PROPERLY DISCONTINUOUS GROUP ACTIONS ON AFFINE HOMOGENEOUS SPACES

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1. Introduction

Let G be a real algebraic group, H an algebraic subgroup of G, and Γ a closed subgroup of G acting on the homogeneous space G/H by left translations. Given $x \in G/H$, Γ_x is the stabilizer of x in Γ. Recall that the action of Γ is *properly discontinuous* (respectively, *free*) if for any compact $K \subset G/H$ the set $\{g \in \Gamma \mid gK \cap K \neq \emptyset\}$ is finite (respectively, Γ_x is trivial for all $x \in G/H$). If Γ acts properly discontinuously and freely on G/H then the manifold of double co-sets $\Gamma\backslash G/H$ is called Clifford-Klein form. The following question is natural and well-known: Which homogeneous manifolds G/H admit nontrivial (respectively, compact) Clifford-Klein forms $\Gamma\backslash G/H$? The question has been studied when the homogeneous spaces G/H is of reductive type, that is, when both G and H are reductive groups (cf.[Be1-2] and [K1-3]). In the present paper we discuss some cases when G/H is never of reductive type. First of all recall the notable Auslander conjecture (cf. [Au]):

Conjecture 1. *Let Γ be a subgroup of the group $\mathrm{Aff}(\mathbb{A}^n)$ of all affine linear transformations of the n-dimensional real affine space $\mathbb{A}^n$. Assume that Γ acts properly discontinuously on $\mathbb{A}^n$ and the quotient $\Gamma\backslash\mathbb{A}^n$ is compact. Then Γ is a virtually solvable group, i.e. Γ contains a solvable subgroup of finite index.*

In other terms, the Auslander conjecture says that if Γ acts properly discontinuously and with compact quotient on $\mathbb{A}^n$ then the Levi subgroup of its Zariski closure in $\mathrm{Aff}(\mathbb{A}^n)$ is trivial. (Recall that the maximal connected semisimple subgroups of G are usually called Levi subgroups of G and they are all conjugated.) The continuous analog of the Auslander conjecture is the following result of T.Kobayashi and R.Lipsman:

Theorem 1.1. ([K1], [L]) *Suppose that H contains a Levi subgroup of G, Γ is a connected algebraic subgroup of G and Γ_x is compact for all $x \in G/H$. Then Γ is a compact extension of a unipotent group.*

In the light of the above discussion the following generalization of Auslander's conjecture is natural:

Conjecture 2. *Suppose that H contains a maximal reductive subgroup of G, Γ acts properly discontinuously on G/H and $\Gamma\backslash G/H$ is compact. Then Γ is virtually solvable.*

It is easy to see that that Conjecture 2 implies Conjecture 1 (cf. Remark 1 in 2.1). Also note that G/H is isomorphic (as a real algebraic variety) to $\mathbb{A}^n$ and G acts on $\mathbb{A}^n$ by regular (polynomial) automorphisms of degree ≥ 1. Conjecture 1 is exactly the case when this action is linear. Some known results about Auslander's conjecture could be extended to Conjecture 2. In §2 and §3 of the present paper we prove the following

Theorem 1.2. *Conjecture 2 is true if the Levi subgroup of G is a product of simple real algebraic groups of ranks ≤ 1.*

The arguments used in the proof of Theorem 1.2 are similar to, but more general than, our arguments in [To1] where the analogous result is proved for Auslander's conjecture. At present we can prove Conjecture 2 for $\dim G/H \leq 4$. (The proof will appear elsewhere.) In a recent paper [D-P] K.Dekimpt and N.Petrosyan formulated an extension of the Auslander conjecture to the case of a contractible real algebraic manifold X with an algebraic group G acting on it. Our Theorem 1.2 improves the result [D-P, Theorem 1.2(a)] about groups G of semisimple rank ≤ 1.

As to the Auslander conjecture, its proof is easy for $n = 2$ and due to D.Fried and W.Goldman [F-G] for $n = 3$. The proof of the conjecture for $n \leq 5$ was announced in [To2] with detailed sketch of the proof for $n = 4$. After the present paper was finished we became aware of the preprint [A-M-S5] where the conjecture is proved for $n \leq 6$. In §4 we give a full proof of Auslander's conjecture for $n \leq 5$ (Theorem 4.5) along the lines in [To2]. Our proof for $n \leq 5$ is different and simpler than the proof in [A-M-S5], in particular, it uses less input. All one needs is the result of G.A.Margulis preprint [Mar3] published as part of [A-M-S3] and the result of [To1][1]. The exact formulation of the result of [Mar3] is given in section 4.1 of the paper (Theorem 4.2(a)).

1.1. **Notation and terminology.** By an algebraic group (resp., algebraic variety) we will mean a *real* linear algebraic group (resp., *real* algebraic variety), that is, the set of all $\mathbb{R}$-rational points of a linear algebraic group (resp., algebraic variety) defined over $\mathbb{R}$. On every algebraic variety we have Hausdorff topology (induced by the topology on $\mathbb{R}$) and also Zariski topology. In order to distinguish the two topologies the topological notions connected with the Zariski topology will be usually used with the prefix "Zariski". (We say: Zariski closed, Zariski closure, Zariski connected, etc.) If M is a subset of an algebraic variety X then $\tilde{M}$ denotes the Zariski closure of M in X and $\overline{M}$ denotes the closure of M in X for the usual (Hausdorff) topology. We denote by G° the connected component of G for the Hausdorff topology and by $\mathrm{R}(G)$ (respectively, $\mathrm{R}_u(G)$) the radical (respectively, unipotent radical) of G. If G acts on a set X and $x \in X$, G_x is the stabilizer of x in G. Given $g \in G$, $g = g_s g_u$ is the Jordan decomposition of g where g_s (resp. g_u) is the semi-simple (resp. unipotent) part

[1]The result of [To1] can be also found in [So1].

of g. We let $< g >$ be the subgroup generated by g. Also, we will denote by Lie(G) the Lie algebra of G. By *rank of* G we mean the common dimension of the maximal $\mathbb{R}$-diagonalizable tori of G. Also, $\mathcal{D}^iG$ is the i-th derived subgroup of G, that is, $\mathcal{D}^0G = G$ and $\mathcal{D}^{i+1}G = [\mathcal{D}^iG, G]$ for all $i \geq 0$.

