

INTEGRATION OF EXACT COURANT ALGEBROIDS

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ABSTRACT. In this paper, we describe an integration of exact Courant algebroids to symplectic 2-groupoids, and we show that the differentiation procedure from [32] inverts our integration.

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1. INTRODUCTION

A Courant algebroid is a Lie 2-algebroid paired with a compatible symplectic structure [26–31]. Therefore, in the study of Courant algebroids two immediate questions are

Question 1: What is the global object “integrating” a Courant algebroid?

Question 2: How do you “differentiate” the global object to recover the Courant algebroid?

Question 1 was first addressed in [31], where, given a Courant algebroid, a construction of a 2-groupoid carrying a symplectic form on the 2-simplices is sketched (see also [35]). Unfortunately, this construction was infinite dimensional. A procedure to differentiate Lie n -groupoids to Lie n -algebroids is described in [32], giving an answer to Question 2. However, this differentiation procedure does not invert the integration procedure described in [31] “on the nose”.

Although the answer to the Question 1 might be somewhat complicated, it has a simple solution for some of the most popular Courant algebroids, namely exact Courant algebroids [33]. For instance, an elegant solution for exact Courant algebroids with trivial characteristic class is presented in [24] in terms of the “bar” construction. In this paper, we present a solution for arbitrary exact Courant algebroids. Our construction results in a (local) Lie 2-groupoid carrying a symplectic form on the space of 2-simplices. Furthermore, we show that if one differentiates our construction (as in [32]), one recovers the original Courant algebroid.

The idea of our construction is as follows. As a differential graded manifold, the standard Courant algebroid over a manifold M is $T[1]T^*[1]M$. Since it is of the form $T[1]X$ for some graded manifold X ($X = T^*[1]M$), the problem of its integration to a local 2-groupoid K_\bullet has a simple solution (see Remark 5). Next we construct a symplectic form on K_2 making K_\bullet into a (local) 2-symplectic 2-groupoid which differentiates to the standard Courant algebroid.

We then prove that an arbitrary exact Courant algebroid over M is isomorphic, as a differential graded manifold, to the standard one. We can therefore use the same 2-groupoid K_\bullet as its integration. The isomorphism is not, however, a symplectomorphism. We compute the modification to the symplectic form on $T[1]T^*[1]M$ and modify the symplectic form on K_2 correspondingly.

In [34], a more conceptual explanation for our construction is given, related to the work of Mehta, Gracia-Saz, Arias Abad, Crainic and Schaetz on actions up to homotopy [1, 2, 15, 16] (see also [35]).

1.0.1. *Acknowledgements.* D.L.-B was supported by an NSERC CGS-D grant, and would like to thank Eckhard Meinrenken for his advice and suggestions, and Chenchang Zhu, and Rajan Mehta for helpful discussions. P. Š. was supported by the Swiss National Science Foundation (grant 200020-120042 and 200020-126817).

We also thank the referee for their valuable recommendations which have made this paper considerably easier to read.

2. BACKGROUND

Remark 1 (A note on Lie groupoids and Lie algebroids). We use the definitions of (local) Lie n -groupoids given by Andre Henriques and Chenchang Zhu [17, 37], in terms of simplicial manifolds. We take the definition of Lie n -algebroids given in [32] in terms of NQ -manifolds.

In this section, we recall the differentiation procedure described in [32] which takes Lie n -groupoids to Lie n -algebroids.

2.1. Simplicial Manifolds. For $n \in \mathbb{N}$, let $[n]$ be the category generated by the directed graph

$$0 \rightarrow 1 \rightarrow \cdots \rightarrow n,$$

and $\mathbf{\Delta}$ the full subcategory of \mathbf{Cat} (the category of small categories) generated by the objects $[0], [1], [2], \dots$. We have the distinguished functors [14]

$$s^j(0 \rightarrow 1 \rightarrow \dots \rightarrow n+1) = (0 \rightarrow 1 \rightarrow \dots \rightarrow j \xrightarrow{\text{id}} j \rightarrow \dots \rightarrow n),$$

(that is, we insert the identity in the j^{th} place), and

$$d^i(0 \rightarrow 1 \rightarrow \dots \rightarrow n-1) = (0 \rightarrow 1 \rightarrow \dots \rightarrow i-1 \rightarrow i+1 \rightarrow \dots \rightarrow n),$$

(that is, we compose $i-1 \rightarrow i \rightarrow i+1$). We denote the corresponding maps in $\mathbf{\Delta}^{\text{op}}$ by s_j and d_i .

Let \mathbf{Man} be the category of smooth manifolds. Then $\mathbf{Man}^{\mathbf{\Delta}^{\text{op}}}$ is the category of simplicial manifolds. More generally, if C is any category, then $C^{\mathbf{\Delta}^{\text{op}}}$ is the category of simplicial objects (in C).

As a quick word on notation, if $X \in C^{\mathbf{\Delta}^{\text{op}}}$, then it is conventional to write $X_n := X([n])$.

Example 1. The *standard n -simplex*, Δ^n , is the contravariant functor

$$\Delta^n := \text{Cat}(\cdot, [n]) \in \text{Set}^{\mathbf{\Delta}^{\text{op}}} \subset \mathbf{Man}^{\mathbf{\Delta}^{\text{op}}}$$

(where sets are viewed as discrete manifolds).

Example 2. The k^{th} horn $\Lambda_k^n \subset \Delta^n$ is defined as

$$(\Lambda_k^n)([j]) = \{f \in \Delta^n([j]) \mid d^k([n]) \not\subseteq f([j])\}.$$

We may think of Λ_k^n as the boundary of the standard n -simplex Δ^n with the k^{th} face removed.

Let M_\bullet be a simplicial manifold. The natural inclusion $\Lambda_k^n \subset \Delta^n$ induces a map

$$(2.1) \quad \mathbf{Man}^{\mathbf{\Delta}^{\text{op}}}(\Delta^n, M_\bullet) \rightarrow \mathbf{Man}^{\mathbf{\Delta}^{\text{op}}}(\Lambda_k^n, M_\bullet).$$

The *Kan conditions* [17] are

$Kan(n, k)$: (2.1) is a surjective submersion, $Kan!(n, k)$: (2.1) is a diffeomorphism.

The following definition is due to Henriques [17].

Definition 1 ([17]). A *Lie n -groupoid* is a simplicial manifold M_\bullet satisfying

- $Kan!(m, k)$ for $m > n$ and $0 \leq k \leq m$, and
- $Kan(m, k)$ for $m > 0$ and $0 \leq k \leq m$.

In particular, a Lie 1-groupoid is the nerve of a Lie groupoid. For instance, the nerve of the pair groupoid (defined explicitly in the following example) is a Lie 1-groupoid.

Example 3. There is a full and faithful functor $E : \mathbf{Man} \rightarrow \mathbf{Man}^{\mathbf{\Delta}^{\text{op}}}$ given by $E_n M = M^{n+1}$ and

$$EM(f) : (z_0, \dots, z_m) \rightarrow (z_{f(0)}, \dots, z_{f(n)}),$$

for any monotone map $f : [n] \rightarrow [m]$.

We will be interested in the local version of n -groupoids, as introduced by Zhu [37]. However, as suggested by a referee, we will reformulate it in terms of the microfolds introduced by Cattaneo, Dherin, and Weinstein [9] and Blohmann, Fernandes, and Weinstein [4] following Milnor [25].

Definition 2. A *microfold* is an equivalence class of pairs (M, S) of manifolds such that $S \subseteq M$ is a closed submanifold. Two such pairs (M_1, S_1) and (M_2, S_2) are said to be equivalent if $S_1 = S_2 = S$ and there exists a third pair (U, S) such that U is simultaneously an open subset of both M_1 and M_2 . We denote the equivalence class by $[M, S]$, and refer to S as the *microfold core* of $[M, S]$.

A morphism between microfolds is a germ of maps between representatives which takes the source microfold core to the target microfold core. Such a morphism is said to be a *surjective submersion* (resp. a *diffeomorphism*) if it is a surjective submersion (resp. a diffeomorphism) for a suitable choice of representatives.

We denote the category of microfolds by Mfold . There is a forgetful functor $F_{core} : \text{Mfold} \rightarrow \text{Man}$ which takes a microfold $[M, S]$ to its microfold core S .

Definition 3. The category $\text{Man}_{loc}^{\Delta^{op}}$ of *local simplicial manifolds* (or *simplicial manifold germs*) is the subcategory of functors $[M_{\bullet}, S_{\bullet}] : \Delta^{op} \rightarrow \text{Mfold}$ such that

- $M_0 = S_0$, and
- the composition $F_{core} \circ [M_{\bullet}, S_{\bullet}] := S_{\bullet}$ with the forgetful functor is a constant functor.

