m - \tfrac{\ell}{|x|^2})^{\frac{\sigma}{2}}w\|_{L^2(\mathbb{R}^m)} \simeq \|(1-\Delta_m)^{\frac{\sigma}{2}}w\|_{L^2(\mathbb{R}^m)}$$

and this follows from Lemma 2.1. □

3. Strichartz estimates for the perturbed equation

We shall need a weighted version of Hardy's inequality:

Proposition 3.1. *Let $n \geq 2$. Let $\alpha \in C^1(0,+\infty)$ be such that $\alpha > 0$ and the integral $\beta(r) := \alpha(r)r^{n-1}\int_0^r \frac{ds}{\alpha(s)s^{n-1}}$ is finite for all $r > 0$. Then the inequality*

$$\int_{\mathbb{R}^n} \frac{|u|^2}{\beta(r)^2}\alpha(r)dx \leq 4\int_{\mathbb{R}^n} |\widehat{x}\cdot\nabla u|^2\alpha(r)dx, \qquad r = |x| \tag{3.1}$$

is valid for all $u \in H^1_{loc}(\mathbb{R}^n \setminus 0)$ such that $\liminf_{r\to 0^+} \frac{\alpha(r)}{\beta(r)}\int_{|x|=r}|u|^2dS = 0$.

Proof. By definition of β we have for the radial derivative $\beta' = \widehat{x}\cdot\nabla\beta(|x|)$

$$\beta' = \frac{n-1}{r}\beta + \frac{\alpha'}{\alpha}\beta + 1$$

which implies the identity

$$\nabla\cdot\left(\widehat{x}\frac{\alpha}{\beta}|u|^2\right) = -\frac{\alpha}{\beta^2}|u|^2 + \frac{\alpha}{\beta}2\Re(u'\overline{u}).$$

Integrate over the difference of two balls $\Omega = B(0,R)\setminus B(0,r)$, $r < R$, to get

$$\int_\Omega \frac{\alpha}{\beta^2}|u|^2dx = 2\Re\int_\Omega \frac{\alpha}{\beta}u'\overline{u}dx + \int_{|x|=r}\frac{\alpha}{\beta}|u|^2dS - \int_{|x|=R}\frac{\alpha}{\beta}|u|^2dS.$$

Drop the last (negative) term and use Cauchy-Schwartz to obtain

$$\int_\Omega \frac{\alpha}{\beta^2}|u|^2dx \leq 2\int_{|x|=r}\frac{\alpha}{\beta}|u|^2dS + 4\int_\Omega \alpha|u'|^2dx$$

which implies (3.1), letting $r \to 0^+$ (on a suitable sequence) and $R \to +\infty$. □

Corollary 3.2. *Let $\zeta \in C^2([0,\infty))$ with $\zeta \geq 0$, $\zeta' > 0$, $\zeta'' \leq 0$, let $\epsilon > 0$ and $n \geq 2$. Then the inequality*

$$\int_{\mathbb{R}^n}[\zeta' + 2\epsilon\zeta]e^{-2\epsilon r}r^{-(n-1)}\frac{|u|^2}{|x|^2}dx \leq 4\int_{\mathbb{R}^n}[\zeta' + 2\epsilon\zeta]e^{-2\epsilon r}r^{-(n-1)}|\widehat{x}\cdot\nabla u|^2dx \tag{3.2}$$

holds for any $u \in H^1_{loc}(\mathbb{R}^n\setminus 0)$ such that $\liminf_{r\to 0^+} r^{-n}\int_{|x|=r}|u|^2dS = 0$.

Proof. Choose $\alpha = [\zeta' + 2\epsilon\zeta]e^{-2\epsilon r}r^{1-n}$ and apply Proposition (3.1). Notice that $(\alpha r^{n-1})' = [\zeta'' - 4\epsilon^2\zeta]e^{-2\epsilon r} \leq 0$ so that αr^{n-1} is nonincreasing, and this implies $\beta \leq r$. Thus we obtain inequality (3.2), provided we can verify the condition $\alpha/\beta \int_{|x|=r} |u|^2 \to 0$. By the assumptions on ζ we get $\zeta'(r) + 2\epsilon\zeta \leq (1+2\epsilon r)\zeta'(0) + \zeta(0)$; notice that $\zeta'(0)$ must be strictly positive. As a consequence,

$$\frac{\alpha}{\beta} = \left(r^{n-1}\int_0^r \frac{e^{2\epsilon s}ds}{\zeta' + 2\epsilon\zeta}\right)^{-1} \leq C_\epsilon \left(r^{n-1}\int_0^r \frac{ds}{1+2\epsilon s}\right)^{-1} \leq C'_\epsilon r^{-n}$$

which concludes the proof. □

Consider now the equation on $\mathbb{R}^n$

$$u'' + a(r)u' + \kappa^2 u - c(r)u = f, \qquad r = |x| \tag{3.3}$$

where $u(r)$ is radial and we write as usual $u' = \widehat{x}\cdot\nabla u(|x|)$ for the radial derivative. The function $a(r)$ will be smooth for $r > 0$ but singular at $r = 0$, the model case being $a = (n-1)/r$. Our next goal is to prove a suitable smoothing estimate for solutions of (3.3). In the following we use the notation $L^2(\gamma(|x|)dx)$ to denote the weighted L^2 space with norm

$$\|u\|_{L^2(\gamma(|x|)dx)} = \left(\int_{\mathbb{R}^n} |u(x)|^2\gamma(|x|)dx\right)^{\frac{1}{2}}.$$

Theorem 3.3. *Let $n \geq 3$ and $\kappa \in \mathbb{C}$ with $\Im\kappa > 0$. Let $a(r) \in C^2(0,+\infty)$, $c(r) \in C^1(0,+\infty)$ and denote with $A(r)$ a function such that $A' = a$ while $\gamma(r) = e^{A(r)}r^{1-n}$.*

Assume that $a(r)$ is bounded for large r, that $\lim_{r\to 0^+}\gamma(r) > 0$ exists, and that for some $0 < \delta_0 < 1$ and some $C > 0$ the function

$$Q(r) = \left(\frac{a'}{2} + \frac{a^2}{4} + c(r)\right)r + \frac{1-\delta_0}{4r}$$

satisfies the conditions

$$0 \leq rQ(r) \leq C, \qquad Q'(r) \leq 0.$$

Then any solution $u \in H^2_{loc}(\mathbb{R}^n)$ of equation (3.3) such that $u, u' \in L^2(\gamma(|x|)dx)$ satisfies the estimate

$$\||x|^{-1}u\|_{L^2(\gamma(|x|)dx)} \leq 4\delta_0^{-1}\||x|f\|_{L^2(\gamma(|x|)dx)}, \qquad \gamma(r) = r^{1-n}e^{A(r)}. \tag{3.4}$$

Proof. Define new functions $v(r), g(r)$ via

$$u(r) = e^{i\kappa r}e^{-A/2}v(r), \qquad f(r) = e^{i\kappa r}e^{-A/2}g(r)$$

and notice that $v(r)$ satisfies the equation

$$v'' + 2i\kappa v' - (\sigma(r) + c(r))v = g, \qquad \sigma(r) := \frac{a'}{2} + \frac{a^2}{4}.$$

Multiply the equation by $2\phi(r)\overline{v'}$, ϕ a weight to be chosen, and take the real part; using the identities

$$\Re(2\phi v''\overline{v'}) = (\phi|v'|^2)' - \phi'|v'|^2, \qquad \Re(4i\phi\kappa v'\overline{v'}) = -4(\Im\kappa)\phi|v'|^2$$

and

$$\Re(-2(\sigma+c)v\phi\overline{v'}) = (-(\sigma+c)\phi|v|^2)' + ((\sigma+c)\phi)'|v|^2$$

we obtain

$$P' + [\phi' + 4(\Im\kappa)\phi]|v'|^2 - ((\sigma+c)\phi)'|v|^2 = \Re(-2g\phi\overline{v'}),$$

where

$$P = (\sigma+c)\phi|v|^2 - \phi|v'|^2.$$

Notice that

$$P' = \widehat{x} \cdot \nabla P = \nabla \cdot \{\widehat{x} P\} - \frac{n-1}{r} P,$$

thus we arrive at the identity

$$\nabla \cdot \{\widehat{x}P\} + \left[\phi' + \tfrac{n-1}{r}\phi + 4(\Im\kappa)\phi\right] \cdot |v'|^2 - \left[((\sigma + c)\phi)' + \tfrac{n-1}{r}(\sigma + c)\phi\right] \cdot |v|^2 = \Re(-2g\phi\overline{v'}). \tag{3.5}$$

We now choose

$$\phi(r) = e^{-2(\Im\kappa)r} r^{-(n-2)}$$

which reduces the identity to

$$\nabla \cdot \{\widehat{x}P\} + [1 + 2(\Im\kappa)r] \cdot \tfrac{\phi}{r}|v'|^2 + [2(\Im\kappa)(c+\sigma)r - ((c+\sigma)r)'] \cdot \tfrac{\phi}{r}|v|^2 = \Re(-2g\phi\overline{v'}). \tag{3.6}$$

We integrate (3.6) on $B(0,R) \setminus B(0,r)$, $r < R$, and we check the behaviour of the boundary terms as $r \to 0$, $R \to +\infty$. Near zero, we must prove that

$$\liminf_{r \to 0^+} \textstyle\int_{|x|=r} [(\sigma + c)\phi|v|^2 - \phi|v'|^2] dS \leq 0$$

thus we can drop the second term $-\phi|v'|^2$ and focus on the first one. Recall that

$$|v|^2 = e^A e^{2(\Im\kappa)r}|u|^2, \qquad \phi|v|^2 = e^A r^{2-n}|u|^2 \sim r|u|^2 \text{ near } 0$$

since by assumption $e^A r^{1-n} \to C$ as $r \to 0$. Noticing that the assumption on Q implies $|\sigma + c| \leq Cr^{-2}$, we see that it is sufficient to prove

$$\liminf_{r \to 0^+} r^{-1} \textstyle\int_{|x|=r} |u|^2 dS = 0. \tag{3.7}$$

The assumption $u, u' \in L^2(\gamma dx)$ implies $u, u' \in L^2_{loc}(\mathbb{R}^n)$ with the standard norm since $\gamma \sim C$ near 0, and hence, by the usual Hardy inequality, we have $u/r \in L^2_{loc}(\mathbb{R}^n)$ which gives (3.7). For future reference we note also that

$$\liminf_{r \to 0^+} \tfrac{1}{r} \textstyle\int_{|x|=r} |u|^2 dS = 0 \quad \Longrightarrow \quad \liminf_{r \to 0^+} r^{-n} \textstyle\int_{|x|=r} |v|^2 dS = 0, \tag{3.8}$$

by definition of v and the assumption $e^A r^{1-n} \to C > 0$. We then consider the boundary term on $\partial B(0,R)$ as $R \to +\infty$; we must prove that

$$\liminf_{R \to +\infty} \textstyle\int_{|x|=R} [(\sigma + c)\phi|v|^2 - \phi|v'|^2] dS \geq 0$$

For the first term we write, recalling that $|\sigma + c| \leq Cr^{-2}$,

$$|(\sigma + c)|\phi|v|^2 = |(\sigma + c)|r\gamma|u|^2 \leq Cr^{-1}\gamma|u|^2$$

and then the assumption $\gamma|u|^2 \in L^1(\mathbb{R}^n)$ implies $\liminf_{R \to +\infty} R^{-1} \int_{|x|=R} \gamma|u|^2 dS = 0$. For the second term, we have

$$\textstyle\int_{|x|=R} \phi|v'|^2 dS \leq CR \int_{|x|=R} (|\kappa|^2 + a^2)(|u|^2 + |u'|^2)\gamma(R) dS$$

and by the assumptions $u, u' \in L^2(\gamma dx)$ and $|a| \leq C$ for large r we have

$$\liminf_{R \to +\infty} R \int_{|x|=R} (|\kappa|^2 + a^2)(|u|^2 + |u'|^2)\gamma(R) dS = 0$$

as required.

