arXiv:1309.7433v1 [math.CV] 28 Sep 2013

COEFFICIENT ESTIMATES AND THE FEKETE-SZEGŐ PROBLEM FOR CERTAIN CLASSES OF POLYHARMONIC MAPPINGS

J. CHEN, A. RASILA, AND X. WANG *

Abstract. We give coefficient estimates for a class of close-to-convex harmonic mappings, and discuss the Fekete-Szegő problem of it. We also introduce two classes of polyharmonic mappings $\mathcal{HS}_p$ and $\mathcal{HC}_p$, consider the starlikeness and convexity of them, and obtain coefficient estimates on them. Finally, we give a necessary condition for a mapping F to be in the class $\mathcal{HC}_p$.

1. Introduction

Let $\mathbb{D}$ denote the unit disk $\{z : |z| < 1, z \in \mathbb{C}\}$, and let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic in $\mathbb{D}$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent. A continuous mapping $f = u + iv$ is a *complex-valued harmonic* mapping in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D, i.e., $\Delta u = \Delta v = 0$, where Δ is the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For a simply connected domain D, we can write f in the form $f = h + \overline{g}$, where h and g are analytic (see [14]). A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all $z \in D$.

A continuous complex-valued mapping F in D is *biharmonic* if the Laplacian of F is harmonic, i.e., F satisfies the equation $\Delta(\Delta F) = 0$. It can be shown that in a simply connected domain D, every biharmonic mapping has the representation

$$F(z) = G_1(z) + |z|^2 G_2(z),$$

where both G_1 and G_2 are harmonic in D.

File: fekete.tex, printed: 1-10-2013, 0.30

2000 *Mathematics Subject Classification.* Primary: 30H10, 30H30; Secondary: 30C20, 30C45.

Key words and phrases. harmonic mapping, polyharmonic mapping, coefficient estimates, the Fekete-Szegő problem.

* Corresponding author.

The research was partly supported by NSF of China (No. 11071063) and Hunan Provincial Innovation Foundation for Postgraduates (No. 125000-4242).

More generally, a complex-valued mapping F of a domain D is called *polyharmonic* (or *p-harmonic*) if F satisfies the equation $\Delta^p F = \Delta(\Delta^{p-1}F) = 0$ for some $p \in \mathbb{N}^+$. In a simply connected domain, a mapping F is polyharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z),$$

where each G_k is harmonic, i.e., $\Delta G_k(z) = 0$ for $k \in \{1, \cdots, p\}$ (see [7, 8]). Obviously, when p=1 (resp. p=2), F is harmonic (resp. biharmonic). The properties of biharmonic mappings have been investigated by many authors (see [2, 3, 4, 12, 18, 20, 23]). We refer to [7, 8, 9, 10, 11] for the discussions on polyharmonic mappings and [13, 14] for the basic properties of harmonic mappings.

We use $\mathcal{S}_H$ to denote the class consisting of univalent harmonic mappings in $\mathbb{D}$. Such mappings can be written in the form

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} \overline{b_j z^j}, \tag{1.2}$$

with $|b_1| < 1$. Let $\mathcal{S}_H^*$ and $\mathcal{C}_H$ be the subclasses of $\mathcal{S}_H$, where the images of $f(\mathbb{D})$ are starlike and convex, respectively. If $b_1 = 0$, then $\mathcal{S}_H$, $\mathcal{S}_H^*$ and $\mathcal{C}_H$ reduce to the classes $\mathcal{S}_H^0$, $\mathcal{S}_H^{0,*}$ and $\mathcal{C}_H^0$, respectively. See also [14].

A classical theorem of Fekete and Szegő [16] states that for $f \in \mathcal{S}$ of the form (1.1), the functional $|a_3 - \lambda a_2^2|$ satisfies the following inequality:

$$|a_3 - \lambda a_2^2| \leq \begin{cases} 3 - 4\lambda, & \lambda \leq 0, \\ 1 + 2e^{-\frac{2\lambda}{1-\lambda}}, & 0 \leq \lambda \leq 1, \\ 4\lambda - 3, & \lambda \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each real λ there exists a function in $\mathcal{S}$ such that equality holds (see [1, 21]). Thus the determination of sharp upper bounds for the nonlinear functional $|a_3 - \lambda a_2^2|$ for any compact family $\mathcal{F}$ of functions in $\mathcal{A}$ is often called the Fekete-Szegő problem for $\mathcal{F}$. Many researchers have studied the Fekete-Szegő problem for analytic close-to-convex functions (see [19, 21, 22]). A natural question is whether we can get similar generalizations to harmonic close-to-convex mappings.

In [13], Clunie and Sheil-Small obtained the following result:

Proposition 1. ([13, Lemma 5.11]) *If $f = h + \overline{g} \in \mathcal{C}_H$, then there exist angles α and β such that*

$$\mathrm{Re}\left\{ \left(e^{i\alpha} h'(z) + e^{-i\alpha} g'(z)\right) \left(e^{i\beta} - e^{-i\beta} z^2\right) \right\} > 0$$

for all $z \in \mathbb{D}$.

Our purpose of this paper is twofold. In Section 3, we obtain coefficient estimates of a class of close-to-convex harmonic mappings and, as an application, show upper bounds for the Fekete-Szegő functionals $|a_3 - \lambda a_2^2|$ and $|b_3 - \lambda b_2^2|$. The main results in this section are Theorems 1 and 2. In Section 4, first we obtain two sufficient conditions for mappings $F \in \mathcal{H}_p$ to be starlike with respect to the origin and convex,

respectively, given in Theorems are 3 and 4. Then, we establish some coefficient estimates for two classes of polyharmonic mappings $\mathcal{HS}_p$ and $\mathcal{HC}_p$. Our main results are Theorems 5 and 6. Finally, we obtain a generalization of Proposition 1 to the class $\mathcal{HC}_p$, which is Theorem 7.