We will denote a local simplicial manifold $[M_{\bullet}, S_{\bullet}] := [M_{\bullet}, M_0]$ simply by $[M_{\bullet}]$.

Working with microfolds in place of manifolds, one obtains direct analogues of Henriques' Kan conditions [17] for local simplicial manifolds. Thus one obtains the following definition of a local Lie n -groupoid, essentially a reformulation of the one by Zhu [37].

Definition 4. A *local Lie n -groupoid* is a local simplicial manifold $[M_{\bullet}]$ satisfying

- $\text{Kan}^l(m, k)$ for $m > n$ and $0 \leq k \leq m$, and
- $\text{Kan}^l(m, k)$ for $m > 0$ and $0 \leq k \leq m$.

There is a forgetful functor $[\cdot] : \text{Man}^{\Delta^{op}} \rightarrow \text{Man}_{loc}^{\Delta^{op}}$ which takes a simplicial manifold M_{\bullet} to its germ $[M_{\bullet}] := [M_{\bullet}, M_0]$. It restricts to a functor from Lie n -groupoids to local Lie n -groupoids.

2.2. NQ -manifolds. An NQ -manifold is a differential non-negatively graded manifold. We recall a reformulation of this definition from [32].

We let SMan denote the category of super-manifolds. Let θ be the standard coordinate on the odd line $\mathbb{R}^{0|1}$. A general map

$$\mathbb{R}^{0|1} \rightarrow \mathbb{R}^{0|1}, \quad \theta \rightarrow \xi + s\theta$$

can be identified with an element $(s, \xi) \in \mathbb{R}^{1|1}$. The corresponding super-semi-group $\underline{\text{End}}(\mathbb{R}^{0|1}) := \underline{\text{SMan}}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \cong \mathbb{R}^{1|1}$ carries the multiplication

$$(s, \xi) \cdot (t, \eta) = (st, \xi + s\eta), \quad s, t \in \mathbb{R}, \quad \xi, \eta \in \mathbb{R}^{0|1}.$$

Definition 5 ([19, 20, 31]). An NQ -manifold [19, 20, 31] is a super-manifold $X \in \text{SMan}$ with an action of $\underline{\text{End}}(\mathbb{R}^{0|1})$ such that $(-1, 0)$ acts as the parity operator (it just changes the sign of the odd coordinates). In this context, a function $f \in C^\infty(X)$ has degree $|f| := k$ if

$$(s, 0) \cdot f = s^k f.$$

Note that the degree of a function is always a non-negative integer. Furthermore, the homological vector field Q on X is defined by

$$Q : f \rightarrow \frac{\partial}{\partial \xi} ((0, \xi) \cdot f).$$

The manifold $(0, 0) \cdot X \subset X$ is called the base of X .

An N -manifold is a super-manifold $X \in \text{SMan}$ with an action of the multiplicative semi-group \mathbb{R} such that $(-1, 0)$ acts as the parity operator.

Finally, a bi- NQ -manifold is a super-manifold $X \in \text{SMan}$ with an action of $(\underline{\text{End}}(\mathbb{R}^{0|1}))^2$ such that $((-1, 0), (-1, 0))$ acts as the parity operator. A function $f \in C^\infty(X)$ has degree $|f| := (k, l)$ if $((s, 0), (t, 0)) \cdot f = s^k t^l f$. Additionally, X is said to be *concentrated in the second grading* if $((0, 0), (1, 0))$ acts by the identity on X .

We let SMan_{NQ} and SMan_N denote the categories of NQ -manifolds and N -manifolds, respectively. Definitions 2 and 3 also extend in the obvious way to define the categories SMfold_{NQ} , SMfold_N , $(\text{SMan}_{NQ})_{loc}^{\Delta^{op}}$ and $(\text{SMan}_N)_{loc}^{\Delta^{op}}$ of NQ -microfolds, N -microfolds, and local simplicial NQ and N -manifolds, respectively.

Example 4. If M is any manifold, then the pre-sheaf $\text{SMan}(\cdot \times \mathbb{R}^{0|1}, M)$ is represented by the super-manifold $T[1]M$. The natural action of $\underline{\text{End}}(\mathbb{R}^{0|1})$ on $\text{SMan}(\cdot \times \mathbb{R}^{0|1}, M)$ is such that $(-1, 0)$ acts as the parity operator. Therefore $T[1]M$ is an NQ -manifold.

In fact $C^\infty(T[1]M) = \Omega^\bullet(M)$ and the homological vector field on $T[1]M$ is just the de Rham differential $Q_{deRham} = d$.

2.3. The Functor $1\text{-Jet} : \text{Man}_{loc}^{\Delta^{op}} \rightarrow \widehat{\text{SMan}}$. In this section, we recall the functor 1-Jet , which maps the category of local simplicial manifolds, $\text{Man}_{loc}^{\Delta^{op}}$, to the category of presheaves of supermanifolds, $\widehat{\text{SMan}} := \text{Sets}^{(\text{SMan}^{op})}$ (where Sets is the category of sets).

Suppose that $[M_\bullet] \in \text{Man}_{loc}^{\Delta^{op}}$, then (following [32]) we define

$$(2.2) \quad 1\text{-Jet}([M_\bullet])(X) = \text{SMfold}^{\Delta^{op}}([X \times E_\bullet \mathbb{R}^{0|1}], [M_\bullet]), \text{ for any } X \in \text{SMan}.$$

Since $E_\bullet \mathbb{R}^{0|1}$ carries a natural action of $\underline{\text{End}}(\mathbb{R}^{0|1})$, so does $1\text{-Jet}([M_\bullet]) \in \widehat{\text{SMan}}$. Furthermore, we have

Proposition 1 ([32]). *If $[M_\bullet]$ is a local Lie n -groupoid, then $1\text{-Jet}([M_\bullet])$ is representable, therefore $1\text{-Jet}([M_\bullet]) \in \text{SMan}_{NQ}$.*

Remark 2. Note that 1-Jet is not a faithful functor. In general, for $n > 1$, there are *more* morphisms between local Lie n -groupoids than between their corresponding NQ -manifolds (see Remark 4 for a relevant example). Furthermore, the composition

$$M_\bullet \rightarrow [M_\bullet] \rightarrow 1\text{-Jet}([M_\bullet])$$

from Lie n -groupoids to NQ -manifolds already fails to be full for $n = 1$ (see the work Crainic and Fernandes [12]).

Definition 6. We say that a local Lie n -groupoid $[M_\bullet]$ *integrates* an NQ -manifold X if $1\text{-Jet}([M_\bullet]) \cong X$. Similarly, we say that a Lie n -groupoid M_\bullet *integrates* an NQ -manifold X if $1\text{-Jet}([M_\bullet]) \cong X$.

Example 5. Since the functor $E : \text{Man} \rightarrow \text{Man}^{\Delta^{\text{op}}}$ from Example 3 is full and faithful, $E_\bullet M$ integrates $T[1]M$.

Example 6. If G is a Lie group and \mathfrak{g} is its Lie algebra, then the nerve of G integrates the NQ -manifold $\mathfrak{g}[1]$. More generally, if Γ is a Lie groupoid and A the corresponding Lie algebroid, then (the nerve of) Γ integrates $A[1]$.

Example 7. Let $\text{Vect}_f(\mathbb{R})$ denote the category of finite dimensional vector spaces over \mathbb{R} , $\text{Ch}^+(\text{Vect}_f(\mathbb{R}))$ the positively graded chain complexes and $V \in \text{Vect}_f(\mathbb{R})$. The Dold-Kan correspondence

$$N : (\text{Vect}_f(\mathbb{R}))^{\Delta^{\text{op}}} \rightleftarrows \text{Ch}^+(\text{Vect}_f(\mathbb{R})) : \Gamma$$

defines the Eilenberg-Mac Lane object $K(V, n) := \Gamma(V[-n])$, which we view as a simplicial manifold.