Thus we are in position to integrate (3.6) on $\mathbb{R}^n$:

$$\textstyle\int [1 + 2(\Im\kappa)r] \cdot \frac{\phi}{r}|v'|^2 dx + \int [2(\Im\kappa)(c + \sigma)r - ((c + \sigma)r)'] \cdot \frac{\phi}{r}|v|^2 dx \leq \int \Re(-2g\phi\overline{v'}) dx.$$

We estimate the right hand side by Cauchy-Schwartz and absorb a term at left, obtaining for any $0 < \delta_0 < 1$

$$\textstyle\int [1 - \delta_0 + 2(\Im\kappa)r] \cdot \frac{\phi}{r}|v'|^2 dx + \int [2(\Im\kappa)(c + \sigma)r - ((c + \sigma)r)'] \cdot \frac{\phi}{r}|v|^2 dx \leq \frac{1}{\delta_0} \int r\phi|g|^2 dx \tag{3.9}$$

Now we apply (3.2) of the previous Corollary with the choice $\zeta(r) = r$; note that the assumption on the behaviour of the function near 0 has already been checked in (3.8). Recalling that $\phi/r = e^{-2(\Im\kappa)r}r^{1-n}$, this gives

$$\tfrac{1}{4}\int[1+2(\Im\kappa)r]\tfrac{\phi}{r}\tfrac{|v|^2}{r^2}dx \le \int[1+2(\Im\kappa)r]\tfrac{\phi}{r}|v'|^2dx.$$

Using this inequality in (3.9) we obtain, for any $0 < \delta_1 < 1$,

$$\int[2(\Im\kappa)Q_0(r) - Q_0'(r)]\tfrac{\phi}{r}|v|^2dx \le \tfrac{1}{\delta_1}\int r\phi|g|^2dx$$

where

$$Q_0(r) = \frac{1-\delta_1}{4r} + (c+\sigma)r.$$

Since $\phi|v|^2 = r\gamma|u|^2$ and $\phi|g|^2 = r\gamma|f|^2$, this is equivalent to

$$\int[2(\Im\kappa)Q_0(r) - Q_0'(r)]|u|^2\gamma dx \le \tfrac{1}{\delta_1}\int r^2|f|^2\gamma dx.$$

Choose $\delta_1 = \delta_0/2$; we have by assumption

$$Q_0(r) \ge Q(r) \ge 0, \qquad -Q_0'(r) = -Q'(r) + \frac{\delta_0-\delta_1}{4r^2} \ge \frac{\delta_0}{8r^2}$$

and this concludes the proof. □

We now specialize the previous estimate to the resolvent equation for a Laplace-Beltrami operator on the manifold M with global metric $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$, restricted to radial functions:

$$u'' + (n-1)\frac{h'}{h}u' + \lambda^2 u = f. \tag{3.10}$$

We have the following result:

Corollary 3.4. *Let $n \ge 3$, $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$, $h \in C^2([0,+\infty))$, with $h > 0$ for $r > 0$, $h(0) = 0$, and $h'(0) = 1$, and define the functions*

$$\mu(r) = \left(\frac{h}{r}\right)^{n-1}, \qquad \widetilde{h}(r) = \frac{n-1}{2}\left(\frac{h''}{h} + \frac{n-3}{2}\frac{h'^2}{h^2}\right). \tag{3.11}$$

Assume that $h_\infty := \lim_{r\to+\infty}\widetilde{h}(r) \ge 0$ exists and that, for some $0 < \delta_0 < 1$ and $C > 0$, the function

$$P(r) = r(\widetilde{h}(r) - h_\infty) + \frac{1-\delta_0}{4r} \tag{3.12}$$

satisfies the conditions

$$0 \le rP(r) \le C, \qquad P'(r) \le 0. \tag{3.13}$$

Then any radial solution $u \in H^2_{loc}(\mathbb{R}^n)$ of equation (3.10) such that $u, u' \in L^2(\mu(|x|)dx)$ satisfies the estimate

$$\||x|^{-1}u\|_{L^2(\mu(|x|)dx)} \le 4\delta_0^{-1}\||x|f\|_{L^2(\mu(|x|)dx)}. \tag{3.14}$$

Proof. Let $w(r) = \mu(r)^{\frac{1}{2}}u(r)$ and $g(r) = \mu(r)^{\frac{1}{2}}f(r)$, then $w(r)$ satisfies the equation

$$w'' + \tfrac{n-1}{r}w' + \lambda^2 w - \left(\widetilde{h} - \tfrac{(n-1)(n-3)}{4r^2}\right)w = g.$$

Setting $\lambda^2 - h_\infty = \kappa^2$ with $\Im\kappa > 0$ (possible since $h_\infty \ge 0$), we rewrite the equation as

$$w'' + \tfrac{n-1}{r}w' + \kappa^2 w - \left(\widetilde{h} - h_\infty - \tfrac{(n-1)(n-3)}{4r^2}\right)w = g.$$

Now we can apply Theorem 3.3 with the choices $a(r) = (n-1)/r$, $A = (n-1)\log r$, $\gamma(r) = 1$, $c(r) = \widetilde{h} - h_\infty - (n-1)(n-3)/(4r^2)$ so that $Q(r) \equiv P(r)$ as one checks immediately. Thus all the assumptions of the Theorem are satisfied and we get the estimate

$$\||x|^{-1}w\|_{L^2(\mathbb{R}^n)} \le 4\delta^{-1}\||x|g\|_{L^2(\mathbb{R}^n)}$$

which coincides with (3.14). □

If we apply the change of variables

$$u(r) = \frac{r^{k+\frac{n-1}{2}}}{h(r)^{\frac{n-1}{2}}} v(r), \qquad f(r) = \frac{r^{k+\frac{n-1}{2}}}{h(r)^{\frac{n-1}{2}}} g(r), \qquad k = 0, 1, 2, \dots$$

we see that $u(r)$ solves (3.10) if and only if $v(r)$ solves the following equation, which we shall regard as a radial equation on $\mathbb{R}^m$:

$$v'' + \frac{m-1}{r} v' + \lambda^2 v - V(r) v = g, \qquad m = 2k + n \tag{3.15}$$

where

$$V(r) = \frac{n-1}{2} \left[\frac{h''}{h} + \frac{n-3}{2} \left(\frac{h'^2}{h^2} - \frac{1}{r^2} \right) \right] + k(k+n-2) \left(\frac{1}{h^2} - \frac{1}{r^2} \right). \tag{3.16}$$

If $h(r)$ satisfies the assumptions of Corollary 3.4, and we apply the previous change of variables in estimate (3.14), we obtain:

Corollary 3.5. *Let $n \geq 3$, $k \geq 0$, $m = 2k + n$, $\lambda \in \mathbb{C}$ with $\Im \lambda > 0$, and let $h(r)$ be as in Corollary 3.4. Then any radial solution $v \in H^2_{loc}(\mathbb{R}^m)$ of equation (3.15)-(3.16) such that $v, v' \in L^2(\mathbb{R}^m)$ satisfies the estimate*

$$\| |x|^{-1} v \|_{L^2(\mathbb{R}^m)} \lesssim \| |x| g \|_{L^2(\mathbb{R}^m)}. \tag{3.17}$$

We notice that, defining

$$W(r) = V(r) - h_\infty \tag{3.18}$$

equation (3.15) can be written

$$\Delta v + (\lambda^2 - h_\infty) v - W(r) v = g$$

where Δ is the Laplace operator on $\mathbb{R}^m$ (restricted to radial functions); moreover, it is easy to check that the first part of assumption (3.13) and the condition $h(r) \geq Cr$ for a $C > 0$ imply

$$\frac{C'}{|x|^2} \geq W(r) \geq -\frac{(m-2)^2 - \delta_0}{4|x|^2}.$$

By Hardy's inequality we obtain

$$\| \nabla u \|_{L^2(\mathbb{R}^m)} \simeq (Hv, v).$$

Thus the operator $H = -\Delta + W(r)$ is selfadjoint and positive definite on $L^2(\mathbb{R}^m)$, and by interpolation and duality we have, as in Lemma 2.1, the equivalence of norms

$$\| H^{s/2} v \|_{L^2(\mathbb{R}^m)} \simeq \| v \|_{\dot{H}^s(\mathbb{R}^m)}, \qquad -1 \leq s \leq 1 \tag{3.19}$$

and analogously, for every $\nu > 0$ (with a constant depending on ν),

$$\| (\nu + H)^{s/2} v \|_{L^2(\mathbb{R}^m)} \simeq \| v \|_{H^s(\mathbb{R}^m)}, \qquad -1 \leq s \leq 1. \tag{3.20}$$

We can now apply Kato's theory to deduce, from the resolvent estimate (3.14), corresponding smoothing estimates for the associated evolution equations. In the terminology of [19], [20], estimate (3.14) implies that multiplication by $|x|^{-1}$ is *supersmoothing* for the operator H, and this implies the estimate

$$\| |x|^{-1} e^{itH} f \|_{L^2(\mathbb{R}^{m+1})} \lesssim \| f \|_{L^2(\mathbb{R}^m)}$$

for the Schrödinger flow e^{itH}. Using the appendix to Kato's theory developed in [10] we obtain the analogous result for the wave flow:

Theorem 3.6. *Let $n \geq 3$, $k \geq 0$, $m = 2k + n$, let $h(r)$ be as in Corollary 3.4 and assume in addition $h(r) \geq cr$ for some $c > 0$. Let $V(r)$ be the function (3.16), $W = V - h_\infty$, and let H be the selfadjoint nonnegative operator on $L^2(\mathbb{R}^m)$ given by $H = -\Delta + W(r)$. Then the wave flow $e^{it\sqrt{H}}$, restricted to radial functions, satisfies the smoothing estimate*

$$\||x|^{-1}e^{it\sqrt{H}}f\|_{L^2(\mathbb{R}^{m+1})} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^m)}, \tag{3.21}$$

and, for any $\nu > 0$, the Klein-Gordon flow $e^{it\sqrt{\nu+H}}$ satisfies the smoothing estimate on radial functions

$$\||x|^{-1}e^{it\sqrt{\nu+H}}f\|_{L^2(\mathbb{R}^{m+1})} \lesssim \|f\|_{H^{1/2}(\mathbb{R}^m)}. \tag{3.22}$$

Proof. By Theorem 2.4 in [10] the operator $|x|^{-1}(H+\nu)^{-\frac{1}{4}}$ is supersmoothing with respect to $\sqrt{H+\nu}$. Then by Kato's theory we deduce the estimate

$$\||x|^{-1}\sqrt{H+\nu}^{-\frac{1}{2}}e^{-t\sqrt{H+\nu}}f\|_{L^2(\mathbb{R}^{m+1})} \lesssim \|f\|_{L^2(\mathbb{R}^m)}.$$

When $\nu = 0$, using (3.19), we obtain (3.21), while for $\nu > 0$, by (3.20), we get (3.22). □

Now, using the method of Rodnianski and Schlag [33], it is a simple task to deduce the full range of non-endpoint Strichartz estimates for the wave and Klein-Gordon equations associated to the operator H. The following is the main result of this section; we sum up in the statement the previous assumptions and notations.

