No invariant line fields on Cantor Julia sets

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Abstract

In this paper, we prove that a rational map with a Cantor Julia set carries no invariant line field on its Julia set. It follows that a structurally stable rational map with a Cantor Julia set is hyperbolic.

1 Introduction and statements

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of the Riemann sphere to itself of degree $d \geqslant 2$. The map f is hyperbolic if there are a smooth conformal metric ρ defined on a neighborhood of the Julia set $J(f)$ and a constant $C > 1$ such that $\|f'(z)\|_\rho > C$ for all $z \in J(f)$. It is equivalent to every critical point of f tends to an attracting periodic cycle under forward iteration. See [20] and [24].

Let Rat_d be the space of all the Möbius equivalence classes of rational maps of degree d. The space Rat_d has dimension $2d-2$. A central problem in holomorphic dynamics is the following.

Conjecture 1 (Density of hyperbolicity). *The set of hyperbolic rational maps is open and dense in the space* Rat_d.

Openness of the set of hyperbolic rational maps is known, but density is only known in the family of real polynomials, see [7], [10], [11], [13], [20] and [29].

A rational map f admits an *invariant line field* on the Julia set $J(f)$ if there is a measurable Beltrami differential $\mu(z)\frac{d\bar{z}}{dz}$ on $\hat{\mathbb{C}}$ such that $f^*\mu = \mu$ a.e., $|\mu| = 1$ on a positive measure subset E of $J(f)$ and $\mu = 0$ on $\hat{\mathbb{C}} \setminus E$.

A rational map f is called a *Lattès example* if it is doubly covered by an integral torus endmorphism. The Julia set of such a rational map is $\hat{\mathbb{C}}$ and $\frac{d\bar{z}}{dz}$ is an invariant line field on $\hat{\mathbb{C}}$.

Conjecture 2. *A rational map f carries no invariant line fields on its Julia set, except when f is a Lattès example.*

This conjecture is stronger than the density of hyperbolic dynamics.

Theorem A ([23]). *The no invariant line field conjecture implies the density of hyperbolic dynamics in the space of all rational maps.*

The absence of invariant line fields on the Julia set is known in the following cases:

(1) Non-infinitely renormalizable quadratic polynomials with no irrational indifferent periodic points, [14], [29] and [32].

(2) Robust infinitely renormalizable quadratic polynomials and real quadratic polynomials, [20].

(3) Quadratic polynomials with a Siegel cycle of bounded type rotation number, [15], [22] and [25].

(4) Real polynomials with only one non-escaping critical point which is real and has odd local degree, [12].

(5) Real rational maps(non Lattès example) whose critical points are all on the extended real axis and have even local degrees, [28].

(6) Summable rational maps with completely invariant Fatou domains, [16].

(7) Summable rational maps with small exponents, [6].

(8) Weakly hyperbolic rational maps, [8].

In this paper we will prove

Theorem 1. *Let f be a rational map with a Cantor Julia set. Then f carries no invariant line fields on its Julia set.*

Remark 1. (1) It is reasonable to conjecture that a Cantor Julia set always has measure zero. Theorem 1 can be regarded as a step towards this conjecture.

(2) As a special case of Theorem 1, we disprove a question in the Chapter 12 of [3].

Let X be a complex manifold. A *holomorphic family* of rational maps $\{f_\lambda(z)\}_{\lambda\in X}$ is a holomorphic map $X\times\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, given by $(\lambda,\, z)\rightarrowtail f_\lambda(z)$. Let $X^s\subset X$ be the set of structurally stable parameters. That is, $a\in X^s$ if and only if there is a neighborhood U of a such that f_a and f_b are topologically conjugate for all $b\in U$. The space $X^{qc}\subset X$ of quasiconformally stable parameters is defined similarly, with conjugacy replaced by quasiconformality.

Using the Harmonic λ-Lemma of Bers and Royden, McMullen and Sullivan proved the following result.

Theorem B ([19] and [23]). *In any holomorphic family of rational maps, X^s is open and dense in X. Moreover, the structurally stable and quasiconformally stable parameters coincide, i.e., $X^s = X^{qc}$.*

Conjecture 3. *A structurally stable rational map is hyperbolic.*

Combine Theorem 1 and the theory of Teichmüller space of rational maps in [23], we can prove the following result.

Theorem 2. *Let f be a rational map with a Cantor Julia set. If f is structurally stable, then it is hyperbolic.*

Remark 2. The same result in Theorem 2 was also proved by Makienko under some additional assumptions, see [17].

A nested sequence of some critical pieces constructed by Kozlovski, Shen, and van Strien in [11], which we shall call "KSS nest", will play a crucial rule in the proof of Theorem 1. Principal nest and modified principal nest are used to study the dynamics of unicritical polynomials, see [2], [4], [9] and [13]. In [13], Lyubich proved the linear growth of its "principal moduli" for quadratic polynomials. This yields the density of hyperbolic maps in the real quadratic family. The same result is also obtained by Graceyk and Świątek in [7]. See also [20] and [29]. Recently, the local connectivity of Julia sets and combinatorial rigidity for unicritical polynomials are proved in [9] and [2] by means of principal nest and modified principal nest.

This paper is organized as follows. In section 2, we present some distortion lemmas which are used in section 4. In section 3, we introduce the Branner-Hubbard puzzle about rational maps with Cantor Julia set and the KSS nest constructed in [11]. By means of the KSS nest and the distortion lemmas, we prove that the shapes of some critical puzzle pieces are bounded in section 4. In section 5, we give the proofs of Theorem 1 and Theorem 2.

2 Distortion lemmas

Any doubly connected domain A on the complex plane is conformally equivalent to one of the following three types of typical domains:

(1) $\mathbb{C}\backslash\{0\}$,

(2) $\Delta\backslash\{0\}$, where $\Delta = \{z \,|\, |z| < 1\}$,

(3) $A_R = \{z \,|\, 1 < |z| < R\}$.

In the case A is conformally equivalent to A_R, the modulus of A is defined as $\mathrm{mod}(A) = \frac{1}{2\pi}\ln R$. In the other two cases, $\mathrm{mod}(A) = \infty$.

For $0 < r < 1$, let $B_r = \Delta \setminus [0, r]$. The modulus $\mathrm{mod}(B_r)$ is decreasing in $(0, 1)$ with $\lim_{r\to 1^-} \mathrm{mod}(B_r) = 0$ and $\lim_{r\to 0^+} \mathrm{mod}(B_r) = +\infty$.

Grötzsch Theorem ([1]). *Let A be a doubly connected domain in Δ which separates the unit circle from the points $\{0, r\}$. Then* $\mathrm{mod}(A) \leqslant \mathrm{mod}(B_r)$.

Denote $d(\cdot,\cdot)$ and diam the distance and diameter with respect to the Euclidean metric respectively.

Lemma 1. *Let $\widetilde{U} \subset\subset U \neq \mathbb{C}$ be two simply connected domains and* $\mathrm{mod}(U \setminus \overline{\widetilde{U}}) \geqslant m > 0$. *Then there exists a constant $c = c(m) > 0$ such that*

$$d(\omega, \partial U) \geqslant c \cdot \mathrm{diam}(\widetilde{U})$$

for any $\omega \in \overline{\widetilde{U}}$.