Concretely,

$$(2.3) \quad K(V, n)_k := \left\{ \{v_f\}_{f:[n] \rightarrow [k]} \mid \begin{array}{l} v_f \in V, v_f = 0 \text{ if } f \text{ is not injective,} \\ \text{and for every } g : [n+1] \rightarrow [k], \sum_{i=0}^{n+1} (-1)^i v_{g \circ d^i} = 0 \end{array} \right\}$$

and for any monotone map $h : [l] \rightarrow [k]$,

$$K(V, n)(h) : K(V, n)_k \rightarrow K(V, n)_l \quad \left(\{v_f\}_{f:[n] \rightarrow [k]} \rightarrow \{v'_{f'} := v_{h \circ f'}\}_{f':[n] \rightarrow [l]} \right).$$

The following picture can be useful. An element of $K(V, n)_k$ is a labelling of the n dimensional faces of the standard k simplex by elements of V , so that the alternating sum around any $n+1$ dimensional face is zero. Note that we have a diffeomorphism $V \cong K(V, n)_n$ given by

$$(2.4) \quad v \rightarrow \left\{ v_f = \left\{ \begin{array}{ll} v & \text{if } f = \text{id} : [n] \rightarrow [n] \\ 0 & \text{otherwise.} \end{array} \right\} \right\}_{f:[n] \rightarrow [n]}$$

$K(V, n)$ integrates the NQ -manifold $V[n]$ (with trivial Q -structure). For any $v \in V[n]$, the corresponding map $E_k \mathbb{R}^{0|1} \rightarrow K(V, n)_k$ is given by

$$(2.5) \quad (\theta_0, \dots, \theta_k) \rightarrow \left\{ v_f := v \left(\sum_{i=0}^n (-1)^i \theta_{f \circ d^i(0)} \cdots \theta_{f \circ d^i(n-1)} \right) \right\}_{f:[n] \rightarrow [k]}.$$

2.3.1. Multiplicative Forms. Following [5–8, 11, 13, 18] we make the following definition of multiplicative forms on a simplicial manifold.

Definition 7. Let M_\bullet be a (local) simplicial manifold. We say that a k -form $\alpha \in \Omega^k(M_n)$ is *multiplicative* if for any $0 \leq i < n$, $s_i^* \alpha = 0$, and $D\alpha = 0$, where

$$(2.6) \quad D := \sum_{i=0}^n (-1)^i d_i^* : \Omega^k(M_n) \rightarrow \Omega^k(M_{n+1})$$

is the simplicial differential.

In the spirit of [5,6,22,23,36] we would like to interpret Definition 7 in terms of morphisms of simplicial manifolds. But first we need to point out that one can extend the functor 1-Jet to (local) simplicial NQ -manifolds,

$$1\text{-Jet} : (\text{SMan}_{NQ})_{loc}^{\Delta^{\text{op}}} \rightarrow \widehat{\text{SMan}},$$

by the same formula (2.2).

If X_\bullet is a (local) simplicial NQ -manifold, then $1\text{-Jet}([X_\bullet])$ carries an action of $(\underline{\text{End}}(\mathbb{R}^{0|1}))^2$. Specifically, in the formula

$$1\text{-Jet}([X_\bullet])(Z) = \text{SMan}^{\Delta^{\text{op}}}([Z \times E_\bullet \mathbb{R}^{0|1}], [X_\bullet]), \quad (\text{for any } Z \in \text{SMan}),$$

$\underline{\text{End}}(\mathbb{R}^{0|1}) \times (1, 0)$ acts directly on the factor X_\bullet and $(1, 0) \times \underline{\text{End}}(\mathbb{R}^{0|1})$ acts directly on the factor $\mathbb{R}^{0|1}$.

Just as certain (local) simplicial manifolds integrate NQ -manifolds, certain (local) simplicial NQ -manifolds integrate bi- NQ -manifolds.

We will be interested in the following example.

Example 8. Suppose that M_\bullet is a (local) simplicial manifold integrating the NQ -manifold X . View X as a bi- NQ -manifold concentrated in the second grading. Then $T[1]M_\bullet$ integrates $T[1, 0]X$. (This follows immediately from the definition of 1-Jet and the fact that $T[1, 0]N$ represents the pre-sheaf $\text{SMan}(\cdot \times \mathbb{R}[-1, 0], N)$ for any $N \in \text{SMan}$).

Recall that a k -form $\alpha \in \Omega^k(M)$ can be thought of as a (grading-preserving) function $\alpha : T[1]M \rightarrow \mathbb{R}[k]$.

Lemma 1. *Let M_\bullet be a (local) simplicial manifold, and α a k -form $\alpha \in \Omega^k(M_n)$. The corresponding map*

$$(2.7) \quad \alpha : T[1]M_n \rightarrow \mathbb{R}[k] \cong K(\mathbb{R}[k], n)_n$$

extends to a simplicial map to the Eilenberg-Mac Lane object (2.3),

$$(2.8) \quad \hat{\alpha} : T[1]M_\bullet \rightarrow K(\mathbb{R}[k], n)_\bullet,$$

if and only if α is multiplicative. In this case, the extension (2.7) is unique.

Proof. First, we will compute the unique extension $\hat{\alpha}$ (assuming it exists).

Using (2.4), we see that for $x \in T[1]M_n$, (2.7) is given by

$$\hat{\alpha}(x) = \left\{ v'_{f'} = \left\{ \begin{array}{ll} \alpha(x) & \text{if } f' := \text{id} \\ 0 & \text{otherwise.} \end{array} \right\} \right\}_{f':[n] \rightarrow [n]}$$

Let $y \in T[1]M_l$, and suppose $\hat{\alpha}(y) = \{v_f\}_{f:[n] \rightarrow [l]}$. Then for an arbitrary map $g : [n] \rightarrow [l]$,

$$(2.9) \quad \begin{aligned} \{v'_{f'} := v_{g \circ f'}\}_{f':[n] \rightarrow [n]} &= K(\mathbb{R}[k], n)(g) \circ \hat{\alpha}(y) \\ &= \hat{\alpha} \circ T[1]M_{\bullet}(g)(y) = \left\{ v'_{f'} := \begin{cases} \alpha \circ T[1]M_{\bullet}(g)(y) & \text{if } f' = \text{id} \\ 0 & \text{otherwise.} \end{cases} \right\}_{f':[n] \rightarrow [n]} \end{aligned}$$

Therefore $v_g = \alpha \circ T[1]M_{\bullet}(g)(y)$. It follows that $\hat{\alpha}$ must be defined by

$$(2.10) \quad \hat{\alpha}(y) = \{v_f := \alpha \circ T[1]M_{\bullet}(f)(y)\}_{f:[n] \rightarrow [l]},$$

for any $y \in T[1]M_l$. We let $v_f(y) := \alpha \circ T[1]M_{\bullet}(f)(y)$.

Note that if $f : [n] \rightarrow [l]$ is not injective, we can factor $f = h \circ s^i$ for some $0 \leq i < n$ and $h : [n-1] \rightarrow [l]$. Hence $v_f(y) = 0$ if $s_i^* \alpha = 0$. It follows that $s_i^* \alpha = 0$ for every $0 \leq i < n$, if and only if $v_f(y) = 0$ for every $y \in T[1]M_l$ and $f : [n] \rightarrow [l]$ non injective.

Using (2.10), we see that for any $g : [n+1] \rightarrow [l]$, and any $y \in T[1]M_l$,

$$\sum_{i=0}^{n+1} (-1)^i v_{g \circ d^i}(y) = \left(\sum_{i=0}^{n+1} (-1)^i \alpha \circ T[1]M_{\bullet}(d^i) \right) \circ T[1]M_{\bullet}(g)(y).$$

So $\sum_{i=0}^{n+1} (-1)^i v_{g \circ d^i}(y) = 0$ for any $g : [n+1] \rightarrow [l]$, and any $y \in T[1]M_l$ if and only if

$$\sum_{i=0}^{n+1} (-1)^i \alpha \circ T[1]M_{\bullet}(d^i) = 0 \Leftrightarrow D\alpha = 0.$$

Comparison with Example 7 shows that (2.10) defines a simplicial map if and only if $s_i^* \alpha = 0$ and $D\alpha = 0$, that is α is multiplicative. \square

As a consequence, if M_{\bullet} integrates an NQ -manifold X , and

$$\alpha \in \Omega^k(M_n), \quad \hat{\alpha} : T[1]M_{\bullet} \rightarrow K(\mathbb{R}[k], n)$$

is a multiplicative k -form, then

$$1\text{-Jet}(\hat{\alpha}) : T[1, 0]X \rightarrow \mathbb{R}[k, n]$$

is a morphism of bi-graded Q -manifolds. That is, $1\text{-Jet}(\hat{\alpha})$ defines a Q -invariant k -form of degree n on X . Moreover, $1\text{-Jet}(\hat{\alpha})$ is closed whenever α is.

2.4. Courant algebroids. Recall [27, 31] that a Courant algebroid is a NQ -manifold X carrying a Q -invariant symplectic form $\omega_X \in \Omega^2(X)$ of degree 2. If M is the base of X , then X is said to be a Courant algebroid over M .

2.4.1. Standard Courant Algebroid. Let M be a manifold. $T^*[2]T[1]M$ carries the canonical symplectic form ω of degree 2. The homological vector field on $T[1]M$, Q_{deRham} , lifts in the canonical way to define a Q -structure on $T^*[2]T[1]M$. Therefore $(T^*[2]T[1]M, Q, \omega)$ is a Courant algebroid, called the *standard Courant algebroid*.