1.2. **Basic affine geometry.** A real affine space $\mathbb{A}^n$ is obtained from a real n-dimensional vector space V, called the *direction* of $\mathbb{A}^n$, by "forgetting" the origin. Most often $V = \mathbb{R}^n$. If x and $y \in \mathbb{A}^n$ we denote by $\overrightarrow{xy}$ the unique vector in V such that $y = x + \overrightarrow{xy}$. An affine automorphism $\gamma \in \mathrm{Aff}(\mathbb{A}^n)$ determines a $\lambda(\gamma) \in \mathrm{GL}(V)$, called *the linear part* of γ, such that for any pair $x, y \in \mathbb{A}^n$ we have $\overrightarrow{\gamma(x)\gamma(y)} = \lambda(\gamma)(\overrightarrow{xy})$. The map $\lambda : \mathrm{Aff}(\mathbb{A}^n) \to \mathrm{GL}(V), \gamma \mapsto \lambda(\gamma)$, is a surjective group homomorphism. Fix an *origin* $p \in \mathbb{A}^n$. If $\overrightarrow{v} \in V$ then $\gamma(p + \overrightarrow{v}) = \gamma(p) + \lambda(\gamma)(\overrightarrow{v})$. So, every $\gamma \in \mathrm{Aff}(\mathbb{A}^n)$ can be decomposed as the linear "vector" transformation $\mathbb{A}^n \to \mathbb{A}^n, p + \overrightarrow{v} \mapsto p + \lambda(\gamma)(\overrightarrow{v})$, followed by the translation by the vector $\overrightarrow{p\gamma(p)}$. Consider the semidirect product of algebraic groups $V \rtimes \mathrm{GL}(V)$ where the action of $\mathrm{GL}(V)$ on V is the natural one. The group $\mathrm{Aff}(\mathbb{A}^n)$ is identified with $V \rtimes \mathrm{GL}(V)$ via the group isomorphism $\mathrm{Aff}(\mathbb{A}^n) \to V \rtimes \mathrm{GL}(V), \gamma \mapsto (\overrightarrow{p\gamma(p)}, \lambda(\gamma))$. The structure of algebraic group on $\mathrm{Aff}(\mathbb{A}^n)$ obtained in this way does not depend on the choice of p. All stabilizers $\mathrm{Aff}(\mathbb{A}^n)_x, x \in \mathbb{A}^n$, are maximal reductive subgroups of $\mathrm{Aff}(\mathbb{A}^n)$ isomorphic to $\mathrm{GL}(V)$ and pairwise vector translation conjugate. Hence every reductive subgroup of $\mathrm{Aff}(\mathbb{A}^n)$ (and, therefore, every semi-simple element in $\mathrm{Aff}(\mathbb{A}^n)$) admits a fixed point. Further on, we will tacitely use this observation.

The group $\mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$ is identified with its image in $\mathrm{GL}_{n+1}(\mathbb{R})$ under the imbedding $(\overrightarrow{v}, m) \mapsto \begin{pmatrix} m & \overrightarrow{v} \\ 0 & 1 \end{pmatrix}$, where the elements from $\mathbb{R}^n$ a vector columns. If $\overrightarrow{v_1}, \cdots, \overrightarrow{v_n}$ is a basis of V, then $\mathcal{F} = \{p; \overrightarrow{v}_1, \cdots, \overrightarrow{v}_n\}$ is a *frame* of $\mathbb{A}^n$. We have an isomorphism $\mathrm{Aff}(\mathbb{A}^n) \to \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R}), \gamma \mapsto \begin{pmatrix} l(\gamma) & \overrightarrow{v}(\gamma) \\ 0 & 1 \end{pmatrix}$, where $l(\gamma)$ is the matrix of $\lambda(\gamma)$ in the basis $\overrightarrow{v_1}, \cdots, \overrightarrow{v_n}$ and $\overrightarrow{v}(\gamma)$ is the vector-column of the coordinates of $\overrightarrow{p\gamma(p)}$ in this basis, called *the matrix representation of* $\mathrm{Aff}(\mathbb{A}^n)$ *in the frame* $\mathcal{F}$. Note that $l(\gamma)$ is the same in any translated frame $\mathcal{F} + \overrightarrow{v} = \{p + \overrightarrow{v}; \overrightarrow{v}_1, \cdots, \overrightarrow{v}_n\}, \overrightarrow{v} \in V$.

2. Rational actions of Γ on $\mathbb{A}^n$

2.1. Let G and Γ be as in the formulation of Conjecture 2. Replacing Γ by a subgroup of finite index, we suppose from now on that G is Zariski connected. Our main goal is to reduce the proof of Conjecture 2 to the case when $\Gamma \cap \mathrm{R}(G) = \{e\}$.

Proposition 2.1. *Assume that G is acting rationally on $\mathbb{A}^n$, the restriction of this action to Γ is properly discontinuous, the quotient $\Gamma \backslash \mathbb{A}^n$ is compact, and* *there exists $x_o \in \mathbb{A}^n$ such that $Sx_o = x_o$ for a maximal reductive subgroup S of G. Then $\mathrm{R}_u(G)$ acts transitively on $\mathbb{A}^n$.*

Proof. Note that $Gx_o = \mathrm{R}_u(G)x_o$ and $\mathrm{R}_u(G)x_o$ is closed and isomorphic as a real algebraic variety to an affine space $\mathbb{A}^k$ (see [Bi] and [Ro]). The group Γ acts properly discontinuously and with compact quotient on $\mathrm{R}_u(G)x_o$. In view of [Se],

$$\mathrm{vcd}(\Gamma) = \dim \mathrm{R}_u(G)x_o = \dim \mathbb{A}^n,$$

where $\mathrm{vcd}(\Gamma)$ denotes the virtual cohomological dimension of Γ. So, $\mathrm{R}_u(G)x_o = \mathbb{A}^n$, completing the proof. □

Remarks: 1. Let Γ be as in the formulation of the Auslander conjecture, G be the Zariski closure of Γ in $\mathrm{Aff}(\mathbb{A}^n)$ and S be a maximal reductive subgroup of G. Let $x_o \in \mathbb{A}^n$ be fixed by S. Put $H = G_{x_o}$. In view of Proposition 2.1, G acts transitively on $\mathbb{A}^n$. So, $\mathbb{A}^n$ can be identified with G/H and, therefore, Conjecture 2 implies Conjecture 1.

2. The co-homological argument used in the proof of Proposition 2.1 is apparently due to Fried and Goldman [F-G]. It is also used in the proof of the analogous Lemma 1.1 in [To1]. As indicated to the author by the referee of [To1], a result similar to [To1, Lemma 1.1] had been proven in [G-H] (Theorem 2.6). Lemma 2.5 in [D-P] corresponds to Lemma 1.1 in [To1].

The following assertion is implicitly contained in the proof of [To1, Proposition 1.4].