Proof. For any $\omega \in \overline{\widetilde{U}}$, let $h_\omega(z)$ be a conformal map from Δ onto U with $h_\omega(0) = \omega$. Then

$$\mathrm{mod}(\Delta \setminus \overline{h_\omega^{-1}(\widetilde{U})}) = \mathrm{mod}(U \setminus \overline{\widetilde{U}}) \geqslant m > 0.$$

From Grötzsch Theorem, there exists a constant $r_0 = r_0(m) < 1$ such that $h_\omega^{-1}(\widetilde{U}) \subset \{z \,|\, |z| < r_0\}$. By Koebe Distortion Theorem, we have

$$d(\omega, \partial U) \geqslant \frac{1}{4}|h'_\omega(0)|$$

and

$$\mathrm{diam}(\widetilde{U}) \leqslant 2|h'_\omega(0)|\frac{r_0}{(1-r_0)^2}.$$

So we can take $c = c(m) = \dfrac{(1-r_0)^2}{8r_0} > 0$, which satisfies the inequality

$$d(\omega, \partial U) \geqslant c \cdot \mathrm{diam}(V).$$

□

Let U be a simply connected domain and $\omega \in U$. The *Shape* of U about ω, denoted by $\mathrm{Shape}(U, \omega)$, is defined as

$$\mathrm{Shape}(U, \omega) = \frac{\max_{z \in \partial U} d(\omega, z)}{\min_{z \in \partial U} d(\omega, z)} = \frac{\max_{z \in \partial U} d(\omega, z)}{d(\omega, \partial U)}.$$

Lemma 2. *Let $g : (\Delta, U, \widetilde{U}) \to (\Delta, V, \widetilde{V})$ be a holomorphic proper map of degree d with $0 \in \widetilde{U} \subset U \subset \Delta$, $0 \in \widetilde{V} \subset V \subset \Delta$. Suppose that*

(1) $\deg(g\,|_{\widetilde{U}}) = \deg(g\,|_U) = \deg(g\,|_\Delta) = d \geqslant 2$,

(2) $\mathrm{mod}(V \setminus \overline{\widetilde{V}}) \geqslant m > 0$.

Then there exists a constant $K = K(m,d) > 0$ such that

$$\mathrm{Shape}(U,0) \leqslant K \cdot \mathrm{Shape}(V,0)^{\frac{1}{d}}.$$

Proof. Let

$$R = \max_{z\in\partial U} |z|,\ r = \min_{z\in\partial U} |z|;$$
$$R' = \max_{\omega\in\partial V} |\omega|, r' = \min_{\omega\in\partial V} |\omega|.$$

There are points $z_R \in \partial U$ and $z_r \in \partial U$ such that $R = |z_R|$ and $r = |z_r|$.

The holomorphic proper map g can be written as

$$g(z) = e^{i\theta} \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a}_j z},$$

where $\theta \in [0, 2\pi)$ and $a_j \in \Delta$, $j = 1, \cdots, d$. By the first assumption, we have $a_j \in \widetilde{U}$ and $|a_j| < |z_R| < 1$ for all j.

Consider the annulus $\{z \,|\, |z| < R\} \setminus \overline{\widetilde{U}}$, we have

$$\mathrm{mod}(\{z \,|\, |z| < R\} \setminus \overline{\widetilde{U}}) \geqslant \mathrm{mod}(U \setminus \overline{\widetilde{U}}) \geqslant \frac{m}{d}.$$

By Grötzsch Theorem, there exists a constant $c_0 = c_0(\frac{m}{d}) > 1$ such that $R = |z_R| \geqslant c_0|a_j|$ for all j. We have

$$R' \geqslant |g(z_R)| = \prod_{j=1}^{d} \frac{|z_R - a_j|}{|1 - \overline{a}_j z_R|} \geqslant \prod_{j=1}^{d} \frac{|z_R| - |a_j|}{1 + |a_j z_R|}$$
$$\geqslant \prod_{j=1}^{d} \frac{|z_R| - |a_j|}{2} \geqslant \prod_{j=1}^{d} (\frac{c_0 - 1}{2c_0})|z_R| = c_1^d R^d$$

with $c_1 = c_1(m,d) = \dfrac{c_0 - 1}{2c_0}$. On the other hand,

$$r' \leqslant |g(z_r)| = \prod_{j=1}^{d} \frac{|z_r - a_j|}{|1 - \overline{a}_j z_r|} \leqslant \prod_{j=1}^{d} \frac{r + |a_j|}{1 - |a_j|} \leqslant \prod_{j=1}^{d} \frac{r + \frac{r}{c}}{1 - r_0} = c_2^d r^d$$

with $c_2 = c_2(m,d) = \dfrac{c+1}{c(1-r_0)}$ in which c and r_0 come from Lemma 1 and its proof. It follows that

$$\mathrm{Shape}(U,0) \leqslant K \cdot \mathrm{Shape}(V,0)^{\frac{1}{d}}$$

with $K = K(m,d) = \dfrac{c_2}{c_1}$. □

3 Branner-Hubbard puzzle and KSS nest

From now on, we always assume that the Julia set $J(f)$ of a rational map f is a cantor set. The Fatou set $F(f)$ has only one component. It is either an attracting basin or a parabolic basin.

We first construct the Branner-Hubbard puzzle.

The attracting case. We assume that ∞ is the fixed attracting point. Take a simply connected neighborhood $U_0 \subset F(f)$ of ∞ such that $U_0 \subset\subset f^{-1}(U_0)$. Let U_n be the component of $f^{-n}(U_0)$ containing ∞. Then $U_n \subset\subset U_{n+1}$ and $F(f) = \cup_{n=0}^{\infty} U_n$. For a large enough integer N_0, $f^{-n}(U_{N_0})$ has only one component for any $n \geqslant 0$. The set $f^{-n}(\hat{\mathbb{C}} \setminus \overline{U}_{N_0})$ is the disjoint union of a finite number of topological disks. For each $n \geqslant 0$, let $\mathbf{P_n}$ be the collection of all components of $f^{-n}(\hat{\mathbb{C}} \setminus \overline{U}_{N_0})$ which are called puzzle pieces of depth n.

For any point $x \in J(f)$ and any $n \geqslant 0$, there is only one $P_n(x) \in \mathbf{P_n}$ containing x. Thus each point $x \in J(f)$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$ and $\cap_{n \geqslant 0} P_n(x) = \{x\}$.

The parabolic case. We suppose that 0 is the parabolic fixed point and ∞ is in the Fatou set. According to the Leau-Fatou Flower Theorem, there is a flower petal $U_0 \subset F(f)$ with $0 \in \partial U_0$ such that $\overline{U_0} \subset f^{-1}(U_0) \cup \{0\}$. We can construct the puzzle as in the attracting case. Each point $x \in J(f) \setminus \cup_{n \geqslant 0} f^{-n}(0)$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$ and $\cap_{n \geqslant 0} P_n(x) = \{x\}$.

Take N_0 large enough such that U_{N_0} contains all critical points in the Fatou set and each puzzle piece contains at most one critical point.

Let

$$\mathrm{Crit} = \{c \in J(f) \,|\, c \text{ is the critical point of } f\}$$

in the attracting case and

$$\mathrm{Crit} = \{c \in J(f) \setminus \cup_{n \geqslant 0} f^{-n}(0) \,|\, c \text{ is the critical point of } f\}$$

in the parabolic case.

For each $x \in J(f)$ (in parabolic case, $x \in J(f) \setminus \cup_{n \geqslant 0} f^{-n}(0)$ respectively), the tableaux $T(x)$ is defined in [3]. It is a two dimension array $\{P_{n,l}(x)\}_{n \geqslant 0, l \geqslant 0}$ with $P_{n,l}(x) = f^l(P_{n+l}(x)) = P_n(f^l(x))$. The position (n, l) is called critical if $P_{n,l}(x)$ contains a critical point of f. If $P_{n,l}(x)$ contains a critical point c, the position (n, l) is called a c-position. The tableau $T(c)$ of a critical point $c \in \mathrm{Crit}$ is called periodic if there is a positive integer k such that $P_n(c) = f^k(P_{n+k}(c))$ for all $n \geqslant 0$. Since the Julia set is a Cantor set, $T(c)$ is not periodic for all $c \in \mathrm{Crit}$.