2. Preliminaries

In [5], Avci and Złotkiewicz introduced the class $\mathcal{HS}$ of univalent harmonic mappings f with the series expansion (1.2) such that

$$\sum_{j=2}^{\infty} j(|a_j|+|b_j|) \leq 1-|b_1|, \quad (0 \leq |b_1| < 1),$$

and the subclass $\mathcal{HC}$ of $\mathcal{HS}$, where

$$\sum_{j=2}^{\infty} j^2(|a_j|+|b_j|) \leq 1-|b_1|, \quad (0 \leq |b_1| < 1).$$

These two classes constitute a harmonic counterparts of classes introduced by Goodman [17]. They are useful in studying questions of so-called δ-neighborhoods originally considered by Ruscheweyh [26] (see also [25]) and in constructing explicit k-quasiconformal extensions of mappings(see Fait et al. [15]).

We denote by $\mathcal{H}_p$ the set of polyharmonic mappings F in $\mathbb{D}$ with the form:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)}\big(h_k(z)+\overline{g_k(z)}\big) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty}(a_{k,j}z^j + \overline{b_{k,j}z^j}), \tag{2.1}$$

where $a_{1,1}=1$, $|b_{1,1}|<1$. We say that a univalent polyharmonic mapping F with $F(0)=0$ is starlike with respect to the origin if the curve $F(re^{i\theta})$ is starlike with respect to the origin for each $r \in (0,1)$. The following result gives a convenient characterization of this property.

Proposition 2. ([24]) *If F is univalent, $F(0)=0$ and $\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big) > 0$ for $z=re^{i\theta}\neq 0$, then F is starlike with respect to the origin.*

A univalent polyharmonic mapping F with $F(0)=0$ and $\frac{\partial}{\partial\theta}F(re^{i\theta})\neq 0$ whenever $r\in(0,1)$, is said to be convex if the curve $F(re^{i\theta})$ is convex for each $r\in(0,1)$.

Proposition 3. ([24]) *If F is univalent, $F(0)=0$, $\frac{\partial}{\partial\theta}F(re^{i\theta})\neq 0$ whenever $r\in(0,1)$, and $\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right] > 0$ for $z=re^{i\theta}\neq 0$, then F is convex.*

In [25], J. Qiao and X. Wang introduced the class $\mathcal{HS}_p$ of polyharmonic mappings F of the form (2.1) satisfying the condition

$$\begin{cases} \displaystyle\sum_{k=1}^{p}\sum_{j=2}^{\infty}\big(2(k-1)+j\big)\big(|a_{k,j}|+|b_{k,j}|\big) \leq 1-|b_{1,1}| - \sum_{k=2}^{p}(2k-1)\big(|a_{k,1}|+|b_{k,1}|\big), \\ \displaystyle 0 \leq |b_{1,1}| + \sum_{k=2}^{p}(2k-1)\big(|a_{k,1}|+|b_{k,1}|\big) < 1, \end{cases} \tag{2.2}$$

and the subclass $\mathcal{HC}_p$ of $\mathcal{HS}_p$, where

(2.3)
$$\begin{cases} \displaystyle\sum_{k=1}^{p}\sum_{j=2}^{\infty}\big(2(k-1)+j^2\big)\big(|a_{k,j}|+|b_{k,j}|\big) \le 1-|b_{1,1}|-\sum_{k=2}^{p}(2k-1)\big(|a_{k,1}|+|b_{k,1}|\big), \\ \displaystyle 0 \le |b_{1,1}|+\sum_{k=2}^{p}(2k-1)\big(|a_{k,1}|+|b_{k,1}|\big) < 1. \end{cases}$$

Obviously, for any $F \in \mathcal{HS}_p$, we have $|F(z)| < 2|z|$ for $z \in \mathbb{D}$. For $p = 1$, the classes $\mathcal{HS}_p$ and $\mathcal{HC}_p$ reduce to $\mathcal{HS}$ and $\mathcal{HC}$, respectively. An important property of these classes is given by the following result.

Theorem A. ([25, Theorem 3.1]) *Suppose $F \in \mathcal{HS}_p$. Then F is univalent and sense preserving in $\mathbb{D}$.*

3. Coefficient estimates for a class of close-to-convex harmonic mappings

In this section, we will consider the coefficient estimates and the Fekete-Szegő problem of mappings of the class $\mathcal{F}$, defined as follows. In [6], Bharanedhar and Ponnusamy obtained the following result:

Theorem B. ([6, Theorem 1]) *Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$, with $h'(0) \neq 0$, which satisfies*

$$g'(z) = e^{i\theta} z h'(z) \text{ and } \operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2} \tag{3.1}$$

for all $z \in \mathbb{D}$. Then f is a univalent close-to-convex mapping in $\mathbb{D}$.