Lemma 2.2. *Let $\Delta \subset \mathrm{GL}_n(\mathbb{R})$ be a discrete solvable subgroup. Then there exists a connected (for the Hausdorff topology on $\mathrm{GL}_n(\mathbb{R})$) solvable subgroup $R \subset \mathrm{GL}_n(\mathbb{R})$ such that $R \cap \Delta$ is a normal subgroup of finite index in Δ and $R/R \cap \Delta$ is compact.*

Proof. The group $\Delta \cap \tilde{\Delta}^\circ$ is a normal subgroup of finite index in Δ. Replacing Δ by $\Delta \cap \tilde{\Delta}^\circ$ it is enough to prove the existence of a connected subgroup R such that $\Delta \subset R \subset \tilde{\Delta}^\circ$ and R/Δ is compact. Note that $\mathcal{D}\Delta$ is a Zariski dense discrete subgroup in the connected unipotent group $\mathcal{D}\tilde{\Delta} = \mathcal{D}\tilde{\Delta}^\circ$. Therefore $\Delta \cap \mathcal{D}\tilde{\Delta}$ is a co-compact lattice in $\mathcal{D}\tilde{\Delta}$. This implies that $\Delta \cdot \mathcal{D}\tilde{\Delta}/\mathcal{D}\tilde{\Delta}$ is a discrete subgroup of $\tilde{\Delta}^\circ/\mathcal{D}\tilde{\Delta}$ (cf. [Rag, Theorem 1.13]). Hence, it is enough to prove the lemma when $\tilde{\Delta}$ is abelian and $\Delta \subset \tilde{\Delta}^\circ$. The Lie group $\tilde{\Delta}^\circ$ is isomorphic to $K \times \mathbb{R}^m$ where K is a compact torus. Let $\pi : \tilde{\Delta}^\circ \to \mathbb{R}^m$ be the natural projection and R' be the linear span of $\pi(\Delta)$. Then Δ is co-compact in $R = \pi^{-1}(R')$. □

2.2. Let G, H and Γ be as in the formulation of Conjecture 2. Using Proposition 2.1 we see that the unipotent radical of the Zariski closure of Γ in G is acting transitively on G/H. So, replacing G by the Zariski closure of Γ we may (as we will) assume that Γ is Zariski dense in G. Put $\Delta = \Gamma \cap \mathrm{R}(G)$. Then $\tilde{\Delta}$ is a normal subgroup of G. Denote by G_1 the Zariski closure of $G/\tilde{\Delta}$, by H_1 the Zariski closure of $H\tilde{\Delta}/\tilde{\Delta}$ in G_1, and by Γ_1 the natural imbedding of Γ/Δ into G_1. Clearly, $\Gamma_1 \cap \mathrm{R}(G_1) = \{e\}$.

The next proposition, which is the central one, allows to "eliminate" the solvable radical when dealing with the Auslander conjecture or with some of its generalizations. Actually, it coincides with [To1, Proposition 1.4(a)]. For reader's convenience we provide a somewhat more detailed than in [To1] proof of the proposition.

Proposition 2.3. *With the above notation and assumptions,* Γ_1 *acts properly discontinuously on* G_1/H_1 *and* $\Gamma_1\backslash G_1/H_1$ *is compact.*

Proof. Since $\tilde{\Delta}$ is a normal subgroup in G, the action of G on G/H permutes the $\tilde{\Delta}$-orbits on G/H. So, we can identify the space of $\tilde{\Delta}$-orbits on G/H with G/H' where $H' = \tilde{\Delta}H$. Let $\tilde{\Delta}_u$ be the unipotent radical of $\tilde{\Delta}$ and T be a maximal reductive subgroup of $\tilde{\Delta}$. Then $\tilde{\Delta}$ is equal to the semidirect product $\tilde{\Delta}_u \rtimes T$. Remark that T is conjugated to a subgroup of H and $\tilde{\Delta}_u$ is normal in G. Hence $H' = \tilde{\Delta}_u H$ and H' is an algebraic subgroup of G.

Let $\phi : \tilde{\Delta} \to \tilde{\Delta}_u$ be the natural projection and R be a connected subgroup of $\tilde{\Delta}$ such that $\Delta \cap R$ is a normal subgroup of finite index in Δ and $R/\Delta \cap R$ is compact (see Lemma 2.2). We will prove that $\phi(R) = \tilde{\Delta}_u$. Denote by $\tilde{\Delta}^\bullet$ the Zariski connected component of $\tilde{\Delta}$. Then $\tilde{\Delta}_u$ is the unipotent radical of $\tilde{\Delta}^\bullet$. Suppose that $\tilde{\Delta}^\bullet$ is abelian. In this case the restriction of ϕ to $\tilde{\Delta}^\bullet$ is a homomorphism of algebraic groups and $\phi(R)$ is connected and , therefore, algebraic subgroup of $\tilde{\Delta}_u$. Since $\phi(R)$ is Zariski dense in $\tilde{\Delta}_u$ we get that $\phi(R) = \tilde{\Delta}_u$. Now, let $\tilde{\Delta}^\bullet$ be arbitrary. Since R is connected, the commutator $\mathcal{D}(R)$ is unipotent and $\phi(R)$ contains $\mathcal{D}(R)$. It is enough to prove that $\mathcal{D}(R) = \mathcal{D}(\tilde{\Delta}^\bullet)$. Indeed, if so, we may factorize by $\mathcal{D}(R)$ and reduce the proof to the case when $\tilde{\Delta}^\bullet$ is abelian. Let us prove that $\mathcal{D}(R)$ contains $\mathcal{D}(\tilde{\Delta}^\bullet)$. (The inclusion $\mathcal{D}(R) \subset \mathcal{D}(\tilde{\Delta}^\bullet)$ is obvious.) Since R is Zariski dense in $\tilde{\Delta}^\bullet$ and $\mathcal{D}(R)$ is an algebraic subgroup of $\tilde{\Delta}^\bullet$ we have that $\mathcal{D}(R)$ is normal in $\tilde{\Delta}^\bullet$. But $R/\mathcal{D}(R)$ is Zariski dense in $\tilde{\Delta}^\bullet/\mathcal{D}(R)$. Therefore $\tilde{\Delta}^\bullet/\mathcal{D}(R)$ is abelian which implies that $\mathcal{D}(R)$ contains $\mathcal{D}(\tilde{\Delta}^\bullet)$, as required.

In view of the above, if $x \in G/H$ then $\tilde{\Delta}x = \tilde{\Delta}_u x = Rx$ is closed and the quotient $\Delta\backslash\tilde{\Delta}x$ is compact. Since Δ acts trivially on G/H', the natural action of Γ on G/H' induces an action of Γ_1 on G/H'. Let us prove that Γ_1 acts properly discontinuously on G/H' and that $\Gamma_1\backslash G/H'$ is compact. Indeed, let $\psi : G/H \to G/H'$ be the natural map, $K_o \subset G/H$ be a compact subset and $K = \psi(K_o)$. Since ψ is Γ-equivariant we have that $\Gamma_1 K = G/H'$ if $\Gamma K_o = G/H$, proving that $\Gamma_1\backslash G/H'$ is compact. Let $\{\gamma_i{}' \mid i \in I\}$ be the set of all elements in Γ_1 such that $\gamma_i{}'K \cap K \neq \emptyset$. For each i we fix a $\gamma_i \in \Gamma$ such that $\gamma_i{}' = \gamma_i\Delta$.

Every fiber of ψ is a $\tilde{\Delta}$-orbit and, by the above, an L-orbit. Therefore for every $i \in I$ there exist $a_i, b_i \in K_o$ and $l_i \in R$ such that $\gamma_i a_i = l_i b_i$. Fix a compact $C \subset R$ such that $R = \Delta C$ and write $l_i = \delta_i c_i$, where $\delta_i \in \Delta$ and $c_i \in C$. Then $(\delta_i^{-1}\gamma_i)a_i = c_i b_i$. But Γ acts properly discontinuously on G/H. Therefore $\{\delta_i^{-1}\gamma_i \mid i \in I\}$ is finite, which implies that Γ_1 acts properly discontinuously on G/H'.