All the tableaus satisfy the following three rules

(T1) If $P_{n,l}(x) = P_n(c)$ for some critical point c, then $P_{i,l}(x) = P_i(c)$ for all $0 \leqslant i \leqslant n$.

(T2) If $P_{n,l}(x) = P_n(c)$ for some critical point c, then $P_{i,l+j}(x) = P_{i,j}(c)$ for $i + j \leqslant n$.

(T3) Let $T(c)$ be a tableau for some critical point c and $T(x)$ be any tableau. Assume

(a) $P_{n+1-l,l}(c) = P_{n+1-l}(c_1)$ for some critical point c_1 and $n > l \geqslant 0$, and $P_{n-i,i}(c)$ contains no critical points for $0 < i < l$.

(b) $P_{n,m}(x) = P_n(c)$ and $P_{n+1,m}(x) \neq P_{n+1}(c)$ for some $m > 0$.

Then $P_{n+1-l,m+l}(x) \neq P_{n+1-l}(c_1)$.

Definition 1. (1) The tableau $T(x)$ for x is *non-critical* if there exists an integer $n_0 \geqslant 0$ such that (n_0, j) is not critical for all $j > 0$.

(2) We write $x \to y$ if for any $n \geqslant 0$, there exists $j > 0$ such that $y \in P_{n,j}(x)$, i.e., $f^j(P_{n+j}(x)) = P_n(y)$. It is clear that $x \to y$ if and only if $y \in \cup_{n>0} f^{-n}(x)$ or $y \in \omega(x)$, the limit set of the forward orbit of x. If $x \to y$ and $y \to z$, then $x \to z$. For each critical point $c \in \mathrm{Crit}$, let

$$F(c) = \{c' \in \mathrm{Crit} \mid c \to c'\}$$

and

$$[c] = \{c' \in \mathrm{Crit} \mid c \to c' \text{ and } c' \to c\}.$$

(3) We say $P_{n+k}(c')$ is a *child* of $P_n(c)$ if $c' \in [c]$, $f^k(P_{n+k}(c')) = P_n(c)$, and $f^{k-1} : P_{n+k-1}(f(c')) \to P_n(c)$ is conformal.

(4) Suppose $c \to c$, i.e., $[c] \neq \emptyset$. We say $T(c)$ is *persistently recurrent* if $P_n(c_1)$ has only finitely many children for all $n \geqslant 0$ and all $c_1 \in [c]$. Otherwise, $T(c)$ is said to be *reluctantly recurrent*.

Take N_0 large enough such that for any $c \in \mathrm{Crit}$, there is no c'-position in the first row of $T(c)$ if $c \not\to c'$.

Let

$$\begin{aligned}
\mathrm{Crit_n} &= \{c \in \mathrm{Crit} \mid T(c) \text{ is non-critical}\},\\
\mathrm{Crit_p} &= \{c \in \mathrm{Crit} \mid T(c) \text{ is persistently recurrent}\},\\
\mathrm{Crit_r} &= \{c \in \mathrm{Crit} \mid T(c) \text{ is reluctantly recurrent}\},\\
\mathrm{Crit_{en}} &= \{c' \in \mathrm{Crit} \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_n}\},\\
\mathrm{Crit_{ep}} &= \{c' \in \mathrm{Crit} \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_p}\},\\
\mathrm{Crit_{er}} &= \{c' \in \mathrm{Crit} \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_r}\}.
\end{aligned}$$

Then

$$\mathrm{Crit} = \mathrm{Crit_n} \cup \mathrm{Crit_p} \cup \mathrm{Crit_r} \cup \mathrm{Crit_{en}} \cup \mathrm{Crit_{ep}} \cup \mathrm{Crit_{er}}.$$

This is not a classification because these sets might intersect.

The following lemma can be found in [27].

Lemma 3. *If $T(c)$ is persistently recurrent, then $F(c) = [c]$.*

Now, we briefly introduce a critical nest which is constructed by Kozlovski, Shen, and van Strien in [11]. Such nest will be called *KSS* nest.

Let A be an open set and $x \in A$. The connected component of A containing x will be denoted by $\mathrm{Comp}_x(A)$. Given a puzzle piece I, let

$$D(I) = \{z \in \mathbb{C} \,|\, \exists k \geqslant 1 \ s.t. \ f^k(z) \in I\} = \bigcup_{k \geqslant 1} f^{-k}(I).$$

For any $z \in D(I)$, let $\mathcal{L}_z(I)$ be the connected component of $D(I)$ containing z. We further define $\hat{\mathcal{L}}_z(I) = I$ if $z \in I$ and $\hat{\mathcal{L}}_z(I) = \mathcal{L}_z(I)$ if $z \in D(I) \setminus I$.

For any $z \in D(I)$, let $k \geqslant 1$ be the integer such that $f^k(\mathcal{L}_z(I)) = I$ and let n_0 be the depth of I. By tableau rules (T1) and (T2), there is at most one c-position on the diagonal

$$\{(n, m) \,|\, n + m = n_0 + k, \quad n_0 < n \leqslant n_0 + k\}$$

in the tableau $T(z)$ for any $c \in \mathrm{Crit}$. Hence

$$\deg(f^k : \mathcal{L}_z(I) \to I) \leqslant D$$

for some constant $D < \infty$ depending only on Crit.

Suppose $T(c_0)$ is persistently recurrent, then $F(c_0) = [c_0]$. Let

$$b = \#[c_0], \quad d_0 = \deg_{c_0} f, \quad d_{max} = \max\{\deg_c f \,|\, c \in [c_0]\},$$

and

$$\mathrm{orb}([c_0]) = \bigcup_{n \geqslant 0} f^n([c_0]).$$

For each puzzle piece $I \ni c_0$, there are pullbacks $\mathcal{A}(I) \subset \mathcal{B}(I)$ of I containing c_0 with the following properties

(P1) $f^t(\mathcal{B}(I)) = I$ and $\deg(f^t|_{\mathcal{B}(I)}) \leqslant {d_{max}}^{b^2}$,

(P2) $\mathcal{A}(I) = \mathrm{Comp}_{c_0} f^{-t}(\mathcal{L}_{f^t(c_0)}(I))$, $f^s(\mathcal{A}(I)) = I$, $s - t \geqslant 1$ is the smallest integer such that $f^{s-t}(f^t(c_0)) \in I$ and $\deg(f^s|_{\mathcal{A}(I)}) \leqslant {d_{max}}^{b^2+b}$,

(P3) $(\mathcal{B}(I) - \mathcal{A}(I)) \cap \mathrm{orb}([c_0]) = \emptyset$.

For details, see [11] or [27].

Definition 2. Given a puzzle piece P containing c_0, a *successor* of P is a piece of the form $\hat{\mathcal{L}}_{c_0}(Q)$, where Q is a child of $\hat{\mathcal{L}}_c(P)$ for some $c \in [c_0]$.

It is clear that $\mathcal{L}_{c_0}(P)$ is a successor of P. Since $T(c_0)$ is persistently recurrent, P has at least two successors and that P has only finitely many successors. Let $\Gamma(P)$ be the last successor of P. Then there exists an integer $q \geqslant 1$, the largest among all of the successors of P, such that $f^q(\Gamma(P)) = P$. From the definition of successor, we have

(P4) $\deg(f^q|_{\Gamma(P)}) \leqslant d_{max}^{2b-1}$.