For later reference, we describe $(T^*[2]T[1]M, Q, \omega)$ explicitly in coordinates. Let x^a ($a = 1, \dots, d = \dim(M)$) be local coordinates on M . We take $\xi^a := dx^a$ to be the

coordinates on $T[1]M$ canonically associated to x^a . On the cotangent bundle $T^*[2]T[1]M$, we take p_a, η_a to be the canonical coordinates associated to x^a, ξ^a .

Remark 3 (Some Shorthand Notation). As a shorthand, we will denote a point with coordinates

$$((x^1, \dots, x^d); (\xi^1, \dots, \xi^d); (p_1, \dots, p_d); (\eta_1, \dots, \eta_d)) \in \mathbb{R}^d \times \mathbb{R}[1]^d \times \mathbb{R}[2]^d \times \mathbb{R}[1]^d$$

by $(x^\alpha, \xi^\alpha, p_\alpha, \eta_\alpha)$.

Furthermore, if $\{\tau_{a_1 \dots a_k}\}_{\{1 \leq a_1, \dots, a_k \leq d\}} \subset C^\infty(M)$, we will often use

$$\tau_{[a_1 \dots a_k]} := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} (-1)^{\text{sign}(\sigma)} \tau_{\sigma(a_1) \dots \sigma(a_k)},$$

as another shorthand, where Σ_k denotes the permutation group of $\{1, \dots, k\}$ and $\text{sign} : \Sigma_k \rightarrow \mathbb{Z}_2$ is the unique non-trivial group morphism.

Finally, we will also invoke Einstein's summation convention, summing over repeated upper and lower indices.

Using Einstein's summation convention, the symplectic form can be written

$$(2.11) \quad \omega := dp_a dx^a + d\eta_a d\xi^a,$$

and the homological vector field is

$$(2.12) \quad Q := \xi^a \partial_{x^a} + p_a \partial_{\eta_a} = X_{\xi^a p_a}.$$

The corresponding Poisson bracket is defined on coordinates by

$$\{p_a, x^b\} = \delta_a^b, \quad \{\eta_a, \xi^b\} = \delta_a^b, \quad \text{where } \delta_a^b := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases}$$

(and the Poisson bracket of any other pair of coordinates is equal to zero).

2.4.2. Exact Courant Algebroids. Exact Courant algebroids were first introduced in [33]. We recall the construction described in [21, 28]. Let $\kappa \in \Omega_{cl}^3(M)$ be a closed 3-form. Under the identification $C^\infty(T[1]M) \cong \Omega^\bullet(M)$, we may view κ as a degree 3 function on $T[1]M$. If $q : T^*[2]T[1]M \rightarrow T[1]M$ is the projection, then $q^* \kappa$ satisfies the Maurer-Cartan equation

$$Qq^* \kappa + \frac{1}{2} \{q^* \kappa, q^* \kappa\} = 0$$

(with both terms vanishing identically).

Therefore, with $X_\kappa := \{q^* \kappa, \cdot\}$,

$$Q_\kappa := Q + X_\kappa$$

defines a new homological vector field on $T^*[2]T[1]M$, preserving the symplectic structure. The triple $(T^*[2]T[1]M, Q_\kappa, \omega)$ is a Courant algebroid, called the *exact Courant algebroid with background 3-form κ* .

2.4.3. *Gauge Transformations.* Suppose that $\beta \in \Omega^2(M)$ is a 2-form. The time-1 Hamiltonian flow along $X_\beta := \{q^*\beta, \cdot\}$ preserves the symplectic structure, but deforms the homological vector field Q_κ to

$$e^{\text{ad}(X_\beta)}(Q_\kappa) = (1 + [X_\beta, \cdot] + \frac{1}{2}[X_\beta, [X_\beta, \cdot]] + \cdots)(Q + X_\kappa) = Q + X_{\kappa - d\beta}.$$

Consequently, the time-1 Hamiltonian flow e^{X_β} defines an isomorphism of Courant algebroids

$$(2.13) \quad e^{X_\beta} : (T^*[2]T[1]M, Q_\kappa, \omega) \rightarrow (T^*[2]T[1]M, Q_{\kappa - d\beta}, \omega).$$

In particular, if β is closed, e^{X_β} defines an automorphism of Courant algebroids. Furthermore, (2.13) shows that the isomorphism class of $(T^*[2]T[1]M, Q_\kappa, \omega)$ depends only on the cohomology class of κ . Any Courant algebroid isomorphic to $(T^*[2]T[1]M, Q_\kappa, \omega)$ is called an *exact Courant algebroid with characteristic class* $[\kappa] \in H^3(M)$.

Definition 8. A *strictly-2-symplectic local Lie 2-groupoid* over a manifold M is a pair $([\hat{X}_\bullet], \omega_{\hat{X}})$, where $[\hat{X}_\bullet]$ is a local Lie 2-groupoid with $[\hat{X}_0] = M$ and $\omega_{\hat{X}} \in \Omega^2([\hat{X}_2])$ is a multiplicative symplectic form.

$([\hat{X}_\bullet], \omega_{\hat{X}})$ *integrates* the Courant algebroid (X, ω_X) if

$$(2.14) \quad 1\text{-Jet}([\hat{X}_\bullet]) = X, \text{ and } 1\text{-Jet}(\hat{\omega}_{\hat{X}}) = \omega_X.$$

Remark 4. Suppose $([\hat{X}_\bullet^1], \omega_{\hat{X}^1})$ and $([\hat{X}_\bullet^2], \omega_{\hat{X}^2})$ are two integrations of a Courant algebroid (X, ω_X) . Since 1-Jet is not faithful, we cannot expect a unique diffeomorphism $\phi : [\hat{X}_\bullet^1] \rightarrow [\hat{X}_\bullet^2]$ to exist such that the following diagram commutes

$$(2.15) \quad \begin{array}{ccc} [T[1, 0]\hat{X}_\bullet^1] & \xrightarrow{d\phi} & [T[1, 0]\hat{X}_\bullet^2] \\ \downarrow \hat{\omega}_{\hat{X}^1} & & \downarrow \hat{\omega}_{\hat{X}^2} \\ K(\mathbb{R}[2], 2) & \longrightarrow & K(\mathbb{R}[2], 2) \end{array}$$

That is, it is possible that no symplectomorphism

$$\phi : [\hat{X}_\bullet^1] \rightarrow [\hat{X}_\bullet^2]$$

satisfying $1\text{-Jet}(\phi) = \text{id}$ exists; and if it does exist, it need not be unique. In particular, the symplectic form $\omega_{\hat{X}}$ on $[\hat{X}_2]$ is not unique, in general.

For example, in § 3, we will construct a strictly-2-symplectic (local) 2-groupoid

$$([\hat{X}_\bullet^1], \omega_{\hat{X}^1}) := ([\mathcal{T}_\bullet^* M], \omega_{\mathcal{T}})$$

integrating $(X, \omega_X) := (T^*[2]T[1]M, \omega)$, the standard Courant algebroid. Meanwhile, Mehta and Tang [24] have constructed a Lie 2-groupoid \hat{X}_\bullet^2 together with a closed multiplicative 2-form $\omega_{\hat{X}^2} \in \Omega^2(\hat{X}_\bullet^2)$ such that (2.14) also holds. Moreover, as we shall explain in Remark 6, there is a canonical map $\phi : [\hat{X}_\bullet^1] \rightarrow [\hat{X}_\bullet^2]$ such that $1\text{-Jet}(\phi) = \text{id}$.

However, the 2-form, $\omega_{\hat{X}^2}$, they construct is degenerate, even along $\hat{X}_0^2 \subset \hat{X}_2^2$. Hence, there exists no map $\phi : [\hat{X}_\bullet^1] \rightarrow [\hat{X}_\bullet^2]$ such that (2.15) commutes.

This suggests that a more flexible definition of (local) symplectic 2-groupoids is needed, which interprets diagrams such as (2.15) in terms of appropriate 2-morphisms. We plan to address this issue in a future paper.

The remainder of this paper will be devoted to constructing strictly-2-symplectic (local) 2-groupoids integrating standard and exact Courant algebroids.

Remark 5. If we ignore the symplectic structure, there is a very simple construction of a simplicial manifold integrating the standard and exact Courant algebroids. However, integrating the symplectic form is slightly more challenging, and the bulk of the work in this note will be spent doing this.

There is a natural isomorphism of NQ -manifolds $T^*[2]T[1]M \cong T[1]T^*[1]M$ (see § 4). In general, if $V \rightarrow M$, $V = V^{-1} \oplus \dots \oplus V^{-m}$, is a graded vector bundle concentrated in negative degrees then it is easy to describe an integration I_\bullet of $T[1]V$ such that I_\bullet is a Lie $m+1$ -groupoid. In the special case $V = T^*[1]M$ the simplicial manifold I_\bullet is (locally) isomorphic to the simplicial manifold \mathcal{T}_\bullet^*M we will construct in § 3.

