

# ON LIPSCHITZ MAPS AND THEIR FLOWS

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**ABSTRACT.** This paper regroups some of the basic properties of Lipschitz maps and their flows. While some of the results presented here have already appeared in other papers, most of them are either only and classically known in the case of smooth maps and needed to be proved in the Lipschitz case for a better understanding of the Lipschitz geometry or are new and necessary to the development of numerical methods for rough paths.

## BASIC NOTATIONS

- $\mathcal{S}_k$  : The symmetric group of order  $k$ .
- $[\gamma]$  : The only integer such that  $0 < \gamma - [\gamma] \leq 1$ ,  $\gamma$  being a real number.
- $\lfloor \gamma \rfloor$  : The integer part of a real number  $\gamma$ , i.e. the only integer such that  $0 \leq \gamma - \lfloor \gamma \rfloor < 1$ .
- $\mathcal{L}_c(E, F)$  : The space of all continuous linear mappings from a normed vector space  $E$  to a normed vector space  $F$ .
- $I_k$  : The identity matrix of rank  $k$ .
- $Id_U$  : The identity map on the set  $U$ .
- $\overline{A}$  : The closure of a subset  $A$  of a topological space.
- $B(x, \alpha)$  : The ball centered at  $x$  of radius  $\alpha$ .

## 1. LIPSCHITZ MAPS

L.C. Young, in [16], uses the concept of  $p$ -variation ( $p \geq 1$ ) to generalize Stieltjes' integration theory to paths of finite  $p$ -variation. In building a theory of differential equations using the aforementioned work, one needs to be able to control the smoothness (in terms of variation) of the image of a path of finite  $p$ -variation under the involved vector fields. It appears that Lipschitz maps, introduced by Stein in [12], are the appropriate type of maps to use in this frame and the wider one of rough paths, introduced by Lyons in [10]. Lipschitz maps also provide a nice regularity structure (see Hairer [7]) which one can use to build solutions to certain types of SPDEs. We give below the definition of Lipschitz maps then set out to answer basic and natural questions about this class of maps: do they have a nice embedding structure? How can they be linked to the more familiar class of  $\mathcal{C}^n$  maps? Are they stable under composition? etc.

### 1.1. Basic definitions and generalities.

**Definition 1.1.** Let  $n \in \mathbb{N}$  and  $0 < \varepsilon \leq 1$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a subset of  $E$ . We will use, without ambiguity, the same notation  $\|\cdot\|$  to designate norms on  $E^{\otimes k}$ , for  $k \in \llbracket 1, n \rrbracket$ , and the norm on  $F$ . For every  $k \in \llbracket 0, n \rrbracket$ , let  $f^k : U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  be a map with values in the space of the symmetric  $k$ -linear mappings from  $E$  to  $F$ . The collection  $f = (f^0, f^1, \dots, f^n)$  is said to be Lipschitz of degree  $n + \varepsilon$  on  $U$  (or in short a *Lip* -  $(n + \varepsilon)$ )

function) if there exist a constant  $M$  and  $n+1$  functions  $R_k : E \times E \rightarrow \mathcal{L}(E^{\otimes k}, F)$ ,  $k \in \llbracket 0, n \rrbracket$  (called the associated remainders) such that for all  $k \in \llbracket 0, n \rrbracket$ :

$$\sup_{x \in U} \|f^k(x)\| \leq M$$

$$\forall x, y \in U, \forall v \in E^{\otimes k} : f^k(x)(v) = \sum_{j=k}^n f^j(y) \left( \frac{v \otimes (x-y)^{\otimes (j-k)}}{(j-k)!} \right) + R_k(x, y)(v),$$

$$\forall x, y \in U : \|R_k(x, y)\| \leq M \|x - y\|^{n+\varepsilon-k}.$$

The smallest constant  $M$  for which the properties above hold is called the *Lip*  $-(n+\varepsilon)$ -norm of  $f$  and is denoted by  $\|f\|_{\text{Lip}-(n+\varepsilon)}$ .

**Remark 1.2.** On any open subset of  $U$  (and in particular on the interior of  $U$ ),  $f^1, \dots, f^n$  are the successive derivatives of  $f^0$ . However, these functions are not necessarily uniquely determined by  $f^0$  on an arbitrary subset of  $U$ . Keeping this in mind, if  $f^0 : U \rightarrow F$  is a function such that there exist  $f^1, \dots, f^n$  such that  $(f^0, f^1, \dots, f^n)$  is *Lip*  $-(n+\varepsilon)$ , we will often say that  $f^0$  is *Lip*  $-(n+\varepsilon)$  with no mention of  $f^1, \dots, f^n$ . We will call the functions  $R_0, R_1, \dots, R_n$  its associated remainders.

One important feature one has to pay attention to when deriving properties of Lipschitz maps is the nature of the norms involved. We study here three types of norms which will enable us to prove the most needed results for the exposition of our work.

**Definition 1.3** (Projective property). Let  $E$  be a normed vector space. Let  $n \in \mathbb{N}^*$ . We say that  $(E^{\otimes k})_{1 \leq k \leq n}$  (respectively  $(E^{\otimes k})_{k \geq 1}$ ) are endowed with norms satisfying the projective property if, for every  $k \in \llbracket 1, n \rrbracket$  (resp.  $k \geq 1$ ) and  $p, q \in \mathbb{N}$  such that  $p+q=k$  and every  $a \in E^{\otimes p}, b \in E^{\otimes q}$ , we have  $\|a \otimes b\| \leq \|a\| \|b\|$ .

Norms satisfying the projective property are abundant, when at least such one exists, in the following sense:

**Lemma 1.4.** Let  $E$  be a normed vector space and  $n \in \mathbb{N}^*$ . Suppose  $(\|\cdot\|_k)_{1 \leq k \leq n}$  are norms on  $(E^{\otimes k})_{1 \leq k \leq n}$  satisfying the projective property, then the norms  $(\alpha^k \|\cdot\|_k)_{1 \leq k \leq n}$  and  $(\beta \|\cdot\|_k)_{1 \leq k \leq n}$ , where  $\alpha > 0$  and  $\beta \geq 1$ , also satisfy the projective property.

**Example.** Let  $E$  be a finite dimensional vector space and let  $(e_1, \dots, e_p)$  be a basis for  $E$ . Let  $n \in \mathbb{N}^*$ . Let  $k \in \llbracket 1, n \rrbracket$  and  $p \geq 1$  and define the norms  $\|\cdot\|_{p,k}$  and  $\|\cdot\|_{\infty,k}$  on  $E^{\otimes k}$  by the following: for  $x \in E^{\otimes k}$ , if  $(\lambda_{i_1, \dots, i_k})_{1 \leq i_1, \dots, i_k \leq p}$  are the coordinates of  $x$  in the basis  $(e_{i_1} \otimes \dots \otimes e_{i_k})_{1 \leq i_1, \dots, i_k \leq p}$  of  $E^{\otimes k}$ , i.e.:

$$x = \sum_{1 \leq i_1, \dots, i_k \leq p} \lambda_{i_1, \dots, i_k} e_{i_1} \otimes \dots \otimes e_{i_k}$$

then:

$$\|x\|_{p,k} = \left( \sum_{1 \leq i_1, \dots, i_k \leq p} |\lambda_{i_1, \dots, i_k}|^p \right)^{1/p} \quad \text{and} \quad \|x\|_{\infty,k} = \max_{1 \leq i_1, \dots, i_k \leq p} |\lambda_{i_1, \dots, i_k}|$$

Then  $(\|\cdot\|_{p,k})_{1 \leq k \leq n}$  and  $(\|\cdot\|_{\infty,k})_{1 \leq k \leq n}$  are norms on  $(E^{\otimes k})_{1 \leq k \leq n}$  satisfying the projective property.

**Definition 1.5.** Let  $E$  and  $F$  be two normed vector spaces and  $u : E \rightarrow F$  be a linear map. Let  $n \in \mathbb{N}^*$ . We define the map  $u^{\otimes n} : E^{\otimes n} \rightarrow F^{\otimes n}$  as the unique linear map satisfying:

$$\forall v_1, \dots, v_n \in E : \quad u^{\otimes n}(v_1 \otimes \dots \otimes v_n) = u(v_1) \otimes \dots \otimes u(v_n)$$

**Definition 1.6** (Compatible norms). Let  $E$  and  $F$  be two normed vector spaces. Let  $n \in \mathbb{N}^*$  and  $C \geq 0$ . We say that  $(E^{\otimes k})_{1 \leq k \leq n}$  and  $(F^{\otimes k})_{1 \leq k \leq n}$  are endowed with  $C$ -compatible norms if, for every bounded linear map  $u : E \rightarrow F$  and every  $k \in \llbracket 1, n \rrbracket$ , we have  $\|u^{\otimes k}\| \leq C\|u\|^k$ . When the value of  $C$  is irrelevant, we may simply say that the norms are compatible.

**Examples.** Let  $E$  be a finite dimensional vector space and let  $(e_1, \dots, e_p)$  be a basis for  $E$ . Let  $n \in \mathbb{N}^*$ . Let  $F$  be a normed vector space. We assume that we have norms on  $(F^{\otimes k})_{1 \leq k \leq n}$  satisfying the projective property. Then:

- The norms  $(\|\cdot\|_{1,k})_{1 \leq k \leq n}$  on  $(E^{\otimes k})_{1 \leq k \leq n}$  are 1-compatible with the norms on  $(F^{\otimes k})_{1 \leq k \leq n}$ .
- The norms  $(\|\cdot\|_{\infty,k})_{1 \leq k \leq n}$  (resp.  $(p^k \|\cdot\|_{\infty,k})_{1 \leq k \leq n}$ ) on  $(E^{\otimes k})_{1 \leq k \leq n}$  are  $p^n$ -compatible (resp. 1-compatible) with the norms on  $(F^{\otimes k})_{1 \leq k \leq n}$ .
- Let  $q > 1$ . The norms  $(\|\cdot\|_{q,k})_{1 \leq k \leq n}$  (resp.  $(p^{k(1-1/q)} \|\cdot\|_{q,k})_{1 \leq k \leq n}$ ) on  $(E^{\otimes k})_{1 \leq k \leq n}$  are  $p^{n(1-1/q)}$ -compatible (resp. 1-compatible) with the norms on  $(F^{\otimes k})_{1 \leq k \leq n}$ .

**Remark 1.7.** As shown in the case of the norms given in the previous examples, given  $C$ -compatible norms, it is always possible to define new norms on  $(E^{\otimes k})_{1 \leq k \leq n}$  so that the new norms are 1-compatible and that the new norms on  $(E^{\otimes k})_{1 \leq k \leq n}$  satisfy the projective property if the original ones do.

**Definition 1.8** (Action of the Symmetric Group on Tensors). Let  $n \in \mathbb{N}^*$ ,  $\sigma \in \mathcal{S}_n$  and  $E$  be a vector space. We define the action of  $\sigma$  on the homogenous tensors of  $E$  of order  $n$  as a linear map by the following:

$$\forall x_1, x_2, \dots, x_n \in E \quad \sigma(x_1 \otimes x_2 \otimes \dots \otimes x_n) = x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \dots \otimes x_{\sigma(n)}$$

**Definition 1.9** (Symmetric norms). Let  $E$  be a vector space and  $n \in \mathbb{N}^*$ . The norm on  $E^{\otimes n}$  is said to be symmetric if:

$$\forall n \in \mathbb{N}^*, \forall \sigma \in \mathcal{S}_n, \forall x \in E^{\otimes n} \quad \|\sigma(x)\| = \|x\|$$

We show now how to control the Lipschitz norm of the Cartesian product of two Lipschitz maps.

**Proposition 1.10.** Let  $\gamma > 0$ . Let  $E, F$  and  $G$  be normed vector spaces. Let  $U$  be a subset of  $E$  and let  $f$  (resp.  $g$ ) be a map defined on  $U$  with values in  $F$  (resp.  $G$ ). Let  $h$  be the map defined on  $U$  by  $h = (f, g)$ . Then:

- If  $f$  and  $g$  are  $\text{Lip} - \gamma$  and  $F \times G$  is endowed with the  $l^1$  (resp.  $l^2, l^\infty$ ) norm, then  $h$  is also  $\text{Lip} - \gamma$  and  $\|h\|_{\text{Lip}-\gamma}$  is less than or equal to the  $l^1$  (resp.  $l^2, l^\infty$ ) norm of  $(\|f\|_{\text{Lip}-\gamma}, \|g\|_{\text{Lip}-\gamma})$ .
- If the norm  $\|\cdot\|_F$  on  $F$  and the norm  $\|\cdot\|_{F \times G}$  on  $F \times G$  are such that there exists  $C > 0$  satisfying:

$$\forall (x, y) \in F \times G : \quad \|x\|_F \leq C\|(x, y)\|_{F \times G}$$

(note that the  $l^1, l^2$  and  $l^\infty$  norms on  $F \times G$  satisfy this property), and if  $h$  is  $\text{Lip} - \gamma$  then  $f$  is  $\text{Lip} - \gamma$  and  $\|f\|_{\text{Lip}-\gamma} \leq C\|h\|_{\text{Lip}-\gamma}$ .