Now we can define the *KSS nest* in the following way: I_0 is a given piece containing c_0 and for $n \geqslant 0$,

$$\begin{aligned}
&L_n = \mathcal{A}(I_n),\\
&M_{n,0} = K_n = \mathcal{B}(L_n),\\
&M_{n,j+1} = \Gamma(M_{n,j}) \text{ for } 0 \leqslant j \leqslant 3b-1,\\
&I_{n+1} = M_{n,3b} = \Gamma^{3b}(K_n) = \Gamma^{3b}(\mathcal{B}(\mathcal{A}(I_n))),
\end{aligned}$$

with $b = \#[c_0]$. See Figure 1.

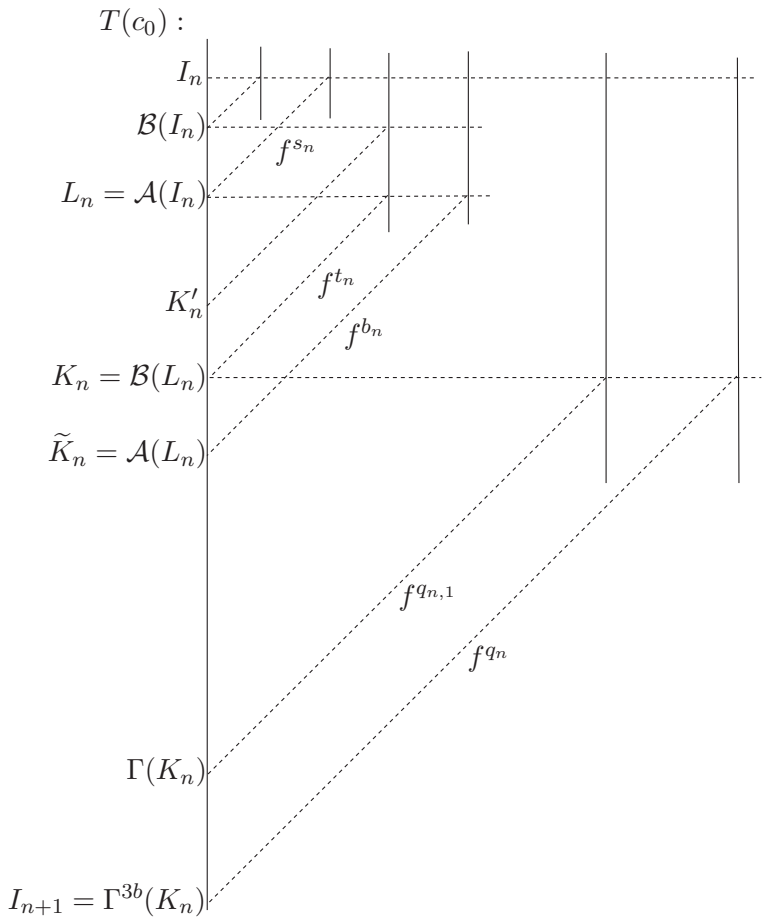

Figure 1 KSS nest

Suppose $f^{s_n}(L_n) = I_n$, $f^{t_n}(K_n) = L_n$, $f^{q_{n,j}}(M_{n,j}) = M_{n,j-1}$ for $1 \leqslant j \leqslant 3b$, and $q_n = \sum_{j=1}^{3b} q_{n,j}$. Let $K'_n = \mathrm{Comp}_{c_0} f^{-t_n}(\mathcal{B}(I_n))$ and $\widetilde{K}_n = \mathcal{A}(L_n)$.

Then $(K'_n \setminus K_n) \cap \mathrm{orb}([c_0]) = \emptyset$ and

$$(K_n \setminus \widetilde{K}_n) \cap \mathrm{orb}([c_0]) = (\mathcal{B}(L_n) \setminus \mathcal{A}(L_n)) \cap \mathrm{orb}([c_0]) = \emptyset.$$

It follows that

$$\deg(f^{t_n}|_{K'_n}) = \deg(f^{t_n}|_{K_n}).$$

See Figure 1.

Let $p_n = q_{n-1} + s_n + t_n$. Then $f^{p_n}(K_n) = K_{n-1}$ and

$$d_0^{3b+2} \leqslant \deg(f^{p_n}|_{K_n}) \leqslant d_1 = d_{max}^{8b^2-2b}.$$

For any puzzle piece J containing c_0 and $z \in J \cap \mathrm{orb}([c_0])$, let $r_z(J) = k(z) \geqslant 1$ be the smallest integer such that $f^{k(z)}(z) \in J$ and

$$r(J) = \min\{k(z) \mid z \in J \cap \mathrm{orb}([c_0])\}.$$

It is obvious that

(1) $r(J_1) \geqslant r(J_2)$ if $J_1 \subset J_2$;

(2) $r(J) \geqslant k$ if $c_0 \in J \subset J'$, $f^k : J \to J'$ and $c_0 \notin f^i(J)$ for $0 < i < k$;

(3) $\mathrm{depth}(\mathcal{A}(J)) - \mathrm{depth}(\mathcal{B}(J)) = r_{f^t(c_0)}(J) \geqslant r(J)$.

The following lemma plays a crucial rule in the proof of our results and in [27].

Lemma 4 ([11]). *For any $n \geqslant 1$,*

(1) $r(I_n) \leqslant s_n \leqslant (b+1)r(L_n)$;

(2) $r(L_n) \leqslant t_n \leqslant br(K_n)$;

(3) $2r(M_{n,j-1}) \leqslant q_{n,j} \leqslant r(M_{n,j})$ for $1 \leqslant j \leqslant 3b$.

The following is an immediately corollary.

Corollary 1. *For any $n \geqslant 1$,*

(1) $s_{n-1} \leqslant r(L_n)$;

(2) $r(I_n) \geqslant 2^{3b} r(I_{n-1})$.

4 Bounded shape of puzzle pieces

We suppose $T(c_0)$ is persistently recurrent and puzzle pieces $c_0 \in \widetilde{K}_n \subset K_n \subset K'_n$ are constructed as in the previous section.

For the polynomial case, the following is the key lemma in [27].

Lemma 5 ([27]). $\liminf_{n\to\infty} \mathrm{mod}(K'_n \setminus \overline{K}_n) > 0$.

Lemma 6. *There exist a constant $m > 0$ depending only on b and d_{max}, and an integer n_0 such that $\mathrm{mod}(K'_n \setminus \overline{K}_n) \geqslant m$ and $\mathrm{mod}(K_n \setminus \overline{\widetilde{K}_n}) \geqslant m$ for all $n \geqslant n_0$.*

Proof. In the attracting case, the annuli $K'_n \setminus \overline{K}_n$ and $K_n \setminus \overline{\widetilde{K}_n}$ are always non-degenerate.

In the parabolic case, there exists an integer n_0 such that $K'_n \setminus \overline{K}_n$ and $K_n \setminus \overline{\widetilde{K}_n}$ are non-degenerate for $n \geqslant n_0$. In fact, there is an integer k_0 such that $P_0(c_0) \setminus \overline{P_{k_0}(c_0)}$ is non-degenerate because $c_0 \notin \cup_{n\geqslant 0} f^{-n}(0)$ and $\cap_{n\geqslant 0} P_n(c_0) = \{c_0\}$. Take $P_{k_0}(c_0)$ as I_0 in the construction of KSS nest. By Corollary 1,

$$\begin{aligned}\mathrm{depth}(K_n) - \mathrm{depth}(K'_n) &= \mathrm{depth}(\mathcal{A}(I_n)) - \mathrm{depth}(\mathcal{B}(I_n)) \\ &\geqslant r(I_n) \to \infty\end{aligned}$$

and

$$\begin{aligned}\mathrm{depth}(\widetilde{K}_n) - \mathrm{depth}(K_n) &= \mathrm{depth}(\mathcal{A}(L_n)) - \mathrm{depth}(\mathcal{B}(L_n)) \\ &\geqslant r(L_n) \geqslant r(I_n) \to \infty.\end{aligned}$$

So there exists an integer n_0 such that

$$\mathrm{depth}(K_n) - \mathrm{depth}(K'_n) \geqslant k_0$$

and

$$\mathrm{depth}(\widetilde{K}_n) - \mathrm{depth}(K_n) \geqslant k_0$$

for $n \geqslant n_0$. This implies that $K'_n \setminus \overline{K}_n$ and $K_n \setminus \overline{\widetilde{K}_n}$ are non-degenerate for $n \geqslant n_0$ because K_n and $\widetilde{K}_n$ are pullbacks of $I_0 = P_{k_0}(c_0)$.