We use $\mathcal{F}$ to denote the class of harmonic mapping f in $\mathbb{D}$ of the form (1.2), satisfying (3.1). Let $\mathcal{H}$ and $\mathcal{G}$ be the subclasses of $\mathcal{F}$, where

$$\mathcal{H} = \{F = h + \overline{g} : F \in \mathcal{F} \text{ and } g \equiv 0\}$$

and

$$\mathcal{G} = \{F = h + \overline{g} : F \in \mathcal{F} \text{ and } h \equiv 0\}.$$

Lemma 1. *The classes $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{F}$ are compact.*

Proof. Suppose that $f_n = h_n + \overline{g_n} \in \mathcal{F}$ and that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. It follows from Hurwitz's theorem that f is harmonic, and therefore it has a canonical representation $f = h + \overline{g}$. It is easy to see that $h_n \to h$ and $g_n \to g$ locally uniformly, and that $h'(0) = 1$ and $g'(z) = e^{i\theta} z h'(z)$. Because

$$\operatorname{Re}\left(1 + z\frac{h_n''(z)}{h_n'(z)}\right) > -\frac{1}{2}$$

in $\mathbb{D}$, it follows that when $n \to +\infty$, then we have

$$\operatorname{Re}\left(\frac{3}{2} + z\frac{h''(z)}{h'(z)}\right) \ge 0 \tag{3.2}$$

for all $z \in \mathbb{D}$. It suffices to show that the equation in (3.2) cannot hold for any $z \in \mathbb{D}$. Obviously, the function $\mathrm{Re}\left(\frac{3}{2} + z\frac{h''(z)}{h'(z)}\right)$ is harmonic in $\mathbb{D}$. By the maximum principle, if $\mathrm{Re}\left(\frac{3}{2} + z_0\frac{h''(z_0)}{h'(z_0)}\right) = 0$ for some $z_0 \in \mathbb{D}$, then $\mathrm{Re}\left(\frac{3}{2} + z\frac{h''(z)}{h'(z)}\right) \equiv 0$, and hence

$$\frac{3}{2} + z\frac{h''(z)}{h'(z)} \equiv iC$$

for some real constant C. That is a contradiction. Hence,

$$\mathrm{Re}\left(\frac{1}{2} + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2}$$

for all $z \in \mathbb{D}$, and the proof is complete. □

Theorem 1. *Let f be of the form* (1.2) *satisfying* (3.1). *Then*

$$|a_j| \leq \frac{j+1}{2} \quad \textit{and} \quad |b_j| \leq \frac{j-1}{2} \tag{3.3}$$

for all $j = 1, 2, \ldots$.

Proof. Since $\mathrm{Re}\left(\frac{3}{2} + z\frac{h''(z)}{h'(z)}\right) > 0$, then there exists an analytic function $p_1(z) = c_0 + c_1 z + c_2 z^2 + \ldots$, such that

$$z\frac{h''(z)}{h'(z)} = p_1(z) - \frac{3}{2} = c_1 z + c_2 z^2 + \ldots, \tag{3.4}$$

and $\mathrm{Re}\{p_1(z)\} > 0$. Then (3.4) implies that

$$j(j+1)a_{j+1}z^j = \big(ja_jc_1 + (j-1)a_{j-1}c_2 + \ldots + a_1c_j\big)z^j$$

for $j = 1, 2, \ldots$, and hence,

$$a_{j+1} = \frac{1}{j(j+1)}\sum_{\gamma=1}^{j}\gamma a_\gamma c_{j+1-\gamma}.$$

Because $p_1(z) = c_0 + c_1 z + c_2 z^2 + \ldots$, and $\mathrm{Re}\ p_1(z) > 0$, then by [14, Lemma 1, p. 50], we have $|c_j| \leq 2\mathrm{Re}\{c_0\} = 3$ for all $j = 1, 2, \ldots$. If we write $a_1 = 1$, it follows that

$$|a_j| \leq \frac{j+1}{2} \quad \text{for all } j = 1, 2, \ldots.$$

By (3.1), $g'(z) = e^{i\theta}zh'(z)$, and therefore $\sum_{j=1}^{\infty} jb_jz^{j-1} = e^{i\theta}\sum_{j=1}^{\infty} ja_jz^j$. Thus, we have

$$b_1 = 0, \quad jb_j = e^{i\theta}(j-1)a_{j-1} \text{ for all } j = 2, 3, \ldots.$$

Then, we obtain

$$|b_j| = \frac{j-1}{j}|a_{j-1}| \leq \frac{j-1}{2} \quad \text{for all } j = 1, 2, \ldots.$$

□

Now, we are ready to establish upper bounds for the Fekete-Szegő functionals $|a_3 - \lambda a_2^2|$ and $|b_3 - \lambda b_2^2|$.

Theorem 2. *Let f be of the form* (1.2) *and satisfy* (3.1). *Then*

$$|a_3 - \lambda a_2^2| \le \max\left\{\frac{1}{2}, \frac{|8-9\lambda|}{4}\right\} \quad \text{and} \quad |b_3 - \lambda b_2^2| \le 1 + \frac{|\lambda|}{4} \tag{3.5}$$

for all $\lambda \in \mathbb{R}$.