**1.2. Local characterization and embeddings.** Once the concept of Lipschitzness understood, one of the first and the most natural questions one may ask is whether  $\text{Lip} - \gamma$  maps are  $\text{Lip} - \gamma'$ , for  $\gamma \geq \gamma' > 0$ . We deal first with the trivial case where the domain of definition of the map is bounded:

**Lemma 1.11.** *Let  $\gamma, \gamma' > 0$  such that  $\gamma' < \gamma$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a bounded subset of  $E$ . Let  $f : U \rightarrow F$  be a  $\text{Lip} - \gamma$  map. We assume that  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are endowed with norms satisfying the projective property. Then  $f$  is  $\text{Lip} - \gamma'$  and if  $L \geq 0$  is larger than or equal to the diameter of  $U$  then:*

$$\|f\|_{\text{Lip} - \gamma'} \leq \|f\|_{\text{Lip} - \gamma} \max \left( 1, \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{L^{j-\gamma'}}{(j - \lfloor \gamma' \rfloor)!} + L^{\gamma-\gamma'} \right)$$

*Proof.* Let  $n, n' \in \mathbb{N}$ ,  $(\varepsilon, \varepsilon') \in (0, 1]^2$  such that  $\gamma = n + \varepsilon$  and  $\gamma' = n' + \varepsilon'$ . Let  $f^1, \dots, f^n$  be maps on  $U$  such that  $(f, f^1, \dots, f^n)$  is  $\text{Lip} - \gamma$  and let  $R_0, \dots, R_n$  be the associated remainders. For  $k \in \llbracket 0, n' \rrbracket$ , define  $S_k : U \times U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  as follows:

$$\forall x, y \in U, \forall v \in E^{\otimes k} : S_k(x, y)(v) = \sum_{j=n'+1}^n f^j(y) \left( \frac{v \otimes (x - y)^{\otimes (j-k)}}{(j - k)!} \right) + R_k(x, y)(v)$$

By a straightforward computation, one gets that, for all  $x, y \in U$ :

$$\|S_k(x, y)\| \leq \|f\|_{\text{Lip} - \gamma} \left( \sum_{j=n'+1}^n \frac{L^{j-\gamma'}}{(j - k)!} + L^{\gamma-\gamma'} \right) \|x - y\|^{\gamma-k}$$

By recognising the  $S_i$ 's in the expansion formulas of the  $f_i$ 's, we see therefore that  $(f, f^1, \dots, f^n)$  is  $\text{Lip} - \gamma'$  with  $S_0, \dots, S_{n'}$  as remainders and:

$$\|f\|_{\text{Lip} - \gamma'} \leq \|f\|_{\text{Lip} - \gamma} \max \left( 1, \sum_{j=n'+1}^n \frac{L^{j-\gamma'}}{(j - n')!} + L^{\gamma-\gamma'} \right)$$

□

**Remark 1.12.** *With the notations of the previous lemma and proof, we have the following inequalities:*

$$\sum_{j=n'+1}^n \frac{L^{j-\gamma'}}{(j - n')!} \leq \max(1, L^{\gamma-\gamma'}) (e - 1)$$

and if  $L > 0$ :

$$\sum_{j=n'+1}^n \frac{L^{j-\gamma'}}{(j - n')!} \leq \frac{e^L - 1}{L^{\varepsilon'}} \leq (e^L - 1) \max(1, \frac{1}{L})$$

The aim now is to be able to go from the case where the domain of definition of the map is bounded to a more general one. This gives us an important local characterization of Lipschitz maps:

**Lemma 1.13.** *Let  $\gamma > 0$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a subset of  $E$ . For every  $k \in \llbracket 0, \lfloor \gamma \rfloor \rrbracket$ , let  $f^k : U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  be a map with values in the space of the symmetric  $k$ -linear mappings from  $E$  to  $F$ . We assume that  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are endowed with norms satisfying the projective property and that there exists  $\delta > 0$  and  $C \geq 0$  such*

that, for every  $x \in U$ ,  $f|_{B(x,\delta) \cap U}$  is Lip  $-\gamma$  with a norm less than or equal to  $C$  ( where  $f = (f^0, \dots, f^{\lfloor \gamma \rfloor})$ ). Then  $f$  is Lip  $-\gamma$  and:

$$\|f\|_{\text{Lip}-\gamma} \leq C \max \left( 1, \max_{0 \leq k \leq \lfloor \gamma \rfloor} \frac{1}{\delta^{\gamma-k}} \left( 1 + \sum_{j=0}^{\lfloor \gamma \rfloor - k} \frac{\delta^j}{j!} \right) \right)$$

*Proof.* Let  $k \in \llbracket 0, \lfloor \gamma \rfloor \rrbracket$ . We already know that  $\sup_{x \in U} \|f^k(x)\| \leq C$ . Define  $R_k : U \times U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  as follows:

$$\forall x, y \in E, \forall v \in E^{\otimes k} : \quad R_k(x, y)(v) = f^k(x)(v) - \sum_{j=k}^{\lfloor \gamma \rfloor} f^j(y) \left( \frac{v \otimes (x-y)^{\otimes (j-k)}}{(j-k)!} \right)$$

Let  $x, y \in U$ . If  $\|x - y\| < \delta$ , then, as  $f|_{B(x,\delta) \cap U}$  is Lip  $-\gamma$ , we have:

$$\|R_k(x, y)\| \leq C \|x - y\|^{\gamma-k}$$

Assume that  $\|x - y\| \geq \delta$ , then, as the  $(E^{\otimes k})_{1 \leq k \leq n}$  are endowed with norms satisfying the projective property, we obtain:

$$\begin{aligned} \frac{\|R_k(x, y)\|}{\|x - y\|^{\gamma-k}} &\leq \frac{\|f^k(x)\|}{\|x - y\|^{\gamma-k}} + \sum_{j=k}^{\lfloor \gamma \rfloor} \frac{\|f^j(y)\|}{\|x - y\|^{\gamma-j} (j-k)!} \\ &\leq C \left( \frac{1}{\delta^{\gamma-k}} + \sum_{j=k}^{\lfloor \gamma \rfloor} \frac{1}{\delta^{\gamma-j} (j-k)!} \right) \\ &\leq \frac{C}{\delta^{\gamma-k}} \left( 1 + \sum_{j=0}^{\lfloor \gamma \rfloor - k} \frac{\delta^j}{j!} \right) \end{aligned}$$

We deduce then that  $f$  is Lip  $-\gamma$  on  $U$  with the suggested upper-bound of  $\|f\|_{\text{Lip}-\gamma}$ .  $\square$

**Remark 1.14.** *With the notations of the previous lemma, we have:*

$$\max \left( 1, \max_{0 \leq k \leq n} \frac{1}{\delta^{\gamma-k}} \left( 1 + \sum_{j=0}^{n-k} \frac{\delta^j}{j!} \right) \right) \leq (1 + e^\delta) \max(1, \frac{1}{\delta^\gamma})$$

We can now state the following natural embedding theorem:

**Theorem 1.15.** *Let  $\gamma, \gamma' > 0$  such that  $\gamma' < \gamma$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a subset of  $E$ . We assume that  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are endowed with norms satisfying the projective property. Let  $f : U \rightarrow F$  be a Lip  $-\gamma$  map. Then  $f$  is Lip  $-\gamma'$  and there exists a constant  $M_{\gamma, \gamma'}$  (depending only on  $\gamma$  and  $\gamma'$ ) such that  $\|f\|_{\text{Lip}-\gamma'} \leq M_{\gamma, \gamma'} \|f\|_{\text{Lip}-\gamma}$*

*Proof.* Let  $\delta > 0$ . Let  $x \in U$ ,  $f$  is Lip  $-\gamma$  on  $B(x, \delta) \cap U$  with a Lip  $-\gamma$  norm less than or equal to  $\|f\|_{\text{Lip}-\gamma}$ . Then, by lemma 1.11,  $f$  is Lip  $-\gamma'$  on  $B(x, \delta) \cap U$  and:

$$\|f\|_{\text{Lip}-\gamma', B(x,\delta) \cap U} \leq \|f\|_{\text{Lip}-\gamma} \max \left( 1, \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{(2\delta)^{j-\gamma'}}{(j - \lfloor \gamma' \rfloor)!} + (2\delta)^{\gamma-\gamma'} \right)$$

Using now lemma 1.13, we deduce that  $f$  is Lip  $-\gamma'$  on  $U$  with a Lip  $-\gamma'$  controlled as follows:

$$\|f\|_{\text{Lip}-\gamma'} \leq \|f\|_{\text{Lip}-\gamma} \max \left( 1, \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{(2\delta)^{j-\gamma'}}{(j-\lfloor \gamma' \rfloor)!} + (2\delta)^{\gamma-\gamma'} \right) \\ \max \left( 1, \max_{0 \leq k \leq \lfloor \gamma' \rfloor} \frac{1}{\delta^{\gamma'-k}} \left( 1 + \sum_{j=0}^{\lfloor \gamma' \rfloor - k} \frac{\delta^j}{j!} \right) \right)$$

The above inequality holding for every  $\delta > 0$ , we can make it sharper by taking the infimum of the right-hand side over all possible positive values of  $\delta$ . This ends the proof.  $\square$

**Remark 1.16.**

$$M_{\gamma, \gamma'} = \inf_{\delta > 0} \max \left( 1, \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{(2\delta)^{j-\gamma'}}{(j-\lfloor \gamma' \rfloor)!} + (2\delta)^{\gamma-\gamma'} \right) \\ \max \left( 1, \max_{0 \leq k \leq \lfloor \gamma' \rfloor} \frac{1}{\delta^{\gamma'-k}} \left( 1 + \sum_{j=0}^{\lfloor \gamma' \rfloor - k} \frac{\delta^j}{j!} \right) \right)$$

By considering the value  $\delta = 1/2$  for the prove above, we get the following estimate:

$$M_{\gamma, \gamma'} \leq 2^{\gamma'} e(1 + e^{1/2}) \leq 2^{\gamma} e(1 + e^{1/2})$$

Which has the additional advantage of being dependent on only one of the variables  $\gamma$  and  $\gamma'$ .

As highlighted for example in [3], a simpler proof of theorem 1.15 can be given when the domain of definition of the map is open and convex. We first give a characterization of Lipschitz maps in this case, which also gives a very useful recursive definition of Lipschitzness. The proof of the following is trivial and can be found if needed in [3] for example:

**Lemma 1.17.** *Let  $n \in \mathbb{N}$ ,  $0 < \varepsilon \leq 1$  and  $C \geq 0$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be a subset of  $E$ . Let  $f : U \rightarrow F$  be a map and for every  $k \in \llbracket 1, n \rrbracket$ , let  $f^k : U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  be a map with values in the space of the symmetric  $k$ -linear mappings from  $E$  to  $F$ . We consider the two following assertions:*

**(A1):**  $(f, f^1, \dots, f^n)$  is Lip  $-(n + \varepsilon)$  and  $\|f\|_{\text{Lip}-(n+\varepsilon)} \leq C$ .

**(A2):**  $f$  is  $n$  times differentiable, with  $f^1, \dots, f^n$  being its successive derivatives.  $\|f\|_{\infty}, \|f^1\|_{\infty}, \dots, \|f^n\|_{\infty}$  are upper-bounded by  $C$  and for all  $x, y \in U : \|f^n(x) - f^n(y)\| \leq C\|x - y\|^\varepsilon$ .

If  $U$  is open then **(A1)**  $\Rightarrow$  **(A2)**. If, furthermore,  $U$  is convex then **(A1)**  $\Leftrightarrow$  **(A2)**.

When the domain of a Lipschitz map is open, convex and bounded, we get a sharper estimate than the one obtained in lemma 1.11:

**Lemma 1.18.** *Let  $\gamma, \gamma' > 0$  such that  $\gamma' < \gamma$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be an open convex bounded subset of  $E$ . Let  $f : U \rightarrow F$  be a Lip  $-\gamma$  function. We assume that  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are endowed with norms satisfying the projective property. Then  $f$  is Lip  $-\gamma'$  and if  $L \geq 0$  is larger than or equal to the diameter of  $U$  then:*

$$\|f\|_{\text{Lip}-\gamma'} \leq \|f\|_{\text{Lip}-\gamma} \max \left( 1, L^{\min(\lfloor \gamma' \rfloor + 1, \gamma) - \gamma'} \right)$$

*Proof.* Uses the characterization in lemma 1.17 and, if  $\lfloor \gamma' \rfloor < \lfloor \gamma \rfloor$ , the fundamental theorem of calculus.  $\square$

Always in the case of an open convex domain, we also get a sharper control of the Lipschitz norm from the uniform local behaviour of the map:

**Lemma 1.19.** *Let  $\gamma > 0$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be an open convex subset of  $E$ . Let  $f : U \rightarrow F$  be a map such that there exists  $\delta > 0$  and  $C \geq 0$  such that, for every  $x \in U$ ,  $f|_{B(x,\delta) \cap U}$  is  $Lip - \gamma$  with a norm less than or equal to  $C$ . Then  $f$  is  $Lip - \gamma$  and:*

$$\|f\|_{Lip-\gamma} \leq C \max\left(1, \frac{2}{\delta^{\gamma - \lfloor \gamma \rfloor}}\right)$$

*Proof.* Uses the characterization in lemma 1.17 and the same technique as in the proof of lemma 1.13. If necessary, a complete proof can be found for example in [3].  $\square$

Theorem 1.15 now becomes:

**Theorem 1.20.** *Let  $\gamma, \gamma' > 0$  such that  $\gamma' < \gamma$ . Let  $E$  and  $F$  be two normed vector spaces and  $U$  be an open convex subset of  $E$ . We assume that  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are endowed with norms satisfying the projective property. Let  $f : U \rightarrow F$  be a  $Lip - \gamma$  map. Then  $f$  is  $Lip - \gamma'$  and there exists a positive constant  $m_{\gamma, \gamma'}$  such that:*

$$\|f\|_{Lip-\gamma'} \leq m_{\gamma, \gamma'} \|f\|_{Lip-\gamma}$$

**Remark 1.21.** *We can choose the constant in the previous theorem such that:*

$$m_{\gamma, \gamma'} = \inf_{\delta > 0} \max\left(1, (2\delta)^{\min(\lfloor \gamma' \rfloor + 1, \gamma) - \gamma'}\right) \max\left(1, \frac{2}{\delta^{\gamma' - \lfloor \gamma' \rfloor}}\right)$$

*And we have in this case  $m_{\gamma, \gamma'} \leq 4$ .*

The following proposition about smooth maps is very useful and comes as an easy consequence of all the above:

**Theorem 1.22.** *Let  $\gamma > 0$ . A map that is  $(\lfloor \gamma \rfloor + 1)$  times continuously differentiable and is such that its derivatives are bounded on a given convex set is Lipschitz- $\gamma$  on that set. Its Lipschitz- $\gamma$  norm can be upper-bounded by the following constant:*

$$L_\gamma = \inf_{\delta > 0} \left( \max(\|f\|_\infty, \|f^1\|_\infty, \dots, \|f^{\lfloor \gamma \rfloor}\|_\infty, \|f^{\lfloor \gamma \rfloor + 1}\|_\infty (2\delta)^{\lfloor \gamma \rfloor + 1 - \gamma}) \cdot \max\left(1, \frac{2}{\delta^{\gamma - \lfloor \gamma \rfloor}}\right) \right)$$

**Remark 1.23.** *It is easy to show that, for example:*

$$L_\gamma \leq 4 \max(\|f\|_\infty, \|f^1\|_\infty, \dots, \|f^{\lfloor \gamma \rfloor}\|_\infty, \|f^{\lfloor \gamma \rfloor + 1}\|_\infty)$$

**1.3. Composition of Lipschitz functions.** As one would expect, a well-defined composition of two Lipschitz maps is also Lipschitz. We start first with the simple case where one of the maps is linear as the derivatives are easier to extract, though, technically, a continuous linear map defined on the whole space is not necessarily Lipschitz (as its values are not necessary uniformly bounded).