By the same proof of Lemma 5, $\mathrm{mod}(K'_n \setminus \overline{K}_n) \geqslant \mu$ for some constant $\mu > 0$ depending only on b and d_{max} when $n \geqslant n_0$. See [27].

Suppose $f^{b_n}(\widetilde{K}_n) = L_n$. Let $h_n = b_n + s_n + q_{n-1}$ and $\widetilde{K}'_n = \mathrm{Comp}_{c_0} f^{-h_n}(K'_{n-1})$. Since

$$\mathrm{depth}(\widetilde{K}_n) - \mathrm{depth}(K_n) = r_{f^{t_n}(c_0)}(L_n) \geqslant r(L_n)$$

and

$$\begin{aligned}\mathrm{depth}(\widetilde{K}_n) - \mathrm{depth}(\widetilde{K}'_n) &= \mathrm{depth}(K_{n-1}) - \mathrm{depth}(K'_{n-1}) \\ &= \mathrm{depth}(\mathcal{A}(I_{n-1})) - \mathrm{depth}(\mathcal{B}(I_{n-1})) \\ &\leqslant s_{n-1},\end{aligned}$$

we conclude that $\widetilde{K}'_n \subset K_n$ from Corollary 1.

From properties (P1), (P2) and (P4),

$$\deg(f^{h_n}|_{\widetilde{K}'_n}) = \deg(f^{h_n}|_{\widetilde{K}_n}) \leqslant D$$

for some $D < \infty$ depending only on b and d_{max}. Hence

$$\begin{aligned} \mathrm{mod}\ (K_n \setminus \overline{\widetilde{K}}_n) &\geqslant \mathrm{mod}\ (\widetilde{K}'_n \setminus \overline{\widetilde{K}}_n) \\ &= \frac{\mathrm{mod}\ (K'_{n-1} \setminus K_{n-1})}{\deg(f^{h_n}|_{\widetilde{K}_n})} \\ &\geqslant \frac{\mu}{D}. \end{aligned}$$

Take $m = \frac{\mu}{D}$. This m depends only on b and d_{max}, and satisfied the conditions set out in this lemma. □

Proposition 1. *There exists a constant $M_1 > 0$ such that*

$$\mathrm{Shape}(K_n, c_0) \leqslant M_1$$

for $n \geqslant n_0$, where n_0 is the integer in Lemma 6.

Proof. Since $(K'_n \setminus K_n) \cap \mathrm{orb}([c_0]) = (K_n \setminus \widetilde{K}_n) \cap \mathrm{orb}([c_0]) = \emptyset$ and $f^{p_{n+1}}(K_{n+1}) = K_n$, it follows $f^{p_{n+1}}(c_0) \in \widetilde{K}_n$. Let $\Omega'_n = \mathrm{Comp}_{c_0} f^{-p_{n+1}}(K'_n)$ and $\widetilde{\Omega}_n = \mathrm{Comp}_{c_0} f^{-p_{n+1}}(\widetilde{K}_n)$. Then

$$2 \leqslant \deg(f^{p_{n+1}}|_{\Omega'_n}) = \deg(f^{p_{n+1}}|_{K_{n+1}}) = \deg(f^{p_{n+1}}|_{\widetilde{\Omega}_n}) \leqslant D_1$$

for some constant $D_1 < \infty$ depending only on b and d_{max}.

Let $\varphi : (\Delta, V, \widetilde{V}) \to (K'_n, K_n, \widetilde{K}_n)$ and $\psi : (\Delta, U, \widetilde{U}) \to (\Omega'_n, K_{n+1}, \widetilde{\Omega})$ be conformal maps with $\varphi(0) = c_0$ and $\psi(0) = c_0$. Let $g = \varphi^{-1} \circ f^{p_{n+1}} \circ \psi$. Then $g : (\Delta, U, \widetilde{U}) \to (\Delta, V, \widetilde{V})$ is a properly holomorphic map with

$$\deg(g\ |_{\widetilde{U}}) = \deg(g\ |_U) = \deg(g\ |_\Delta) = D_n.$$

By Koebe Distortion Theorem and Lemma 2, there exists a constant K depending only on b and d_{max} such that

$$\mathrm{Shape}(K_{n+1}, c_0) \leqslant K \cdot \mathrm{Shape}(K_n, c_0)^{\frac{1}{2}}$$

hold for all $n \geqslant n_0$. We conclude that

$$\mathrm{Shape}(K_n, c_0) \leqslant M_1$$

for some constant $M_1 > 0$ when $n \geqslant n_0$. □

5 Proofs of Theorem 1 and Theorem 2

Recall the definition of Crit in Section 3. Let $X_1 = \cup_{n\geqslant 0} f^{-n}(\mathrm{Crit})$ in the attracting case and

$$X_1 = (\cup_{n\geqslant 0} f^{-n}(\mathrm{Crit})) \bigcup (\cup_{n\geqslant 0} f^{-n}(0))$$

in the parabolic case, where 0 is the parabolic fixed point.

For any $x \in J(f) \setminus X_1$, let

$$\begin{aligned}
&\mathrm{Crit}(x) = \{c \in \mathrm{Crit} \mid x \to c\}\\
&\mathrm{Crit_n}(x) = \{c \in \mathrm{Crit}(x) \mid T(c) \text{ is non-critical}\},\\
&\mathrm{Crit_p}(x) = \{c \in \mathrm{Crit}(x) \mid T(c) \text{ is persistently recurrent}\},\\
&\mathrm{Crit_r}(x) = \{c \in \mathrm{Crit}(x) \mid T(c) \text{ is reluctantly recurrent}\},\\
&\mathrm{Crit_{en}}(x) = \{c' \in \mathrm{Crit}(x) \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_n}(x)\},\\
&\mathrm{Crit_{ep}}(x) = \{c' \in \mathrm{Crit}(x) \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_p}(x)\},\\
&\mathrm{Crit_{er}}(x) = \{c' \in \mathrm{Crit}(x) \mid c' \not\to c' \text{ and } c' \to c \text{ for some } c \in \mathrm{Crit_r}(x)\}.
\end{aligned}$$

Then

$$\mathrm{Crit}(x) = \mathrm{Crit_n}(x) \cup \mathrm{Crit_p}(x) \cup \mathrm{Crit_r}(x) \cup \mathrm{Crit_{en}}(x) \cup \mathrm{Crit_{ep}}(x) \cup \mathrm{Crit_{er}}(x).$$

Further let

$$\begin{aligned}
&X_2 = \{x \in J(f) \setminus X_1 \mid \mathrm{Crit}(x) = \emptyset \text{ or } \mathrm{Crit_n}(x) \cup \mathrm{Crit_r}(x) \neq \emptyset\},\\
&X_3 = \{x \in J(f) \setminus X_1 \mid \mathrm{Crit}(x) = \mathrm{Crit_p}(x) \cup \mathrm{Crit_{ep}}(x),\ \mathrm{Crit_{ep}}(x) \neq \emptyset\},\\
&X_4 = \{x \in J(f) \setminus X_1 \mid \mathrm{Crit}(x) = \mathrm{Crit_p}(x) \neq \emptyset\}.
\end{aligned}$$

Then

$$J(f) = \bigcup_{i=1}^{4} X_i.$$

Lemma 7. *For any $x \in X_2 \cup X_3$, there exist a puzzle piece P_0 of depth 0 and infinitely many i_n such that*

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leqslant D$$

for some constant $D < \infty$ depending on x.