In order to complete the proof of the proposition it remains to notice that G/H' and G_1/H_1 are both canonically homeomorphic to the affine variety $\mathrm{R}_u(G)/\widetilde{\Delta}_u \cdot \mathrm{R}_u(H)$. □

We will use Proposition 2.3 together with the following:

Proposition 2.4. *With the notation and assumptions of Conjecture 2, additionally assume that $\Gamma \cap \mathrm{R}(G) = \{e\}$ and Γ is Zariski dense in G. Denote by L a Levi subgroup of G and by K a maximal compact subgroup of L. Then*

$$\dim(G/H) \leq \dim(L/K). \tag{1}$$

Proof. Since Γ acts properly discontinuously and with compact quotient on the affine space G/H, we have

$$\dim(G/H) = \mathrm{vcd}(\Gamma).$$

On the other hand, the projection of Γ into $G/\mathrm{R}(G)$ is injective and the connected component of its closure in $G/\mathrm{R}(G)$ is solvable by a result of Auslander (cf.[Rag, Theorem 8.24]). Therefore the projection of Γ into $G/\mathrm{R}(G)$ is discrete. This implies that Γ acts properly discontinuously on the symmetric space of L. Therefore,

$$\mathrm{vcd}(\Gamma) \leq L/K,$$

completing the proof. □

The following proposition is useful.

Proposition 2.5. *With G and H as in the formulation of Conjecture 2, let $g \in G$. Let $U = \widetilde{< g_u >}$ where g_u is the unipotent part of g. Then there exists $p \in G/H$ with the following properties:*

(i) *The orbit Up is closed and g-invariant;*
(ii) *g_s fixes Up element-wise;*
(iii) *$\dim Up = 1$ if $g_u \neq e$ and $gp = p$ if $g_u = e$.*

Proof. Since H contains a maximal reductive subgroup of G there exists a $\sigma \in G$ such that $g_s \in \sigma H \sigma^{-1}$. Let $p = \sigma H$. It follows from $g_s g_u = g_u g_s$ that g_s fixes Up element-wise and that Up if g-invariant. It is well known (and easy to prove) that $\dim U = 1$ if $g_u \neq e$. Finally, Up is closed as a unipotent orbit on an affine algebraic variety (cf.[Bi]). □

Remark: If $g \in \mathrm{Aff}(\mathbb{A}^n)$ and all eigenvalues of $\lambda(g)$ are different from 1 and pairwise different then $g = g_s$ and according to (iii) (applied to $G = \mathrm{Aff}(\mathbb{A}^n)$

and $H = \mathrm{GL}_n(\mathbb{R})$) there exists a $p \in \mathbb{A}^n$ such that $gp = p$. This assertion is well-known and is easy to prove directly. Using it one proves easily that if Γ acts properly discontinuously on $\mathbb{A}^2$ then Γ is virtually solvable, in particlar, the Auslander conjecture holds when $n = 2$.

Finally, let us also mention:

Proposition 2.6. *Let G and H be as in the formulation of Conjecture 2 and S be a maximal reductive subgroup of G contained in H. Then the action of S on G/H by left translations is linearizable.*

Proof. Put $U = \mathrm{R}_u(G)$. Then $U_1 = H \cap U$ is the unipotent radical of H and G/H is rationally isomorphic to U/U_1. Since U and U_1 are $\mathrm{Int}(S)$-invariant there exists $\mathrm{Ad}(S)$-invariant vector subspace $\mathcal{W} \subset \mathrm{Lie}(U)$ such that $\mathrm{Lie}(U) = \mathcal{W} \bigoplus \mathrm{Lie}(U_1)$ and $W = \exp \mathcal{W}$ is a regular cross-section for U/U_1 (i.e. the map $W \times U_1 \to U$, $(x, y) \to xy$, is a regular isomorphism of real algebraic varieties), cf. [Bo-Spr, 9.13]. Since $\exp \circ \mathrm{Ad}(x) = \mathrm{Int}(x) \circ \exp$ for any $x \in S$, we have that the map $\mathcal{W} \to G/H$, $w \to (\exp w)H$, is S-equivariant isomorphism of algebraic varieties. Therefore, the action of S on G/H is linearizable. □

Corollary 2.7. *Suppose that the action of S on G/H is irreducible. Then the action of G on G/H is linearizable, that is, there exists an isomorphism $\varphi : G/H \to \mathbb{A}^n$ such that if $g \in G$ and $l_g : G/H \to G/H, xH \mapsto gxH$, then $\varphi \circ l_g \circ \varphi^{-1} \in \mathrm{Aff}(\mathbb{A}^n)$ for all g.*

Proof. We use the notation U and U_1 as in the proof of Proposition 2.6. Let $\mathcal{N}_U(U_1)$ be the normalizer of U_1 in U. We suppose that $\dim G/H > 0$. Then $\mathcal{N}_U(U_1) \supsetneqq U_1$. Since $\mathcal{N}_U(U_1)$ is S-invariant and the action of S on G/H is irreducible, U_1 is a normal subgroup of U and, therefore, of G too. Factorizing G and H by U_1 we reduce the proof to the case when $U_1 = \{e\}$, i.e., when $H = S$. Since $\mathcal{D}(U) \cdot S$ is a proper subgroup of G and the action of S on G/H is irreducible, $\mathcal{D}(U) = \{e\}$. Hence G is a semidirect product of S and the vector group U on which S acts linearly, implying the corollary. □

Corollary 2.7 shows that Conjectures 1 and 2 coincide for irreducible actions of S on G/H.

3. Proof of Theorem 1.2

3.1. **Some representation theory.** Let G, H, Γ be as in the formulation of Theorem 1.2. Let L be a Levi subgroup of H. Then L is an almost direct product of simple algebraic groups $L_1, \cdots, L_r$ each of rank ≤ 1.

Using Proposition 2.3 we reduce the proof of the theorem to the case when $\Gamma \cap \mathrm{R}(G)$ is trivial and Γ is Zariski dense in G. Moreover, by a theorem of Selberg (see [S]) Γ contains a torsion free subgroup of finite index. Hence, we may (and will) suppose that Γ is torsion free. In view of Proposition 2.4 the relation (1) holds. We will denote by V the tangent space of G/H at the origin and by ρ the representation of L on V induced by the action of L on G/H by left translations (Proposition 2.6). Since the kernel of the action of G on G/H is a normal algebraic subgroup N of G contained in H, factorizing G and H by N we may (and will) suppose that G acts faithfully on G/H. In this case the representation ρ is also faithful.

The following proposition is an improved version of [To1, 2.5].

Proposition 3.1. *With the above notation and assumptions, $L = S_1 \times S_2 \times ... \times S_m$, where $S_i = \mathrm{SL}_2(\mathbb{R})$ or $S_i = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, and $V = V_1 \bigoplus V_2 \bigoplus ... \bigoplus V_m$ where each V_i is an L-module such that each $S_j, j \neq i$, acts trivially on V_i, V_i is the standard representation of $\mathrm{SL}_2(\mathbb{R})$ if $S_i = \mathrm{SL}_2(\mathbb{R})$, and V_i is the tensor product of two standard representations of $\mathrm{SL}_2(\mathbb{R})$ if $S_i = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.*

The next lemma is derived from [H, Table 5, p.518].