Theorem 3.7. *Let $h \in C^2([0,+\infty))$ with $h(r) \geq cr$ for some $c > 0$, $h(0) = 0$ and $h'(0) = 1$. Define for $n \geq 3$*

$$\widetilde{h}(r) = \frac{n-1}{2}\left(\frac{h''}{h} + \frac{n-3}{2}\frac{h'^2}{h^2}\right). \tag{3.23}$$

Assume that $h_\infty := \lim_{r\to+\infty}\widetilde{h}(r) \geq 0$ exists and that, for some $0 < \delta_0 < 1$ and $C > 0$, the function

$$P(r) = r(\widetilde{h}(r) - h_\infty) + \frac{1-\delta_0}{4r} \tag{3.24}$$

satisfies the conditions

$$0 \leq rP(r) \leq C, \qquad P'(r) \leq 0. \tag{3.25}$$

Finally, let $k \geq 0$, define $V(r)$ as

$$V(r) = \frac{n-1}{2}\left[\frac{h''}{h} + \frac{n-3}{2}\left(\frac{h'^2}{h^2} - \frac{1}{r^2}\right)\right] + k(k+n-2)\left(\frac{1}{h^2} - \frac{1}{r^2}\right), \tag{3.26}$$

and let H be the selfadjoint operator on $L^2(\mathbb{R}^m)$, with $m = 2k + n$, defined by $H = -\Delta + W(|x|)$, $W(r) := V(r) - h_\infty$.

Then the wave flow $e^{it\sqrt{H}}$ on $\mathbb{R}^t \times \mathbb{R}^m$ satisfies the following Strichartz estimates: for radial f,

$$\||D|^{\frac{1}{q}-\frac{1}{p}}e^{it\sqrt{H}}f\|_{L^p_tL^q_x} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^m)}, \qquad |D| = (-\Delta)^{\frac{1}{2}}, \tag{3.27}$$

provided (p,q) satisfy

$$\frac{2}{p} + \frac{m-1}{q} = \frac{m-1}{2}, \qquad 2 < p \leq \infty, \qquad 2 \leq q < \frac{2(m-1)}{m-3} \tag{3.28}$$

while for fixed $\nu > 0$ the Klein-Gordon flow $e^{it\sqrt{H+\nu}}$ on $\mathbb{R}^t \times \mathbb{R}^m$ satisfies, for radial f,

$$\|\langle D\rangle^{\frac{1}{q}-\frac{1}{p}}e^{it\sqrt{\nu+H}}f\|_{L^p_tL^q_x} \lesssim \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^m)}, \qquad \langle D\rangle = (1-\Delta)^{\frac{1}{2}}, \tag{3.29}$$

provided (p,q) *satisfy either* (3.35) *or*

$$\frac{2}{p}+\frac{m}{q}=\frac{m}{2}, \qquad 2<p\le\infty, \qquad 2\le q<\frac{2m}{m-2}. \tag{3.30}$$

Proof. By Duhamel's formula one can represent the flow $u(t,x)=e^{it\sqrt{H}}f$ in terms of the unperturbed flow as

$$e^{it\sqrt{H}}f=\cos(t|D|)f+i\sin(t|D|)|D|^{-1}\sqrt{H}f-\int_0^t\frac{\sin((t-s)|D|)}{|D|}W(r)uds.$$

For the first two terms, by the standard Strichartz estimates for the unperturbed wave equation we have

$$\||D|^{\frac{1}{q}-\frac{1}{p}}e^{it|D|}f\|_{L^p_tL^q_x}\lesssim\|f\|_{\dot H^{\frac12}}$$

and

$$\||D|^{\frac{1}{q}-\frac{1}{p}}|D|^{-1}e^{it|D|}\sqrt{H}f\|_{L^p_tL^q_x}\lesssim\|\sqrt{H}f\|_{\dot H^{-\frac12}}\lesssim\|f\|_{\dot H^{\frac12}}$$

with (p,q) as in (3.28). In order to handle the Duhamel term, one uses the following mixed estimate for the free flow

$$\left\||D|^{\frac{1}{q}-\frac{1}{p}}\int_0^t\frac{e^{i(t-s)|D|}}{|D|}F(s,x)ds\right\|_{L^p_tL^q_x}\lesssim\||x|F\|_{L^2(\mathbb{R}^{m+1})}. \tag{3.31}$$

This estimate is proved in a standard way as follows: first by the Christ-Kiselev lemma the estimate is equivalent to the similar estimate for the untruncated integral (provided $p>2$, which excludes the endpoint case); then the estimate is split into the homogeneous estimate for the free flow

$$\||D|^{\frac{1}{q}-\frac{1}{p}}e^{it|D|}|D|^{-\frac12}f\|_{L^p_tL^q_x}\lesssim\|f\|_{L^2}$$

composed with the dual smoothing estimate for the free flow

$$\|\textstyle\int|D|^{-\frac12}e^{-is|D|}G(s,x)ds\|_{L^2}\lesssim\||x|G(t,x)\|_{L^2_tL^2_x}$$

(dual of (3.21) for the unperturbed wave equation). Now, plugging $F=Wu$ inside the right hand side of (3.31) and noticing that $|W|\le C|x|^{-2}$, we obtain also for the Duhamel term

$$\left\||D|^{\frac{1}{q}-\frac{1}{p}}\int_0^t\frac{e^{i(t-s)|D|}}{|D|}Wuds\right\|_{L^p_tL^q_x}\lesssim\||x|Wu\|_{L^2(\mathbb{R}^{m+1})}\le C\||x|^{-1}u\|_{L^2(\mathbb{R}^{m+1})}$$

which is bounded by $\|f\|_{L^2(\mathbb{R}^m)}$ using the smoothing estimate (3.21). The three estimates together give (3.27). The proof for Klein-Gordon is identical; the estimates for the free flow which are required in the proof have the form

$$\|\langle D\rangle^{\frac{1}{q}-\frac{1}{p}}e^{it\sqrt{1-\Delta}}f\|_{L^p_tL^q_x}\lesssim\|f\|_{H^{\frac12}(\mathbb{R}^m)}.$$

Such estimates hold both if the couple (p,q) is wave admissible, i.e. satisfies (3.28), and if it is Schrödinger admissible, i.e. satisfies (3.30). A complete proof in the second case can be found for instance in the Appendix of [11]. On the other hand for the first case the proof follows from the estimate

$$j\ge1,\quad \phi_j\in\mathscr{S},\quad \operatorname{spt}\widehat{\phi_j}=\{|\xi|\sim2^j\}\quad\Longrightarrow\quad\|e^{it\sqrt{1-\Delta}}\phi_j\|_{L^\infty(\mathbb{R}^m)}\lesssim|t|^{-\frac{m-1}{2}}2^{\frac{m+1}{2}}$$

(due to Brenner [4]) by the standard Ginibre-Velo procedure; note that we do not need the endpoint estimate. □

Using again the Christ-Kiselev lemma and a TT^* argument, we deduce in a standard way the nonhomogeneous Strichartz estimates from the previous homogeneous estimates, at least in the non endpoint case. We obtain

Corollary 3.8. *With the notations and the assumptions of the previous two Theorems, one has the estimate*

$$\left\| |D|^{\frac{1}{q}-\frac{1}{p}} \int_0^t \frac{e^{i(t-s)\sqrt{H}}}{\sqrt{H}} F(s)ds \right\|_{L_t^p L_x^q} \lesssim \||D|^{-\frac{1}{\widetilde{q}}+\frac{1}{\widetilde{p}}} F\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}} \tag{3.32}$$

for all $F(t,x)$ radial in the space variable, and all couples (p,q) and $(\widetilde{p},\widetilde{q})$ as in (3.28). Similarly, we have for $\nu > 0$

$$\left\| \langle D\rangle^{\frac{1}{q}-\frac{1}{p}} \int_0^t \frac{e^{i(t-s)\sqrt{\nu+H}}}{\sqrt{\nu+H}} F(s)ds \right\|_{L_t^p L_x^q} \lesssim \|\langle D\rangle^{-\frac{1}{\widetilde{q}}+\frac{1}{\widetilde{p}}} F\|_{L_t^{\widetilde{p}'} L_x^{\widetilde{q}'}} \tag{3.33}$$

for all $F(t,x)$ radial in the space variable, and all couples (p,q) and $(\widetilde{p},\widetilde{q})$ satisfying either (3.35) or (3.30).

Remark 3.1. In the Klein-Gordon case, by interpolation one can obtain a wider range of admissible couples (p,q). We omit the details since in the following we shall only need the wave admissible case.

Remark 3.2. Recall that we can write the fractional Sobolev embedding on $\mathbb{R}^m$ in the form

$$\||D|^{\frac{m}{r}} v\|_{L^r} \lesssim \||D|^{\frac{m}{q}} v\|_{L^q}, \qquad 1 < q \le r < \infty$$

which includes essentially all cases with the exception of some endpoints. On the other hand we can write (3.27) in the equivalent form

$$\||D|^{\frac{1}{p}+\frac{m}{q}-\frac{m-1}{2}} e^{it\sqrt{H}} f\|_{L_t^p L_x^q} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^m)}.$$

Nesting the two we obtain immediately the following *Sobolev-Strichartz estimates*:

$$\||D|^{\frac{1}{p}+\frac{m}{r}-\frac{m-1}{2}} e^{it\sqrt{H}} f\|_{L_t^p L_x^r} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^m)} \tag{3.34}$$

provided the couple (p,r) satisfies

$$0 < \frac{1}{r} \le \frac{1}{2} - \frac{2}{m-1}\frac{1}{p}, \qquad 2 < p \le \infty. \tag{3.35}$$

A similar extension of the range holds then also for the nonhomogeneous estimate (3.32), where we can replace (p,q) and $(\widetilde{p},\widetilde{q})$ with couples (p,r) and $(\widetilde{p},\widetilde{r})$ satisfying the extended condition (3.35).

Similar extensions hold also for the Klein-Gordon equation. In the following, we shall only use the case of wave-type estimates. Recall that

$$\|v\|_{L^p} \lesssim \|\langle D\rangle^{\frac{m}{q}} v\|_{L^q} \quad \text{for all} \quad q \le p < \infty$$

and together with the fractional Sobolev embedding this implies the inequality

$$\|\langle D\rangle^{\frac{m}{r}} v\|_{L^r} \lesssim \|\langle D\rangle^{\frac{m}{q}} v\|_{L^q}, \qquad 1 < q \le r < \infty.$$

Thus we can proceed exactly as for (3.34) and we obtain the extended estimates

$$\|\langle D\rangle^{\frac{1}{p}+\frac{m}{r}-\frac{m-1}{2}} e^{it\sqrt{\nu+H}} f\|_{L_t^p L_x^r} \lesssim \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^m)}, \tag{3.36}$$

provided the couple (p,r) satisfies (3.35). Note that we can obtain Sobolev-Strichartz estimates also in the Schrödinger admissible range of indices, but we shall not need this.

In a similar way, in estimate (3.33) we can replace (p,q) and $(\widetilde{p},\widetilde{q})$ with any couples (p,r) and $(\widetilde{p},\widetilde{r})$ satisfying the extended condition (3.35).

4. The fixed point argument

We begin by recalling some basic nonlinear estimates for later use. The first one is a well-known Hölder inequality for fractional derivatives:

Lemma 4.1 (Kato-Ponce). *For any test functions u, v, any $s \geq 0$ and $1 < p < \infty$ one has*

$$\||D|^s(uv)\|_{L^p} \lesssim \||D|^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|u\|_{L^{p_3}}\||D|^s v\|_{L^{p_4}} \tag{4.1}$$

and

$$\|\langle D\rangle^s(uv)\|_{L^p} \lesssim \|\langle D\rangle^s u\|_{L^{p_1}}\|v\|_{L^{p_2}} + \|u\|_{L^{p_3}}\|\langle D\rangle^s v\|_{L^{p_4}} \tag{4.2}$$

provided $p_1, p_2, p_3, p_4 \in]1, \infty]$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$

In particular, this gives

$$\|u^3\|_{\dot{H}^s_p} \lesssim \|u\|_{\dot{H}^s_r}\|u\|^2_{L^q}$$

provided $p^{-1} = r^{-1} + 2q^{-1}$.