Proof. There are four possibilities.

(1) $T(x)$ is non-critical, i.e. $\mathrm{Crit}(x) = \emptyset$. There exists an integer $n_0 \geqslant 0$ such that (n_0, j) is not a critical position for all $j > 0$. For any $n \geqslant 1$, $\deg(f^n|_{P_{n_0+n}(x)}) \leqslant \deg(f|_{P_{n_0+1}(x)})$. The degrees of these maps

$$f^{n_0+n} : P_{n_0+n}(x) \to P_0(f^{n_0+n}(x))$$

has an upper bound $D < \infty$. Take a subsequence i_n of $n_0 + n$ such that $P_0(f^{i_n}(x)) = P_0$ for some fixed puzzle piece P_0. Then

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leqslant D$$

for all n.

(2) $x \to c$ for some $c \in \mathrm{Crit_n}(x)$. From (1), there are a puzzle piece P_0, a positive integer N_1 and infinitely many j_n such that

$$\deg(f^{j_n} : P_{j_n}(c) \to P_0) \leqslant N_1$$

for all n. For each n, let l_n be the first moment such that $f^{l_n}(x) \in P_{j_n}(c)$, *i.e.*, (j_n, l_n) is the first c-position on the j_n-th row in $T(x)$. By tableau rules (T1) and (T2), there is at most one c'-position on the diagonal

$$\{(k, m) \mid k + m = j_n + l_n, \quad j_n < k \leqslant j_n + l_n\}$$

for any $c' \in \mathrm{Crit}(x) - \{c\}$. There exists a positive integer N_2 depending on $\mathrm{Crit}(x)$ such that

$$\deg(f^{l_n} : P_{j_n+l_n}(x) \to P_{j_n}(c)) \leqslant N_2.$$

Take $i_n = j_n + l_n$ and $D = N_1 + N_2$. Then

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leqslant D$$

for all n.

(3) $x \to c$ for some $c \in \mathrm{Crit_r}(x)$. There exist an integer $n_0 \geq 0$, $c' \in [c]$, $c_1 \in [c]$ and infinitely many integers $k_n \geqslant 1$ such that $\{P_{n_0+k_n}(c')\}_{n\geqslant 1}$ are children of $P_{n_0}(c_1)$. Since $c' \in [c]$, we have $x \to c'$. For each n, let m_n be the first moment such that $f^{m_n}(x) \in P_{n_0+k_n}(c')$. There is at most one $\widetilde{c}$-position on the diagonal

$$\{(n, m) \mid n + m = n_0 + k_n + m_n, \quad n_0 + k_n < n \leqslant n_0 + k_n + m_n\}$$

in $T(x)$ for any $\widetilde{c} \in \mathrm{Crit}(x) - \{c'\}$. Therefore, $f^{m_n+k_n}(P_{n_0+k_n+m_n}(x)) = P_{n_0}(c_1)$ and there is an integer N_3 independent of n such that

$$\deg(f^{m_n+k_n}|_{P_{n_0+k_n+m_n}(x)}) \leqslant N_3 < \infty$$

for any $n \geqslant 1$. Take $i_n = n_0 + k_n + m_n$, $D = N_3 + \deg(f^{n_0}|_{P_{n_0}(c_1)})$ and $P_0 = f^{n_0}(P_{n_0}(c_1))$. Then

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leqslant D$$

for all n.

(4) $x \in X_3$, i.e. $\mathrm{Crit_n}(x) \cup \mathrm{Crit_r}(x) = \emptyset$ and $\mathrm{Crit_{ep}}(x) \neq \emptyset$. Take $c_0 \in \mathrm{Crit_{ep}}(x)$. Let$\{(0, j_n\}_{n \geqslant 1}$ be all c_0-positions in $T(x)$. We claim that there is at most one c-position on the diagonal

$$\{(n, m) \mid n + m = j_n, \quad 0 < n \leqslant j_n\}$$

for all $c \in \mathrm{Crit}(x)$. If this is false, there are at least two c-positions on this diagonal for some $c \in \mathrm{Crit}(x)$. This means $c \in \mathrm{Crit_p}(x)$ and $c_0 \in F(c)$. By Lemma 1, $F(c) = [c]$ and $c_0 \in \mathrm{Crit_p}(x)$. It contradicts with $c_0 \in \mathrm{Crit_{ep}}(x)$. So the above claim is true. There exists a positive integer D such that

$$\deg(f^{j_n} : P_{j_n}(x) \to P_0(f^{j_n}(x))) \leqslant D.$$

Take a subsequence $\{i_n\}$ of $\{j_n\}$ such that $P_0(f^{i_n}(x)) = P_0$ for a fixed P_0. Then

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leqslant D$$

for all n. □

Proposition 2. $\mathrm{mes}(X_1 \cup X_2 \cup X_3) = 0$, *where* mes *denotes the Lebesgue measure on the complex plane* $\mathbb{C}$.

Proof. It is sufficient to prove that any point $x \in X_2 \cup X_3$ is not a density point of $J(f)$.

From Lemma 7, for any point $x \in X_2 \cup X_3$, there exist a puzzle piece P_0 and infinitely many i_n such that

$$\deg(f^{i_n} : P_{i_n}(x) \to P_0) \leqslant D$$

for some constant $D < \infty$ depending on x.

The attracting case. There is a subsequence of $\{i_n\}$, say itself, such that $\{f^{i_n}(x)\}$ converges to some point $x_0 \in J(f)$. We assume that $f^{i_n}(x) \in P_1(x_0)$ for all n. It is obvious that there is a disk $D(y, r_0)$ in $P_1(x_0) \cap F(f)$ for some constant $r_0 > 0$. By distortion results for holomorphic p-valent mappings(see [5], [8],[28],[30] and [31]), there are constants $1 \leqslant M < \infty$ and $0 < \lambda < 1$ depending on x such that

$$\mathrm{Shape}(P_{i_n+1}(x), x) \leqslant M$$

and

$$\frac{\mathrm{mes}(P_{i_n+1}(x) \cap J(f))}{\mathrm{mes}(P_{i_n+1}(x))} \leqslant \lambda$$

for all n. Since $\cap_{n \geqslant 0} P_{i_n+1}(x) = \{x\}$, the point x is not a density point of $J(f)$.

The parabolic case. If there are a puzzle piece P_{n_0} of depth n_0 compactly contained in P_0 and infinitely many i_n such that $f^{i_n}(x) \in P_{n_0}$, we can prove that x is not a density point of $J(f)$ by the same argument as in

the attracting case. Otherwise, there is a subsequence of $\{f^{i_n}(x)\}$, say itself, converges to some point $x_0 \in \partial P_0$. The point x_0 belongs to $\cup_{n\geqslant 0} f^{-n}(0)$. We assume that 0 is the parabolic fixed point as before. In the construction of the Branner-Hubbard puzzle, the flower petal U_0 can be chosen such that there exists a sector $S \subset P_0$ with the vertex at x_0. For all n, let $r_n > 0$ be the distance from the point $f^{i_n}(x)$ to the boundary of the sector S. Let

$$\widetilde{\Omega}_n(x) = \mathrm{Comp}_x(f^{-i_n}(D(f^{i_n}(x), r_n)))$$

and

$$\Omega_n(x) = \mathrm{Comp}_x(f^{-i_n}(D(f^{i_n}(x), \frac{1}{2}r_n))).$$

Then

$$\deg(f^{i_n} : \widetilde{\Omega}_n(x) \to D(f^{i_n}(x), r_n)) \leqslant D.$$

By the Leau-Fatou Flower Theorem, there is a disk

$$D(y_n, \frac{1}{4}r_n) \subset D(f^{i_n}(x), \frac{1}{2}r_n) \cap F(f)$$

for large n. By the same argument as above, x is not a density point of $J(f)$. □

Recall the definition of invariant line fields in Section 1.