The simplicial manifold I_\bullet is defined as follows: a point in I_k is a $k+1$ -tuple (x_0, \dots, x_k) of points of M together with a choice of $v_\phi \in V_{x_\phi(0)}^{-l}$ for every monotone map $\phi : [l] \rightarrow [k]$ (for every l), such that $v_\phi = 0$ for every non-injective ϕ .

As explained by Mehta and Tang [24], this construction has the following more conceptual explanation: V can be viewed as a bundle of non-negatively graded chain complexes (with trivial differential). Using the Dold-Kan correspondence pointwise (cf. Example 7), we get a bundle $\Gamma(V)_\bullet$ of simplicial vector spaces over M . Next we note that the bisimplicial manifold $E_\bullet\Gamma(V)_\bullet$ integrates the bi- NQ -manifold $T[1,0]V$. Finally, one obtains the simplicial manifold $I_\bullet := \overline{W}(E_\bullet\Gamma(V)_\bullet)$ by applying the *bar construction* functor $\overline{W} : \text{Man}^{\Delta^{2\text{op}}} \rightarrow \text{Man}^{\Delta^{\text{op}}}$ described in [3, 10, 24].

The fact that $I_\bullet := \overline{W}(E_\bullet\Gamma(V)_\bullet)$ integrates $T[1]V$ follows directly from the fact that 1-Jet intertwines the functor \overline{W} with the ‘total-grading’ functor from bi- NQ -manifolds to NQ -manifolds induced by the diagonal inclusion $\underline{\text{End}}(\mathbb{R}^{0|1}) \subset \underline{\text{End}}(\mathbb{R}^{0|1})^2$.

3. INTEGRATION OF THE STANDARD COURANT ALGEBROID

Let M be a manifold. In this section we construct a local strictly-2-symplectic 2-groupoid integrating $T^*[2]T[1]M$, the standard Courant algebroid over M . To do this, we first choose a connection on M . Let $U \subset M \times M$ be a symmetric neighbourhood of the diagonal such that for any $(x, y) \in U$ there exists a unique geodesic

$$\gamma_{x,y} : [0, 1] \rightarrow M, \text{ such that } \gamma_{x,y}(0) = x, \gamma_{x,y}(1) = y, \text{ and } (x, \gamma_{x,y}(t)) \in U.$$

For $t \in [0, 1]$ let $q_t : U \rightarrow M$ be given by $q_t(x, y) = \gamma_{x,y}(t)$. We define $W := q_{1/2}^*T^*M$. Notice that, using the parallel transport, we can identify W with $W_t := q_t^*T^*M$ for any $t \in [0, 1]$.

Let $U_2 \subset M^3$ be given by $U_2 = \{(x, y, z) | (x, y), (x, z), (y, z) \in U\}$. We have a map $m : U_2 \rightarrow M^3$ given by

$$m : (x, y, z) \mapsto (q_{1/2}(x, y), q_{1/2}(y, z), q_{1/2}(z, x)).$$

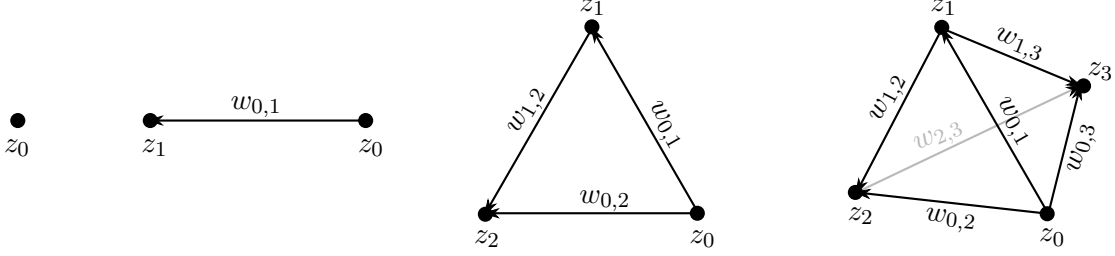


FIGURE 1. The local 2-groupoid $[\mathcal{T}_\bullet^* M]$. (Here $z_i \in M$ and $w_{i,j} \in W_{(z_i, z_j)}$.)

We shall suppose that U is such that m is an injective local diffeomorphism.

Similarly, we let $U_n \subset M^{n+1} = E_n M$ be given by $U_n = \{(z_0, \dots, z_n) \mid (z_i, z_j) \in U, \forall 0 \leq i, j \leq n\}$. It is clear that $U_\bullet \subset E_\bullet M$ is a simplicial submanifold.

3.1. The 2-groupoid. We now construct a simplicial manifold $\mathcal{T}_\bullet^* M$ which integrates $T^*[2]T[1]M$. We define it by

$$\mathcal{T}_n^* M := \{(z_0, \dots, z_n); \{w_{i,j}\}_{0 \leq i, j \leq n}\}$$

such that $(z_i, z_j) \in U$, and $w_{i,j} \in W_{(z_i, z_j)}$ for all $0 \leq i, j \leq n$ and $w_{i,j} = -w_{j,i}$ for all i, j .

For any monotone map $f : [n] \rightarrow [m]$ we define $\mathcal{T}^*(f) : \mathcal{T}_m^* M \rightarrow \mathcal{T}_n^* M$ by

$$\begin{aligned} ((z_0, \dots, z_m); \{w_{i,j}\}_{0 \leq i, j \leq m}) &\mapsto ((z_{f(0)}, \dots, z_{f(n)}); \{w'_{i,j}\}_{0 \leq i, j \leq n}) \\ &\text{where } w'_{i,j} := w_{f(i), f(j)}. \end{aligned}$$

In the sequel, when describing an element of $\mathcal{T}_n^* M$, we will often omit the terms $w_{ii} \equiv 0$, since they are always 0. We can visualize $\mathcal{T}_\bullet^* M$ graphically as in Figure 1. It is clear that $[\mathcal{T}_\bullet^* M]$ is a local 2-groupoid since simplices are fully determined by their 0 and 1 dimensional faces.

Recall the simplicial manifold $E_\bullet M$ defined in Example 3. We remark that $\mathcal{T}_\bullet^* M$ is a simplicial vector bundle over U_\bullet , an open simplicial submanifold of $E_\bullet M$, with the projection

$$(3.1) \quad q_{\mathcal{T}} : ((z_0, \dots, z_n); \{w_{i,j}\}_{0 \leq i, j \leq n}) \rightarrow (z_0, \dots, z_n).$$

Remark 6. If one replaces the vector bundle $W := W_{\frac{1}{2}}$ with $T^*M \times M \supseteq W_0$, the construction of $\mathcal{T}_\bullet^* M$ described above is formally identical to the construction of the simplicial manifold I_\bullet described in Remark 5. Since parallel transport defines an isomorphism $W \cong W_0$, we get a canonical embedding of simplicial manifolds $\mathcal{T}_\bullet^* M \subseteq I_\bullet$. Note that I_\bullet is precisely the Lie 2-groupoid described by Mehta and Tang [24].

Remark 7. Recall that one can integrate $T^*[2]T[1]M$ to an infinite dimensional simplicial manifold X_\bullet [31], whose n -simplices are

$$X_n = \text{SMan}_{NQ} (T[1]|\Delta^n|, T^*[2]T[1]M),$$

where $|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1 \text{ and } x_i \geq 0\}$ is the geometric n -simplex. Using the canonical isomorphism $\omega_{T^*[1]M}^b \circ L : T^*[2]T[1]M \rightarrow T[1]T^*[1]M$ described in (4.1), together with Lemma 2, one sees that

$$X_n \cong \text{SMan}_N(T[1]|\Delta^n|, T^*[1]M) \cong \text{VBund}(T|\Delta^n|, T^*M),$$

where VBund is the category of vector bundles.

We may view our simplicial manifold \mathcal{T}_\bullet^*M as a simplicial submanifold of $\text{VBund}(T|\Delta^\bullet|, T^*M)$. Indeed,

$$\mathcal{T}_0^*M = \text{VBund}(T|\Delta^0|, T^*M) = M.$$

Meanwhile, we can realize elements of \mathcal{T}_n^*M ($n > 0$) inductively as harmonic maps¹ in $\text{VBund}(T|\Delta^n|, T^*M)$ which satisfy the following boundary conditions

- they restrict to elements of $\mathcal{T}_{n-1}^*M \subset \text{VBund}(T|\Delta^{n-1}|, T^*M)$ along the boundary $T\partial|\Delta^n| \subset T|\Delta^n|$, and
- they map horizontal vectors at points in $(T\partial|\Delta^n|)^\perp \subset T|\Delta^n|$ to horizontal vectors (horizontal with respect to the Levi-Cevita connections).