**Proposition 1.24.** *Let  $E, F$  and  $G$  be three normed vector spaces and  $U$  be a subset of  $E$ . Let  $\gamma > 0$  and let  $f : U \rightarrow F$  be a  $Lip - \gamma$  map. Let  $u : F \rightarrow G$  a bounded linear map. Then  $u \circ f$  is  $Lip - \gamma$  and  $\|u \circ f\|_{Lip-\gamma} \leq \|u\| \|f\|_{Lip-\gamma}$ .*

*Proof.* Let  $n \in \mathbb{N}$  such that  $\gamma \in (n, n+1]$ . Let  $f^1, \dots, f^n$  be maps on  $U$  such that  $(f, f^1, \dots, f^n)$  is Lip  $-\gamma$  and let  $R_0, \dots, R_n$  be the associated remainders. Let  $g = u \circ f$  and for every  $k \in \llbracket 1, n \rrbracket$ , let  $g^k$  and  $S_k$  be defined as follows:

$$\forall x, y \in E, \forall v \in E^{\otimes k} : g^k(x)(v) = u(f^k(x)(v)), \quad S_k(x, y)(v) = u(R_k(x, y)(v))$$

Then it is easy to check that  $(g, g^1, \dots, g^n)$  is Lip  $-\gamma$  with  $S_0, \dots, S_n$  as remainders and with a Lip  $-\gamma$  norm upper-bounded by  $\|u\| \|f\|_{\text{Lip}-\gamma}$ .  $\square$

**Remark 1.25.** *Although a linear map in general is not Lipschitz, we can restrict ourselves, in the previous example, to a bounded domain of  $F$  so that the restriction of  $u$  on that domain is Lipschitz. We will be then in the case of a composition of two Lipschitz maps but we don't get a control of the Lipschitz norm as sharp as the one in proposition 1.24.*

The image of a product of Lipschitz maps by a bilinear map is also Lipschitz:

**Proposition 1.26.** *Let  $E, F, G$  and  $H$  be normed vector spaces and  $U$  be a subset of  $E$ . Let  $\gamma > 0$  and let  $f : U \rightarrow F$  and  $g : U \rightarrow G$  be two Lip  $-\gamma$  maps. Let  $B : F \times G \rightarrow H$  a continuous bilinear map. We assume that  $(E^{\otimes k})_{k \geq 1}$  are endowed with norms satisfying the projective and symmetric properties. Then  $B(f, g) : U \rightarrow H$  is Lip  $-\gamma$  and there exists a constant  $C$  (depending only on  $\gamma$ ) such that  $\|B(f, g)\|_{\text{Lip}-\gamma} \leq C \|B\| \|f\|_{\text{Lip}-\gamma} \|g\|_{\text{Lip}-\gamma}$ .*

The idea of this proof is rather simple but contains notions and ideas that will be very important to the proof of the main theorem of this section.

*Proof.* Let  $\varepsilon \in (0, 1]$ . We prove by induction the following statement:

For all  $n \in \mathbb{N}$ , for any normed vector spaces  $E, F, G$  and  $H$  and any subset  $U$  of  $E$ , there exists a real constant  $C_{n, \varepsilon}$  (depending only on  $n$  and  $\varepsilon$ ) such that if  $f : U \rightarrow F$  and  $g : U \rightarrow G$  are Lip  $-(n + \varepsilon)$  maps and  $B : F \times G \rightarrow H$  is a continuous bilinear map, then  $(B(f, g), B^1, \dots, B^n)$  is Lip  $-(n + \varepsilon)$  where, for  $k \in \llbracket 1, n \rrbracket$ ,  $x \in U$  and  $v \in E^{\otimes k}$ :

$$B^k(x)(v) = \sum_{\substack{i \in \llbracket 0, k \rrbracket \\ \sigma \in \mathcal{S}_k}} \frac{B(f^i(x), g^{k-i}(x))}{i!(k-i)!} \sigma(v)$$

And:

$$\|B(f, g)\|_{\text{Lip}-(n+\varepsilon)} \leq C_{n, \varepsilon} \|B\| \|f\|_{\text{Lip}-(n+\varepsilon)} \|g\|_{\text{Lip}-(n+\varepsilon)}$$

For  $n = 0$ , the proof of the statement is trivial and is left as an exercise. Let  $n \in \mathbb{N}$ . We assume the statement true for  $n$  and let us prove it for  $n + 1$ . Let  $E, F, G$  and  $H$  be normed vector spaces and  $U$  be a subset of  $E$ , and let  $f : U \rightarrow F$  and  $g : U \rightarrow G$  be two Lip  $-(n + 1 + \varepsilon)$  maps and  $B : F \times G \rightarrow H$  be a continuous bilinear map. We will show that  $(Z, Z^1, \dots, Z^{n+1})$  is Lip  $-(n + 1 + \varepsilon)$  where,  $Z := B(f, g)$  and for  $k \in \llbracket 1, n + 1 \rrbracket$ ,  $x \in U$  and  $v \in E^{\otimes k}$ :

$$Z^k(x)(v) = \sum_{\substack{i \in \llbracket 0, k \rrbracket \\ \sigma \in \mathcal{S}_k}} \frac{B(f^i(x), g^{k-i}(x))}{i!(k-i)!} \sigma(v)$$

For  $k \in \llbracket 0, n + 1 \rrbracket$ , let  $R_k$  (resp.  $S_k, T_k$ ) be the remainder of order  $k$  associated to  $f$  (resp.  $g, Z$ ). Let  $x, y \in U$ . Writing the Taylor expansion of  $f$  and  $g$  and using the bilinearity of  $B$ ,



we get:

$$Z(x) = \sum_{i=0}^{n+1} Z^i(y) \left( \frac{(x-y)^{\otimes i}}{i!} \right) + T_0(x, y)$$

Where:

$$T_0(x, y) = B(R_0(x, y), g(x)) + B(f(x), S_0(x, y)) + B(R_0(x, y), S_0(x, y)) + \sum_{\substack{i, j \in \llbracket 0, n+1 \rrbracket \\ i+j > n+1}} B(f, g) \left( \frac{(x-y)^{\otimes (i+j)}}{i!j!} \right)$$

It is obvious then that we can bound the  $Z^k(x)$ 's and  $T_0(x, y)$  adequately (appropriate exponents can be obtained by reasoning over balls of the same size and then using lemma 1.13).

We prove now that  $Z^1$  is Lip  $-(n + \varepsilon)$  with a Taylor expansion that we can identify with that of  $Z$  and with a well bounded Lip  $-(n + \varepsilon)$  norm.

$f^1$  and  $g$  (resp.  $f$  and  $g^1$ ) are both Lip  $-(n + \varepsilon)$ . Therefore, by the induction hypothesis,  $B(f^1, g)$  (resp.  $B(f, g^1)$ ) is Lip  $-(n + \varepsilon)$ . Hence  $(Z^1, (Z^1)^1 \dots, (Z^1)^n)$  is Lip  $-(n + \varepsilon)$ , where, for  $k \in \llbracket 1, n \rrbracket$ ,  $x \in U$  and  $v \in E^{\otimes k}$ :

$$(Z^1)^k(x)(v) = \sum_{\substack{i \in \llbracket 0, k \rrbracket \\ \sigma \in \mathcal{S}_k}} \frac{B((f^1)^i(x), g^{k-i}(x)) + B(f^i(x), (g^1)^{k-i}(x))}{i!(k-i)!} \sigma(v)$$

and there exists a constant  $c_{n, \varepsilon}$  such that:

$$\|Z^1\|_{\text{Lip}-(n+\varepsilon)} \leq c_{n, \varepsilon} \|B\| \|f\|_{\text{Lip}-(n+1+\varepsilon)} \|g\|_{\text{Lip}-(n+1+\varepsilon)}$$

To end the proof now, we only have to make the identification between  $(Z^1)^k$  and  $Z^{k+1}$ , for all  $k \in \llbracket 1, n \rrbracket$ . Let then  $k \in \llbracket 1, n \rrbracket$ ,  $x \in U$  and  $v_1, \dots, v_k, v_{k+1} \in E$  and define:  $v = v_1 \otimes \dots \otimes v_k$ . Studying the position of  $v_{k+1}$  in  $Z^{k+1}(x)(v \otimes v_{k+1})$ , we are naturally led into dividing the sum into the two following parts:

$$\begin{aligned} Z^{k+1}(x)(v \otimes v_{k+1}) &= \sum_{\substack{i \in \llbracket 0, k+1 \rrbracket \\ \sigma \in \mathcal{S}_{k+1}}} \frac{B(f^i(x)(\sigma^{1,i}(v \otimes v_{k+1})), g^{k+1-i}(x)(\sigma^{2,i}(v \otimes v_{k+1})))}{i!(k+1-i)!} \\ &= \sum_{i=1}^{k+1} \left( \sum_{\substack{\sigma \in \mathcal{S}_{k+1} \\ \sigma^{-1}(k+1) \leq i}} \frac{B(f^i(x)(\sigma^{1,i}(v \otimes v_{k+1})), g^{k+1-i}(x)(\sigma^{2,i}(v \otimes v_{k+1})))}{i!(k+1-i)!} \right) + \\ &\quad \sum_{i=0}^k \left( \sum_{\substack{\sigma \in \mathcal{S}_{k+1} \\ \sigma^{-1}(k+1) > i}} \frac{B(f^i(x)(\sigma^{1,i}(v \otimes v_{k+1})), g^{k+1-i}(x)(\sigma^{2,i}(v \otimes v_{k+1})))}{i!(k+1-i)!} \right) \end{aligned}$$

Where, for  $\sigma \in \mathcal{S}_{k+1}$  and  $i \in \llbracket 0, k+1 \rrbracket$ ,  $\sigma^{1,i}(v \otimes v_{k+1})$  and  $\sigma^{2,i}(v \otimes v_{k+1})$  are the only elements of  $E^{\otimes i}$  and  $E^{\otimes(k+1-i)}$  respectively such that

$$\sigma(v \otimes v_{k+1}) = \sigma^{1,i}(v \otimes v_{k+1}) \otimes \sigma^{2,i}(v \otimes v_{k+1})$$

Then notice, using the symmetry of the  $f^i(x)$ 's, that we can write, for  $i \in \llbracket 1, k+1 \rrbracket$ :

$$\begin{aligned} & \sum_{\sigma \in \mathcal{S}_{k+1}} \frac{B(f^i(x)(\sigma^{1,i}(v \otimes v_{k+1})), g^{k-i}(x)(\sigma^{2,i}(v \otimes v_{k+1})))}{i!(k+1-i)!} \\ &= \sum_{\tau \in \mathcal{S}_k} \frac{B(f^i(x)(\tau^{1,i-1}(v) \otimes v_{k+1}), g^{k+1-i}(x)(\tau^{2,i-1}(v)))}{(i-1)!(k+1-i)!} \end{aligned}$$

We deal with the other term by using the same idea. We finally get:

$$\begin{aligned} Z^{k+1}(x)(v \otimes v_{k+1}) &= \sum_{i \in \llbracket 0, k \rrbracket} \frac{B(f^{i+1}(x)(\sigma^{1,i}(v) \otimes v_{k+1}), g^{k-i}(x)(\sigma^{2,i}(v)))}{i!(k-i)!} + \\ & \sum_{\sigma \in \mathcal{S}_k} \frac{B(f^i(x)(\sigma^{1,i}(v)), g^{k+1-i}(x)(\sigma^{2,i}(v) \otimes v_{k+1}))}{i!(k-i)!} \end{aligned}$$

Which is exactly  $(Z^1)^k(x)(v)(v_{k+1})$ . This ends this proof.  $\square$

**Remark 1.27.** *The real-valued Lip- $\gamma$  functions form an algebra under point-wise multiplication.*

**Remark 1.28.** *If  $E \otimes F$  is endowed with a norm satisfying the projective property, then the tensor product of an  $E$ -valued Lipschitz map by an  $F$ -valued Lipschitz map is also Lipschitz as a direct consequence of proposition 1.26.*