Let $\mathcal{H}(f)$ be the collection of all holomorphic maps $h : U \to V$, where U, V are open sets such that there exist $i,\ j \in \mathbb{N}$ with $f^i \circ h = f^j$ on U.

The following proposition due to Weixiao Shen is a criterion to test whether a rational map carries invariant line field on its Julia set or not.

Proposition 3 ([28]). *Let f be a rational map of degree $\geqslant 2$ and x be a point in $J(f)$. If there are a constant $C > 1$, a positive integer $N \geqslant 2$ and a sequence $h_n : U_n \to V_n$ in $\mathcal{H}(f)$ with the following properties:*

(1) U_n, V_n are topological disks and

$$\mathrm{diam}(U_n) \to 0 \text{ and } \mathrm{diam}(V_n) \to 0$$

as $n \to \infty$.

(2) h_n is a proper map of degree between 2 and N.

(3) For some $u \in U_n$ such that $h_n'(u) = 0$ and for $v = h_n(u)$ we have Shape$(U_n, u) \leqslant C$ and Shape$(V_n, v) \leqslant C$.

(4) $d(U_n, x) \leqslant C \cdot \mathrm{diam}(U_n)$ and $d(V_n, x) \leqslant C \cdot \mathrm{diam}(V_n)$.

Then for any f-invariant line field μ, $\mu(x) = 0$ or μ is not almost continuous at x.

Proposition 4. *Suppose μ is an invariant line field on the Julia set $J(f)$. If $x \in X_4$, then $\mu(x) = 0$ or μ is not almost continuous at x.*

Proof. For $x \in X_4$, $\mathrm{Crit}(x) = \mathrm{Crit_p}(x) \neq \emptyset$. Let $c_0 \in \mathrm{Crit_p}(x)$ and let $\widetilde{K}_n \subset K_n \subset K'_n$ be puzzle pieces around c_0 as in Section 3.

For any $n \geqslant n_0$, the annuli $K'_n \setminus K_n$ and $K_n \setminus \widetilde{K}_n$ are non-degenerate. Let l_n be the first moment such that $f^{l_n}(x) \in \widetilde{K}_n$. Let $\widetilde{V}_n(x) = \mathrm{Comp}_x(f^{-l_n}(\widetilde{K}_n))$, $V_n(x) = \mathrm{Comp}_x(f^{-l_n}(K_n))$ and $V'_n(x) = \mathrm{Comp}_x(f^{-l_n}(K'_n))$. For large n, the puzzle piece $V'_n(x)$ contains no critical points. Let $v_n > 0$ be the smallest integer such that $f^{v_n}(\widetilde{V}_n(x))$ contains a critical point $c \in [c_0]$. Set $\widetilde{\Lambda}_n = f^{v_n}(\widetilde{V}_n(x))$, $\Lambda_n = f^{v_n}(V_n(x))$ and $\Lambda'_n = f^{v_n}(V'_n(x))$. See Figure 2.

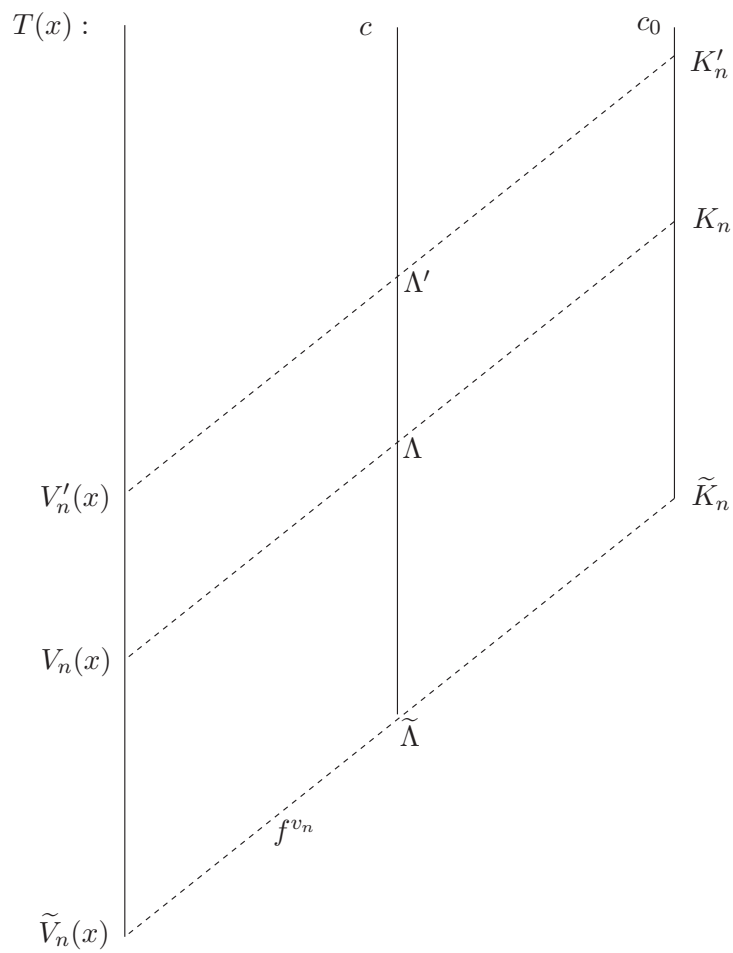

Figure 2

From the conditions

$$(K'_n \setminus K_n) \cap \mathrm{orb}([c_0]) = (K_n \setminus \widetilde{K}_n) \cap \mathrm{orb}([c_0]) = \emptyset$$

and $\mathrm{Crit}(x) = \mathrm{Crit_p}(x)$, we know that $f^{v_n} : V_n'(x) \to \Lambda_n'$ is conformal and

$$2 \leqslant \deg(f^{l_n}|_{V_n'(x)}) = \deg(f^{l_n}|_{V_n(x)}) = \deg(f^{l_n}|_{\widetilde{V}_n(x)}) \leqslant D_2$$

for some constant D_2 depending only on $\mathrm{Crit}(x)$. By Proposition 1 and Lemma 2,

$$\mathrm{Shape}(V_n(x), x) \leqslant M_2$$

and

$$\mathrm{Shape}(\Lambda_n, c) \leqslant M_2$$

for some constant $M_2 > 0$.