The second Lemma is a standard fractional Moser type inequality:

Lemma 4.2. *Assume $F(r)$ is in $C^N(\mathbb{R})$, $N \geq 1$ integer, and let $0 < s < N$, $1 < p < \infty$. If $F(0) = 0$, then there exists a function $\phi(r)$ such that for any test functions u, v one has*

$$\|F(u)\|_{H^s_p} \leq \phi(\|u\|_{L^\infty})\|u\|_{H^s_p}, \tag{4.3}$$

$$\|F(u) - F(v)\|_{H^s_p} \leq \phi(R)\left[\|u - v\|_{H^s_p} + \|u - v\|_{L^\infty}\right], \tag{4.4}$$

where $R = \|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u\|_{H^s_p} + \|v\|_{H^s_p}$.

The third Lemma is a Strauss type inequality, i.e., an improved weighted Sobolev embedding for radial functions. For a proof and a comprehensive treatment of such inequalities, we refer to [15].

Lemma 4.3. *Let $m \geq 2$, $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} \leq s < \frac{m}{p}$. For any radial function $u(x)$ on $\mathbb{R}^m$ one has*

$$\||x|^{\frac{m}{p} - \frac{m}{q} - s}u\|_{L^q} \lesssim \|u\|_{\dot{H}^s_p}. \tag{4.5}$$

If the condition on s is restricted to $\frac{1}{p} - \frac{1}{q} < s < \frac{m}{p}$, then the previous estimate holds for all $1 \leq p \leq q \leq \infty$.

The following consequence of Lemma 4.3 will be a crucial ingredient in the proof of the main result:

Lemma 4.4. *Let $m \geq 2$, $1 < p \leq q < \infty$, $\sigma \geq 0$ and $\frac{1}{p} - \frac{1}{q} \leq s < \frac{m}{p} - \sigma$. Assume the function $\gamma(r) \in C^{[\sigma]+1}(]0, +\infty[, \mathbb{R})$ satisfies for $r > 0$*

$$|\gamma^{(j)}(r)| \lesssim r^{\frac{m}{p} - \frac{m}{q} - s - j}, \qquad j = 0, \dots, [\sigma] + 1. \tag{4.6}$$

Then for any radial function $u(x)$ on $\mathbb{R}^m$ one has the estimate

$$\|\gamma(|x|)u\|_{\dot{H}^\sigma_q} \lesssim \|u\|_{\dot{H}^{s+\sigma}_p}. \tag{4.7}$$

Proof. Write $\gamma(r) = \rho(r)|x|^{\frac{m}{p} - \frac{m}{q} - s}$ so that

$$\rho(r) := \gamma(r)|x|^{-\frac{m}{p} + \frac{m}{q} + s} \quad \Longrightarrow \quad |\rho^{(j)}(r)| \lesssim r^{-j}, \qquad j = 0, \dots, [\sigma] + 1.$$

We shall prove the estimate

$$\||D|^\sigma(\rho(r)|x|^{\frac{m}{p} - \frac{m}{q} - z}u)\|_{L^q} \lesssim \||D|^{\sigma + z}u\|_{L^p} \tag{4.8}$$

for all $1 < p \le q < \infty$, $\sigma \ge 0$ and all $z \in \mathbb{C}$ in the complex strip $\frac{1}{p} - \frac{1}{q} \le \Re z < \frac{m}{p} - \sigma$. Note that the right hand side is equivalent to the $\dot H_q^{\Re z+\sigma}$ norm, by the well known property (see e.g. [41] or [6])

$$\||D|^{iy} v\|_{L^q} \simeq \|v\|_{L^q}, \qquad 1 < q < \infty.$$

When σ is a nonnegative integer, the claim is proved directly writing

$$\||D|^{\sigma}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}u)\|_{L^q} \simeq \textstyle\sum_{|\alpha|=\sigma} \|\partial^{\alpha}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}u)\|_{L^q},$$

expanding the derivatives by the chain rule, and applying to each term estimate (4.5).

Consider now the case of a real $\sigma > 0$. By complex interpolation between the integer cases

$$\||D|^{[\sigma]}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}u)\|_{L^q} \lesssim \|D^{s+[\sigma]}u\|_{L^p}, \qquad \tfrac{1}{p} - \tfrac{1}{q} \le \Re z < \tfrac{m}{p} - [\sigma]$$

and

$$\||D|^{[\sigma]+1}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}u)\|_{L^q} \lesssim \|D^{s+[\sigma]+1}u\|_{L^p}, \qquad \tfrac{1}{p} - \tfrac{1}{q} \le \Re z < \tfrac{m}{p} - [\sigma] - 1$$

we obtain that (4.8) is true provided

$$\tfrac{1}{p} - \tfrac{1}{q} \le \Re z < \tfrac{m}{p} - [\sigma] - 1. \tag{4.9}$$

On the other hand, if we first use Sobolev embedding

$$\||D|^{\sigma}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}u)\|_{L^q} \lesssim \||D|^{[\sigma]+1}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}u)\|_{L^r}, \qquad [\sigma]+1-\tfrac{m}{r} = \sigma - \tfrac{m}{q}$$

and then apply again the inequality for the integer case, we obtain that (4.8) is true for $\Re z$ in the range

$$\tfrac{1}{p} - \tfrac{1}{r} \le \Re z + \sigma - ([\sigma]+1) < \tfrac{m}{p} - ([\sigma]+1)$$

which is equivalent to

$$\tfrac{1}{p} - \tfrac{1}{q} + (1 - \{\sigma\})\tfrac{m-1}{m} \le \Re z < \tfrac{m}{p} - \sigma, \qquad \{\sigma\} := \sigma - [\sigma]. \tag{4.10}$$

Now we define the analytic family of operators

$$T_z v := |D|^{\sigma}(\rho(r)|x|^{\frac{m}{p}-\frac{m}{q}-z}|D|^{-z-\sigma}v)$$

and we note that we have proved that $T_z : L^p \to L^q$ is bounded for $\Re z$ in the range (4.9) and also in the range (4.10). By Stein-Weiss interpolation we obtain that T_z is bounded for all $\Re z$ in the range $\frac{1}{p} - \frac{1}{q} \le \Re z < \frac{m}{p} - \sigma$ as claimed, and this concludes the proof. □

We are now ready to prove the main result of this section.

Let $n \ge 3$ and let $h \in C^{[\frac{n-1}{2}]+2}([0,+\infty))$ with the properties

$$h(0) = h''(0) = 0, \qquad h'(0) = 1, \qquad h(r) > cr \text{ for some } c > 0. \tag{4.11}$$

Define

$$\widetilde{h}(r) = \frac{n-1}{2}\left(\frac{h''}{h} + \frac{n-3}{2}\frac{h'^2}{h^2}\right). \tag{4.12}$$

Assume that $h_\infty := \lim_{r\to+\infty} \widetilde{h}(r) \ge 0$ exists and that, for some $0 < \delta_0 < 1$ and $C > 0$, the function

$$P(r) = r(\widetilde{h}(r) - h_\infty) + \frac{1-\delta_0}{4r} \tag{4.13}$$

satisfies the conditions

$$0 \le rP(r) \le C, \qquad P'(r) \le 0. \tag{4.14}$$

Finally, let $k \ge 1$ and define the potential $V(r)$ as

$$V(r) = \frac{n-1}{2}\left[\frac{h''}{h} + \frac{n-3}{2}\left(\frac{h'^2}{h^2} - \frac{1}{r^2}\right)\right] + k(k+n-2)\left(\frac{1}{h^2} - \frac{1}{r^2}\right). \tag{4.15}$$

and assume that

$$|V^{(j)}(r)| \lesssim r^{-1} \quad \text{for large } r, \qquad 1 \le j \le [\tfrac{n-1}{2}]. \tag{4.16}$$

We consider the following Cauchy problem on $\mathbb{R}_t \times \mathbb{R}^m_x$:

$$\psi_{tt} - \Delta\psi + V(|x|)\psi = \alpha(|x|)^2 Z\big(\beta(|x|)\psi\big) \cdot \psi^3, \tag{4.17}$$

$$\psi(0,x) = f(x), \qquad \partial_t\psi(0,x) = g(x). \tag{4.18}$$

with spherically symmetric data.

Theorem 4.5 (Global well posedness with small data)**.** *Let $n \ge 3$, $k \ge 1$ and $m = 2k+n$. Let $h \in C^{[\frac{n-1}{2}]+3}([0,+\infty))$ be a real valued function satisfying (4.11), (4.14) while the potential $V(r)$ defined by (4.15) satisfies (4.16). Moreover, let $\alpha, \beta \in C^{[\frac{n-1}{2}]+1}([0,+\infty))$ be such that*

$$r|\alpha^{(j)}(r)| + |\beta^{(j)}(r)| \lesssim r^{k-j}, \qquad 0 \le j \le [\tfrac{n-1}{2}]+1. \tag{4.19}$$

Consider the Cauchy problem (4.17), (4.18) on $\mathbb{R}_t \times \mathbb{R}^m_x$ with spherically symmetric data and $Z \in C^{[\frac{n-1}{2}]+1}(\mathbb{R},\mathbb{R})$.

In the case $h_\infty > 0$, if $\|f\|_{H^{\frac{n}{2}}} + \|g\|_{H^{\frac{n}{2}-1}}$ is sufficiently small, Problem (4.17), (4.18) has a unique global solution $\psi \in L^\infty H^{\frac{n}{2}} \cap CH^{\frac{n}{2}} \cap L^p H_q^{\frac{n-1}{2}}$, where $p = \frac{4(m+1)}{m+3}$, $q = \frac{4m(m+1)}{2m^2-m-5}$.

In the case $h_\infty = 0$, if $\||D|^{\frac{1}{2}} f\|_{H^{\frac{n-1}{2}}} + \||D|^{-\frac{1}{2}} g\|_{H^{\frac{n-1}{2}}}$ is sufficiently small, Problem (4.17), (4.18) has a unique global solution ψ with $|D|^{\frac{1}{2}}\psi \in L^\infty H^{\frac{n-1}{2}} \cap CH^{\frac{n-1}{2}}$ and $\psi \in \cap L^p H_q^{\frac{n-1}{2}}$, with p, q as before.

Proof. It is clear that the assumptions of Theorem 3.7 are satisfied. In particular, the function $h_1 = \frac{h(r)-r}{r^3}$ is of class $C^{[\frac{n-1}{2}]}$ as it can be verified by direct computation using properties (4.11), and writing $h(r) = r + r^3 h_1(r)$ one checks easily that the potential $V(r)$ is of class $C^{[\frac{n-1}{2}]}$ also at the origin. In view of (4.16), we see that the assumptions of Lemmas 2.1, 2.2, 2.3 and 2.4 are satisfied with the choice $c(x) = V(|x|)$. Thus we are in position to use the Strichartz estimates from that Theorem, and also the consequences in Corollary 3.8 and Remark 3.2.

Note also that the proof of the two cases $h_\infty = 0$ and > 0 is almost identical; indeed, the Strichartz estimates that we use in the following are valid both in the wave and in the Klein-Gordon case. We shall perform the proof only in the first (slightly harder) case and leave to the reader to check that the argument works also in the second case with minimal modifications.