**Proposition 1.29.** *Let  $\gamma > 0$  and  $E, F$  and  $G$  be three normed vector spaces. We assume that  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  and  $(F^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are endowed with compatible norms. Let  $f : F \rightarrow G$  be a Lip- $\gamma$  map and  $u : E \rightarrow F$  a bounded linear map. Then  $f \circ u$  is Lip- $\gamma$  and  $\|f \circ u\|_{\text{Lip-}\gamma} \leq \|f\|_{\text{Lip-}\gamma} \max(1, \|u\|^\gamma)$ .*

*Proof.* Let  $f^1, \dots, f^n$  be maps defined on  $F$  such that  $(f, f^1, \dots, f^n)$  is Lip- $\gamma$  and let  $R_0, \dots, R_n$  be the associated remainders. Let  $g = f \circ u$  and, for  $k \in \llbracket 1, n \rrbracket$ , let  $g^k : E \rightarrow \mathcal{L}(E^{\otimes k}, G)$  and  $S_k : E \times E \rightarrow \mathcal{L}(E^{\otimes k}, G)$  be the maps defined by:

$$\forall x, y \in E, \forall v \in E^{\otimes k} : g^k(x)(v) = f^k(u(x))(u^{\otimes k}(v)), S_k(x, y)(v) = R_k(u(x), u(y))(u^{\otimes k}(v))$$

Let  $k \in \llbracket 0, n \rrbracket$ ,  $x, y \in E$  and  $v \in E^{\otimes k}$ . Then we have, using the previous definitions:

$$\begin{aligned} g^k(x)(v) &= f^k(u(x))(u^{\otimes k}(v)) \\ &= \sum_{j=k}^n f^j(u(y)) \left( \frac{u^{\otimes k}(v) \otimes (u(x-y))^{\otimes(j-k)}}{(j-k)!} \right) + R_k(u(x), u(y))(u^{\otimes k}(v)) \\ &= \sum_{j=k}^n f^j(u(y)) \left( \frac{u^{\otimes k}(v) \otimes u^{\otimes(j-k)}(x-y)^{\otimes(j-k)}}{(j-k)!} \right) + S_k(x, y)(v) \\ &= \sum_{j=k}^n f^j(u(y)) \left( u^{\otimes j} \left( \frac{v \otimes (x-y)^{\otimes(j-k)}}{(j-k)!} \right) \right) + S_k(x, y)(v) \\ &= \sum_{j=k}^n g^j(y) \left( \frac{v \otimes (x-y)^{\otimes(j-k)}}{(j-k)!} \right) + S_k(x, y)(v) \end{aligned}$$

And  $\|g^k(x)\| \leq \|f\|_{\text{Lip-}\gamma} \|u\|^k$  and  $\|S_k(x, y)\| \leq \|f\|_{\text{Lip-}\gamma} \|u\|^\gamma \|x-y\|^{\gamma-k}$ . Hence,  $(g, g^1, \dots, g^n)$  is Lip- $\gamma$  (with  $(S_0, \dots, S_n)$  as remainders) and

$$\begin{aligned} \|g\|_{\text{Lip-}\gamma} &\leq \max(\|f\|_{\text{Lip-}\gamma}, \|f\|_{\text{Lip-}\gamma} \|u\|, \dots, \|f\|_{\text{Lip-}\gamma} \|u\|^n, \|f\|_{\text{Lip-}\gamma} \|u\|^\gamma) \\ &\leq \|f\|_{\text{Lip-}\gamma} \max(1, \|u\|^\gamma) \end{aligned}$$

$\square$

**Remark 1.30.** *If  $(E^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  and  $(F^{\otimes k})_{1 \leq k \leq \lfloor \gamma \rfloor}$  are not necessarily endowed with compatible norms, then  $f \circ u$  is still  $\text{Lip} - \gamma$  and:*

$$\|f \circ u\|_{\text{Lip} - \gamma} \leq \|f\|_{\text{Lip} - \gamma} \max_{0 \leq k \leq \lfloor \gamma \rfloor} \|u^{\otimes k}\| (1 \vee \|u\|^{\gamma - k})$$

We are ready now to show that a well-defined composition of Lipschitz maps is itself Lipschitz. We start first with the following simple case:

**Lemma 1.31.** *Let  $E, F$  and  $G$  be three normed vector spaces. Let  $U$  be a subset of  $E$  and  $V$  be a subset of  $F$ . Let  $\varepsilon \in (0, 1]$ . We assume that  $(E^{\otimes k})_{k \geq 1}$  and  $(F^{\otimes k})_{k \geq 1}$  are endowed with norms satisfying the projective property. Let  $f : U \rightarrow F$  and  $g : V \rightarrow G$  be two  $\text{Lip} - (1 + \varepsilon)$  maps such that  $f(U) \subseteq V$ . Then  $g \circ f$  is  $\text{Lip} - (1 + \varepsilon)$  and, there exists a constant  $C_\varepsilon$  (depending only on  $\varepsilon$ ) such that:*

$$\|g \circ f\|_{\text{Lip} - (1 + \varepsilon)} \leq C_\varepsilon \|g\|_{\text{Lip} - (1 + \varepsilon)} \max(\|f\|_{\text{Lip} - (1 + \varepsilon)}^{(1 + \varepsilon)}, 1)$$

*Proof.* Let  $f = (f^0, f^1) : U \rightarrow F$  and  $g = (g^0, g^1) : V \rightarrow G$  be two  $\text{Lip} - (1 + \varepsilon)$  maps such that  $f(U) \subseteq V$ , and with remainders denoted by  $(R_0, R_1)$  and  $(S_0, S_1)$  respectively. In this case,  $(g \circ f)^1$  is simply the map given by the chain rule and which maps every element  $x$  in  $U$  to  $g^1(f^0(x)) \circ f^1(x)$ . Let  $x, y \in U$ . Then the Taylor expansions of  $g$  and  $f$  enable us to write  $g^0(f^0(x))$  in function of  $g^0(f^0(y))$  and  $(g \circ f)^1(y)$  as follows:

$$g^0(f^0(x)) = g^0(f^0(y)) + g^1(f^0(y))(f^1(y)(x - y)) + g^1(f^0(y))(R_0(x, y)) + S_0(f^0(x), f^0(y))$$

Let  $T_0 : U \times U \rightarrow G$  and  $T_1 : U \times U \rightarrow \mathcal{L}(E, G)$  be defined as follows:

$$\begin{aligned} \forall x, y \in U, \forall v \in E : \quad T_0(x, y) &= g^1(f^0(y))(R_0(x, y)) + S_0(f^0(x), f^0(y)) \\ T_1(x, y) &= (g \circ f)^1(x) - (g \circ f)^1(y) \end{aligned}$$

Let  $x, y \in U$ . It is easy to see that:

$$\|(g^0 \circ f^0)(x)\| \leq \|g\|_{\text{Lip} - (1 + \varepsilon)}$$

and that:

$$\|(g \circ f)^1(x)\| \leq \|g\|_{\text{Lip} - (1 + \varepsilon)} \|f\|_{\text{Lip} - (1 + \varepsilon)}$$

Furthermore, we have:

$$\|g^1(f^0(y))(R_0(x, y))\| \leq \|g\|_{\text{Lip} - (1 + \varepsilon)} \|f\|_{\text{Lip} - (1 + \varepsilon)} \|x - y\|^{1 + \varepsilon}$$

By theorem 1.15,  $f$  is also  $\text{Lip} - 1$  and there exists a constant  $M_{1, \varepsilon}$  depending only on 1 and  $\varepsilon$  such that  $\|f\|_{\text{Lip} - 1} \leq M_{1, \varepsilon} \|f\|_{\text{Lip} - (1 + \varepsilon)}$ . Consequently, we can write:

$$\|S_0(f^0(x), f^0(y))\| \leq \|g\|_{\text{Lip} - (1 + \varepsilon)} M_{1, \varepsilon}^{1 + \varepsilon} \|f\|_{\text{Lip} - (1 + \varepsilon)}^{1 + \varepsilon} \|x - y\|^{1 + \varepsilon}$$

Hence:

$$\|T_0(x, y)\| \leq \|g\|_{\text{Lip} - (1 + \varepsilon)} (\|f\|_{\text{Lip} - (1 + \varepsilon)} + M_{1, \varepsilon}^{1 + \varepsilon} \|f\|_{\text{Lip} - (1 + \varepsilon)}^{1 + \varepsilon}) \|x - y\|^{1 + \varepsilon}$$

By writing

$$T_1(x, y) = g^1(f^0(x))(f^1(x) - f^1(y)) + (g^1(f^0(x)) - g^1(f^0(y)))(f^1(y))$$

and by using similar techniques as above, we get the inequality:

$$\|T_1(x, y)\| \leq \|g\|_{\text{Lip} - (1 + \varepsilon)} (\|f\|_{\text{Lip} - (1 + \varepsilon)} + M_{1, \varepsilon}^\varepsilon \|f\|_{\text{Lip} - (1 + \varepsilon)}^{1 + \varepsilon}) \|x - y\|^\varepsilon$$

Therefore,  $g \circ f$  is  $\text{Lip} - (1 + \varepsilon)$  (with the suggested Taylor expansion) and by defining, for example,  $C_{1,\varepsilon} = 2 \max(1, M_{1,\varepsilon}^{1+\varepsilon})$ , we obtain:

$$\|g \circ f\|_{\text{Lip}-(1+\varepsilon)} \leq C_{1,\varepsilon} \|g\|_{\text{Lip}-(1+\varepsilon)} \max(\|f\|_{\text{Lip}-(1+\varepsilon)}^{1+\varepsilon}, 1)$$

□

**Notation.** For any finite set  $\{\alpha_1, \dots, \alpha_r\}$ ,  $\mathcal{S}_{\alpha_1, \dots, \alpha_r}$  denotes the set of all bijections from  $\{\alpha_1, \dots, \alpha_r\}$  onto itself.

We will need the combinatorial result stated in the next lemma before we can proceed:

**Lemma 1.32.** Let  $V$  be a non-empty set. For  $j \in \mathbb{N}^*$  and  $w \in V^{\otimes j}$ , we will denote by  $\text{Sym}(w)$  the symmetric part of  $w$ . Let  $N \in \mathbb{N}^*$  and  $v_1, \dots, v_N \in V$ , then:

$$\sum_{\sigma \in \mathcal{S}_N} \sum_{\tau \in \mathcal{S}_{\sigma(1), \dots, \sigma(i)}} v_{\tau \circ \sigma(1)} \otimes \cdots \otimes v_{\tau \circ \sigma(i)} \otimes \text{Sym}(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(N)})$$

equals:

$$i!(N-i)! \sum_{1 \leq r_1 < \cdots < r_i \leq N} \sum_{\tau \in \mathcal{S}_{r_1, \dots, r_i}} v_{\tau(r_1)} \otimes \cdots \otimes v_{\tau(r_i)} \otimes v_{r_{i+1}} \otimes \cdots \otimes v_{r_N}$$

Where, in order to have a compact expression, for every  $1 \leq r_1 < \cdots < r_i \leq N$ , we mean by  $r_{i+1}, \dots, r_N$  the integers that are such that  $r_{i+1} < \cdots < r_N$  and  $\{r_{i+1}, \dots, r_N\} = \llbracket 1, N \rrbracket - \{r_1, \dots, r_i\}$ .

*Proof.* For  $\sigma, \tilde{\sigma} \in \mathcal{S}_k$  such that  $\sigma(\llbracket 1, i \rrbracket) = \tilde{\sigma}(\llbracket 1, i \rrbracket)$ , we have:

$$\sum_{\tau \in \mathcal{S}_{\sigma(1), \dots, \sigma(i)}} v_{\tau \circ \sigma(1)} \otimes \cdots \otimes v_{\tau \circ \sigma(i)} = \sum_{\tau \in \mathcal{S}_{\tilde{\sigma}(1), \dots, \tilde{\sigma}(i)}} v_{\tau \circ \tilde{\sigma}(1)} \otimes \cdots \otimes v_{\tau \circ \tilde{\sigma}(i)}$$

and:

$$\text{Sym}(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(N)}) = \text{Sym}(v_{\tilde{\sigma}(i+1)} \otimes \cdots \otimes v_{\tilde{\sigma}(N)})$$

In the light of the previous identities, we rewrite  $\mathcal{S}_N$  as the following disjoint union:

$$\mathcal{S}_N = \bigcup_{1 \leq r_1 < \cdots < r_i \leq N} \{\sigma \in \mathcal{S}_N \mid \sigma(\llbracket 1, i \rrbracket) = \{r_1, \dots, r_i\}\}$$

Notice that, for  $1 \leq r_1 < \cdots < r_i \leq N$ , the set  $\{\sigma \in \mathcal{S}_N \mid \sigma(\llbracket 1, i \rrbracket) = \{r_1, \dots, r_i\}\}$  has  $i!(N-i)!$  elements. Therefore:

$$\sum_{\sigma \in \mathcal{S}_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)} \otimes \text{Sym}(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(N)})$$

is equal to:

$$\sum_{1 \leq r_1 < \cdots < r_i \leq N} \sum_{\tau \in \mathcal{S}_{r_1, \dots, r_i}} v_{\tau(r_1)} \otimes \cdots \otimes v_{\tau(r_i)} \otimes \sum_{\sigma \in \mathcal{S}_N, \sigma(\llbracket 1, i \rrbracket) = \{r_1, \dots, r_i\}} \text{Sym}(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(N)})$$

which itself is equal to:

$$i!(N-i)! \sum_{1 \leq r_1 < \cdots < r_i \leq N} \sum_{\tau \in \mathcal{S}_{r_1, \dots, r_i}} v_{\tau(r_1)} \otimes \cdots \otimes v_{\tau(r_i)} \otimes v_{r_{i+1}} \otimes \cdots \otimes v_{r_N}$$