Let $\widetilde{\Gamma}_n = \mathcal{L}_c(\widetilde{\Lambda}_n)$, $\widetilde{U}_n = \mathcal{L}_x(\widetilde{\Gamma}_n)$ and $f^{u_n}(\widetilde{U}_n) = \widetilde{\Lambda}_n$. Further let $\widetilde{U}_n(x) = \mathrm{Comp}_x f^{-u_n}(\widetilde{\Lambda}_n)$, $U_n(x) = \mathrm{Comp}_x f^{-u_n}(\Lambda_n)$ and $U_n'(x) = \mathrm{Comp}_x f^{-u_n}(\Lambda_n')$. Then

$$2 \leqslant \deg(f^{u_n}|_{U_n'(x)}) = \deg(f^{u_n}|_{U_n(x)}) = \deg(f^{u_n}|_{\widetilde{U}_n(x)}) \leqslant D_3$$

for some constant D_3 depending only on Crit. There exists a positive constant $M_3 > 0$ such that

$$\mathrm{Shape}(U_n, x) \leqslant M_3.$$

For each large n, define $h_n = f^{-v_n} \circ f^{u_n}$. Then $h_n : U_n(x) \to V_n(x)$ is a properly holomorphic mapping of degree between 2 and some constant N. All conditions in Proposition 3 are satisfied, hence the lemma holds. □

Proof of Theorem 1. If f has an invariant line field μ on the Julia set $J(f)$, then there exists a positive measure subset E of $J(f)$ such that $\mathrm{support}(\mu) = E$. Since μ is measurable in $\mathbb{C}$, almost every point in $\mathbb{C}$ is almost continuous. From Proposition 4 and Proposition 2, $\mathrm{mes}(X_4 \cap \mathrm{support}(\mu)) = 0$ $\mathrm{mes}(\mathrm{support}(\mu)) = 0$. It is a contradiction. So f carries no invariant line field on its Julia set. □

Recall that the Teichmüller space of a rational map f of degree d is defined by

$$\mathrm{Teich}(\hat{\mathbb{C}}, f) = \{g \in \mathrm{Rat}_d \,|\, g \text{ is quasiconformally conjugated with } f\}/\mathrm{Aut}(\hat{\mathbb{C}}).$$

C. McMullen and D. Sullivan have given a formula to compute the dimension of $\mathrm{Teich}(\hat{\mathbb{C}}, f)$ in [23].

Theorem C ([23]). *The dimension of the Teichmüller space of a rational map is given by $N = N_{AC} + N_{HR} + N_{LF} - N_P$, in which*

- *N_{AC} is the number of foliated equivalence classes of acyclic critical point in the Fatou set;*
- *N_{HR} is the number of cycles of Herman rings;*
- *N_{LF} is the number of ergodic line field on the Julia set;*
- *N_P is the number of the parabolic cycles.*

Proof of Theorem 2. Let f be a structurally stable rational map with a Cantor Julia set. From Theorem B in Section 1, $\dim(\mathrm{Teich}(\hat{\mathbb{C}}, f)) = 2d - 2$.

By the implicit function theorem, any rational map with indifferent cycles is structurally unstable, therefore f has no Siegel disks or parabolic basins. Moreover, any rational map with Herman rings is also structurally unstable by a result due to Mañé, see [18]. These imply that $N_{HR} = N_P = 0$. From Theorem 1, $N_{LF} = 0$. We conclude that $N_{AC} = 2d - 2$ and all critical points are in the attracting Fatou components. It means that f is hyperbolic. □

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References

[1] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand Co., 1966.

[2] A. Avila, J. Kahn, M. Lyubich, and W. Shen, *Combinatorial rigidity for unicritical polynomials*, Stony Brook IMS, Preprint, 2005/05; Arxiv: math.DS/0507213.

[3] B. Branner and J. H. Hubbard, *The iteration of cubic polynomials, Part II: Patterns and parapatterns*, Acta Math., 169 (1992), 229-325.

[4] H. Bruin, W. Shen, and S. van Strien, *Existence of unique SRB-measures is typical for unimodal families*, Ann. Sci. Éc. Norm. Sup., to appear.

[5] L. Carlson, P. Jones, and C. -J.Yoccoz, *Julia and John*, Bol. Soc. Brasil mat.(N.S.), 25(1994), 1-30.

[6] J. Graczyk and S. Smirnov, *Non uniform hyperbolicity in complex dynamics I, II*, Prepublication d'Orsay, 2001-36.

[7] J. Graczyk and G. Świątek, *Generic hyperbolicity in the logistic family*, Ann. of Math., 146(1997), 1-52.

[8] P. Haissinsky, *Ridigity and expansion for rational maps*, J. London Math. Soc., 63(2001), 128-140.

[9] J. Kahn and M. Lyubich, *Local connectivity of Julia set for unicritical polynomials*, Stony Brook IMS, Preprint, 2005/03; ArXiv: math.DS/0505191.

[10] O. Kozlovski, W. Shen, and S. van Strien, *Density of hyperbolicity in real one-dimensional dynamics*, Ann. of Math., to appear.

[11] O. Kozlovski, W. Shen, and S. van Strien, *Rigidity for real polynomials*, Ann. of Math., to appear.

[12] G. Levin and S. van Strien, *Total disconnectedness and absence of invariant linefields for real polynomials*, Asteriques, 261(2000), 161-172.

[13] M. Lyubich, *Dynamics of quadratic polynomials, I-II*, Acta Math., 178(1997), 185-247, 247-297.

[14] M. Lyubich, *On the Lebesque measure of the Julia set of a quadratic polynomial*, Stony Brook IMS, Preprint, 1991/10; arXiv:math.DS/9201285.

[15] M. Lyubich and M. Yampolsky, *Dynamics of quadratic polynomials: Complex bounds for real maps*, Ann. Inst. Fourier, 47(1997), 1219-1255.

[16] P. M. Makienko, *Remarks on the Ruelle operator and the invariant line fields problem, II*, Ergodic Theory Dynam. Systems, 25(2005), 1561-1581.

[17] P. M. Makienko, *Non-persistently recurrent points, qc-surgery and instability of rational maps with totally disconnected Julia sets*, Preprint, ArXiv: math.DS/0506141.

[18] R. Mañé, *On the instability of Herman rings*, Invent. math. 81(1985), 459-471.

[19] R. Mañé, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. Éc. Morm. Sup. 16(1983), 193-217.

[20] C. McMullen, *Complex Dynamics and Renormalization*, Ann. of Math. Stud., No.135, Princeton University Press, 1994.

[21] C. McMullen, *Frontiers in complex dynamics*, Bull. of Amer. Math. Soc., 31(1994), 155-172.

[22] C. McMullen, *Self-similarity of Siegel disks and Hausdorff dimension of Julia sets*, Acta Math., 180(1998), 247-292.

[23] C.McMullen and D. Sullivan, *Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system*, Adv. Math., 135(1998), 351-395.

[24] J. Milnor, *Dynamics in One Complex Variable*, 2nd edition, Vieweg, 2000.

[25] C. L. Petersen, *Local connectivity of some Julia sets containing a circle with an irrational rotation*, Acta Math., 177(1996), 163-224.

[26] C. Pommerenke, *Univalent Functions*, Vanderhoeck & Ruprecht in Göttingen, 1975.

[27] W. Qiu and Y. Yin, *Proof of the Branner-Hubbard conjecture on Cantor Julia sets*, Preprint, 2006; ArXiv: math.DS/0608045.

[28] W. Shen, *On the measurable dynamics of real rational functions*, Ergodic Theory Dynam. Systems, 23(2003), 957-983.

[29] M. Shishikura, *Yoccoz puzzle, τ-functions and their applications*, Unpublished, 1998.

[30] M. Shishikura and Tan Lei, *An alternative proof of Mañé's theorem on non-expanding Julia sets*. In: The Mandelbrot set, Theme and Variations, edited by Tan Lei, 265-279, London Math. Soc. Lecture note Ser., No 274, Cambrige University Press, Cambrige, 2000.

[31] Y. Yin, *Geometry and dimension of Julia sets*. In: The Mandelbrot set, Theme and Variations, edited by Tan Lei, 281-287, London Math. Soc. Lecture note Ser., No 274, Cambrige University Press, Cambrige, 2000.

[32] J. -C. Yoccoz, *On the local connectivity of the Mandelbrot set*, Unpublished, 1990.

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