In the construction of \mathcal{T}_\bullet^*M , we only considered ‘harmonic’ simplicies in X_\bullet which are suitably close to being degenerate. However, given any element of $\text{VBund}(T|\Delta^n|, T^*M)$, one could first deform it to a homotopic ‘energy minimizing’ map, and then look at nearby ‘harmonic’ simplicies. Patching these neighbourhoods together, one might hope to construct a global integration of $T^*[2]T[1]M$.

3.2. The 2-form. Next we construct a multiplicative symplectic form on \mathcal{T}_2^*M which integrates the symplectic structure on $T^*[2]T[1]M$.

The map $m : U_2 \rightarrow M^3$ lifts to a local diffeomorphism $\tilde{m} : \mathcal{T}_2^*M \rightarrow T^*M^3$,

$$\tilde{m} : ((x_0, x_1, x_2); (w_{01}, w_{02}, w_{12})) \mapsto (m(x_0, x_1, x_2); (w_{01}, w_{12}, w_{20})).$$

The symplectic form $\omega_{\mathcal{T}}$ on \mathcal{T}_2^*M is defined to be the pullback of the symplectic form on T^*M^3 .

To check that $\omega_{\mathcal{T}}$ is multiplicative, i.e. that $D\omega = 0$, it suffices to show that $\omega_{\mathcal{T}} = D\alpha$ for some $\alpha \in \Omega^2(\mathcal{T}_1^*M)$. Let $\tilde{q}_{1/2} : \mathcal{T}_1^*M \rightarrow T^*M$ denote the map defined by $(x, y; w) \mapsto (q_{1/2}(x, y), w)$. We can use $\alpha = \tilde{q}_{1/2}^* \omega_{T^*M}$, where ω_{T^*M} is the symplectic form on T^*M .

3.3. Proof that $([\mathcal{T}_\bullet^*M], \omega_{\mathcal{T}})$ integrates $T^*[2]T[1]M$.

Theorem 1. *The strictly-2-symplectic local 2-groupoid $([\mathcal{T}_\bullet^*M], \omega_{\mathcal{T}})$ integrates the standard Courant algebroid $(T^*[2]T[1]M, Q, \omega)$.*

Proof. First we calculate $1\text{-Jet}([\mathcal{T}_\bullet^*M])$. Since (3.1) defines a simplicial map $q_{\mathcal{T}} : \mathcal{T}_\bullet^*M \rightarrow E_\bullet M$, it induces a map

$$q_{\mathcal{T}} : \text{SMan}^{\Delta^{\text{op}}}(X \times E_\bullet \mathbb{R}^{0|1}, \mathcal{T}_\bullet^*M) \rightarrow \text{SMan}^{\Delta^{\text{op}}}(X \times E_\bullet \mathbb{R}^{0|1}, E_\bullet M),$$

¹Note that since the fibres of vector bundles are totally geodesic and linear maps are totally geodesic, a vector bundle morphism $f : T|\Delta^n| \rightarrow T^*M$ is harmonic if and only if its restriction $f \circ \sigma : |\Delta^n| \rightarrow T^*M$ to any flat section $\sigma : |\Delta^n| \rightarrow T|\Delta^n|$ is harmonic.

which is natural in X , and therefore a map

$$\tilde{q}_{\mathcal{T}} : 1\text{-Jet}([\mathcal{T}_{\bullet}^*M]) \rightarrow 1\text{-Jet}([E_{\bullet}M]) = T[1]M,$$

where the identity $1\text{-Jet}([E_{\bullet}M]) = T[1]M$ was described in Example 5.

For $v \in T[1]_xM$, the corresponding map $\text{ev}_v : E_1\mathbb{R}^{0|1} \rightarrow E_1M$ is given by

$$\text{ev}_v : (\theta_0, \theta_1) \rightarrow (x + \theta_0v, x + \theta_1v).$$

We would like to look at all possible maps $E_{\bullet}\mathbb{R}^{0|1} \rightarrow \mathcal{T}_{\bullet}^*M$ lying over the map ev_v .

Now a map $F_v : E_1\mathbb{R}^{0|1} \rightarrow \mathcal{T}_1^*M$ such that $\tilde{q}_{\mathcal{T}}(F_v) = \text{ev}_v$ must be of the form

$$(3.2) \quad F_v : (\theta_0, \theta_1) \rightarrow ((x + \theta_0v, x + \theta_1v); (w_{0,1} = a\theta_0 + b\theta_1 + c\theta_0\theta_1)),$$

for $a, b \in W_{x+\theta_0v, x+\theta_1v}[1] \cong T^*[1]_xM$ and $c \in W_{x+\theta_0v, x+\theta_1v}[2] \cong T^*[2]_xM$.

If we would like to extend $F_v : E_1\mathbb{R}^{0|1} \rightarrow \mathcal{T}_1^*M$ to a map of simplicial (super)-manifolds, the extension must be equivariant with respect to the action of the category $\mathbf{\Delta}^{\text{op}}$. In particular, we must have

$$(3.3) \quad F_v \circ s_0 = s_0 \circ \text{ev}_v \Rightarrow a = -b.$$

However (3.3) is the only obstruction to extending F_v . A routine check shows that

$$(\theta_0, \dots, \theta_n) \rightarrow ((x + \theta_0v, \dots, x + \theta_nv); (w_{ij} = b(\theta_j - \theta_i) + c\theta_i\theta_j))$$

is the unique extension of (3.2) to a map $E_{\bullet}\mathbb{R}^{0|1} \rightarrow \mathcal{T}_{\bullet}^*M$.

Consequently simplicial maps $E_{\bullet}\mathbb{R}^{0|1} \rightarrow \mathcal{T}_{\bullet}^*M$ are parametrized by the super-manifold $T^*[2]T[1]M$. In the coordinates described in § 2.4, the maps are

$$\text{ev}_{(x^\alpha, \xi^\alpha, p_\alpha, \eta_\alpha)} : (\theta_0, \dots, \theta_n) \rightarrow ((x^\alpha + \theta_0\xi^\alpha, \dots, x^\alpha + \theta_n\xi^\alpha); (w_{ij} = \eta_\alpha(\theta_j - \theta_i) + p_\alpha\theta_i\theta_j)),$$

see Figure 2. It follows that $1\text{-Jet}([\mathcal{T}_{\bullet}^*M]) = T^*[2]T[1]M$ as a super-manifold.

To show that $1\text{-Jet}([\mathcal{T}_{\bullet}^*M])$ is the NQ -manifold $T^*[2]T[1]M$, we must show that the Q structure on $1\text{-Jet}([\mathcal{T}_{\bullet}^*M])$ is given by (2.12). The natural action of $\mathbb{R}^{0|1}$ on $E_{\bullet}\mathbb{R}^{0|1}$ is

$$\theta \cdot (\theta_0, \dots, \theta_n) = (\theta_0 + \theta, \dots, \theta_n + \theta).$$

So we see that

$$\begin{aligned} \text{ev}_{(x^\alpha, \xi^\alpha, p_\alpha, \eta_\alpha)} : \theta \cdot (\theta_0, \dots, \theta_n) &\rightarrow ((x^\alpha + \theta\xi^\alpha + \theta_0\xi^\alpha, \dots, x^\alpha + \theta\xi^\alpha + \theta_n\xi^\alpha); \\ &(w_{ij} = (\eta_\alpha + \theta p_\alpha)(\theta_j - \theta_i) + p_\alpha\theta_i\theta_j)), \end{aligned}$$

which descends to the action

$$(3.4) \quad \theta \cdot (x^\alpha, \xi^\alpha, p_\alpha, \eta_\alpha) = (x^\alpha + \theta\xi^\alpha, \xi^\alpha, p_\alpha, \eta_\alpha + \theta p_\alpha)$$

on $1\text{-Jet}([\mathcal{T}_{\bullet}^*M]) = T^*[2]T[1]M$. Since (3.4) is just the flow along the vector field (2.12), it follows that $1\text{-Jet}([\mathcal{T}_{\bullet}^*M]) = T^*[2]T[1]M$ as an NQ -manifold.