Lemma 3.2. *Let Q be a simple real algebraic group and $\mathrm{rank}_{\mathbb{R}} Q \leq 1$. Let d be the dimension of the minimal nontrivial representation of Q and s be the dimension of the symmetric space of Q. Then $d \geq s$ and $d = s$ if and only if Q is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and $d = s = 2$.*

Proof of Proposition 3.1. Let $V = V_1 \bigoplus V_2 \bigoplus ... \bigoplus V_m$ be a direct sum of irreducible L-submodules. Each V_i is a tensor product of irreducible nontrivial L_{ij}-modules V_{ij} (i.e. $V_i = \bigotimes_{1 \leq j \leq r_i} V_{ij}, r_i \in \mathbb{N}$), where $L_{ij} \in \{L_1,\ L_2, ..., L_r\}$. Put $n = \dim V$ and $n_{ij} = \dim V_{ij}$. Then

$$n = \sum_{1 \leq i \leq m} (\prod_{1 \leq j \leq r_i} n_{ij}). \tag{2}$$

For every L_i, we let d_i (respectively, s_i) be the dimension of the minimal nontrivial real representation of L_i (respectively, the dimension of the symmetric space of L_i). Remark that $s_1 + s_2 + ... + s_r$ is the dimension of the symmetric space of L. Using (1), (2) and the faithfulness of ρ, we get

$$d_1 + d_2 + ... + d_r \leq n \leq s_1 + s_2 + ... + s_r. \tag{3}$$

According to Lemma 3.2 $d_i \geq s_i$ for all i. It follows from (3) that $d_i = s_i = 2$ and $L_i = \mathrm{SL}_2(\mathbb{R})$ for all i. In particular,

$$n = 2r. \tag{4}$$

Since $\sum_{1 \leq i \leq m} r_i \geq r$ and $n_{ij} \geq 2$ for all i and j, it follows from (2) and (4) that $\sum_{1 \leq i \leq m} r_i = r$, $1 \leq r_i \leq 2$ and all $n_{ij} = 2$ (i.e. each V_{ij} is a standard $\mathrm{SL}_2(\mathbb{R})$-module). Moreover, we see that V is a faithful representation of $L_1 \times \cdots \times L_r$. This implies that $L = S_1 \times S_2 \times ... \times S_m$ as in the formulation of the proposition. □

3.2. **End of the proof.** Let ρ be as in the formulation of Proposition 3.1. The isomorphism $\mathcal{D}G/\mathrm{R}_u(\mathcal{D}G) \cong L$ gives a natural surjective homomorphism $\pi : \mathcal{D}G \to L$. Put $\phi = \rho \circ \pi$. Let $\gamma \in \mathcal{D}G$. By Proposition 2.5 γ_s fixes element-wise a smooth curve on G/H. There exists $g \in \mathcal{D}G$ such that $g\gamma_s g^{-1} \in L$. Hence $g\gamma_s g^{-1}$ fixes element-wise a smooth curve on G/H passing through the origin. So, 1 is an eigenvalue of $\phi(g\gamma_s g^{-1})$ and, therefore, of $\phi(\gamma_s)$ and $\phi(\gamma)$ too. But $\phi(\Gamma)$ is Zariski dense in L. Therefore 1 is an eigenvalue of $\rho(s)$ for every $s \in L$. In view of Proposition 3.1 L is trivial, that is, Γ is solvable. □

4. On Auslander's conjecture

4.1. **Some known results.** First we formulate a general result which is often useful in tackling Auslander's conjecture. So, let S be a real, connected, non-compact, and semi-simple algebraic group. An element $g \in S$ is said to be $\mathbb{R}$-regular if the number of eigenvalues, counted with multiplicity of modulus 1, of $\mathrm{Ad}(g)$ is minimal possible. Note that every $\mathbb{R}$-regular element in a semi-simple (or reductive) group is semi-simple. It is known (and can be checked by direct computation) that if $S = \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{Sp}_{2n}(\mathbb{R}), n \geq 2$, (the cases arising in section 4.2) then $g \in S$ is $\mathbb{R}$-regular if and only if all its eigenvalues are real and their moduli are distinct. The following theorem is proved by different methods in [Be-L], [P] and [A-M-S1]. (Concerning to its second part, we refer to [P, Remark, p.545].)

Theorem 4.1. *Any Zariski dense sub-semigroup Δ of S contains an $\mathbb{R}$-regular element. Moreover, the set of $\mathbb{R}$-regular elements in Γ is dense in G in the Zariski topology.*

Note that if $S = \mathrm{SL}_n(\mathbb{R})$ or $\mathrm{Sp}_{2n}(\mathbb{R}), n \geq 2$, then the set of elements in S with all eigenvalues different from 1 is Zariski open and non-empty which implies that the set of $\mathbb{R}$-regular elements in Δ with all eigenvalues different from 1 is Zariski dense in G. We will use this assertion in the course of our proof in 4.2 of the Auslander conjecture for $n \leq 5$ [2].

Further on, we denote by $\mathrm{SO}_{p,q}(\mathbb{R})$ the special orthogonal group of a quadratic form on $\mathbb{R}^n$ of signature (p, q), $n = p + q$, and by $\mathrm{Sp}_{2n}(\mathbb{R})$ the symplectic subgroup of $\mathrm{SL}_{2n}(\mathbb{R})$. If $n = p$ we use the standard notation $\mathrm{SO}_n(\mathbb{R})$ instead of $\mathrm{SO}_{n,0}(\mathbb{R})$.

Now, let $\lambda : \mathrm{Aff}(\mathbb{A}^n) \to \mathrm{GL}_n(\mathbb{R})$ be the natural projection (see 1.2) and H be an algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$. A subgroup $\Gamma \subset \mathrm{Aff}(\mathbb{A}^n)$ is called H-linear if $\lambda(\Gamma) \subset H$. If $H = \mathrm{SO}_n(\mathbb{R})$, i.e. if Γ consists of Euclidean transformations of $\mathbb{A}^n$, the Auslander conjecture follows from the classical Bieberbach theorem. Goldman and Kamishima (see [G-K]) proved the conjecture for Lorentz space

[2] Note that Theorem 4.1 is not indispensable for the proof of Auslander's conjecture for ≤ 5. It could be replaced by a weaker claim in the spirit of [Ti, Lemmas 2.1-2.5] but this would make the proof less natural and somewhat more complicate.

forms, i.e. for $H = \mathrm{SO}_{n-1,1}(\mathbb{R})$, and Grunewald and Margulis proved it when H is a reductive group of real rank ≤ 1 (see [Gr-Mar]).

Recall the following results of Abels, Margulis and Soifer.