□

The following composition result in the general case has already appeared in [3] but with a slight mistake in the control of the Lipschitz norm of the composition map that we correct here along with giving a full and detailed proof that differs from the one suggested in the aforementioned paper:

**Theorem 1.33.** *Let  $E, F$  and  $G$  be three normed vector spaces. Let  $U$  be a subset of  $E$  and  $V$  be a subset of  $F$ . Let  $\gamma > 0$ . We assume that  $(E^{\otimes k})_{k \geq 1}$  and  $(F^{\otimes k})_{k \geq 1}$  are endowed with norms satisfying the projective property. Let  $f : U \rightarrow F$  and  $g : V \rightarrow G$  be two  $\text{Lip} - \gamma$  maps such that  $f(U) \subseteq V$ . Then  $g \circ f$  is  $\text{Lip} - \gamma$  and, if  $\gamma \geq 1$ , there exists a constant  $C_\gamma$  (depending only on  $\gamma$ ) such that:*

$$\|g \circ f\|_{\text{Lip} - \gamma} \leq C_\gamma \|g\|_{\text{Lip} - \gamma} \max(\|f\|_{\text{Lip} - \gamma}^\gamma, 1)$$

*Proof.* The idea of the proof is very simple but rather technical and long. We leave the case  $\gamma \leq 1$  as an easy and straightforward exercise. Let  $\varepsilon \in (0, 1]$ . We will prove the following by induction:

**Claim.** *For all  $n \in \mathbb{N}^*$ , for any normed vector spaces  $E, F, G$  and  $H$ , such that  $(E^{\otimes k})_{k \geq 1}$  and  $(F^{\otimes k})_{k \geq 1}$  are endowed with norms satisfying the projective property, and any subsets  $U$  of  $E$  and  $V$  of  $F$ , there exists a real constant  $C_{n, \varepsilon}$  (depending only on  $n$  and  $\varepsilon$ ) such that if  $f = (f^0, \dots, f^n) : U \rightarrow F$  and  $g = (g^0, \dots, g^n) : V \rightarrow G$  are two  $\text{Lip} - (n + \varepsilon)$  maps such that  $f(U) \subseteq V$ , then  $g \circ f = (g^0 \circ f^0, (g \circ f)^1, \dots, (g \circ f)^n)$  is  $\text{Lip} - (n + \varepsilon)$  and:*

$$\|g \circ f\|_{\text{Lip} - (n + \varepsilon)} \leq C_{n, \varepsilon} \|g\|_{\text{Lip} - (n + \varepsilon)} \max(\|f\|_{\text{Lip} - (n + \varepsilon)}^{n + \varepsilon}, 1)$$

where, for every  $k \in \llbracket 1, n \rrbracket$ ,  $y \in U$ , and  $v \in E^{\otimes k}$ ,  $(g \circ f)^k(y)(v)$  is given by the following formula:

$$(g \circ f)^k(y)(v) = \sum_{j=1}^k \frac{g^j(f(y))}{j!} \sum_{\substack{1 \leq i_1, \dots, i_j \leq n \\ i_1 + \dots + i_j = k}} \frac{f^{i_1}(y) \otimes \dots \otimes f^{i_j}(y)}{i_1! \dots i_j!} \left( \sum_{\sigma \in \mathcal{S}_k} (\sigma(v)) \right)$$

The case  $n = 1$  has been proved in lemma 1.31.

Let now  $n \in \mathbb{N}^*$ . We assume that the assertion is true for  $n$  and let us prove it for  $n + 1$ . Let  $f = (f^0, \dots, f^{n+1}) : U \rightarrow F$  and  $g = (g^0, \dots, g^{n+1}) : V \rightarrow G$  be two  $\text{Lip} - (n + 1 + \varepsilon)$  functions such that  $f(U) \subseteq V$ , and with remainders denoted by  $R_0, \dots, R_{n+1}$  and  $S_0, \dots, S_{n+1}$  respectively. Let  $x, y \in U$ . Using the Taylor expansion of  $g$ , we have:

$$g^0(f^0(x)) = g^0(f^0(y)) + \sum_{j=1}^{n+1} g^j(f^0(y)) \left( \frac{(f^0(x) - f^0(y))^{\otimes j}}{j!} \right) + S_0(f^0(x), f^0(y))$$

Define  $P_0(x, y) = R_0(x, y)$ , and for every  $k \in \llbracket 1, n + 1 \rrbracket$ :  $P_k(x, y) = f^k(y) \frac{(x-y)^{\otimes k}}{k!}$ . Having in mind these notations and the Taylor expansion of  $f$  and the one we want to get for  $g \circ f$ , define:

$$T_0(x, y) = S_0(f^0(x), f^0(y)) + \sum_{j=1}^{n+1} \frac{g^j(f^0(y))}{j!} \left( \sum_{\substack{0 \leq i_1, \dots, i_j \leq n+1 \\ i_1 + \dots + i_j = j}} P_{i_1}(x, y) \otimes \dots \otimes P_{i_j}(x, y) + \sum_{\substack{1 \leq i_1, \dots, i_j \leq n+1 \\ i_1 + \dots + i_j > n+1}} f^{i_1}(y) \otimes \dots \otimes f^{i_j}(y) \frac{(x-y)^{\otimes (i_1 + \dots + i_j)}}{i_1! \dots i_j!} \right)$$

Then, we can simply write:

$$\begin{aligned} g^0(f^0(x)) &= g^0(f^0(y)) + T_0(x, y) + \\ &\quad \sum_{k=1}^{n+1} \sum_{j=1}^k \frac{g^j(f^0(y))}{j!} \left( \sum_{\substack{1 \leq i_1, \dots, i_j \leq n+1 \\ i_1 + \dots + i_j = k}} f^{i_1}(y) \otimes \dots \otimes f^{i_j}(y) \frac{(x-y)^{\otimes k}}{i_1! \dots i_j!} \right) \\ &= g^0(f^0(y)) + \sum_{k=1}^{n+1} (g \circ f)^k(y) \frac{(x-y)^{\otimes k}}{k!} + T_0(x, y) \end{aligned}$$

Assume that  $\|x - y\| < 2$ . It is an easy exercise then to show that there exists a constant  $M_{n,\varepsilon}$  (depending only on  $n$  and  $\varepsilon$ ) such that:

$$\|T_0(x, y)\| \leq M_{n,\varepsilon} \|g\|_{\text{Lip}-(n+1+\varepsilon)} \max(\|f\|_{\text{Lip}-(n+1+\varepsilon)}^{(n+1+\varepsilon)}, 1) \|x - y\|^{n+1+\varepsilon}$$

We have also that  $\|g^0 \circ f^0\|_\infty \leq \|g\|_{\text{Lip}-(n+1+\varepsilon)}$ . All that remains to do then to end the proof is to show that  $((g \circ f)^1, \dots, (g \circ f)^{n+1})$  and the associated remainders satisfy the appropriate Taylor expansion with well controlled uniform convergence norms. This will show that  $g \circ f$  is  $\text{Lip} - (n + 1 + \varepsilon)$  on every intersection of a ball of radius 1 with  $U$  with a uniformly controlled  $\text{Lip} - (n + 1 + \varepsilon)$  norm and we can then conclude using lemma 1.13.

Define the following maps:

$$\begin{aligned} \varphi : U &\rightarrow \mathcal{L}_c(E, F) \times \mathcal{L}_c(F, G) \\ x &\mapsto (f^1(x), g^1(f^0(x))) \end{aligned}$$

and:

$$\begin{aligned} \psi : \mathcal{L}_c(E, F) \times \mathcal{L}_c(F, G) &\rightarrow \mathcal{L}_c(E, G) \\ (u, v) &\mapsto v \circ u \end{aligned}$$

As  $g^1, f^0$  and  $f^1$  are all  $\text{Lip} - (n + \varepsilon)$ , then  $\varphi$  is also  $\text{Lip} - (n + \varepsilon)$ .  $\psi$  is a continuous bilinear map with norm 1 (it is also smooth and is therefore  $\text{Lip} - (n + \varepsilon)$  on any bounded set). As  $(g \circ f)^1 = \psi \circ \varphi$  then  $((g \circ f)^1, \dots, ((g \circ f)^1)^n)$  is  $\text{Lip} - (n + \varepsilon)$  and there exists a constant  $C_{n,\varepsilon}$  depending only on  $n$  and  $\varepsilon$  (using proposition 1.26) such that:

$$\|(g \circ f)^1\|_{\text{Lip}-(n+\varepsilon)} \leq C_{n,\varepsilon} \|g\|_{\text{Lip}-(n+\varepsilon+1)} \max(\|f\|_{\text{Lip}-(n+\varepsilon+1)}^{n+\varepsilon+1}, 1)$$

Now, we only have to identify, for every  $k \in \llbracket 1, n \rrbracket$ ,  $((g \circ f)^1)^k$  with  $(g \circ f)^{k+1}$ . Let  $k \in \llbracket 1, n \rrbracket$ . Let  $x \in U$  and  $v \in E^{\otimes k}$ . By the induction hypothesis, we have:

$$((g \circ f)^1)^k(x)(v) = \sum_{j=1}^k \frac{\psi^j(\varphi(x))}{j!} \sum_{\substack{1 \leq i_1, \dots, i_j \leq n \\ i_1 + \dots + i_j = k}} \frac{\varphi^{i_1}(x) \otimes \dots \otimes \varphi^{i_j}(x)}{i_1! \dots i_j!} \left( \sum_{\sigma \in \mathcal{S}_k} \sigma(v) \right)$$

Which, by studying the successive derivatives of  $\psi$  (which is straight-forward since it is a bilinear map) and  $\varphi$ , gives two simple formulas depending on whether  $k = 1$  or  $k > 1$ . For  $k = 1$ , it is an easy exercise to see that, for every  $v_1, v_2 \in E$ , we have:

$$(((g \circ f)^1)^1(x)(v_1))(v_2) = (g \circ f)^2(x)(v_1 \otimes v_2)$$

Assume then that  $k > 1$  and for  $v = v_1 \otimes \cdots \otimes v_k$ ,  $v_1, \dots, v_k \in E$ ,  $((g \circ f)^1)^k(x)(v)$  is the sum of four terms:

$$\begin{aligned} & (g^1 \circ f)(x) \circ (f^1)^k(x)(v) + (g^1 \circ f)^k(x)(v) \circ f^1(x) + \\ & \frac{1}{2} \sum_{\sigma \in \mathcal{S}_k} \sum_{i=1}^{k-1} \frac{1}{i!(k-i)!} \\ & ((g^1 \circ f)^i(x)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \circ (f^1)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)}) + \\ & (g^1 \circ f)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)}) \circ (f^1)^i(x)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)})) \end{aligned}$$

Let  $v_1, \dots, v_{k+1} \in E$ . We use the formula for the successive derivatives given by the induction hypothesis to simplify the four terms in the sum above:

First term:

$$(g^1 \circ f)(x) \circ (f^1)^k(x)(v_1 \otimes \cdots \otimes v_k)(v_{k+1}) = g^1(f(x))(f^{k+1}(x)(v_1 \otimes \cdots \otimes v_{k+1}))$$

Second term:

$$\begin{aligned} & ((g^1 \circ f)^k(x)(v_1 \otimes \cdots \otimes v_k) \circ f^1(x))(v_{k+1}) \\ & = \sum_{j=1}^k \frac{g^{j+1}(f(x))}{j!} \left( \sum_{\substack{1 \leq i_1, \dots, i_j \leq n \\ i_1 + \dots + i_j = k}} \frac{f^{i_1}(x) \otimes \cdots \otimes f^{i_j}(x) \otimes f^1(x)}{i_1! \cdots i_j!} \left( \sum_{\sigma \in \mathcal{S}_k} \sigma(v_1 \otimes \cdots \otimes v_k) \otimes v_{k+1} \right) \right) \end{aligned}$$

Third and fourth term: They are dealt with in the same manner and are actually equal. Consequently, we will admit below that they are indeed equal and show how to deal with the third term only. Let  $i \in \llbracket 1, k-1 \rrbracket$  and  $\sigma \in \mathcal{S}_k$ . Using the induction hypothesis for  $g^1 \circ f$ , we get that  $(g^1 \circ f)^i(x)(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)})$  is equal to:

$$\sum_{p=1}^i \frac{(g^1)^p(f(x))}{p!} \left( \sum_{\substack{1 \leq m_1, \dots, m_p \leq n \\ m_1 + \dots + m_p = i}} \frac{f^{m_1}(x) \otimes \cdots \otimes f^{m_p}(x)}{m_1! \cdots m_p!} \left( \sum_{\tau \in \mathcal{S}_{\sigma(1), \dots, \sigma(i)}} v_{\tau \circ \sigma(1)} \otimes \cdots \otimes v_{\tau \circ \sigma(i)} \right) \right)$$

As  $g^{p+1}(f(x))$  is symmetric, this equals, when composed with  $(f^1)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)})(v_{k+1})$ :

$$\begin{aligned} & \sum_{p=1}^i \frac{g^{p+1}(f(x))}{p!} \left( \sum_{\substack{1 \leq m_1, \dots, m_p \leq n \\ m_1 + \dots + m_p = i}} \frac{f^{m_1}(x) \otimes \cdots \otimes f^{m_p}(x) \otimes f^{k-i+1}(x)}{m_1! \cdots m_p!} \right. \\ & \left. \left( \sum_{\tau \in \mathcal{S}_{\sigma(1), \dots, \sigma(i)}} v_{\tau \circ \sigma(1)} \otimes \cdots \otimes v_{\tau \circ \sigma(i)} \otimes v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)} \otimes v_{k+1} \right) \right) \end{aligned}$$

As  $f^{k-i+1}(x)$  is symmetric, by using lemma 1.3, we get that the third term:

$$\sum_{\substack{1 \leq i \leq k-1 \\ \sigma \in \mathcal{S}_k}} \frac{(g^1 \circ f)^i(x)}{i!(k-i)!} (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}) \circ (f^1)^{k-i}(x)(v_{\sigma(i+1)} \otimes \cdots \otimes v_{\sigma(k)})(v_{k+1})$$

is equal to:

$$\sum_{p=1}^{k-1} \frac{g^{p+1}(f(x))}{p!} \sum_{i=p}^{k-1} \left( \sum_{\substack{1 \leq m_1, \dots, m_p \leq n \\ m_1 + \dots + m_p = i}} \frac{f^{m_1}(x) \otimes \dots \otimes f^{m_p}(x) \otimes f^{k-i+1}(x)}{m_1! \dots m_p!} \right. \\ \left. \left( \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{\tau \in \mathcal{S}_{r_1, \dots, r_i}} v_{\tau(r_1)} \otimes \dots \otimes v_{\tau(r_i)} \otimes v_{r_{i+1}} \otimes \dots \otimes v_{r_k} \otimes v_{k+1} \right) \right)$$

Now, the strategy is to compare the terms of the same degree in  $(g \circ f)^{k+1}(x)(v)$  and  $((g \circ f)^1)^k(x)(v)$  (to simplify, we call degree the integer  $i$  in the expression  $g^i(f(x))$ ).