What remains is to check that $\omega_{\mathcal{T}} \in \Omega^2(\mathcal{T}_2^*M)$ integrates the canonical symplectic form on the cotangent bundle $T^*[2]T[1]M$. For this purpose we need to find the pullback of $\omega_{\mathcal{T}}$ by the evaluation map $\text{ev}_2 : T^*[2]T[1]M \times E_2\mathbb{R}^{0|1} \rightarrow \mathcal{T}_2^*M$. We have

$$q_{1/2}(x^\alpha + \theta_i\xi^\alpha, x^\alpha + \theta_j\xi^\alpha) = x^\alpha + \frac{\theta_i + \theta_j}{2}\xi^\alpha.$$

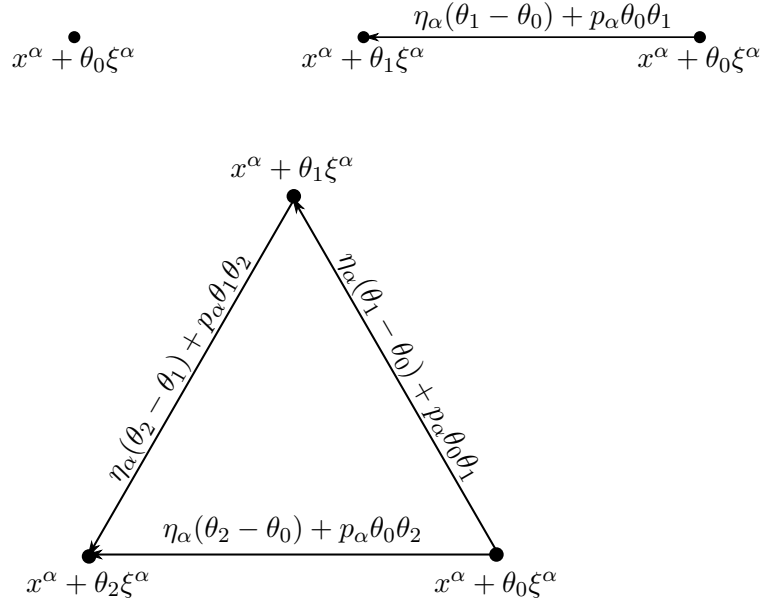


FIGURE 2. The Maps $E_{\bullet}\mathbb{R}^{0|1} \rightarrow \mathcal{T}_{\bullet}^*M$ parametrized by $T^*[2]T[1]M$.

As a result,

$$\text{ev}_2^* \omega_{\mathcal{T}} = \sum_{(i,j)=(0,1),(1,2),(2,0)} d\left(x^{\alpha} + \frac{\theta_i + \theta_j}{2} \xi^{\alpha}\right) d\left(\eta_{\alpha}(\theta_j - \theta_i) + p_{\alpha} \theta_i \theta_j\right).$$

The $(2,0)$ -bihomogeneous part of $\text{ev}_2^* \omega_{\mathcal{T}}$ (the part which is a 2-form on $T^*[2]T[1]M$ parametrized by $E_2\mathbb{R}^{0|1}$) is equal to

$$(\text{ev}_2^* \omega_{\mathcal{T}})^{(2,0)} = (dp_{\alpha} dx^{\alpha} + d\eta_{\alpha} d\xi^{\alpha})(\theta_0 \theta_1 + \theta_1 \theta_2 + \theta_2 \theta_0).$$

Comparison with (2.5) shows that $1\text{-Jet}(\hat{\omega}_{\mathcal{T}}) = dp_{\alpha} dx^{\alpha} + d\eta_{\alpha} d\xi^{\alpha}$. Since the symplectic form on $T^*[2]T[1]M$ is given by (2.11), this concludes the proof. \square

Remark 8. In [24], Rajan Mehta and Xiang Tang construct a (global) Lie 2-groupoid by applying the bar construction to the bi-simplicial manifold integrating the Lie bialgebroid (T^*M, TM) . As explained in Remark 6, $[\mathcal{T}_{\bullet}^*M]$ is isomorphic to the local version of the Lie 2-groupoid they build.

Furthermore, they construct a closed (degenerate) two-form $\omega'_{\mathcal{T}}$ on the 2-simplices of \mathcal{T}_2^*M . A calculation similar to the proof above shows that $1\text{-Jet}(\omega'_{\mathcal{T}}) = \omega_{T^*[2]T[1]M}$, therefore, their construction differentiates to the standard Courant algebroid.

The requirement in Definition 8 that the 2-form be non-degenerate “on the nose”, is (at least at first glance) less complicated than the corresponding non-degeneracy requirements in [24, § 6.2]. However, it would be interesting to know if Mehta and Tang’s requirements

on their 2-form are equivalent to the 1-Jet of their 2-form being non-degenerate “on the nose”. Additionally, our 2-form is only defined on a local Lie 2-groupoid, while theirs is defined on a genuine Lie 2-groupoid.

Notice that, unlike the construction in [24], $\mathcal{T}_\bullet^*M : \Delta^{\text{op}} \rightarrow \text{Man}$ extends naturally to a functor $\text{FSets}^{\text{op}} \rightarrow \text{Man}$ (where FSets is the category of finite sets).

4. INTEGRATION OF EXACT COURANT ALGEBROIDS

In this section, we show that as NQ -manifolds, any exact Courant algebroid over M is isomorphic to the standard Courant algebroid over M (see also [34]). Therefore *once we modify the symplectic structure*, the construction in § 3 also integrates exact Courant algebroids.

Lemma 2. *Let $F : \text{SMan}_{NQ} \rightarrow \text{SMan}_N$ be the forgetful functor. Then $T[1] : \text{SMan}_N \rightarrow \text{SMan}_{NQ}$ is a right adjoint for F . In particular, for $X \in \text{SMan}_{NQ}$ and $Z \in \text{SMan}_N$,*

$$\text{SMan}_{NQ}(X, T[1]Z) \cong \text{SMan}_N(X, Z).$$

Proof. Since $T[1]Z$ represents the pre-sheaf $\text{SMan}(\cdot \times \mathbb{R}^{01}, Z)$, we have

$$\text{SMan}_{NQ}(X, T[1]Z) = \text{SMan}_N(X \times_{\mathbb{R}^{01}} \mathbb{R}^{01}, Z),$$

but the action of \mathbb{R}^{01} on X defines an isomorphism $X \times_{\mathbb{R}^{01}} \mathbb{R}^{01} \cong X$.

It is clear that this bijection $\text{SMan}_{NQ}(X, T[1]Z) \cong \text{SMan}_N(X, Z)$ is natural in both X and Z , which concludes the proof.

Let us describe the unit and co-unit of the adjunction explicitly. Note that the homological vector field Q_X on X defines a section $Q_X : X \rightarrow T[1]X$. Furthermore, $Q_X : X \rightarrow T[1]X$ is a NQ -morphism with respect to the de Rham vector field on $T[1]X$. The unit of the adjunction is the natural transformation

$$1 \rightarrow T[1] \circ F, \quad X \rightarrow (Q_X : X \rightarrow T[1]X).$$

Meanwhile, if $q_Z : T[1]Z \rightarrow Z$ is the canonical projection, then the co-unit is the natural transformation

$$F \circ T[1] \rightarrow 1, \quad Z \rightarrow (q_Z : T[1]Z \rightarrow Z).$$

In particular, if $f : X \rightarrow Z$ is any N -morphism,

$$T[1]f \circ Q_X : X \rightarrow T[1]Z$$

is the corresponding NQ -morphism. In the other direction, if $\tilde{f} : X \rightarrow T[1]Z$ is an NQ -morphism, the corresponding N -morphism is just $q_Z \circ \tilde{f}$. \square

Example 9 (The Anchor Map). Suppose X is any NQ -manifold over the base $B(X) := (0, 0) \cdot X$ (here $(0, 0) \in \underline{\text{End}}(\mathbb{R}^{01})$, see Definition 5). Multiplication by $(0, 0)$ defines the canonical projection $q_X : X \rightarrow B(X)$. It follows from Lemma 2 that there is a unique NQ -morphism

$$a_X : X \rightarrow T[1]B(X)$$

over $q_X : X \rightarrow B(X)$, called the *anchor map* for X .

Furthermore, Lemma 2 implies that a_X is a natural transformation from the identity functor on SMan_{NQ} to $T[1] \circ B : \text{SMan}_{NQ} \rightarrow \text{SMan}_{NQ}$. In fact $T[1] : \text{Man} \rightarrow \text{SMan}_{NQ}$ is a right adjoint for $B : \text{SMan}_{NQ} \rightarrow \text{Man}$, with unit a_X and co-unit the identity.

Let

$$(4.1a) \quad L : T^*[2]T[1]M \rightarrow T^*[2]T^*[1]M$$

denote the Legendre transform and

$$(4.1b) \quad \omega_{T^*[1]M}^b : T^*[2]T^*[1]M \rightarrow T[1]T^*[1]M$$

be the diffeomorphism defined by the canonical symplectic structure $\omega_{T^*[1]M}$ on $T^*[1]M$ [22, 27]. Note that the composition $\omega_{T^*[1]M}^b \circ L$ is an NQ -morphism with respect to the de Rham vector field on $T[1]T^*[1]M$ [27]. Finally, let $s : T[1]T^*[1]M \rightarrow T^*[1]M$ denote the canonical projection.