Theorem 4.2. ([A-M-S3, Theorems A and B]) *Suppose that $n = 2k+1 \geq 3$. Then the following holds:*

(a) *if k is even there is no Γ acting properly discontinuously on $\mathbb{A}^r$ with $\lambda(\Gamma)$ Zariski dense in $SO_{k+1,k}(\mathbb{R})$;*
(b) *if k is odd there are free groups Γ acting properly discontinuously on $\mathbb{A}^r$ with $\lambda(\Gamma)$ Zariski dense in $SO_{k+1,k}(\mathbb{R})$.*

Theorem 4.2(a) was proved in the 1991 Margulis' preprint [Mar3]. Theorem 4.2(b) is a generalization of Margulis' results [Mar1] and [Mar2] where Theorem 4.2(b) is proved for $SO_{2,1}(\mathbb{R})$ disproving a conjecture of J.Milnor [Mi]. Different aspects of [Mar1] and [Mar2] were developed in [Dr1], [Dr2] and [Dr-G].

The results from [A-M-S3] are sharpen by the following:

Theorem 4.3. ([A-M-S4, Theorems A,B and C]) *Suppose that $\lambda(\Gamma) \subset \mathrm{SO}_{p,q}(\mathbb{R})$, $n = p+q$. Denote by H the Zariski closure of $\lambda(\Gamma)$ in $\mathrm{GL}_n(\mathbb{R})$. Then:*

(a) *Γ can not act properly discontinuously on $\mathbb{A}^n$ if $|p-q| \geq 2$ and $H \supseteq \mathrm{SO}_{p,q}(\mathbb{R})$;*
(b) *Γ can not act properly discontinuously on $\mathbb{A}^n$ if q is even and the homogeneous space $\mathrm{SO}_{p,q}(\mathbb{R})/H$ is compact;*
(c) *Γ is virtually solvable if $q = 2$ and $\Gamma\backslash\mathbb{A}^n$ is compact.*

4.2. **Proof of Auslander's conjecture for $n \leq 5$.** From now on Γ is a subgroup of $\mathrm{Aff}(\mathbb{A}^n)$, $n \leq 5$, and G is its Zariski closure in $\mathrm{Aff}(\mathbb{A}^n)$. We will prove the following:

Proposition 4.4. *If G contains a simple algebraic group of rank ≥ 2 then Γ does not act properly discontinuously on $\mathbb{A}^n$.*

In view of Theorem 1.2 (or [To1]), Proposition 4.4 implies immediately:

Theorem 4.5. *The Auslander conjecture is true for $n \leq 5$.*

4.2.1. We will use the notation and the terminology of section 1.2. In order to prove Proposition 4.4 we need a particular case of the following general

Proposition 4.6. (*cf.*[To2, Lemma 4.2]) *Let H be a Zariski connected algebraic subgroup of $\mathrm{Aff}(\mathbb{A}^m)$, S be a maximal reductive subgroup of H and L be the Levi subgroup of S. Then there exists a decomposition $V = W_1 \bigoplus \cdots \bigoplus W_k$, where W_i are irreducible $\lambda(S)$-modules, such that for every $i \geq 1$ the sum $W_1 \bigoplus \cdots \bigoplus W_i$ is $\lambda(H)$-invariant. Hence, for every $p \in \mathbb{A}^m$ there exists a* *frame with origin p in which the matrix representation of H is of the form:*

$$\begin{pmatrix} \rho_1 & & & * & & & | & * \\ & \rho_2. & & & & & | & \\ & & & & & & | & . \\ & & . & & & & | & . \\ & 0 & & . & & & | & . \\ & & & & . & \rho_k & | & \\ - & - & - & - & - & - & | & - \\ 0 & 0 & . & . & . & 0 & | & 1 \end{pmatrix},$$

where ρ_i *,* $i = 1, ..., k$*, are irreducible matrix representations of both* $\lambda(H)$ *and* $\lambda(S)$*. Moreover, if some restriction* $\rho_{i|\lambda(L)}$ *is not irreducible then* $\rho_{i|\lambda(L)} = \sigma_i \bigoplus \sigma_i$ *where* σ_i *is an irreducible representation of* $\lambda(L)$*.*

Proof. The existence of the decomposition $V = W_1 \bigoplus \cdots \bigoplus W_k$ as in the formulation of the proposition will be proved by induction on $\dim V$. Denote by U the unipotent radical of $\lambda(H)$. Let W_1 be an irreducible $\lambda(H)$-submodule of V. Since U is a unipotent group there exists a U-invariant vector $\overrightarrow{v} \in W_1 \setminus \{0\}$. For every $g \in \lambda(S)$, $g\overrightarrow{v}$ is also U-invariant. Since $\lambda(H) = \lambda(S) \ltimes U$, W_1 consists of U-invariant vectors and, therefore, is an irreducible $\lambda(S)$-module. Suppose that $W_1, \cdots, W_i$ are irreducible nontrivial $\lambda(S)$-submodules of V such that $W_1 + \cdots + W_i = W_1 \oplus \cdots \oplus W_i \subsetneqq V$ and $W_1 + \cdots + W_j$ is $\lambda(H)$-invariant for every $1 \leq j \leq i$. Let W'_{i+1} be a $\lambda(H)$-submodule of V containing $W_1 + \cdots + W_i$ and such that $W'_{i+1}/W_1 + \cdots + W_i$ is a non-trivial irreducible $\lambda(H)$-submodule of $V/W_1 + \cdots + W_i$. By the complete reducibility of the action of $\lambda(S)$ on W'_{i+1} there exists an irreducible $\lambda(S)$-submodule W_{i+1} of W'_{i+1} such that $W'_{i+1} = W_1 \oplus \cdots \oplus W_{i+1}$, completing the proof of the existence of the decomposition.

Now, suppose that $\rho : S \to \mathrm{GL}(W)$ is an irreducible representation but $\rho_{|L}$ is not. Let Z be the center of S. If W' is an irreducible Z-module then $\dim W' = 2$ and W is a direct sum of translations of W' by elements from S. This implies that if W'' is an irreducible L-module then there exists a $c \in Z$ such that $W = W'' \oplus cW''$, proving the last assertion of the proposition. □

Remark. It is worth mentioning that $\lambda(S)$ is isomorphic to S and the representations $\{\rho_{i|\lambda(S)} : i = 1, \cdots, k\}$, do not depend (up to isomorphism) on the choice of S.

4.2.2. *Proof of Proposition 4.4.* We divide the proof in two steps: the first being mostly routine and the second containing the main ingredients of the proof.

Step 1. Replacing Γ by a subgroup of finite index we may (and will) assume that the algebraic group G is Zariski connected. We will apply Proposition 4.6 with $H = G$. We fix a frame $\{p; \overrightarrow{v_1}, \cdots, \overrightarrow{v_n}\}$ of $\mathbb{A}^n, n \leq 5$, such that the matrix representation of G in this frame is as given by Proposition 4.6 and we keep the notation S, L and ρ_i, $i = 1, \cdots, k$, from its formulation. Since $\lambda_{|S}$ is injective, when this does not lead to confusion, we also write ρ_i instead of $\rho_i \circ \lambda$.