Using the notations $|D_V| = (-\Delta + V)^{\frac{1}{2}}$ and $\langle D_V\rangle = (1 - \Delta + V)^{\frac{1}{2}}$, we define the nonlinear map

$$\Lambda(\psi(t,r)) := \cos(t|D_V|)f + \sin(t|D_V|)|D_V|^{-1}g + \Box_V^{-1}F$$

where

$$\Box_V^{-1}F = \int_0^t \frac{\sin((t-s)|D_V|)}{|D_V|}F(s)ds, \qquad F = \alpha(r)^2 Z(\beta(r)\psi)\cdot\psi^3.$$

We shall perform a Picard iteration in a suitably defined space. Let

$$a = \frac{2(m+1)}{m-1}, \qquad a' = \frac{2(m+1)}{m+3}, \qquad b = \frac{4m(m+1)}{2m^2-m-5} \quad \left(\iff \quad \frac{m}{b} = \frac{m-1}{2} - \frac{1}{2a'}\right)$$

and define the space X of functions $u(t,x)$ on $\mathbb{R}_t \times \mathbb{R}^m_x$, spherically symmetric in x, such that the following norm is finite:

$$\|u(t,x)\|_X := \|u\|_{L_t^{2a'} H_b^{\frac{n-1}{2}}} + \||D|^{\frac{1}{2}}u\|_{L_t^\infty H^{\frac{n-1}{2}}}.$$

Note that the couple $(2a', b)$ satisfies the extended condition (3.35) since $m \geq 5$, thus we can apply the Sobolev-Strichartz estimate (3.34) with the choice $(p, r) = (2a', b)$. In the following computations we use the notations

$$\langle D \rangle = (1-\Delta)^{\frac{1}{2}}, \qquad \langle \widetilde{D} \rangle = (M-\Delta)^{\frac{1}{2}}$$

where M is chosen large enough (with respect to V) that we can apply Lemma 2.2. In a similar way, we write

$$\langle D_V \rangle = (1-\Delta+V)^{\frac{1}{2}}, \qquad \widetilde{\langle D_V \rangle} = (M-\Delta+V)^{\frac{1}{2}}.$$

We must estimate $\|\Lambda(\psi)\|_X$. The first term is

$$\|\cos(t|D_V|)f\|_X = \|\langle D \rangle^{\frac{n-1}{2}} \cos(t|D_V|)f\|_{L^{2a'}L^b} + \||D|^{\frac{1}{2}} \langle D \rangle^{\frac{n-1}{2}} \cos(t|D_V|)f\|_{L^\infty L^2}$$

which, by (2.6), (2.4) and (2.9), is equivalent to

$$\simeq \|\cos(t|D_V|)\widetilde{\langle D_V \rangle}^{\frac{n-1}{2}} f\|_{L^{2a'}L^b} + \||D|^{\frac{1}{2}} \cos(t|D_V|)|\widetilde{\langle D_V \rangle}^{\frac{n-1}{2}} f\|_{L^\infty L^2}.$$

Using the Strichartz-Sobolev estimate (3.34) for the first term, and directly (3.27) for the second term, we obtain

$$\lesssim \||D|^{\frac{1}{2}} \widetilde{\langle D_V \rangle}^{\frac{n-1}{2}} f\|_{L^2} \simeq \||D|^{\frac{1}{2}} \langle D \rangle^{\frac{n-1}{2}} f\|_{L^2}$$

where in the last step we used again (2.9) from Lemma 2.4 and (2.6). In a similar way, for the second term in $\Lambda(\psi)$ we obtain

$$\|\sin(t|D_V|)|D_V|^{-1} g\|_X \lesssim \||D|^{-\frac{1}{2}} \langle D \rangle^{\frac{n-1}{2}} g\|_{L^2}.$$

Next, for the last term in $\Lambda(\psi)$ we can write, proceeding as before,

$$\|\square_V^{-1} F\|_X \simeq \|\square_V^{-1} \widetilde{\langle D_V \rangle}^{\frac{n-1}{2}} F\|_{L^{2a'}L^b} + \||D|^{\frac{1}{2}} \square_V^{-1} \widetilde{\langle D_V \rangle}^{\frac{n-1}{2}} F\|_{L^\infty L^2}$$

and using (3.32) (with the extension in Remark 3.2), (2.4) and (2.6)

$$\lesssim \|\widetilde{\langle D_V \rangle}^{\frac{n-1}{2}} F\|_{L^{a'}_{t,x}} \simeq \|\langle \widetilde{D} \rangle^{\frac{n-1}{2}} F\|_{L^{a'}_{t,x}} \simeq \|\langle D \rangle^{\frac{n-1}{2}} F\|_{L^{a'}_{t,x}}$$

since (a, a) is an admissible couple satisfying (3.28). Summing up, we have proved

$$\|\Lambda(\psi)\|_X \lesssim \||D|^{\frac{1}{2}} \langle D \rangle^{\frac{n-1}{2}} f\|_{L^2} + \||D|^{-\frac{1}{2}} \langle D \rangle^{\frac{n-1}{2}} g\|_{L^2} + \|\langle D \rangle^{\frac{n-1}{2}} F\|_{L^{a'}_{t,x}}. \tag{4.20}$$

It remains to estimate the nonlinear term $F = \alpha(r)^2 Z(\beta(r)\psi) \cdot \psi^3$. We claim that

$$\|\langle D \rangle^{\frac{n-1}{2}} F\|_{L^{a'}_{t,x}} \lesssim \|\psi\|_X^3 \cdot \Phi_1(\|\psi\|_X) \quad \text{with} \quad a' = \tfrac{2(m+1)}{m+3}, \ m = 2k+n \tag{4.21}$$

for some continous function $\Phi_1(s)$. Define $\widetilde{Z}(r) := Z(r) - Z(0)$ and write

$$\|\langle D \rangle^{\frac{n-1}{2}} F\|_{L^{a'}_{t,x}} = |Z(0)| \cdot \|\langle D \rangle^{\frac{n-1}{2}} (\alpha^2 \psi^3)\|_{L^{a'}_{t,x}} + \|\langle D \rangle^{\frac{n-1}{2}} (\widetilde{Z}(\beta\psi)\alpha^2\psi^3)\|_{L^{a'}_{t,x}}. \tag{4.22}$$

For the first term in (4.22) we have, by the Kato-Ponce inequality (4.2),

$$\|\langle D \rangle^{\frac{n-1}{2}} (\alpha^2\psi^3)\|_{L^{a'}_{t,x}} \lesssim \|\langle D \rangle^{\frac{n-1}{2}} \psi\|_{L^\infty L^{p_1}} \|\alpha\psi\|^2_{L^{2a'}L^{2p_2}} + \|\langle D \rangle^{\frac{n-1}{2}} (\alpha^2\psi^2)\|_{L^{2a'}L^{p_3}} \|\psi\|_{L^{2a'}L^{p_4}}$$

where we have chosen

$$\tfrac{1}{p_1} = \tfrac{m-1}{2m}, \qquad \tfrac{1}{p_2} = \tfrac{1}{a'} - \tfrac{m-1}{2m}, \qquad \tfrac{1}{p_3} = \tfrac{2m+1}{2ma'} - \tfrac{k}{m}, \qquad \tfrac{1}{p_4} = \tfrac{k}{m} - \tfrac{1}{2ma'}.$$

By Sobolev embedding and by (4.5) we have, since $k + \frac{n-1}{2} - \frac{m}{b} = \frac{1}{2a'} = 1 - \frac{m}{2p_2}$,

$$\|v\|_{L^{p_1}} \lesssim \||D|^{\frac{1}{2}} v\|_{L^2}, \qquad \|\alpha v\|_{L^{2p_2}} \lesssim \|r^{k-1} v\|_{L^{2p_2}} \lesssim \||D|^{\frac{n-1}{2}} v\|_{L^b}$$

and these inequalities imply

$$\|\langle D \rangle^{\frac{n-1}{2}} \psi\|_{L^\infty L^{p_1}} \|\alpha\psi\|^2_{L^{2a'}L^{2p_2}} \lesssim \|\psi\|_X^3. \tag{4.23}$$

Also by Sobolev embedding, since $\frac{m}{p_4} = k - \frac{1}{2a'} = \frac{m}{b} - \frac{n-1}{2}$, we have

$$\|\psi\|_{L^{2a'}L^{p_4}} \lesssim \|\langle D\rangle^{\frac{n-1}{2}}\psi\|_{L^{2a'}L^b} \leq \|\psi\|_X.$$

On the other hand, if we define

$$\frac{1}{p_5} = \frac{n+1}{2m} \equiv \frac{m+1}{2m} - \frac{k}{m}$$

we have $\frac{1}{p_3} = \frac{1}{p_5} + \frac{1}{2p_2}$ and by the Kato-Ponce inequality we can write

$$\|\langle D\rangle^{\frac{n-1}{2}}(\alpha^2\psi^2)\|_{L^{2a'}L^{p_3}} \lesssim \|\langle D\rangle^{\frac{n-1}{2}}(\alpha\psi)\|_{L^\infty L^{p_5}}\|\alpha\psi\|_{L^{2a'}L^{2p_2}};$$

the last factor is bounded by $\|\psi\|_X$ as above, while for the other one we have

$$\|\langle D\rangle^{\frac{n-1}{2}}(\alpha\psi)\|_{L^\infty L^{p_5}} \lesssim \||D|^{\frac{1}{2}}\langle D\rangle^{\frac{n-1}{2}}\psi\|_{L^\infty L^2} \leq \|\psi\|_X;$$

in the last step we wrote $\|\langle D\rangle^{\frac{n-1}{2}}(\alpha\psi)\|_{L^{p_5}} \simeq \|\alpha\psi\|_{L^{p_5}} + \||D|^{\frac{n-1}{2}}\alpha\psi\|_{L^{p_5}}$ and applied (4.7) to each term. Summing up, we have proved that the first term in (4.22) can be estimated with

$$\|\langle D\rangle^{\frac{n-1}{2}}(\alpha^2\psi^3)\|_{L^{a'}_{t,x}} \lesssim \|\psi\|_X^3. \tag{4.24}$$

We now estimate the second term of (4.22). By Kato-Ponce we have

$$\|\langle D\rangle^{\frac{n-1}{2}}[\widetilde{Z}\alpha^2\psi^3]\|_{L^{a'}_{t,x}} \lesssim \|\langle D\rangle^{\frac{n-1}{2}}(\alpha^2\psi^3)\|_{L^{a'}_{t,x}}\|\widetilde{Z}\|_{L^\infty_{t,x}} + \|\alpha^2\psi^3\|_{L^{a'}L^{q_1}}\|\langle D\rangle^{\frac{n-1}{2}}\widetilde{Z}\|_{L^\infty L^{q_2}} \tag{4.25}$$

where we choose

$$\frac{1}{q_1} = \frac{1}{a'} + \frac{k}{m} - \frac{m-1}{2m} \equiv \frac{1}{a'} - \frac{n-1}{2m}, \qquad \frac{1}{q_2} = \frac{1}{a'} - \frac{1}{q_1} \equiv \frac{m-1}{2m} - \frac{k}{m} \equiv \frac{n-1}{2m}.$$