Suppose that $(T^*[2]T[1]M, Q_\kappa, \omega)$ is an exact Courant algebroid with background 3-form $\kappa \in \Omega^3(M)$. Then by Lemma 2, there is a NQ -morphism $\psi_\kappa : (T^*[2]T[1]M, Q_\kappa) \rightarrow (T^*[2]T[1]M, Q)$ corresponding to the N -morphism $s \circ \omega_{T^*[1]M}^b \circ L$.

Proposition 2. *The NQ -morphism $\psi_\kappa : (T^*[2]T[1]M, Q_\kappa) \rightarrow (T^*[2]T[1]M, Q)$ is given by translation along the fibres of $q : T^*[2]T[1]M \rightarrow T[1]M$ by the degree-2 1-form*

$$\frac{1}{2} \iota_{Q_{dR}} \iota_{Q_{dR}} q_M^* \kappa,$$

where $q_M : T[1]M \rightarrow M$ is the projection, and Q_{dR} is the de Rham vector field on $T[1]M$. In particular, ψ_κ is a diffeomorphism.

Proof. A routine calculation shows that, in coordinates, both maps are equal to

$$\psi_\kappa : (x^\alpha, \xi^\alpha, p_\alpha, \eta_\alpha) \rightarrow (x^\alpha, \xi^\alpha, p_\alpha + 3\kappa_{[abc]} \xi^b \xi^c, \eta_\alpha),$$

where $\kappa = \kappa_{[abc]} dx^a dx^b dx^c$. □

It follows that, under the diffeomorphism ψ_κ , the canonical symplectic form ω on $T^*[2]T[1]M$ transforms as

$$(\psi_\kappa^{-1})^* \omega = \omega - \frac{1}{2} q^* d \iota_{Q_{dR}} \iota_{Q_{dR}} q_M^* \kappa = \omega - q^* \mathcal{L}_{Q_{dR}} \iota_{Q_{dR}} q_M^* \kappa.$$

Therefore, we need to modify the symplectic form on $\mathcal{T}_2^* M$ by a multiplicative term which is mapped to $-\mathcal{L}_{Q_{dR}} \iota_{Q_{dR}} q_M^* \kappa$ by 1-Jet.

4.1. Modification of the symplectic form on $\mathcal{T}_2^* M$. To lift this modification of the symplectic form to $\mathcal{T}_2^* M$, we make use of the connection on M to integrate κ along geodesics in M .

Recall from § 3.1 that \mathcal{T}_\bullet^*M is a simplicial vector bundle over the simplicial manifold $U_\bullet \subseteq E_\bullet M$. We define a 2-form $\mu \in \Omega^2(U_1)$

$$(4.2) \quad \mu := \int_{[0,1]} \Gamma^* \kappa$$

by integration over the fibres of

$$(4.3) \quad \Gamma : [0, 1] \times U_1 \rightarrow M, \quad \Gamma(t, x, y) = \gamma_{x,y}(t).$$

Let $\varkappa \in \Omega^2(U_2)$ be defined by

$$\varkappa = D\mu := \sum_{i=0}^2 (-1)^i d_i^* \mu.$$

We see immediately that $D\varkappa = 0$, so it is multiplicative. Furthermore,

$$d \int_{[0,1]} \Gamma^* \kappa - \int_{[0,1]} d\Gamma^* \kappa = d_1^* \kappa - d_0^* \kappa.$$

Since κ is closed, this implies that $d\mu = d_1^* \kappa - d_0^* \kappa$. Consequently \varkappa is closed.

Lemma 3. $1\text{-Jet}(\hat{\varkappa}) = \mathcal{L}_{Q_{dR}} \iota_{Q_{dR}} q_M^* \kappa$

The two form $\omega_{\mathcal{T}} - q_{\mathcal{T}}^* \varkappa$ is non-degenerate since we are modifying the standard symplectic form on of T^*M^3 by a closed 2-form on (an open subset of) M^3 . Thus, we immediately get the following theorem.

Theorem 2. *The strictly-2-symplectic local 2-groupoid $([\mathcal{T}_\bullet^*M], \omega_{\mathcal{T}} - q_{\mathcal{T}}^* \varkappa)$ integrates the exact Courant algebroid $(T^*[2]T[1]M, Q_\kappa, \omega)$ with background 3-form $\kappa \in \Omega^3(M)$.*

proof of Lemma 3. We do the calculation in coordinates. Recall from Example 5 that $T[1]M$ parametrizes the simplicial maps $E_\bullet \mathbb{R}^{0|1} \rightarrow E_\bullet M$. A point $(x^\alpha, \xi^\alpha) \in T[1]M$ parametrizes the map

$$\text{ev}_{(x^\alpha, \xi^\alpha)} : (\theta_0, \dots, \theta_n) \rightarrow (x^\alpha + \theta_0 \xi^\alpha, \dots, x^\alpha + \theta_n \xi^\alpha).$$

Composition with $\Gamma : [0, 1] \times E_1 M \rightarrow M$ is given by

$$\Gamma : (t, \text{ev}_{(x^\alpha, \xi^\alpha)}(\theta_0, \theta_1)) \rightarrow (x^\alpha + (t(\theta_1 - \theta_0) + \theta_0) \xi^\alpha).$$

Therefore, with $\kappa = \kappa_{[abc]} dx^a dx^b dx^c$,

$$\begin{aligned} & \Gamma^* \kappa_{[abc]} dx^a dx^b dx^c \\ &= \kappa_{[abc]} (x^\alpha + (\dots) \xi^\alpha) d(x^\alpha + (\dots) \xi^\alpha) d(x^b + (\dots) \xi^b) d(x^c + (\dots) \xi^c) \\ &= \left(\kappa_{[abc]}(x^\alpha) + \frac{\partial \kappa_{[abc]}}{\partial x^d}(x^\alpha) (t(\theta_1 - \theta_0) + \theta_0) \xi^d + \dots \right) \\ & \quad (3(\theta_1 - \theta_0) \xi^a dt dx^b dx^c + 6\theta_0 \theta_1 \xi^a dt d\xi^b dx^c + \dots) \\ &= \left(\kappa_{[abc]} (3(\theta_1 - \theta_0) \xi^a dx^b dx^c - 6\theta_0 \theta_1 \xi^a d\xi^b dx^c) - 3 \frac{\partial \kappa_{[abc]}}{\partial x^d} \theta_0 \theta_1 \xi^d \xi^a dx^b dx^c \right) dt + \dots \end{aligned}$$

Where we have omitted the terms that do not contain dt . So at the point $\text{ev}_{(x^\alpha, \xi^\alpha)}(\theta_0, \theta_1)$, μ is given by

$$\mu(\text{ev}_{(x^\alpha, \xi^\alpha)}(\theta_0, \theta_1)) = \kappa_{[abc]}(3(\theta_1 - \theta_0)\xi^a dx^b dx^c - 6\theta_0\theta_1\xi^a d\xi^b dx^c) - 3\frac{\partial\kappa_{[abc]}}{\partial x^d}\theta_0\theta_1\xi^d\xi^a dx^b dx^c$$

Therefore, at the point $\text{ev}_{(x^\alpha, \xi^\alpha)}(\theta_0, \theta_1, \theta_2)$, the 2-form \varkappa is given by

$$\varkappa(\text{ev}_{(x^\alpha, \xi^\alpha)}(\theta_0, \theta_1, \theta_2)) = -3(2\kappa_{[abc]}\xi^a d\xi^b dx^c + \frac{\partial\kappa_{[abc]}}{\partial x^d}\xi^d\xi^a dx^b dx^c)(\theta_0\theta_1 + \theta_1\theta_2 + \theta_2\theta_0).$$

Comparison with (2.5) shows that

$$1\text{-Jet}(\hat{\varkappa}) = -6\kappa_{[abc]}\xi^a d\xi^b dx^c - 3\frac{\partial\kappa_{[abc]}}{\partial x^d}\xi^d\xi^a dx^b dx^c.$$

Using $d\kappa = 0$, we calculate,

$$\begin{aligned} 1\text{-Jet}(\hat{\varkappa}) &= 6\kappa_{[abc]}d\xi^a\xi^b dx^c + 3\frac{\partial\kappa_{[abc]}}{\partial x^d}dx^d\xi^a\xi^b dx^c \\ &= 3d(\kappa_{[abc]}\xi^a\xi^b dx^c) \\ &= \frac{1}{2}d\iota_{Q_{dR}}\iota_{Q_{dR}}q_M^*\kappa \\ &= \mathcal{L}_{Q_{dR}}\iota_{Q_{dR}}q_M^*\kappa \end{aligned}$$

□

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