There exists l such that $\mathcal{D}^l G = \mathcal{D}^{l+1} G$. Since $\mathcal{D}^l \Gamma$ is Zariski dense in $\mathcal{D}^l G$ and $\mathcal{D}^l G$ contains a simple algebraic group of rank ≥ 2 (because G does), replacing Γ by $\mathcal{D}^l \Gamma$, we reduce the proof of Proposition 4.4 to the case when $G = \mathcal{D}G$. In this case $L = S$. In view of the classification in [Bou, Table 2] of the dimensions of the irreducible representations of the simple algebraic groups, if H is a simple algebraic group of rank $d > 0$ and $\rho : H \to \mathrm{SL}(W)$ is its non-trivial representation then $\dim W \geq d+1$ and $\dim W = d+1$ if and only if ρ is an isomorphism [3]. It follows from Proposition 4.6 and the assumption that L contains a simple group of rank ≥ 2 that: (a) if $n = 3$ then $L = \mathrm{SL}_3(\mathbb{R})$, and (b) if L is not simple then $n = 5$ and $L = \mathrm{SL}_3(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Using Theorem 4.1, we get that in both cases (a) and (b) there exists an element $\gamma \in \Gamma$ of infinite order such that all eigenvalues of $\lambda(\gamma)$ are different from 1 and pairwise different. According to Proposition 2.5(iii), γ admits a fixed point proving that Γ does not act properly discontinuously on $\mathbb{A}^n$.

So, it remains to consider the case when $4 \leq n \leq 5$ and L is a simple algebraic group of rank ≥ 2. It is enough to prove that if Proposition 4.4 is valid for $n-1$ it is also valid for n. Assume to the contrary that Γ acts properly discontinuously on $\mathbb{A}^n$ and Proposition 4.4 is true for $n-1$. If one of the representations $\rho_i, 1 \leq i \leq k$, as in the formulation of Proposition 4.6, is not trivial then $\dim \rho_i \geq 3$ and, since $n \leq 5$, all remaining representations $\rho_j, j \neq i$, are trivial. Also, if ρ_k is trivial it follows from $G = \mathcal{D}G$ that Γ acts properly discontinuously of the hyperplane $p + \mathbb{R}\overrightarrow{v_1} + \cdots + \mathbb{R}\overrightarrow{v}_{n-1}$ contradicting the assumption that Proposition 4.4 is true for $n-1$. Hence ρ_k is a non-trivial representation, $3 \leq \dim \rho_k \leq 5$, and all ρ_i, $1 \leq i \leq k-1$, are trivial. Moreover, it follows from the remark after Proposition 2.5 that $\lambda(L)$ should not contain an element with pairwise different eigenvalues all different from 1. Now, let $k = 1$. In view of [Bou, Table 2], $L = \mathrm{SO}_{3,2}(\mathbb{R})$, $n = 5$ and $\rho_{1|L}$ is the standard representation. By Margulis' Theorem 4.2(a), in this case Γ does not act properly discontinuously on $\mathbb{A}^5$. Therefore $k > 1$ and $3 \leq \dim \rho_k \leq 4$. Since $\mathrm{rank}(L) \geq 2$, [Bou, Table 2] implies that ρ_k is one of the standard representations of $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_4(\mathbb{R})$ or $\mathrm{Sp}_4(\mathbb{R})$. So, in order to complete the proof it remains to consider the possibilities (i), (ii), (iii) and (iv) below. (In the formulations of (i)−(iv) we use the notation: $\mathrm{M}_{i,j}(\mathbb{R})$ is the set of all real matrices with i rows and j columns, $0_{i,j}$ is the zero in $\mathrm{M}_{i,j}(\mathbb{R})$ and I_i is the unit matrix in $\mathrm{M}_{i,i}(\mathbb{R})$.)

(i) $\lambda(G) \subseteq \left\{ \begin{pmatrix} \mathrm{I}_1 & m_{1,4} \\ 0_{4,1} & g \end{pmatrix} \mid g \in \mathrm{SL}_4(\mathbb{R}), m_{1,4} \in \mathrm{M}_{1,4}(\mathbb{R}) \right\}$, $L = \mathrm{SL}_4(\mathbb{R})$;

(ii) $\lambda(G) \subseteq \left\{ \begin{pmatrix} \mathrm{I}_1 & m_{1,4} \\ 0_{4,1} & g \end{pmatrix} \mid g \in \mathrm{Sp}_4(\mathbb{R}), m_{1,4} \in \mathrm{M}_{1,4}(\mathbb{R}) \right\}$, $L = \mathrm{Sp}_4(\mathbb{R})$;

[3] Formally, [Bou, Table 2] concerns the representations of the algebraic groups over $\mathbb{C}$ but since the absolute rank (over $\mathbb{C}$) of an algebraic group over $\mathbb{R}$ is greater than or equal to its real rank the assertion remains true for real algebraic groups as in the context of this paper.

(iii) $\lambda(G) \subseteq \left\{ \begin{pmatrix} \mathrm{I}_2 & m_{2,3} \\ 0_{3,2} & g \end{pmatrix} \mid g \in \mathrm{SL}_3(\mathbb{R}), m_{2,3} \in \mathrm{M}_{2,3}(\mathbb{R}) \right\}$, $L = \mathrm{SL}_3(\mathbb{R})$;

(iv) $\lambda(G) \subseteq \left\{ \begin{pmatrix} \mathrm{I}_1 & m_{1,3} \\ 0_{3,1} & g \end{pmatrix} \mid g \in \mathrm{SL}_3(\mathbb{R}), m_{1,3} \in \mathrm{M}_{1,3}(\mathbb{R}) \right\}$, $L = \mathrm{SL}_3(\mathbb{R})$.

Step 2. We treat the cases (i) − (iv) simultaneously. By Theorem 4.1 there exists $\gamma \in \Gamma$ such that $\rho_k(\gamma)$ is $\mathbb{R}$-regular in the Zariski closure of $\rho_k(\Gamma)$ and all its eigenvalues are different from 1. (Note that this property of γ remains valid for any choice of L in *Step 1.*) Let $\gamma = \gamma_s\gamma_u$ be the Jordan decomposition of γ. Choose L such that $\gamma_s \in L$. Fix a $p_\circ \in \mathbb{A}^n$ with $Lp_\circ = p_\circ$.