The term of degree 1 in  $(g \circ f)^{k+1}(x)(v_1 \otimes \dots \otimes v_{k+1})$  is:

$$g^1(f(x)) \left( \frac{f^{k+1}(x)}{(k+1)!} \left( \sum_{\sigma \in \mathcal{S}_{k+1}} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k+1)} \right) \right)$$

and as  $f^{k+1}(x)$  is symmetric and  $\text{card}(\mathcal{S}_{k+1}) = (k+1)!$ , this is equal to the term of degree 1 in  $((g \circ f)^1)^k(x)(v_1 \otimes \dots \otimes v_k)(v_{k+1})$  (which corresponds to the first term here).

The term of degree  $k+1$  in  $(g \circ f)^{k+1}(x)(v_1 \otimes \dots \otimes v_{k+1})$  is:

$$\frac{g^{k+1}(f(x))}{(k+1)!} (f^1(x) \otimes \dots \otimes f^1(x)) \left( \sum_{\sigma \in \mathcal{S}_{k+1}} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k+1)} \right)$$

As  $g^{k+1}(f(x))$  is symmetric, then this term is equal to

$$g^{k+1}(f(x))(f^1(x) \otimes \dots \otimes f^1(x))(v_1 \otimes \dots \otimes v_{k+1})$$

and the same argument proves that is exactly the term of degree  $k+1$  in

$$((g \circ f)^1)^k(x)(v_1 \otimes \dots \otimes v_k)(v_{k+1})$$

(which only appears in the second term here).

Finally, let  $j \in \llbracket 1, k-1 \rrbracket$ . The term of degree  $j+1$  in the second and third term of  $(g \circ f)^{k+1}(x)(v_1 \otimes \dots \otimes v_{k+1})$  is the image under  $\frac{g^{j+1}(f(x))}{j!}$  of the compactly written sum:

$$\sum_{i=j}^k \left( \sum_{\substack{1 \leq m_1, \dots, m_j \leq n \\ m_1 + \dots + m_j = i}} \frac{f^{m_1}(x) \otimes \dots \otimes f^{m_j}(x) \otimes f^{k-i+1}(x)}{m_1! \dots m_j!} \right. \\ \left. \left( \sum_{1 \leq r_1 < \dots < r_i \leq k} \sum_{\tau \in \mathcal{S}_{r_1, \dots, r_i}} v_{\tau(r_1)} \otimes \dots \otimes v_{\tau(r_i)} \otimes v_{r_{i+1}} \otimes \dots \otimes v_{r_k} \otimes v_{k+1} \right) \right)$$

Using the symmetries of both  $g^{j+1}(f(x))$  and the  $f^i(x)$ 's, this can be shown to have the same image under  $g^{j+1}(f(x))$  as:

$$\frac{1}{j+1} \sum_{\substack{1 \leq m_1, \dots, m_{j+1} \leq n \\ m_1 + \dots + m_{j+1} = k+1}} \frac{f^{m_1}(x) \otimes \dots \otimes f^{m_{j+1}}(x)}{m_1! \dots m_{j+1}!} \left( \sum_{\sigma \in \mathcal{S}_{k+1}} (\sigma(v_1 \otimes \dots \otimes v_{k+1})) \right)$$

Which ends the proof.  $\square$

**Remark 1.34.** *We can obtain an easier proof for theorem 1.33 by using the extension theorem that will be introduced in subsection 1.5. The inequality will still prove hard to get and will involve a constant depending on the dimension of the spaces, an inconvenient that we don't have in the proof presented above.*



**1.4. A quantitative estimate.** In this section, we give some more precise local quantitative estimates (in the Lipschitz norm) if the value of a Lipschitz map at a point is known.

**Theorem 1.35.** *Let  $\gamma, \gamma' > 0$  such that  $\gamma' < \gamma$ . Let  $E$  and  $F$  be two normed vector spaces,  $U$  be a subset of  $E$  and  $x_0 \in U$ . Let  $f = (f^0, \dots, f^n)$  be a Lip  $-\gamma$  map on  $U$  with values in  $F$ . Assume that for all  $k \in \llbracket 0, n \rrbracket$  :  $f^k(x_0) = 0$ . We also assume that  $(E^{\otimes k})_{1 \leq k \leq n}$  are endowed with norms satisfying the projective property. Then for all  $\delta > 0$ , one has:*

$$\|f\|_{\text{Lip}-\gamma', B(x_0, \delta) \cap U} \leq \|f\|_{\text{Lip}-\gamma} \max \left( \delta^\gamma, \delta^{\gamma - \lfloor \gamma' \rfloor}, \delta^{\gamma - \gamma'} \left( \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{2^{j-\gamma'}}{(j - \lfloor \gamma' \rfloor)!} + 2^{\gamma - \gamma'} \right) \right)$$

*Proof.* Let  $n, n' \in \mathbb{N}$  and  $\varepsilon, \varepsilon' \in (0, 1]$  such that  $\gamma = n + \varepsilon$  and  $\gamma' = n' + \varepsilon'$ . Denote by  $(R_0, \dots, R_n)$  the remainders associated to  $f$ . Let  $\delta > 0$  and let  $k \in \llbracket 0, n' \rrbracket$ . Let  $x \in B(x_0, \delta) \cap U$  and  $v \in E^{\otimes k}$ . Then, as  $f$  is Lip  $-\gamma$  and that  $f^k(x_0) = 0$  for all  $k \in \llbracket 0, n \rrbracket$ , we get:

$$\begin{aligned} \|f^k(x)(v)\| &= \left\| \sum_k^n f^j(x_0) \left( \frac{v \otimes (x - x_0)^{\otimes (j-k)}}{(j-k)!} \right) + R_k(x, x_0)(v) \right\| \\ &= \|R_k(x, x_0)(v)\| \\ &\leq \|f\|_{\text{Lip}-\gamma} \|x - x_0\|^{\gamma-k} \|v\| \\ &\leq \|f\|_{\text{Lip}-\gamma} \delta^{\gamma-k} \|v\| \end{aligned}$$

Therefore,  $\sup_{x \in B(x_0, \delta) \cap U} \|f^k(x)\| \leq \|f\|_{\text{Lip}-\gamma} \delta^{\gamma-k}$ , for all  $k \in \llbracket 0, n' \rrbracket$ .

Let  $k \in \llbracket 0, n' \rrbracket$ . We define  $S_k : U \times U \rightarrow \mathcal{L}(E^{\otimes k}, F)$  (the new remainder) by:

$$S_k(x, y)(v) = f^k(x)(v) - \sum_{j=k}^{n'} f^j(y) \left( \frac{v \otimes (x - y)^{\otimes (j-k)}}{(j-k)!} \right)$$

Let  $x, y \in B(x_0, \delta) \cap U$  and  $v \in E^{\otimes k}$ . Writing the Taylor expansion of  $f$  as a Lip  $-\gamma$  map, we get the following identity:

$$S_k(x, y)(v) = \sum_{j=n'+1}^n f^j(y) \left( \frac{v \otimes (x - y)^{\otimes (j-k)}}{(j-k)!} \right) + R_k(x, y)(v)$$

Which, using our new upper-bound for  $\|f^k\|_{\infty, B(x_0, \delta) \cap U}$ , leads to the inequality:

$$\begin{aligned} \|S_k(x, y)(v)\| &\leq \|f\|_{\text{Lip}-\gamma} \left( \sum_{j=n'+1}^n \delta^{\gamma-j} \frac{\|x-y\|^{j-k}}{(j-k)!} + \|x-y\|^{\gamma-k} \right) \|v\| \\ &\leq \|f\|_{\text{Lip}-\gamma} \|x-y\|^{\gamma'-k} \left( \sum_{j=n'+1}^n \delta^{\gamma-j} \frac{\|x-y\|^{j-\gamma'}}{(j-k)!} + \|x-y\|^{\gamma-\gamma'} \right) \|v\| \end{aligned}$$

Therefore:

$$\sup_{x, y \in B(x_0, \delta) \cap U} \|S_k(x, y)\| \leq \|f\|_{\text{Lip}-\gamma} \|x-y\|^{\gamma'-k} \delta^{\gamma-\gamma'} \left( \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{2^{j-\gamma'}}{(j - \lfloor \gamma' \rfloor)!} + 2^{\gamma-\gamma'} \right)$$

Which ends the proof.  $\square$

**Remark 1.36.**

$$\max \left( \delta^\gamma, \delta^{\gamma - \lfloor \gamma' \rfloor}, \delta^{\gamma - \gamma'} \left( \sum_{j=\lfloor \gamma' \rfloor + 1}^{\lfloor \gamma \rfloor} \frac{2^{j - \gamma'}}{(j - \lfloor \gamma' \rfloor)!} + 2^{\gamma - \gamma'} \right) \right) \leq (\delta^\gamma \vee \delta^{\gamma - \gamma'}) (e^2 - 1 + 2^{\gamma - \gamma'})$$

**Remark 1.37.** *With the notations of the previous theorem, if we only have  $f^k(x_0) = 0$  for  $k \in \llbracket 0, n' \rrbracket$ , the result remains essentially true but with a slightly different upper-bound (but that still converges to 0 as  $\delta$  goes to 0). However, in the cases where  $\gamma' = \gamma$  or if there exists  $k \in \llbracket 0, n' \rrbracket$  such that  $f^k(x_0) \neq 0$  then we cannot get a better control of  $\|f\|_{\text{Lip}-\gamma', B(x_0, \delta) \cap U}$  than  $\|f\|_{\text{Lip}-\gamma}$  (we can, nevertheless, improve the control of  $\|f^k\|_{\infty, B(x_0, \delta) \cap U}$  for all  $k \in \llbracket 0, n \rrbracket$  in the first case), as the example of the function  $x \mapsto x$  defined on an open bounded interval containing 0 shows in the cases ( $\gamma' = \gamma = 1$ ) or when ( $\gamma' \in (1, 2)$  and  $\gamma' < \gamma \leq 2$ ).*

Using theorem 1.35, one can easily then compare in the  $\text{Lip}-\gamma'$  norm two  $\text{Lip}-\gamma$  maps, when  $\gamma' < \gamma$ , which values and “successive derivatives” values (in the sense of a Lipschitz map) agree at one point.

**1.5. Extension theorems.** One of the most interesting and still open problems in Lipschitz geometry is about the existence of extensions of Lipschitz maps to the whole space and the control of the Lipschitz norm of the extension. We state below two examples of such results: one in which one can extend Lipschitz maps of any degree to the whole space, but at the cost of amplifying the Lipschitz norm; and another one where the extension has the same Lipschitz norm as the map we start with but which is currently only obtained for Lipschitz-1 maps in the framework of Hilbert spaces.

**Theorem 1.38** (Stein [12]). *Let  $\gamma \geq 1$ . Let  $E$  and  $F$  be two finite dimensional vector spaces and  $K$  be a closed subset of  $E$ . There exists continuous a linear map sending every  $F$ -valued  $\text{Lip}-\gamma$  map  $f$  defined on  $K$  to an  $F$ -valued  $\text{Lip}-\gamma$  map  $\tilde{f}$  defined on  $E$  such that  $\tilde{f}|_K = f$ . Moreover, the norm of the linear extension map depends only on  $\gamma$  and the dimensions of  $E$  and  $F$ .*

**Theorem 1.39** (Kirszbraun [8]). *Let  $H_1$  and  $H_2$  be two Hilbert spaces. Let  $A$  be a subset of  $H_1$ ,  $K \geq 0$  and  $f : A \rightarrow H_2$  be a map such that:*

$$\forall x, y \in A : \quad \|f(x) - f(y)\| \leq K \|x - y\|$$

*( $f$  is 1-Hölder). Then there exists a map  $\tilde{f} : H_1 \rightarrow H_2$  such that  $\tilde{f}|_A = f$  and:*

$$\forall x, y \in H_1 : \quad \|\tilde{f}(x) - \tilde{f}(y)\| \leq K \|x - y\|$$

*Moreover, if  $f$  is bounded ( $f$  is  $\text{Lip}-1$ ) then  $\tilde{f}$  can be chosen to be bounded and such that  $\sup_A \|f\| = \sup_{H_1} \|\tilde{f}\|$ .*

It is also of importance to mention the more general problems of extension and approximation known as the Whitney’s extension problem. Among numerous articles that deal with these questions from different angles, let us mention in particular Whitney’s own works in [13], [14] and [15] and Fefferman’s in [4], [5] and [6].