Proof. Let

$$p_2(z) = \frac{2}{3}\left(\frac{3}{2} + z\frac{h''(z)}{h'(z)}\right),$$

where $h(z) = z + \sum_{j=1}^{\infty} a_j z^j$. Then by simple calculations, we obtain

$$p_2(z) = 1 + \frac{2}{3}\big(2a_2 z + (6a_3 - 4a_2^2)z^2 + \dots\big). \tag{3.6}$$

Write $p_2(z) = 1 + u_1 z + u_2 z^2 + \dots$. Because Re $p_2(z) > 0$, then by [24, formula (10), p. 166], we have

$$\left|u_2 - \frac{u_1^2}{2}\right| \le 2 - \frac{|u_1|^2}{2}.$$

It follows from (3.6) that

$$a_2 = \frac{3}{4}u_1 \ \text{ and } \ a_3 = \frac{1}{4}u_2 + \frac{3}{8}u_1^2.$$

Hence,

$$\begin{aligned} |a_3 - \lambda a_2^2| &= \left|\frac{1}{4}u_2 + \frac{3}{8}u_1^2 - \frac{9}{16}\lambda u_1^2\right| \\ &= \frac{1}{4}\left|u_2 - \frac{1}{2}u_1^2 + \frac{8-9\lambda}{4}u_1^2\right| \\ &\le \frac{1}{4}\left(2 - \frac{1}{2}|u_1|^2 + \frac{|8-9\lambda|}{4}|u_1|^2\right). \end{aligned} \tag{3.7}$$

If $\frac{|8-9\lambda|}{4} < \frac{1}{2}$, then (3.7) implies

$$|a_3 - \lambda a_2^2| \le \frac{1}{2}.$$

Equality is attained if we choose $a_2 = 0$ and $a_3 = \pm\frac{1}{2}$.

If $\frac{|8-9\lambda|}{4} \ge \frac{1}{2}$, then it follows from [14, Lemma 1, p. 50] that $|u_1| \le 2\mathrm{Re}\{p_2(0)\} = 2$ and (3.7) that

$$|a_3 - \lambda a_2^2| \le \frac{1}{2} + \frac{1}{4}\left(\frac{|8-9\lambda|}{4} - \frac{1}{2}\right)|u_1|^2 \le \frac{|8-9\lambda|}{4}.$$

Choosing $a_2 = \pm\frac{3}{2}$ and $a_3 = 2$ in (3.7) shows that the result is sharp.

Since $g'(z) = e^{i\theta} z h'(z)$, we have

$$\sum_{j=1}^{\infty} j b_j z^{j-1} = e^{i\theta} \sum_{j=1}^{\infty} j a_j z^j.$$

Obviously, $b_2 = \frac{e^{i\theta}}{2} a_1 = \frac{e^{i\theta}}{2}$ and $b_3 = \frac{2e^{i\theta}}{3} a_2$. Hence, (3.3) implies

$$|b_3 - \lambda b_2^2| = \left| \frac{2e^{i\theta}}{3} a_2 - \frac{\lambda e^{2i\theta}}{4} \right| \leq \frac{2}{3}|a_2| + \frac{|\lambda|}{4} \leq 1 + \frac{|\lambda|}{4}.$$

If $\lambda \geq 0$, then equality is attained when $b_3 = -e^{2i\theta}$, i.e. $a_2 = -\frac{3}{2} e^{i\theta}$. If $\lambda < 0$, then equality is attained when $b_3 = e^{2i\theta}$, i.e. $a_2 = \frac{3}{2} e^{i\theta}$. □

Remark 1. Both equalities in (3.5) are attained when $a_2 = \frac{3}{2}$ and $b_3 = e^{i\theta}$ or $a_2 = -\frac{3}{2}$ and $b_3 = -e^{i\theta}$, but only in the case $|8 - 9\lambda| \geq 2$ and $\theta = 2k\pi$, where $k \in \mathbb{Z}$.

4. Coefficient estimates for two classes of polyharmonic mappings

Let L denote the following differential operator defined on the class of complex-valued C^1 functions:

$$L = z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}}.$$

An important property of the operator $L[F]$ is that it behaves with respect to polyharmonic mappings much like the operator $zf'(z)$ defined for analytic functions (see [4, Corollary 1(3)]).

Theorem 3. *Each mapping $F \in \mathcal{HS}_p$ is starlike with respect to the origin.*

Proof. Let $F \in \mathcal{HS}_p$ be of the form (2.1). It follows from Theorem A that

$$J_F(0) = 1 - |b_{1,1}|^2 > 0. \tag{4.1}$$

By computation, we have

$$\begin{aligned} \frac{\partial}{\partial \theta}\big(\arg F(re^{i\theta})\big) &= \frac{\partial}{\partial \theta}\left[\mathrm{Im}\big(\log F(re^{i\theta})\big)\right] \\ &= \mathrm{Im}\left[\frac{\partial}{\partial \theta}\big(\log F(re^{i\theta})\big)\right] \\ &= \mathrm{Im}\frac{izF_z(z) - i\overline{z}F_{\overline{z}}(z)}{F(z)} \\ &= \mathrm{Re}\frac{L[F(z)]}{F(z)} \\ &= \mathrm{Re}\frac{\sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(j a_{k,j} z^j - j\overline{b_{k,j} z^j}\right)}{\sum_{k=1}^{p} |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(a_{k,j} z^j + \overline{b_{k,j} z^j}\right)}. \end{aligned} \tag{4.2}$$

By (4.1) and (4.2), we obtain

$$\begin{aligned}\lim_{z\to 0}\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big) &= \lim_{z\to 0}\mathrm{Re}\frac{1-\overline{b_{1,1}}\overline{z}/z}{1+\overline{b_{1,1}}\overline{z}/z}\\ &= \lim_{z\to 0}\mathrm{Re}\frac{\big(1-\overline{b_{1,1}}\overline{z}/z\big)\,(1+b_{1,1}z/\overline{z})}{\big|1+\overline{b_{1,1}}\overline{z}/z\big|^2}\\ &= \lim_{z\to 0}\frac{1-|b_{1,1}|^2}{|1+\overline{b_{1,1}}\overline{z}/z|^2}\\ &\geq \frac{1-|b_{1,1}|^2}{(1+|b_{1,1}|)^2}>0.\end{aligned} \tag{4.3}$$