We have $V = W_\circ \oplus W$ where $W_\circ = \{\overrightarrow{x} \in V | \lambda(\gamma_s)(\overrightarrow{x}) = \overrightarrow{x}\} = \{\overrightarrow{x} \in V | \lambda(g)(\overrightarrow{x}) = \overrightarrow{x}$ for all $g \in G\}$ and W is $\lambda(\gamma_s)$-invariant. Using $\lambda(\gamma_s)\lambda(\gamma_u) = \lambda(\gamma_u)\lambda(\gamma_s)$ and the choice of γ, one proves easily that γ_u is a translation belonging to the center of G. Indeed, let $\lambda(\gamma) = \begin{pmatrix} \mathrm{I}_i & m_{i,j} \\ 0_{j,i} & g \end{pmatrix}$ and $g = g_s g_u$ be the Jordan decomposition of g. By the choice of L and Proposition 4.6, we have $\lambda(\gamma_s) = \begin{pmatrix} \mathrm{I}_i & 0_{i,j} \\ 0_{j,i} & g_s \end{pmatrix}$ and $\lambda(\gamma_u) = \begin{pmatrix} \mathrm{I}_i & m_{i,j} \\ 0_{j,i} & g_u \end{pmatrix}$. Since the eigenvalues of g_s are pairwise different we get that $g_u = I_j$ and since they are all different from 1 we get that $m_{i,j} = 0_{i,j}$. Therefore, $\lambda(\gamma_u) = \mathrm{Id}_V$, equivalently, γ_u is a translation by a vector $\overrightarrow{v_\gamma}$. We have

$$\gamma_s\gamma_u p_\circ = p_\circ + \lambda(\gamma_s)(\overrightarrow{v_\gamma}) = \gamma_u\gamma_s p_\circ = p_\circ + \overrightarrow{v_\gamma}.$$

Hence $\overrightarrow{v_\gamma} \in W_\circ$. Let $h \in G$. Then $\lambda(h)\overrightarrow{v_\gamma} = \overrightarrow{v_\gamma}$. So, if $x \in \mathbb{A}^n$ then

$$h\gamma_u(x) = h(x + \overrightarrow{v_\gamma}) = h(x) + \overrightarrow{v_\gamma} = \gamma_u h(x),$$

proving that γ_u belongs to the center of G.

Put

$$E^\circ(\gamma) = \{p \in \mathbb{A}^n \mid \gamma_s p = p\}.$$

Then $E^\circ(\gamma) = p_\circ + W_\circ$. Denote

$$V^+(\gamma) = \{\overrightarrow{v} \in V \mid \lim_{n \to -\infty} \gamma_s^n(p_\circ + \overrightarrow{v}) = p_\circ\}$$

and

$$V^-(\gamma) = \{\overrightarrow{v} \in V \mid \lim_{n \to +\infty} \gamma_s^n(p_\circ + \overrightarrow{v}) = p_\circ\}.$$

Let

$$E^+(\gamma) = E^\circ(\gamma) + V^+(\gamma) \quad \text{and} \quad E^-(\gamma) = E^\circ(\gamma) + V^-(\gamma).$$

Since $E^\pm(\gamma) = E^\mp(\gamma^{-1})$, we may (and will) assume that

$$\dim E^+(\gamma) \geq \dim E^-(\gamma).$$

Let $\delta \in \Gamma$. Since γ_u is central, we have $\gamma_u = (\delta\gamma\delta^{-1})_u$. Also,

$$E^\circ(\delta\gamma\delta^{-1}) = \delta E^\circ(\gamma) \quad \text{and} \quad E^\pm(\delta\gamma\delta^{-1}) = \delta E^\pm(\gamma).$$

Remark that $E^\circ(\gamma)$ and $E^\circ(\delta\gamma\delta^{-1})$ are parallel subspaces of $\mathbb{A}^n$ directed by $W_\circ$. Suppose that $E^\circ(\gamma) = \delta E^\circ(\gamma)$ for all $\delta \in \Gamma$. Recall that every subgroup of $\mathrm{Aff}(\mathbb{A}^m), m \leq 2$, acting properly discontinuously on $\mathbb{A}^m$ is virtually solvable. Since G is connected, $G = \mathcal{D}G$ and $\dim E^\circ(\gamma) \leq 2$, we get that the action of Γ on $E^\circ(\gamma)$ is trivial which contradicts the assumption that Γ acts properly discontinuously on $\mathbb{A}^n$. Therefore there exists $\delta \in \Gamma$ such that

$$E^\circ(\gamma) \cap E^\circ(\delta\gamma\delta^{-1}) = \emptyset. \tag{5}$$

With such a δ, $E^+(\gamma) \cap E^+(\delta\gamma\delta^{-1})$ contains a line $l = q + \mathbb{R}\overrightarrow{v_\gamma}$. We can write $q = q_1 + \overrightarrow{w_1}$, where $q_1 \in E^\circ(\gamma)$ and $\overrightarrow{w_1} \in V^+(\gamma) \setminus \{\overrightarrow{0}\}$, and $q = q_2 + \overrightarrow{w_2}$, where $q_2 \in E^\circ(\delta\gamma\delta^{-1})$ and $\overrightarrow{w_2} \in V^+(\delta\gamma\delta^{-1}) \setminus \{\overrightarrow{0}\}$.

Put $p_m = q + m\overrightarrow{v_\gamma}, m \in \mathbb{N}$. Then

$$\lim_{m\to+\infty} \gamma^{-m}(p_m) = q_1 \quad \text{and} \quad \lim_{m\to+\infty} (\delta\gamma\delta^{-1})^{-m}(p_m) = q_2.$$

Note that the set $X = \{\gamma^{-m}(p_m) \mid m \in \mathbb{N}\} \cup \{(\delta\gamma\delta^{-1})^{-m}(p_m) \mid m \in \mathbb{N}\}$ is relatively compact and $(\delta\gamma^{-m}\delta^{-1}\gamma^m)X \cap X \neq \emptyset$ for all m. It follows from (5) that $\delta\gamma^{-m_1}\delta^{-1}\gamma^{m_1} \neq \delta\gamma^{-m_2}\delta^{-1}\gamma^{m_2}$ if $m_1 \neq m_2$ which contradicts the assumption that Γ acts properly discontinuously on $\mathbb{A}^n$. This completes our proof. □

Remark: As noted in [To2], the element δ can be chosen in such a way that the subgroup spanned by γ^m and $\delta\gamma^m\delta^{-1}$ is free if m is sufficiently large. This can be achieved by a well-known argument of Tits [Ti]. Indeed, let $\varphi : G \to L$ be the natural projection and L be identified with its image in $\mathrm{SL}(W)$. Let $\mathrm{P}(W)$ be the projective space of W. If $g \in G$ and $\varphi(g)$ has pairwise different positive eigenvalues $\alpha_1 > \cdots > \alpha_t > 0$ with respective eigenvectors $\overrightarrow{w_1}, \cdots, \overrightarrow{w_t}$, we denote by $\mathrm{A}(g)$ (resp.$\mathrm{A}'(g)$) the projectivization of the vector space $\mathbb{R}\overrightarrow{w_1}$ (resp. $\mathbb{R}\overrightarrow{w_2} + \cdots + \mathbb{R}\overrightarrow{w_t}$). Since L acts irreducibly on W, we can choose δ in such a way that $\mathrm{A}(\gamma) \cup \mathrm{A}(\gamma^{-1}) \subset \mathrm{P}(W) \setminus (\mathrm{A}'(\delta\gamma\delta^{-1}) \cup \mathrm{A}'(\delta\gamma^{-1}\delta^{-1}))$ and $\mathrm{A}(\delta\gamma\delta^{-1}) \cup \mathrm{A}(\delta\gamma^{-1}\delta^{-1}) \subset \mathrm{P}(W) \setminus (\mathrm{A}'(\gamma) \cup \mathrm{A}'(\gamma^{-1}))$. Now, it follows from the "ping-pong lemma" that the subgroup spanned by γ^m and $\delta\gamma^m\delta^{-1}$ is free if m is sufficiently large.

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