## 2. CONSTANT RANK THEOREMS FOR LIPSCHITZ MAPS

The two versions of the constant rank theorem in this section and the related techniques are classical in the case of smooth maps and the literature is abundant in this matter (see for example [9]). As the reader may notice, we will only be assuming that the derivatives are Lipschitz (instead of the maps themselves) as this is a less demanding requirement to get our quantitative estimates.

We will be working in the finite-dimensional case and will assume that the norms on tensor spaces satisfy all the norm properties presented in the previous section. Norms of continuous linear maps are computed as subordinate norms.

**2.1. The inverse function theorem.** We will need the following well-known lemma:

**Lemma 2.1.** *Let  $E$  and  $F$  be two normed vector spaces. Let  $U$  and  $V$  be two open subsets of  $E$  and  $F$  respectively and let  $\varphi : U \rightarrow V$  be a homeomorphism. Let  $x \in U$  and assume that  $\varphi$  is differentiable at  $x$  and that  $d\varphi(x)$  is invertible and continuous. Then  $\varphi^{-1}$  is differentiable at  $\varphi(x)$  and  $d\varphi^{-1}(\varphi(x)) = (d\varphi(x))^{-1}$ .*

When working with Lipschitz maps, one can quantify the size of the domain on which a map stays of maximal rank around a point where one knows it is of maximal rank as follows:

**Lemma 2.2.** *Let  $\gamma \geq 1$ . Let  $E$ ,  $F$  and  $G$  be normed vector spaces. Let  $U$  be a subset of  $E$  and let  $f : U \rightarrow \mathcal{L}(F, G)$  be a  $Lip\text{-}\gamma$ . Let  $x_0 \in U$  and  $M_1, M_2 \in (0, \infty)$  and assume that  $\|f\|_{Lip\text{-}\gamma} \leq M_1$ :*

(1) *There exists  $\delta > 0$  depending only on  $\gamma$  and  $M_1 M_2$  such that:*

$$\forall x \in \overline{B(x_0, \delta)} \cap U : \|f(x) - f(x_0)\| \leq \frac{1}{2M_2}$$

*In particular, if  $f(x_0)$  is invertible and  $\|f(x_0)^{-1}\| \leq M_2$ , then  $f$  is invertible on  $\overline{B(x_0, \delta)} \cap U$ .*

(2) *Assume that  $F$  and  $G$  are finite dimensional of dimension  $p$  and  $m$  respectively, that  $f$  is of rank less than or equal to  $k \in \mathbb{N}$  and that  $f(x_0)$  is of maximal rank  $k$ . Identify  $f$  with a matrix of functions  $(f_{i,j})_{(i,j) \in \llbracket 1, m \rrbracket \times \llbracket 1, p \rrbracket}$ . Let  $(i_1, \dots, i_k)$  and  $(j_1, \dots, j_k)$  be, respectively, strictly ordered subsets of  $\llbracket 1, m \rrbracket$  and  $\llbracket 1, p \rrbracket$  such that  $M = (f_{i_r, j_l}(x_0))_{1 \leq r, l \leq k}$  is invertible. We assume that  $\|M^{-1}\| \leq M_2$ . Then there exists  $\delta > 0$  that depends only on  $\gamma$  and  $M_1 M_2$  such that, for all  $x \in \overline{B(x_0, \delta)} \cap U$ ,  $f(x)$  is of rank  $k$ .*

*Proof.* (1) Let  $n \in \mathbb{N}$  and  $\varepsilon \in (0, 1]$  such that  $\gamma = n + \varepsilon$ . Let  $R : U \times U \rightarrow \mathcal{L}(F, G)$  be such that for all  $x, y \in U$ :

$$f(x) = f(y) + \sum_{k=1}^n f^k(y) \left( \frac{(x-y)^{\otimes k}}{k!} \right) + R(x, y)$$

and

$$\|R(x, y)\| \leq M_1 \|x - y\|^{n+\varepsilon}$$

We get from the above that, for  $x \in U$ :

$$\|f(x) - f(x_0)\| \leq M_1 \left( \sum_{k=1}^n \frac{\|x - x_0\|^k}{k!} + \|x - x_0\|^{n+\varepsilon} \right)$$

It suffices to choose  $\delta$  such that:

$$\forall 0 \leq t \leq \delta : \sum_{k=1}^n \frac{t^k}{k!} + t^{n+\varepsilon} \leq \frac{1}{2M_1M_2}$$

Which proves the claim.

- (2) The previous result insures that we can find  $\delta > 0$  that depends only on  $\gamma$  and  $M_1M_2$  such that the square matrix  $(f_{i_r, j_l}(x))_{1 \leq r, l \leq k}$  is invertible for all  $x \in \overline{B(x_0, \delta)} \cap U$ . Therefore the rank of  $(f_{i,j}(x))_{(i,j) \in [1, m] \times [1, p]}$  is larger than or equal to  $k$  on  $\overline{B(x_0, \delta)} \cap U$ . Since the rank of  $f$  is always less than or equal to  $k$ , then  $f(x)$  is necessarily of rank  $k$  on  $\overline{B(x_0, \delta)} \cap U$ . □

**Definition 2.3.** Let  $\gamma > 0$ . Let  $E$  and  $F$  be two normed vector spaces,  $U$  be a subset of  $E$  and  $V$  a subset of  $F$ . A map  $f : U \rightarrow V$  is said to be a Lipschitz diffeomorphism of degree  $\gamma$  (a *Lip* -  $\gamma$  diffeomorphism in short) if  $f$  is *Lip* -  $\gamma$  and bijective and  $f^{-1}$  is also *Lip* -  $\gamma$ .

**Theorem 2.4** (Inverse Function). Let  $n \in \mathbb{N}^*$  and  $\varepsilon \in (0, 1]$ . Let  $U$  be an open subset of a Banach space  $E$  and let  $\varphi : U \rightarrow E$  be a differentiable map such that  $d\varphi$  is *Lip* -  $(n + \varepsilon - 1)$ . Let  $x_0 \in U$  and assume that  $d\varphi(x_0)$  is invertible. Then, for every  $M_1 > 0$  and  $M_2 > 0$  such that  $\|d\varphi\|_{\text{Lip}-(n+\varepsilon-1)} \leq M_1$  and  $\|d\varphi(x_0)^{-1}\| \leq M_2$ , there exists a constant  $\delta$ , depending only on  $n, \varepsilon$  and  $M_1M_2$ , such that for every  $\alpha \in (0, \delta]$  satisfying  $\overline{B(x_0, \alpha)} \subseteq U$  there exists a constant  $c$  depending only on  $n, \varepsilon, \|x_0\|, M_1$  and  $M_2$  such that:

- $\varphi : B(x_0, \alpha) \cap \varphi^{-1}(V_0) \rightarrow V_0$  is a *Lip* -  $(n + \varepsilon)$ -diffeomorphism, where  $V_0 = \varphi(x_0) + d\varphi(x_0)(B(0, \alpha/2))$ .
- $B(x_0, \alpha/3) \subseteq B(x_0, \alpha) \cap \varphi^{-1}(V_0)$ .
- $\|\varphi_{|V_0}^{-1}\|_{\text{Lip}-(n+\varepsilon)} \leq c$ .

*Proof.* Let  $M_1, M_2 > 0$  and assume that  $\|d\varphi\|_{\text{Lip}-(n+\varepsilon-1)} \leq M_1$  and  $\|d\varphi(x_0)^{-1}\| \leq M_2$ . We start by proving that, on a well-chosen bounded subset of  $U$ ,  $\varphi$  is injective and satisfies some key inequalities. As  $d\varphi$  is Lipschitz, then by lemma 2.2, we can find  $\delta > 0$  depending only on  $n + \varepsilon$  and  $M_1M_2$  such that:

$$\forall x \in \overline{B(x_0, \delta)} \cap U : \|d\varphi(x) - d\varphi(x_0)\| \leq \frac{1}{2M_2} \left( \leq \frac{1}{2\|d\varphi(x_0)^{-1}\|} \right)$$

Let  $0 < \alpha \leq \delta$  be such that  $\overline{B(x_0, \alpha)} \subseteq U$ . For all  $x, \tilde{x} \in \overline{B(x_0, \alpha)}$ , we have then

$$(1) \quad \|d\varphi(x_0)^{-1} \circ d\varphi(x) - \text{Id}\| \leq \frac{1}{2}$$

and consequently

$$(2) \quad \|d\varphi(x_0)^{-1}(\varphi(x) - \varphi(\tilde{x})) - (x - \tilde{x})\| \leq \frac{1}{2}\|x - \tilde{x}\|$$

$$(3) \quad \frac{1}{2}\|x - \tilde{x}\| \leq \|d\varphi(x_0)^{-1}(\varphi(x) - \varphi(\tilde{x}))\| \leq \frac{3}{2}\|x - \tilde{x}\|$$

Thus,  $\varphi$  is injective on  $\overline{B(x_0, \alpha)}$ .

Denote  $V_0 = \varphi(x_0) + d\varphi(x_0)B(0, \alpha/2)$  and let  $y \in V_0$ . We prove now that there exists a unique point  $x \in B(x_0, \alpha)$  such that  $\varphi(x) = y$ :

Let  $G$  be the map:

$$G : \begin{array}{ccc} \overline{B(x_0, \alpha)} & \longrightarrow & E \\ x & \longmapsto & x + d\varphi(x_0)^{-1}(y - \varphi(x)) \end{array}$$

Note that  $x \in \overline{B(x_0, \alpha)}$  is a fixed point of  $G$  if and only if  $\varphi(x) = y$ . Let  $x \in \overline{B(x_0, \alpha)}$ , then by (2):

$$\|G(x) - x_0\| \leq \|d\varphi(x_0)^{-1}(y - \varphi(x_0))\| + \|x - x_0 + d\varphi(x_0)^{-1}(\varphi(x) - \varphi(x_0))\| < \alpha$$

Therefore  $G(\overline{B(x_0, \alpha)}) \subseteq \overline{B(x_0, \alpha)}$  and by (2), we conclude that  $G$  is a contraction. It has therefore a unique fixed point, denote it by  $\tilde{x}$ . Then we have:

$$\|\tilde{x} - x_0\| = \|G(\tilde{x}) - x_0\| < \alpha$$

Hence  $\tilde{x} \in B(x_0, \alpha)$ . Which proves the claim.

We show now that  $\varphi$  is a homeomorphism when restricted to a specific domain.  $\varphi : B(x_0, \alpha) \cap \varphi^{-1}(V_0) \rightarrow V_0$  is continuous and bijective. Therefore,  $\varphi^{-1} : V_0 \rightarrow B(x_0, \alpha) \cap \varphi^{-1}(V_0)$  exists and is 1-Hölder by (3). Hence  $\varphi$  is a homeomorphism (from  $B(x_0, \alpha) \cap \varphi^{-1}(V_0)$  onto  $V_0$ ).

Note that the inequality (3) shows that  $B(x_0, \alpha/3) \subseteq B(x_0, \alpha) \cap \varphi^{-1}(V_0)$ .

Now lemma 2.1 (together with the inequality (1)) shows that  $\varphi^{-1}$  is differentiable at every point of  $V_0$  and that for every  $y \in V_0$ ,  $d\varphi^{-1}(y) = (d\varphi(\varphi^{-1}(y)))^{-1}$ . We will show by induction that  $\varphi^{-1}$  is Lip  $-(n + \varepsilon)$ . More precisely, we will prove that for every  $k \in \llbracket 1, n \rrbracket$ ,  $\varphi^{-1}$  is Lip  $-(k + \varepsilon)$  and that there exists a constant  $H_k$  depending only on  $n, \varepsilon, M_1$  and  $M_2$  such that  $\|d\varphi^{-1}\|_{\text{Lip}-(k+\varepsilon-1)} \leq H_k$ . But let us first make some remarks:

- $V_0$  being open and convex, we can then use the criteria in lemma 1.17 to show that  $\varphi^{-1}$  is Lip  $-(n + \varepsilon)$ .
- $\|\varphi^{-1}\|_{\infty, V_0} \leq \alpha + \|x_0\| \leq \delta + \|x_0\|$ .
- If we denote by  $i$  the inversion map on  $\overline{B_{\mathcal{L}(E)}(d\varphi(x_0), \frac{1}{2M_2})}$  (which is a smooth map and thus Lipschitz),  $d\varphi^{-1}$  can then be seen as the composition map of  $\varphi^{-1}$ ,  $d\varphi$  and  $i$ :

$$d\varphi^{-1} : V_0 \xrightarrow{\varphi^{-1}} B(x_0, \alpha) \xrightarrow{d\varphi} \overline{B_{\mathcal{L}(E)}(d\varphi(x_0), \frac{1}{2M_2})} \xrightarrow{i} \mathcal{L}(E)$$

For  $\gamma > 0$ , let  $C_\gamma$  denote the Lip  $-(\gamma)$  norm of  $i$ .