It follows from Theorem A that each $F\in\mathcal{HS}_p$ is univalent in $\mathbb{D}$. Then, we have that $F(z)\neq 0$ for $z\in\mathbb{D}\setminus\{0\}$, and the function $\mathrm{Re}\frac{L[F(z)]}{F(z)}$ is continuous in $\mathbb{D}\setminus\{0\}$. Therefore, by (4.2), (4.3) and the continuity of F in $\mathbb{D}\setminus\{0\}$, we see the condition $\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big)>0$ for all $z\in\mathbb{D}\setminus\{0\}$ is equivalent to

$$\frac{L[F(z)]}{F(z)}\neq\frac{\zeta-1}{\zeta+1} \tag{4.4}$$

for all $z\in\mathbb{D}\setminus\{0\}$ and all $\zeta\in\mathbb{C}$ with $|\zeta|=1$ and $\zeta\neq -1$. Hence, (4.4) holds if and only if

$$\Phi(z):=(\zeta+1)L[F(z)]-(\zeta-1)F(z)\neq 0$$

for all $z\in\mathbb{D}\setminus\{0\}$ and all $|\zeta|=1$. Calculations show that

$$\begin{aligned}|\Phi(z)| &= \Bigg|(\zeta+1)\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}j\big(a_{k,j}z^j-\overline{b_{k,j}z^j}\big)\\ &\quad -(\zeta-1)\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\big(a_{k,j}z^j+\overline{b_{k,j}z^j}\big)\Bigg|\\ &= \Bigg|\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\Big(\big(j+1+\zeta(j-1)\big)a_{k,j}z^j-\big(j-1+\zeta(j+1)\big)\overline{b_{k,j}z^j}\Big)\Bigg|.\end{aligned}$$

If F is the identity, obviously, we have $|\Phi(z)|=2|z|$. If $F(z)=z+\overline{b_{1,1}z}$, then

$$|\Phi(z)|=2\left|z-\zeta\overline{b_{1,1}z}\right|\geq 2|z|(1-|b_{1,1}|).$$

If F is not an affine mapping, then

$$|\Phi(z)|>|z|\left(2-2|b_{1,1}|-2\sum_{k=1}^{p}\sum_{j=2}^{\infty}j(|a_{k,j}|+|b_{k,j}|)-2\sum_{k=2}^{p}(|a_{k,1}|+|b_{k,1}|)\right).$$

Hence, each mapping $F\in\mathcal{HS}_p$ is starlike with respect to the point $z=0$. □

Theorem 4. *Each mapping $F\in\mathcal{HC}_p$ is convex.*

Proof. Let $F \in \mathcal{HC}_p$ be of the form (2.1). By (4.2), we have

$$\begin{aligned}\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right] &= \mathrm{Re}\frac{L\left[\frac{\partial}{\partial\theta}F(re^{i\theta})\right]}{\frac{\partial}{\partial\theta}F(re^{i\theta})} \\ &= \mathrm{Re}\frac{L[L[F(z)]]}{L[F(z)]} \\ &= \mathrm{Re}\frac{\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\left(j^2a_{k,j}z^j + j^2\overline{b_{k,j}z^j}\right)}{\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\left(ja_{k,j}z^j - j\overline{b_{k,j}z^j}\right)}.\end{aligned} \tag{4.5}$$

Then by (4.1) and (4.5), we have

$$\begin{aligned}\lim_{z\to 0}\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right] &= \lim_{z\to 0}\mathrm{Re}\frac{1+\overline{b_{1,1}}\overline{z}/z}{1-\overline{b_{1,1}}\overline{z}/z} \\ &= \lim_{z\to 0}\mathrm{Re}\frac{\left(1+\overline{b_{1,1}}\overline{z}/z\right)\left(1-b_{1,1}z/\overline{z}\right)}{\left|1-\overline{b_{1,1}}\overline{z}/z\right|^2} \\ &= \lim_{z\to 0}\frac{1-|b_{1,1}|^2}{|1-\overline{b_{1,1}}\overline{z}/z|^2} \\ &\geq \frac{1-|b_{1,1}|^2}{(1+|b_{1,1}|)^2} > 0.\end{aligned} \tag{4.6}$$

If F is the identity, obviously, we have $|L[F(z)]| = |z|$. If $F(z) = z + \overline{b_{1,1}z}$, then

$$|L[F(z)]| = |z - \overline{b_{1,1}z}| \geq |z|(1-|b_{1,1}|).$$

If F is not an affine mapping, then

$$\begin{aligned}|L[F(z)]| &= \left|\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\left(ja_{k,j}z^j - j\overline{b_{k,j}z^j}\right)\right| \\ &> |z|\left(1-|b_{1,1}|-\sum_{k=1}^{p}\sum_{j=2}^{\infty}j\big(|a_{k,j}|+|b_{k,j}|\big)-\sum_{k=2}^{p}j\big(|a_{k,1}|+|b_{k,1}|\big)\right).\end{aligned}$$