By (4.5) we have

$$\|\beta\psi\|_{L^\infty_{t,x}} \lesssim \|r^k\psi\|_{L^\infty} \lesssim \|\psi\|_{L^\infty\dot{H}^{\frac{m}{2}-k}} = \|\psi\|_{L^\infty\dot{H}^{\frac{n}{2}}} \leq \|\psi\|_X$$

and this implies, for some continuous $\Phi_1(s)$,

$$\|\widetilde{Z}(\beta\psi)\|_{L^\infty_{t,x}} \lesssim \Phi_1(\|\psi\|_X).$$

Further, by Lemma 4.2, we have

$$\|\langle D\rangle^{\frac{n-1}{2}}\widetilde{Z}(\beta\psi)\|_{L^\infty L^{q_2}} \lesssim \Phi_2(\|\beta\psi\|_{L^\infty_{t,x}})\|\langle D\rangle^{\frac{n-1}{2}}(\beta\psi)\|_{L^\infty L^{q_2}}$$

while by (4.7) (writing as above the nonhomogeneous Sobolev norm as a sum of homogeneous terms and applying (4.7) to each term)

$$\|\langle D\rangle^{\frac{n-1}{2}}(\beta\psi)\|_{L^\infty L^{q_2}} \lesssim \||D|^{\frac{1}{2}}\langle D\rangle^{\frac{n-1}{2}}\psi\|_{L^\infty L^2} \leq \|\psi\|_X$$

which gives

$$\|\langle D\rangle^{\frac{n-1}{2}}\widetilde{Z}(\beta\psi)\|_{L^\infty L^{q_2}} \lesssim \Phi_2(\|\psi\|_X)\|\psi\|_X.$$

Finally, we have by Sobolev embedding

$$\|\alpha^2\psi^3\|_{L^{a'}L^{q_1}} \lesssim \|\langle D\rangle^{\frac{n-1}{2}}(\alpha^2\psi^3)\|_{L^{a'}_{t,x}}$$

and coming back to (4.25) we obtain, recalling also (4.24),

$$\|\langle D\rangle^{\frac{n-1}{2}}[\widetilde{Z}\alpha^2\psi^3]\|_{L^{a'}_{t,x}} \lesssim \|\psi\|_X^3 \cdot \Phi_1(\|\psi\|_X) + \|\psi\|_X^4 \cdot \Phi_2(\|\psi\|_X).$$

Putting everything together, we obtain the claim (2.11) and in conclusion we have proved

$$\|\Lambda(\psi)\|_X \lesssim \||D|^{\frac{1}{2}}\langle D\rangle^{\frac{n-1}{2}}f\|_{L^2} + \||D|^{-\frac{1}{2}}\langle D\rangle^{\frac{n-1}{2}}g\|_{L^2} + \|\psi\|_X^3 \cdot \Phi(\|\psi\|_X)$$

In a similar way we can prove the estimate

$$\|\Lambda(\psi_1) - \Lambda(\psi_2)\|_X \lesssim \|\psi_1 - \psi_2\|_X \cdot [\|\psi_1\|_X^2 + \|\psi_2\|_X^2] \cdot \Phi_3(\|\psi_1\|_X + \|\psi_2\|_X)$$

and this is sufficient to deduce the existence of a fixed point for Λ in X, provided the quantity
$$\||D|^{\frac{1}{2}}\langle D\rangle^{\frac{n-1}{2}}f\|_{L^2}+\||D|^{-\frac{1}{2}}\langle D\rangle^{\frac{n-1}{2}}g\|_{L^2}$$
is sufficiently small. Note that continuity in time of the solution follows from the fact that the fixed point can be obtained as the limit in the space X of a sequence of smooth Picard iterates. This concludes the proof of the Theorem. □

Remark 4.1. We show how to modify the previous proof in order to deduce local existence with large data, as mentioned in Remark 1.2. Let $T>0$ and define a version X_T of the space X in which the norms $L^{2a'}_t H^{\frac{n-1}{2}}_b$ and $L^\infty_t H^{\frac{n-1}{2}}$ are now restricted to the time interval $t\in[0,T]$; we denote the restricted norms with $L^{2a'}_T H^{\frac{n-1}{2}}_b$ and $L^\infty_T H^{\frac{n-1}{2}}$ respectively. The operator Λ is defined as above, and we look for a fixed point in the closed ball of X_T
$$B_\epsilon=\{\psi\in X_T:\|\psi-\psi_{lin}\|_{L^{2a'}_T H^{\frac{n-1}{2}}_b}+\||D|^{\frac{1}{2}}(\psi-\psi_{lin})\|_{L^\infty_T H^{\frac{n-1}{2}}}\le\epsilon\}$$
where
$$\psi_{lin}:=\cos(t|D_V|)f+\sin(t|D_V|)|D_V|^{-1}g.$$
Note that if $\psi\in B_\epsilon$ we have
$$\||D|^{\frac{1}{2}}\psi\|_{L^\infty_T H^{\frac{n-1}{2}}}\le\epsilon+E_0$$
where
$$E_0:=\||D|^{\frac{1}{2}}f\|_{H^{\frac{n-1}{2}}}+\||D|^{-\frac{1}{2}}g\|_{H^{\frac{n-1}{2}}},\tag{4.26}$$
while
$$\|\psi\|_{L^{2a'}_T H^{\frac{n-1}{2}}_b}\le 2\epsilon$$
provided T is sufficiently small, since the $L^{2a'}_T H^{\frac{n-1}{2}}_b$ norm of ψ_{lin} is bounded and hence tends to 0 as $T\to 0$. We have now, for $\psi\in B_\epsilon$,
$$\|\Lambda(\psi)-\psi_{lin}\|_{X_T}=\|\Box_V^{-1}F\|_{L^{2a'}_T H^{\frac{n-1}{2}}_b}+\|\Box_V^{-1}F\|_{L^\infty_T H^{\frac{n-1}{2}}}.$$
Repeating the steps of the previous proof, but using now time localized versions of the Strichartz estimates, we prove that
$$\|\Lambda(\psi)-\psi_{lin}\|_{X_T}\lesssim\|\psi\|^2_{L^{2a'}_T H^{\frac{n-1}{2}}_b}\||D|^{\frac{1}{2}}\psi\|_{L^\infty_T H^{\frac{n-1}{2}}}\cdot\Phi(\|\psi\|_{X_T})\le\epsilon/2$$
provided ϵ is chosen small enough; in particular, Λ takes B_ϵ into itself. A corresponding estimate can be proved for $\Lambda(\psi_1)-\Lambda(\psi_2)$, with $\psi_1,\psi_2\in B_\epsilon$:
$$\|\Lambda(\psi_1)-\Lambda(\psi_2)\|_{X_T}\lesssim[\|\psi_1\|_{L^{2a'}_T H^{\frac{n-1}{2}}_b}+\|\psi_2\|_{L^{2a'}_T H^{\frac{n-1}{2}}_b}]\cdot\Phi_3(\|\psi_1\|_{X_T}+\|\psi_2\|_{X_T})\cdot\|\psi_1-\psi_2\|_{X_T};$$
since the factor in square brackets is bounded by 4ϵ, we see that Λ is a contraction if ϵ is sufficiently small.

An inspection of the previous argument shows that the small data assumption in the main Theorem can be weakened, by assuming only that the linear part of the flow u_{lin} is sufficiently small.

It is possible to improve the uniqueness part of the previous result by a slight increase of the regularity of the initial data:

Theorem 4.6 (Regularity and unconditional uniqueness)**.** *Consider Problem* (4.17), (4.18) *under the same assumptions of Theorem 4.5.*

In the case $h_\infty>0$, if for some $0\le\delta<k$ the quantity $\|f\|_{H^{\frac{n}{2}+\delta}}+\|g\|_{H^{\frac{n}{2}-1+\delta}}$ is sufficiently small, then the problem has a global solution ψ with $\psi\in L^\infty H^{\frac{n}{2}+\delta}\cap$

$CH^{\frac{n}{2}+\delta} \cap L^p H_q^{\frac{n-1}{2}}$, *with* p, q *as in Theorem 4.5. If* $\delta \geq \frac{1}{m+1}$, *this is the unique solution in* $CH^{\frac{n}{2}+\delta}$.

In the case $h_\infty = 0$, *if for some* $0 \leq \delta < k$ *the quantity* $\||D|^{\frac{1}{2}} f\|_{H^{\frac{n-1}{2}+\delta}} + \||D|^{-\frac{1}{2}} g\|_{H^{\frac{n-1}{2}+\delta}}$ *is sufficiently small, then the problem has a unique global solution* ψ *with* $|D|^{\frac{1}{2}}\psi \in L^\infty H^{\frac{n-1}{2}+\delta} \cap CH^{\frac{n-1}{2}+\delta}$ *and* $\psi \in L^p H_q^{\frac{n-1}{2}}$. *If* $\delta \geq \frac{1}{m+1}$, *this is the unique solution with* $|D|^{\frac{1}{2}}\psi \in CH^{\frac{n-1}{2}+\delta}$.

Proof. As usual we give the detail of the proof only in the case $h_\infty = 0$ which is more delicate. Let X_δ be the space with norm $(n = m - 2k)$

$$\|u(t,x)\|_{X_\delta} := \|u\|_{L_t^{2a'} H_b^{\frac{n-1}{2}+\delta}} + \||D|^{\frac{1}{2}} u\|_{L_t^\infty H^{\frac{n-1}{2}+\delta}}, \qquad \delta \geq 0$$

so that the space used in the previous proof is X_0. Following the same steps we arrive at the estimate

$$\|\Lambda(\psi)\|_{X_\delta} \lesssim \||D|^{\frac{1}{2}} \langle D\rangle^{\frac{n-1}{2}\delta} f\|_{L^2} + \||D|^{-\frac{1}{2}} \langle D\rangle^{\frac{n-1}{2}+\delta} g\|_{L^2} + \|\langle D\rangle^{\frac{n-1}{2}+\delta} F\|_{L^{a'}_{t,x}}.$$

with $F = \alpha^2 Z(\beta\psi) \cdot \psi^3$. The proof of the nonlinear estimate proceeds as before; instead of (4.23) we get

$$\|\langle D\rangle^{\frac{n-1}{2}+\delta}\psi\|_{L^\infty L^{p_1}} \|\alpha\psi\|^2_{L^{2a'} L^{2p_2}} \lesssim \|\psi\|_{X_\delta} \|\psi\|^2_{X_0} \lesssim \|\psi\|^3_{X_\delta}.$$

with the same choice of indices, since $\|\psi\|_{X_0} \leq \|\psi\|_{X_\delta}$. In a similar way we have, with the same indices as before,

$$\|\langle D\rangle^{\frac{n-1}{2}+\delta}(\alpha^2\psi^2)\|_{L^{2a'} L^{p_3}} \lesssim \|\langle D\rangle^{\frac{n-1}{2}+\delta}(\alpha\psi)\|_{L^\infty L^{p_5}} \|\alpha\psi\|_{L^{2a'} L^{2p_2}}$$

where the last factor is bounded by $\|\psi\|_{X_0}$; on the other hand,

$$\|\langle D\rangle^{\frac{n-1}{2}+\delta}(\alpha\psi)\|_{L^\infty L^{p_5}} \lesssim \|\alpha\psi\|_{L^\infty L^{p_5}} + \||D|^{\frac{n-1}{2}+\delta}(\alpha\psi)\|_{L^\infty L^{p_5}}$$

where $\|\alpha\psi\|_{L^\infty L^{p_5}} \lesssim \|\psi\|_{X_0}$ while, using Lemma 4.4,

$$\||D|^{\frac{n-1}{2}+\delta}(\alpha\psi)\|_{L^\infty L^{p_5}} \lesssim \||D|^{\frac{n}{2}+\delta}\psi\|_{L^\infty L^2}$$

provided

$$\tfrac{1}{2} - \tfrac{1}{p_5} \leq \tfrac{1}{2} < \tfrac{m}{2} - \tfrac{n-1}{2} - \delta \equiv k + \tfrac{1}{2} - \delta$$

i.e., $\delta < k$. In conclusion we have

$$\|\langle D\rangle^{\frac{n-1}{2}+\delta}(\alpha^2\psi^3)\|_{L^{a'}_{t,x}} \lesssim \|\psi\|^3_{X_\delta} \quad \text{provided} \quad \delta < k.$$

The estimate of the full nonlinear term $F = \alpha^2\psi^3 Z(\beta\psi)$ is similar. Thus we obtain global existence and uniqueness in X_δ for all $0 \leq \delta < k$. This proves the regularity part of the statement.