Therefore, $F \in \mathcal{HC}_p$ implies $L[F(z)] \neq 0$ for $z \in \mathbb{D}\setminus\{0\}$, and hence the function $\mathrm{Re}\frac{L[L[F(z)]]}{L[F(z)]}$ is continuous in $\mathbb{D}\setminus\{0\}$. Therefore, by (4.5), (4.6) and the continuity of $L[F(z)]$ in $\mathbb{D}\setminus\{0\}$, we see the condition $\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta}F(re^{i\theta})\right)\right] > 0$ for all $z \in \mathbb{D}\setminus\{0\}$ is equivalent to

$$\mathrm{Re}\frac{L[L[F(z)]]}{L[F(z)]} \neq \frac{\zeta-1}{\zeta+1} \tag{4.7}$$

for all $z \in \mathbb{D}\setminus\{0\}$ and all $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ and $\zeta \neq -1$. Hence, (4.7) holds if and only if

$$\Psi(z) := (\zeta+1)L[L[F(z)]] - (\zeta-1)L[F(z)] \neq 0$$

for all $z \in \mathbb{D} \setminus \{0\}$ and all $|\zeta| = 1$. Calculations show that

$$\begin{aligned}|\Psi(z)| = \Bigg|&(\zeta+1)\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty} j^2\big(a_{k,j}z^j + \overline{b_{k,j}z^j}\big)\\ &-(\zeta-1)\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty} j\big(a_{k,j}z^j - \overline{b_{k,j}z^j}\big)\Bigg|\\ =&\sum_{k=1}^{p}|z|^{2(k-1)}\sum_{j=1}^{\infty}\Big(\big(j^2+j+\zeta(j^2-j)\big)a_{k,j}z^j + \big(j^2-j+\zeta(j^2+j)\big)\overline{b_{k,j}z^j}\Big).\end{aligned}$$

If F is the identity, obviously, we have $|\Psi(z)| = 2|z|$. If $F(z) = z + \overline{b_{1,1}z}$, then

$$|\Psi(z)| = 2\left|z + \zeta\overline{b_{1,1}z}\right| \geq 2|z|(1-|b_{1,1}|).$$

If F is not an affine mapping, then

$$|\Psi(z)| > |z|\left(2 - 2|b_{1,1}| - 2\sum_{k=1}^{p}\sum_{j=2}^{\infty} j^2\big(|a_{k,j}| + |b_{k,j}|\big) - 2\sum_{k=2}^{p}\big(|a_{k,1}| + |b_{k,1}|\big)\right).$$

It follows that each mapping $F \in \mathcal{HC}_p$ is convex. □

Example 1. Let $F_1(z) = z + \frac{1}{3}\overline{z} + \frac{1}{6}|z|^2\overline{z}$. Then F_1 is convex. See also Figure 1.

It is well known that the coefficients of every starlike mapping $f \in \mathcal{S}_H^{*,0}$ of the form (1.2) satisfy the sharp inequalities

$$|a_j| \leq \frac{(2j+1)(j+1)}{6}, \quad |b_j| \leq \frac{(2j-1)(j-1)}{6}, \quad \big||a_j| - |b_j|\big| \leq j$$

for $j = 2, 3, \ldots$ (see [27]). The coefficients of each mapping $f \in \mathcal{C}_H^0$ satisfy the sharp inequalities

$$|a_j| \leq \frac{j+1}{2}, \quad |b_j| \leq \frac{j-1}{2}, \quad \text{and} \quad \big||a_j| - |b_j|\big| \leq 1$$

for $j = 2, 3, \ldots$ (see [13]).

Next, we obtain similar results for mappings in $\mathcal{HS}_p$ and $\mathcal{HC}_p$.

Theorem 5. *The coefficients of every mapping $F \in \mathcal{HS}_p$ satisfy the sharp inequalities*

$$\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j} \tag{4.8}$$

for all $j = 2, 3, \ldots$.

Proof. Let $F \in \mathcal{HS}_p$ be of the form (2.1). By (2.2), we have

$$\sum_{k=1}^{p} j(|a_{k,j}| + |b_{k,j}|) \leq \sum_{k=1}^{p}\sum_{j=2}^{\infty}\big(2(k-1)+j\big)\big(|a_{k,j}| + |b_{k,j}|\big) \leq 1.$$

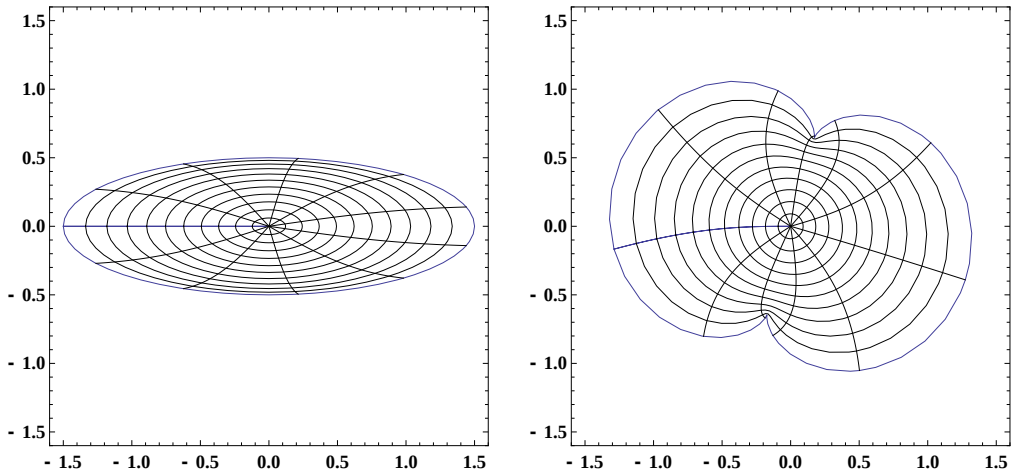

FIGURE 1. The images of $\mathbb{D}$ under the mappings $F_1(z) = z + \frac{1}{3}\overline{z} + \frac{1}{6}|z|^2\overline{z}$ (left) and $F_2(z) = z + \frac{z^3}{3}e^{\frac{\pi}{6}i}$.

It follows that

$$\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j}$$

for $j = 2, 3, \ldots$. □

Example 2. Let $F_2(z) = z + \frac{z^j}{j}e^{i\varphi}$ for all $j = 2, 3, \ldots$ and $\varphi \in \mathbb{R}$. Then $F_2 \in \mathcal{HS}$ is univalent, sense preserving and starlike with respect to the origin. Obviously, the coefficients of F_2 satisfy (4.8). See Figure 1 for the case where $j = 3$ and $\varphi = \pi/6$.

The above example shows that the coefficient estimate (4.8) is sharp for $p = 1$.

Theorem 6. *The coefficients of each mapping $F \in \mathcal{HC}_p$ satisfy the sharp inequalities*

$$\sum_{k=1}^{p}(|a_{k,j}| + |b_{k,j}|) \leq \frac{1}{j^2} \tag{4.9}$$

for $j = 2, 3, \ldots$.

Proof. The proof of Theorem 6 is similar to the proof of Theorem 5, and we will omit it. □

Example 3. Let $F_3(z) = z + \frac{z^j}{j^2}e^{i\varphi}$ for all $j = 2, 3, \ldots$ and $\varphi \in \mathbb{R}$. Then $F_3 \in \mathcal{HC}$ is a univalent, sense preserving and convex harmonic mapping. Obviously, the coefficients of F_3 satisfy (4.9). See Figure 2 for the case where $j = 3$ and $\varphi = \pi/6$.

This example shows that the coefficient estimate (4.9) is sharp for $p = 1$.

Now, we are ready to generalize Proposition 1 to the polyharmonic mappings of the class $\mathcal{HC}_p$.

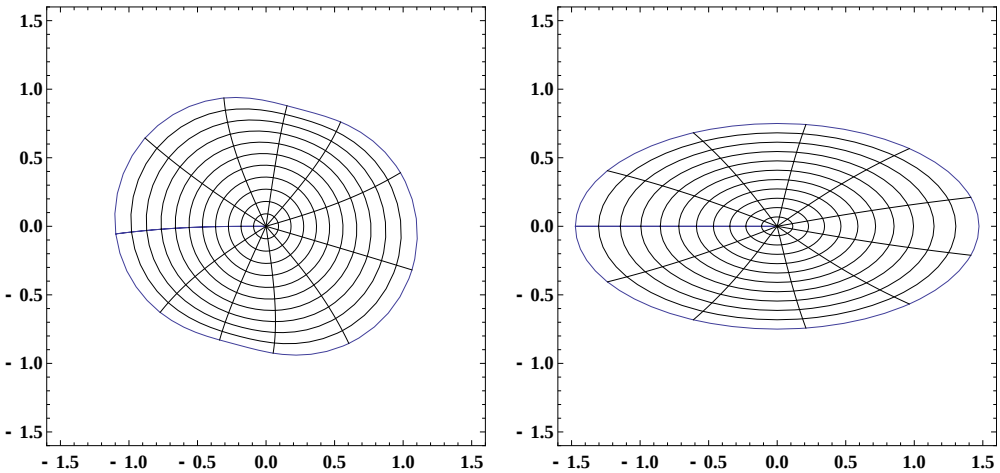

FIGURE 2. The images of $\mathbb{D}$ under the mappings $F_3(z) = z + \frac{z^3}{9}e^{\frac{\pi}{6}i}$ (left) and $F_4(z) = z + \frac{1}{9}|z|^2 z + \frac{1}{4}\overline{z} + \frac{1}{9}|z|^2\overline{z}$.

Theorem 7. *If $F \in \mathcal{HC}_p$ and $a_{k,1} = 0$ for $k \in \{2, \dots, p\}$, then there exist angles α and β such that*

$$\text{Re}\left\{\left(e^{i\alpha}\sum_{k=1}^{p}|z|^{2(k-1)}h_k'(z) + e^{-i\alpha}\sum_{k=1}^{p}|z|^{2(k-1)}g_k'(z)\right)(e^{i\beta} - e^{-i\beta}z^2)\right\} > 0 \tag{4.10}$$

for all $z \in \mathbb{D}$.

Proof. Let $F \in \mathcal{HC}_p$ be of the form (2.1), fix $r \in (0,1)$, and let

$$F_r(z) = \sum_{k=1}^{p} r^{2(k-1)}\big(h_k(z) + \overline{g_k(z)}\big) = \sum_{j=1}^{\infty}\sum_{k=1}^{p}\left(a_{k,j}r^{2(k-1)}z^j + \overline{b_{k,j}}r^{2(k-1)}\overline{z^j}\right),\ z \in \mathbb{D}.$$

Then F_r is harmonic. By the hypothesis and (2.3), $F \in \mathcal{HC}_p$ implies

$$\sum_{j=2}^{\infty} j^2\left|\sum_{k=1}^{p} a_{k,j}r^{2(k-1)}\right| + \sum_{j=2}^{\infty} j^2\left|\sum_{k=1}^{p} b_{k,j}r^{2(k-1)}\right| \le 1 - \left|\sum_{k=1}^{p} b_{k,j}r^{2(k-1)}\right|,$$

i.e., $F_r \in \mathcal{C}_H$ (see [5]). Then Proposition 1 implies that there exist angles α and β such that

$$\text{Re}\left\{\left(e^{i\alpha}\sum_{k=1}^{p}r^{2(k-1)}h_k'(z) + e^{-i\alpha}\sum_{k=1}^{p}r^{2(k-1)}g_k'(z)\right)(e^{i\beta} - e^{-i\beta}z^2)\right\} > 0$$

for all $z \in \mathbb{D}$. Let $r = |z|$. The result is proved. □

Example 4. Obviously, the mapping $F_1(z) = z + \frac{1}{3}\overline{z} + \frac{1}{6}|z|^2\overline{z} \in \mathcal{HC}_2$ (see Figure 1). Let $\alpha = \beta = 0$. Then F_1 satisfies the inequality (4.10).