To prove unconditional uniqueness, consider two solutions $\psi, \widetilde{\psi}$ belonging to $C([0,T]; H^{\frac{n}{2}+\delta})$ for some $T > 0$; we shall prove that if $\delta \geq \frac{1}{m+1}$ then $\psi \equiv \widetilde{\psi}$ on some smaller interval $t \in [0,\epsilon]$, and this will conclude the proof. The difference $\chi = \widetilde{\psi} - \psi$ satisfies the equation

$$\chi'' - \Delta\chi + V\chi = F_1 + F_2, \qquad \chi(0,r) = \chi_t(0,r) = 0$$

where

$$F_1 = \alpha^2(\chi^3 + 3\chi^2\psi + 3\chi\psi^2) Z(\beta\widetilde{\psi})$$

and

$$F_2 = \alpha^2\psi^3[Z(\beta\widetilde{\psi}) - Z(\beta\psi)] = \alpha^2\psi^3\beta\chi \cdot \textstyle\int_0^1 Z((1-s)\beta\widetilde{\psi} + s\beta\psi)ds.$$

Note that

$$Z_0(t,r) := Z(\beta\widetilde{\psi}), \qquad Z_1(t,r) := \textstyle\int_0^1 Z((1-s)\beta\widetilde{\psi} + s\beta\psi)ds$$

are bounded functions by estimate (4.5). We now apply the Strichartz estimate (3.32) for the special case $p = q = \widetilde{p} = \widetilde{q}$:

$$\| \int_0^t H^{-\frac{1}{2}} e^{i(t-s)\sqrt{H}} F ds\|_{L_{t,x}^{\frac{2(m+1)}{m-1}}} \lesssim \|F\|_{L_{t,x}^{\frac{2(m+1)}{m+3}}}. \tag{4.27}$$

Localizing the estimate on a time interval $I = [0, \epsilon]$ to be chosen, we obtain

$$\|\chi\|_{L_{t\in I,x}^{\frac{2(m+1)}{m-1}}} \lesssim \|F_1 + F_2\|_{L_{t\in I,x}^{\frac{2(m+1)}{m+3}}}.$$

By Hölder's inequality and (4.5) we have

$$\|\chi \cdot (\alpha\chi)^2 \cdot Z_0\|_{L^{\frac{2(m+1)}{m+3}}} \lesssim \|\chi\|_{L^{\frac{2(m+1)}{m-1}}} \|\alpha\chi\|^2_{L^{m+1}} \lesssim \|\chi\|_{L^{\frac{2(m+1)}{m-1}}} \|\chi\|^2_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}}.$$

Similar estimates hold for the other two terms in F_1:

$$\|(\alpha\chi)(\alpha\psi)\chi \cdot Z_0\|_{L^{\frac{2(m+1)}{m+3}}} \lesssim \|\chi\|_{L^{\frac{2(m+1)}{m-1}}} \|\psi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}} \|\chi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}}$$

and

$$\|\chi \cdot (\alpha\psi)^2 \cdot Z_0\|_{L^{\frac{2(m+1)}{m+3}}} \lesssim \|\chi\|_{L^{\frac{2(m+1)}{m-1}}} \|\psi\|^2_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}}.$$

and in conclusion

$$\|F_1\|_{L^{\frac{2(m+1)}{m+3}}} \lesssim \|\chi\|_{L^{\frac{2(m+1)}{m-1}}} \cdot (\|\psi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}} + \|\chi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}})^2.$$

To estimate F_2 we write analogously

$$\|(\alpha\psi)^2\chi \cdot (\beta\psi)Z_1\|_{L^{\frac{2(m+1)}{m+3}}} \lesssim \|\chi\|_{L^{\frac{2(m+1)}{m-1}}} \|\psi\|^2_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}}.$$

Summing up, the function χ satisfies the following estimate on the interval $t \in I = [0, \epsilon]$:

$$\|\chi\|_{L_{t\in I,x}^{\frac{2(m+1)}{m-1}}} \lesssim (\|\psi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}} + \|\chi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}})^2 \cdot \|\chi\|_{L_{t\in I,x}^{\frac{2(m+1)}{m-1}}}$$

Since we know a priori that the $L^\infty(I; H^{\frac{n}{2}+\frac{1}{m+1}})$ norm of χ and ψ is bounded, we deduce

$$\|\psi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}} + \|\chi\|_{L^{m+1}_{t\in I}\dot{H}^{\frac{n}{2}+\frac{1}{m+1}}} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Hence for ϵ sufficiently small we obtain $\|\chi\| \leq \frac{1}{2}\|\chi\|$ which implies $\chi \equiv 0$ for $t \in [0, \epsilon]$, as claimed. ☐

5. The equivariant wave map equation

This final section of the paper contains the main application of Theorem 4.5 to the equivariant wave map equation. The assumptions on the base manifolds define a class of manifolds which for the sake of exposition we call *admissible*. Note that if a smooth and rotationally symmetric manifold M^n has a global metric $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$, then $h(r)$ must be the restriction to $\mathbb{R}^+$ of a C^∞ odd function, with $h(0) = 0$ and $h'(0) = 1$. Note also that our result applies to C^k manifolds with $k = [\frac{n-1}{2}] + 3$, but for simplicity we confine ourselves to the smooth case.

Definition 5.1 (Admissible manifold)**.** We say that a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ is *admissible* if it is the restriction of a smooth odd function with $h'(0) = 1$ and in addition:

- *(i)* There exists $h_\infty \geq 0$ such that $H(r) := h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})'' = h_\infty + O(r^{-2})$ for $r \gg 1$.
- *(ii)* $H^{(j)}(r) = O(r^{-1})$ and $(h^{-\frac{1}{2}})^{(j)} = O(r^{-\frac{1}{2}-j})$ for $r \gg 1$ and $1 \leq j \leq [\frac{n-1}{2}]$.
- *(iii)* There exist $c, \delta_0 > 0$ such that for $r > 0$ we have $h(r) \geq cr$ while the function $P(r) = rH(r) - rh_\infty + \frac{1-\delta_0}{4r}$ satisfies the condition $P(r) \geq 0 \geq P'(r)$.

We say that a manifold M^n is *admissible* if it has a global metric of the form $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$ with $h(r)$ admissible.

The simplest admissible manifolds are the flat space, with $h(r) = r$ and $h_\infty = 0$, and the real hyperbolic space $\mathbb{H}^n$, with $h(r) = \sinh(r)$ and $h_\infty > 0$. However the class is substantially larger, and we shall exhibit a few interesting examples below.

Consider now the equivariant wave map equation

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + k(n-2+k)\frac{g(\phi)g'(\phi)}{h(r)^2} = 0, \tag{5.1}$$

with initial data

$$\phi(0,x) = \phi_0(x), \qquad \partial_t\phi(0,x) = \phi_1(x) \tag{5.2}$$

from the admissible, n-dimensional base manifold M^n with metric $dr^2+h(r)^2d\omega^2_{\mathbb{S}^{n-1}}$ to a target, ℓ-dimensional manifold N^ℓ, with metric $d\rho^2 + g(\rho)^2 d\omega^2_{\mathbb{S}^{\ell-1}}$. Note that the base manifold is noncompact while the target can arbitrary, thus $g : [0,A) \to \mathbb{R}^+$ with A finite or infinite.

In the following statement we use the notation $|D_M| = (-\Delta_M)^{\frac{1}{2}}$. The Sobolev space H^s on M^n is defined through the norm

$$\|\phi\|_{H^s} := \|(1-\Delta_M)^{\frac{s}{2}} v\|_{L^2(M^n)}$$

while the weighted Sobolev spaces $H^s_q(w)$ on M^n are defined through the norms

$$\|v\|_{H^s_q(w)} := \|w^{-1}v\|_{H^s_q(\mathbb{R}^m)}, \qquad w(r) := \frac{r^{\frac{m-1}{2}}}{h(r)^{\frac{n-1}{2}}}.$$

Then we have:

Corollary 5.2. *Let $n \geq 3$, $k \geq 1$ and $0 \leq \delta < k$. Let M^n and N^ℓ be two rotationally invariant manifolds of dimension n and ℓ respectively, with M^n admissible, and let h_∞ be as in Definition 5.1.*

If $h_\infty > 0$ and $\|\phi_0\|_{H^{\frac{n}{2}+\delta}} + \|\phi_1\|_{H^{\frac{n}{2}-1+\delta}}$ is sufficiently small, Problem (5.1), (5.2) has a unique global solution $\phi \in L^\infty H^{\frac{n}{2}+\delta} \cap CH^{\frac{n}{2}+\delta} \cap L^p H_q^{\frac{n-1}{2}+\delta}(w)$, with p,q as in Theorem 4.5. Moreover, if $\delta \geq \frac{1}{m+1}$, this is the unique solution in $CH^{\frac{n}{2}+\delta}$.

If $h_\infty = 0$ and $\||D_M|^{\frac{1}{2}}\phi_0\|_{H^{\frac{n-1}{2}+\delta}(M)} + \||D_M|^{-\frac{1}{2}}\phi_1\|_{H^{\frac{n-1}{2}+\delta}(M)}$ is sufficiently small, Problem (5.1), (5.2) has a unique global solution ϕ with $|D_M|^{\frac{1}{2}}\phi \in L^\infty H^{\frac{n-1}{2}+\delta} \cap CH^{\frac{n-1}{2}+\delta}$ and $\phi \in L^p H_q^{\frac{n-1}{2}+\delta}(w)$. Moreover, if $\delta \geq \frac{1}{m+1}$, this is the unique solution with $|D_M|^{\frac{1}{2}}\phi \in CH^{\frac{n-1}{2}+\delta}$.

Proof. The proof is just a transposition of Theorems 4.5 and 4.6 via the change of variables $\phi(t,r) = w(r)\psi(t,r)$, using the equivalence of norms given by Lemma 2.6. □

We conclude the paper with a discussion of the class of admissible manifolds. It is not difficult to come up with explicit examples, notably the flat space $\mathbb{R}^n$, the real hyperbolic spaces $\mathbb{H}^n$ for $n \geq 3$, and some spaces with polynomial growth of the metric; this already shows that admissibility does not impose a constraint on the growth rate of the metric at infinity. The examples are discussed in detail below.