However, the mapping $F_4(z) = z + \frac{1}{9}|z|^2 z + \frac{1}{4}\overline{z} + \frac{1}{9}|z|^2\overline{z} \in \mathcal{HC}_2$ (see Figure 2) also satisfies the inequality (4.10) for $\alpha = \beta = 0$ with $a_{2,1} = \frac{1}{9}$.

Remark 2. The proof of Theorem 7 requires a somewhat unnatural additional assumption concerning the coefficients $a_{k,1}$. It is not obvious if the result holds without this assumption.

References

1. H. R. Abdel, D. Gawad and K. Thomas, The Fekete-Szegő problem for strong close-to-convex functions. *Proc. Amer. Math. Soc.* **114** (1992), 345-349.
2. Z. Abdulhadi and Y. Abu Muhanna, Landau's theorem for biharmonic mappings. *J. Math. Anal. Appl.* **338** (2008), 705-709.
3. Z. Abdulhadi, Y. Abu Muhanna and S. Khuri, On univalent solutions of the biharmonic equation. *J. Inequal. Appl.* **5** (2005), 469-478.
4. Z. Abdulhadi, Y. Abu Muhanna and S. Khuri, On some properties of solutions of the biharmonic equation. *Appl. Math. Comput.* **177** (2006), 346-351.
5. Y. Avci and E. Złotkiewicz, On harmonic univalent mappings. *Ann. Univ. Mariae Curie Skłodowska (Sect A)* **44** (1990), 1-7.
6. S. V. Bharanedhar and S. Ponnusamy, Coefficient conditions for harmonic univalent mappings and hypergeometric mappings. arXiv:1109.0925v2.
7. Sh. Chen, S. Ponnusamy and X. Wang, Bloch and Landau's theorems for planar p-harmonic mappings. *J. Math. Anal. Appl.* **373** (2011), 102-110.
8. Sh. Chen, S. Ponnusamy and X. Wang, Landau's theorem for p-harmonic mappings in several complex variables. *Ann. Polon. Math.* **103** (2012), 67-87.
9. J. Chen, A. Rasila and X. Wang, On polyharmonic univalent mappings. In press, *Math. Rep. (Bucur.)* arXiv:1302.2018.
10. J. Chen, A. Rasila and X. Wang, Starlikeness and convexity of polyharmonic mappings. In press, *Bull. Belg. Math. Soc. Simon Stevin.* arXiv:1302.2398.
11. J. Chen, A. Rasila and X. Wang, Landau's theorem for polyharmonic mappings. *J. Math. Anal. Appl.* **409** (2014), 934-945.
12. J. Chen and X. Wang, On certain classes of biharmonic mappings defined by convolution. *Abstr. Appl. Anal.* **2012**, Article ID 379130, 10 pages. doi:10.1155/2012/379130
13. J. G. Clunie and T. Sheil-Small, Harmonic univalent functions. *Ann. Acad. Sci. Fenn. Ser. A. I.* **9** (1984), 3-25.
14. P. Duren, *Harmonic Mappings in the Plane*. Cambridge University Press, Cambridge, 2004.
15. M. Fait, J. Krzyż and J. Zygmunt, Explicit quasiconformal extensions for some classes of univalent functions. *Comment. Math. Helv.* **51** (1976), 279-285.
16. M. Fekete and G. Szegő, Eine bemerkung über ungerade schlichte functionen. *J. London Math. Soc.* **8** (1933), 85-89.
17. A. W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.* **8** (1957), 588-601.
18. J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*. Prentice-Hall, Englewood Cliffs, NJ, USA, 1965.
19. F. R. Keogh and E. P. Merkes, A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.* **20** (1969), 8-12.
20. S. A. Khuri, Biorthogonal series solution of Stokes flow problems in sectorial regions. *SIAM J. Appl. Math.* **56** (1996), 19-39.
21. W. Koepf, On the Fekete-Szegő problem for close-to-convex functions. *Proc. Amer. Math. Soc.* **101** (1987), 89-95.
22. W. Koepf, On the Fekete-Szegő problem for close-to-convex functions II. *Arch. Math.* **49** (1987), 420-433.
23. W. E. Langlois, *Slow Viscous Flow*. Macmillan, New York, NY, USA, 1964.
24. C. Pommerenke, *Univalent Functions*. Vandenhoeck and Ruprecht, Gőttingen, 1975.

25. J. Qiao and X. Wang, On p-harmonic univalent mappings (in Chinese). *Acta Math. Sci.* **32A** (2012), 588-600.
26. S. Ruscheweyh, Neighborhoods of univalent functions. *Proc. Amer. Math. Soc.* **18** (1981), 521-528.
27. T. Sheil-Small, Constants for planar harmonic mappings. *J. London Math. Soc.* **42A** (1990), 237-248.

J. Chen, Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China.
E-mail address: jiaolongchen@sina.com

A. Rasila, Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China, and Department of Mathematics and Systems Analysis, Aalto University, P. O. Box 11100, FI-00076 Aalto, Finland.
E-mail address: antti.rasila@iki.fi

X. Wang, Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China.
E-mail address: xtwang@hunnu.edu.cn