However, we first give some stability criteria which show that manifolds sufficiently close to an admissible manifold are also admissible. A simple criterion is the following:

Proposition 5.3. *Let h, h_ϵ be restrictions to $\mathbb{R}^+$ of smooth odd functions, with $h'(0) = h'_\epsilon(0) = 1$, and let $H(r) := h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})''$ and $H_\epsilon(r) := h_\epsilon^{\frac{1-n}{2}}(h_\epsilon^{\frac{n-1}{2}})''$. Assume the following conditions:*

(1) $h_\epsilon > cr$ *for some* $c > 0$ *and all* $r > 0$*;*
(2) $|H - H_\epsilon| \leq \frac{\epsilon}{r^2}$ *for some* $\epsilon > 0$ *and all* $r > 0$*;*
(3) $|H' - H'_\epsilon| \leq \frac{\epsilon}{r^3}$ *for some* $\epsilon > 0$ *and all* $r > 0$*;*
(4) $H^{(j)} - H_\epsilon^{(j)} = O(r^{-1})$ *for* $r \gg 1$*,* $j \leq [\frac{n-1}{2}]$*;*
(5) $(h_\epsilon^{-\frac{1}{2}})^{(j)} = O(r^{-j-\frac{1}{2}})$ *for* $r \gg 1$*,* $j \leq [\frac{n-1}{2}]$*.*

If h *is admissible, then* h_ϵ *is also admissible, provided* ϵ *is sufficiently small.*

Proof. The criterion is a restatement of Definition 5.1. Note in particular that, by (2), the limit h_∞ at infinity of H and H_ϵ is the same, and if we define $P_\epsilon = r(H_\epsilon - h_\infty) + \frac{1-\delta_1}{4r}$ with $0 < \delta_1 < \delta_0$, the property $P_\epsilon \geq 0 \geq P'_\epsilon$ follows immediately from the corresponding property for P and assumptions (2), (3), provided ϵ is small enough. ☐

We can make the criterion easier to apply by introducing the functions

$$\sigma_j(r) := \frac{h_\epsilon^{(j)}}{h_\epsilon}, \qquad p_j(r) := \frac{h_\epsilon^{(j)} - h^{(j)}}{h}. \tag{5.3}$$

Then the difference $H_\epsilon - H$ can be expressed as

$$H_\epsilon - H = \tfrac{n-1}{2}(p_2 - \sigma_2 p_0) + \tfrac{(n-1)(n-3)}{4}(2\sigma_1 + \sigma_1 p_0 - p_1)(p_1 - \sigma_1 p_0) \tag{5.4}$$

while by Faa' di Bruno's formula we have

$$(h_\epsilon^{-\frac{1}{2}})^{(j)} = \sum_{\nu=0}^{j} \sum_{j_1+\dots+j_\nu=j} C h_\epsilon^{-\frac{1}{2}} \sigma_{j_1} \cdots \sigma_{j_\nu} \tag{5.5}$$

where the constants $C = C(\nu, j_1, \dots, j_\nu)$ may be different for each term of the sum. Note also the recursive relations

$$\sigma'_j = \sigma_{j+1} - \sigma_j \sigma_1 \qquad p'_j = p_{j+1} + p_j(p_1 - \sigma_1 p_0 - \sigma_1). \tag{5.6}$$

We can now give an effective criterion which is useful in case the metric $h(r)$ grows exponentially, so that we can not expect any decay at infinity for σ_j. The following conditions are not sharp but easy to check on concrete examples:

Proposition 5.4. *Let* $h, h_\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ *be restrictions of smooth odd functions with* $h'(0) = h'_\epsilon(0) = 1$*. Assume that*
(i) $h_\epsilon(r) \gtrsim r + r^n$ *for* $r > 0$
(ii) $|h_\epsilon^{(j)}| \lesssim h_\epsilon$ *for* $j \leq [\frac{n-1}{2}] + 2$ *and* $r \gg 1$
(iii) $|h^{(j)} - h_\epsilon^{(j)}|/h \leq \epsilon \langle r \rangle^{-3}$ *for* $j \leq 3$ *and* $r > 0$
(iv) $|h^{(j)} - h_\epsilon^{(j)}|/h \lesssim r^{-1}$ *for* $j \leq [\frac{n-1}{2}] + 2$ *and* $r \gg 1$*.*

If h *is admissible, then* h_ϵ *is also admissible, provded* ϵ *is small enough.*

Proof. We use the notations (5.3). To prove the claim it is sufficient to check conditions (1)–(5) in Proposition 5.3.

Applying (ii) for large r, and the definition and smoothness of h_ϵ near zero, we have

$$|\sigma_j| \lesssim \frac{1}{r} + 1$$

for all $r > 0$ and $j \leq [\frac{n-1}{2}] + 2$. We have also by (iii)

$$|p_j| \leq \epsilon \langle r \rangle^{-3} \quad \text{for} \quad r > 0,\ j \leq 3.$$

Then by (5.4) we obtain easily

$$|H_\epsilon - H| \lesssim \epsilon r^{-2}, \qquad |H'_\epsilon - H'| \lesssim \epsilon r^{-3}$$

which are conditions (2), (3), while (1) is implied by (i). Moreover, using the recursions (5.6) and assumption (iv), we see that σ_j is bounded for large r and

$j \le [\frac{n-1}{2}]+2$; this implies that condition (4) is satisfied. Finally, recalling (5.5) and using (i), we have for large r

$$|(h_\epsilon^{-\frac12})^{(j)}| \lesssim h_\epsilon^{-\frac12} \lesssim r^{-\frac{n}{2}}$$

since the σ_j are bounded, and this implies (5).

Note that condition (i) can be relaxed to $h \gtrsim r + r^{n-1}$ when n is even, and that in condition (iii) we could allow some singularity at 0, for instance it is sufficient to assume that $|p_3| \lesssim \epsilon r^{-3}$. □

When the metric $h(r)$ has polynomial growth, it is more convenient to use the following set of conditions to check for admissibility:

Proposition 5.5. *Let $h, h_\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ be restrictions of smooth odd functions with $h'(0) = h'_\epsilon(0) = 1$. Assume that*

(i) $h_\epsilon(r) \gtrsim cr$ for $r > 0$ and some $c > 0$

(ii) $|h_\epsilon^{(j)}| \lesssim h_\epsilon r^{-j}$ for $j \le [\frac{n-1}{2}]+2$ and $r \gg 1$

(iii) $|h^{(j)} - h_\epsilon^{(j)}|/h \le \epsilon r^{-j}$ for $j \le 3$ and $r > 0$

(iv) $|h^{(j)} - h_\epsilon^{(j)}|/h \lesssim r^{-1}$ for $1 \le j \le [\frac{n-1}{2}]+2$ and $r \gg 1$.

If h is admissible, then h_ϵ is also admissible, provded ϵ is small enough.

Proof. The proof is very similar to the previous one and again based on Proposition 5.3. Note in particular that we have now

$$|(h_\epsilon^{-\frac12})^{(j)}| \lesssim \sum h_\epsilon^{-\frac12} r^{-j_1-\dots-j_\nu} \lesssim r^{-\frac12-j}$$

by assumptions (i), (ii). □

5.1. **Asymptotically flat manifolds.** For the flat metric $h(r) = r$ we have $H(r) = \frac{(n-1)(n-3)}{4r^2}$ and it is elementary to check that all conditions of Definition 5.1 are satisfied, so that flat $\mathbb{R}^n$ is an admissible manifold. Consider now a rotationally symmetric manifold M^n whose metric is a perturbation of the flat space, of the form

$$h_\epsilon(r) = r + \mu(r) \tag{5.7}$$

with $\mu(r)$ odd, smooth, with $\mu'(0) = 0$, satisfying the following assumptions:

$$|\mu(r)| + r|\mu'(r)| + r^2|\mu''(r)| + r^3|\mu'''(r)| \le \epsilon r \quad \text{for all} \quad r > 0 \tag{5.8}$$

and

$$|\mu^{(j)}(r)| \lesssim r^{1-j} \quad \text{for} \quad r \gg 1, \quad j \le [\tfrac{n-1}{2}]+2. \tag{5.9}$$

Then Proposition 5.5 implies immediately that the metric h_ϵ is admissible if ϵ is sufficiently small.

Note that in dimension $n = 4$ this result is essentially a corollary of Theorem 1.1 in [27]. In that paper the global existence of small wave maps is proved on four dimensional, asymptotically flat manifolds without symmetry assumptions.

5.2. **Perturbations of hyperbolic spaces.** For real hyperbolic spaces $M = \mathbb{H}^n$ we have

$$h(r) = \sinh(r), \qquad h_\infty = \frac{(n-1)^2}{4}, \qquad H(r) = h_\infty + \frac{(n-1)(n-3)}{4\sinh^2 r}$$

so that (i), (ii) of Definition 5.1 are satisfied. Moreover

$$P(r) = \frac{(n-1)(n-3)}{4}\frac{r}{\sinh^2 r} + \frac{1-\delta_0}{4r}$$

and it is easy to check that $P \ge 0 \ge P'$ if $\delta_0 < 1$. Thus Corollary 5.2 implies global existence of equivariant wave maps $\phi : \mathbb{R}\times\mathbb{H}^n \to N^\ell$ for small data in the critical space $H^{\frac{n}{2}}$, for $n \geq 3$. Note that this result could be also obtained by applying the sharp Strichartz estimates in [1], [2] for linear flows on real hyperbolic spaces.

However, we are able to treat the more general case of a perturbation of the hyperbolic metric

$$h_\mu(r) = \sinh r + \mu(r) \tag{5.10}$$

under rather bland conditions on the perturbation $\mu(r)$. For instance, we may assume that, for all $r > 0$,

$$|\mu(r)| + |\mu'(r)| + |\mu''(r)| + |\mu'''(r)| \leq \epsilon \langle r \rangle^{-3} \sinh r$$

and that, for sufficiently large $r \gg 1$,

$$|\mu^{(j)}(r)| \leq C r^{-1} e^r.$$

Note that the perturbation $\mu(r)$ can be unbounded as $r \to +\infty$. Then, by Proposition 5.4, the function h_μ is also admissible provided ϵ is small enough, and we obtain the global existence of small equivariant wave maps of critical regularity from a base manifold with metric $dr^2 + h_\mu(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$.

5.3. **Manifolds with prescribed growth.** It is not difficult to construct examples of admissible manifolds if one allows a singularity at the south pole $r = 0$. The singularity can be smoothed out by flattening the manifold near zero with a suitable cutoff. We illustrate the procedure with two examples, one metric with polynomial growth and one with exponential growth.

The choice

$$h(r) = r(1 + \sqrt{r})^M, \qquad M > 0$$

gives rise to

$$H(r) = \frac{n-1}{16(1+\sqrt{r})^2 r^2} H_0(r)$$

with

$$H_0(r) = 4(n-3) + (4n(M+2) - 6(M+4))\sqrt{r} + (M+2)(M(n-1) + 2(n-3))r$$

and

$$P(r) = \frac{Q_0 + Q_1\sqrt{r} + Q_2 r}{16(1+\sqrt{r})^2 r},$$

where

$$Q_0 = 4(n-2)^2 - 4\delta_0, \qquad Q_1 = 2M(n-1)(2n-3) + 8(n-2)^2 - 8\delta_0,$$
$$Q_2 = (Mn + 2n - M - 4)^2 - 4\delta_0.$$

Since $0 < \delta_0 < 1$, we see that $P(r)$ is a sum of positive, decreasing functions, and all properties of Definition 5.1 are satisfied, with the exception of $h(r)$ being the restriction of a smooth odd function, due to the singularity at $r = 0$. Now we modify the definition of $h(r)$ as follows:

$$h_\epsilon(r) = r(1 + \sqrt{r} \cdot e^{-\frac{\epsilon}{r}})^M$$

where $\epsilon > 0$ is a small parameter. Then h_ϵ is still admissible provided ϵ is small enough, as it follows from Proposition 5.5, and is of course smooth at $r = 0$ and can be extended to an odd function on $\mathbb{R}$.

In a similar way, the metric with exponential growth

$$h(r) = e^r - 1$$

gives

$$H(r) = h_\infty + \frac{(n-1)(2(n-2)e^r - n + 1)}{4(e^r - 1)^2}, \qquad h_\infty = \frac{(n-1)^2}{4}$$

so that

$$P(r) = \frac{(n-1)(n-2)}{2}\frac{r}{e^r - 1} + \frac{(n-1)(n-3)}{4}\frac{r}{(e^r-1)^2} + \frac{1-\delta_0}{4r}$$

is a sum of positive, decreasing terms. Thus all conditions in Definition 5.1 are trivially satisfied, with the exception of $h(r)$ being the restriction of an odd smooth function, but it is sufficient to modify it near zero as above to obtain an admissible metric.

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