We start now our induction. For  $k = 1$ , we know that  $\varphi^{-1}$  is bounded. Let  $y, \tilde{y} \in V_0$ , then we have, using that  $d\varphi$  is  $\varepsilon$ -Hölder and that  $\varphi^{-1}$  is 1-Hölder:

$$\begin{aligned} \|d\varphi^{-1}(y) - d\varphi^{-1}(\tilde{y})\| &\leq \|i(d\varphi(\varphi^{-1}(y))) - i(d\varphi(\varphi^{-1}(\tilde{y})))\| \\ &\leq C_1 \|d\varphi(\varphi^{-1}(y)) - d\varphi(\varphi^{-1}(\tilde{y}))\| \\ &\leq C_1 \|d\varphi\|_{\text{Lip}-\varepsilon} \|\varphi^{-1}(y) - \varphi^{-1}(\tilde{y})\|^\varepsilon \\ &\leq (2M_2)^\varepsilon C_1 \|d\varphi\|_{\text{Lip}-\varepsilon} \|y - \tilde{y}\|^\varepsilon \end{aligned}$$

Hence,  $d\varphi^{-1}$  is  $\varepsilon$ -Hölder. Written as a composition map, we see that  $d\varphi^{-1}$  is bounded (by  $C_1$ ). Consequently,  $\varphi^{-1}$  is Lip  $-(1 + \varepsilon)$ . Following theorem 1.15, let  $m > 0$  be constant dependent only on  $n$  and  $\varepsilon$ , such that  $\|d\varphi\|_{\text{Lip}-\varepsilon} \leq m \|d\varphi\|_{\text{Lip}-(n+\varepsilon-1)}$ . Then:

$$\|d\varphi^{-1}\|_{\text{Lip}-\varepsilon} \leq H_1, \text{ where } H_1 = C_1 \max(1, (2M_2)^\varepsilon m M_1)$$

Let  $k \in \llbracket 1, n-1 \rrbracket$ . We assume that  $\varphi^{-1}$  is  $\text{Lip}-(k+\varepsilon)$  and that there exists a constant  $H_k$  depending only on  $n, \varepsilon, M_1$  and  $M_2$  such that:

$$\|\text{d}\varphi^{-1}\|_{\text{Lip}-(k+\varepsilon-1)} \leq H_k$$

As  $\text{d}\varphi^{-1} = i \circ \text{d}\varphi \circ \varphi^{-1}$  and  $i, \text{d}\varphi$  and  $\varphi^{-1}$  are all  $\text{Lip}-(k+\varepsilon)$ , then, by lemma 1.33,  $\text{d}\varphi^{-1}$  is  $\text{Lip}-(k+\varepsilon)$  with a Lipschitz norm less than a constant  $H_{k+1}$  depending only on  $k+\varepsilon, C_{k+\varepsilon}$  (which depends only on  $k, \varepsilon, M_1$  and  $M_2$ ),  $\|\text{d}\varphi\|_{\text{Lip}-(k+\varepsilon)}$  (which can be controlled using only  $M_1, k, n$  and  $\varepsilon$  by corollary 1.15) and  $\|\varphi^{-1}\|_{\text{Lip}-(k+\varepsilon)}$  (which is less than  $H_k \vee (\delta + \|x_0\|)$ );  $\delta$ , we recall, depends only on  $n, \varepsilon$  and  $M_1 M_2$ ). Which ends the induction. Consequently,  $\varphi : B(x_0, \alpha) \cap \varphi^{-1}(V_0) \rightarrow V_0$  is a  $\text{Lip}-(n+\varepsilon)$  diffeomorphism.  $\square$

## 2.2. The constant rank theorem.

**Definition 2.5** (Local Inverse). *Let  $E$  and  $F$  be two topological spaces. Let  $U$  be a subset of  $E$  and  $\varphi : U \rightarrow F$  be a map. We say that an  $E$ -valued map  $\hat{\varphi}$  defined on a subset of  $F$  containing  $\varphi(U)$  is a local inverse of  $\varphi$  on  $U$  if  $\hat{\varphi} \circ \varphi|_U = \text{Id}_U$ .*

**Definition 2.6** (Immersion). *Let  $E$  and  $F$  be two topological vector spaces. Let  $U$  be a subset of  $E$  and  $\varphi : U \rightarrow F$  be a differentiable map. We say that  $\varphi$  is an immersion if, for every  $x \in U$ ,  $\text{d}\varphi(x)$  is injective.*

In the following theorem, we will use the  $l^\infty$  norms. The statement of the theorem and the subsequent proof adapt easily in the case of other norms.

**Theorem 2.7** (Constant Rank). *Let  $(n, p, q, k) \in (\mathbb{N}^*)^3 \times \mathbb{N}$ ,  $\varepsilon \in (0, 1]$  and  $M_1$  and  $M_2$  be two positive real numbers. Let  $U$  be an open subset of  $\mathbb{R}^p$ . Let  $\varphi = (\varphi_1, \dots, \varphi_q) : U \rightarrow \mathbb{R}^q$  be a differentiable map of constant rank  $k$  such that  $\text{d}\varphi$  is  $\text{Lip}-(n+\varepsilon-1)$ . Let  $x_0 \in U$ ,  $(i_1, \dots, i_k)$  and  $(j_1, \dots, j_k)$  be, respectively, strictly ordered subsets of  $\llbracket 1, q \rrbracket$  and  $\llbracket 1, p \rrbracket$  such that  $M = (\frac{\partial \varphi_{i_r}}{\partial x_{j_l}}(x_0))_{1 \leq r, l \leq k}$  is invertible. We assume that  $\|\text{d}\varphi\|_{\text{Lip}-(n+\varepsilon-1)} \leq M_1$  and  $\|M^{-1}\| \leq M_2$ . Then, there exist two constants  $\delta$  and  $c$ , depending only on  $n, \varepsilon, M_1$  and  $M_2$  such that for every  $\alpha \in (0, \delta]$  such that  $\overline{B(x_0, \alpha)} \subseteq U$ , we have:*

- A  $\text{Lip}-(n+\varepsilon)$  diffeomorphism  $f : U_0 \rightarrow H$  centered at  $x_0$ , where  $U_0$  and  $H$  are two open subsets of  $\mathbb{R}^p$  and  $B(x_0, \alpha/3) \subseteq U_0 \subseteq U$ .
- A diffeomorphism  $g$  defined on an open subset  $W$  of  $\mathbb{R}^q$  centered at  $\varphi(x_0)$  and containing  $\varphi(U_0)$  such that  $\text{d}g$  is  $\text{Lip}-(n+\varepsilon-1)$  and the restriction of  $g$  to  $\varphi(U_0)$  or to any bounded set of  $W$  is  $\text{Lip}-(n+\varepsilon)$ .

such that, for all  $(x_1, \dots, x_p) \in H$ :

$$g \circ \varphi \circ f^{-1}(x_1, \dots, x_p) = (x_1, \dots, x_k, 0, \dots, 0)$$

Moreover, if  $k = p$ , then  $\varphi|_{U_0}$  is an injective immersion and admits a local inverse  $\hat{\varphi}$  on  $U_0$  that is  $\text{Lip}-(n+\varepsilon)$  and such that

$$\|\text{d}\hat{\varphi}\|_{\text{Lip}-(n+\varepsilon-1)} \leq C_{n+\varepsilon} c \|\text{d}\varphi\|_{\text{Lip}-(n+\varepsilon-1)}$$

Where  $C_{n+\varepsilon}$  is the constant specified in theorem (1.33).

$$\begin{array}{ccc} (B(x_0, \alpha/3) \subseteq) U_0 (\subseteq \mathbb{R}^p) & \xrightarrow{\varphi} & (\varphi(U_0) \subseteq) W (\subseteq \mathbb{R}^q) \\ f \downarrow & & \downarrow g \\ H & \xrightarrow{\pi} & g(W) \end{array}$$

*Proof.* We start first by two changes of variables that will enable us later to see  $\varphi$  almost as a projection of the first  $k$  variables. We will indentify  $\mathbb{R}^p$  (resp.  $\mathbb{R}^q$ ) with  $\mathbb{R}^k \oplus \mathbb{R}^{p-k}$  (resp.  $\mathbb{R}^k \oplus \mathbb{R}^{q-k}$ ). For  $x \in \mathbb{R}^p$ , we denote by  $\overline{(x_{j_l})_{1 \leq l \leq k}}$  the image of  $x$  by the projection onto  $\mathbb{R}^{p-k}$  which kernel is the span of  $\overline{((e_{j_l})_{1 \leq l \leq k})}$ , where  $(e_i)_{1 \leq i \leq p}$  is the canonical basis of  $\mathbb{R}^p$ . We define in a similar way the vector  $\overline{(z_{i_r})_{1 \leq r \leq k}}$  for  $z \in \mathbb{R}^q$ . Now, let  $f_1$  and  $g_1$  be the two following diffeomorphisms:

$$\begin{aligned} f_1 : U &\rightarrow \tilde{U} (= f_1(U)) \\ x &\mapsto \overline{((x_{j_l})_{1 \leq l \leq k}, (x_{j_l})_{1 \leq l \leq k})} \end{aligned}$$

and:

$$\begin{aligned} g_1 : \mathbb{R}^q &\rightarrow \mathbb{R}^q \\ z &\mapsto \overline{((z_{i_r})_{1 \leq r \leq k}, \overline{(z_{i_r})_{1 \leq r \leq k}})} \end{aligned}$$

We also define  $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_q) := g_1 \circ \varphi \circ f_1^{-1}$ ,  $A = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_k)$  and  $B = (\tilde{\varphi}_{k+1}, \dots, \tilde{\varphi}_q)$ . Then,  $d\tilde{\varphi}$ ,  $dA$  and  $dB$  are Lip  $-(n + \varepsilon - 1)$  and we have:

$$\max(\|dA\|_{\text{Lip}-(n+\varepsilon-1)}, \|dB\|_{\text{Lip}-(n+\varepsilon-1)}) = \|d\tilde{\varphi}\|_{\text{Lip}-(n+\varepsilon-1)} = \|d\varphi\|_{\text{Lip}-(n+\varepsilon-1)}$$

Let  $f_2$  be the map defined on  $\tilde{U} (\subseteq \mathbb{R}^k \oplus \mathbb{R}^{p-k})$  by:

$$f_2(a, b) = (A(a, b) - A(f_1(x_0)), b - ((f_1(x_0))_j)_{k-p \leq j \leq p})$$

Then  $f_2$  is differentiable at every point of  $\tilde{U}$  and  $df_2$  is Lip  $-(n + \varepsilon - 1)$  with:

$$\|df_2\|_{\text{Lip}-(n+\varepsilon-1)} \leq \max(1, \|d\varphi\|_{\text{Lip}-(n+\varepsilon-1)})$$

The representation matrix of  $df_2(f_1(x_0))$  in the canonical basis of  $\mathbb{R}^p$  is under the form:

$$\begin{pmatrix} M & \tilde{M} \\ (0) & I_{p-k} \end{pmatrix}$$

Where  $\tilde{M}$  is some matrix in  $\mathcal{M}_{k, p-k}$ . Hence  $df_2(f_1(x_0))$  is invertible and

$$\begin{aligned} \|df_2(f_1(x_0))^{-1}\| &\leq \max(1, \|M^{-1}\|, \|M^{-1}\tilde{M}\|) \\ &\leq \max(1, \|M^{-1}\| \max(1, C_{p,q} \|d\varphi\|_{\text{Lip}-(n+\varepsilon-1)})) \end{aligned}$$

Where  $C_{p,q}$  is an integer depending only on  $p$  and  $q$ . Using theorem 2.4, let  $\delta$  (resp.  $c$ ) be a constant depending on  $n, \varepsilon, M_1$  and  $M_2$  (resp. depending on  $n, \varepsilon, M_1, M_2$  and  $\|f_1(x_0)\|$ ) such that, for every  $\alpha \in (0, \delta]$  such that  $\overline{B(f_1(x_0), \alpha)} \subseteq \tilde{U}$ , the map:

$$f_2 : B(f_1(x_0), \alpha) \cap f_2^{-1}(H) \rightarrow H$$

is a Lip  $-(n + \varepsilon)$  diffeomorphism, where:

$$H = df_2(f_1(x_0))(B(0, \alpha/2))$$

(we drop again the restriction signs  $\cdot|_H$ ) and:

$$\|f_2^{-1}\|_{\text{Lip}-(n+\varepsilon)} \leq c$$

We also have  $B(f_1(x_0), \alpha/3) \subseteq B(f_1(x_0), \alpha) \cap f_2^{-1}(H)$ . We can define  $U_0 = f_1^{-1}(B(f_1(x_0), \alpha) \cap f_2^{-1}(H))$ .

We prove now that  $\tilde{\varphi} \circ f_2^{-1}$  is independent of the second variable (when identifying  $\mathbb{R}^p$  with  $\mathbb{R}^k \oplus \mathbb{R}^{p-k}$ ). Write  $f_2^{-1}$  under the form:

$$f_2^{-1}(y_1, y_2) = (C(y_1, y_2), D(y_1, y_2))$$

Then the identity  $f_2 \circ f_2^{-1} = \text{Id}$  shows that for every  $(y_1, y_2) \in H$ :

$$A(C(y_1, y_2), D(y_1, y_2)) - A(f_1(x_0)) = y_1$$

which allows writing  $\tilde{\varphi} \circ f_2^{-1}$  under the form:

$$\tilde{\varphi} \circ f_2^{-1}(y_1, y_2) = (y_1 + A(f_1(x_0)), B(f_2^{-1}(y_1, y_2)))$$

For lighter expressions, we define  $\tilde{B}$  on  $H$  by  $\tilde{B} = B \circ f_2^{-1}$ . As  $d\tilde{\varphi}$  is of rank  $k$ ,  $d(\tilde{\varphi} \circ f_2^{-1})$  is of rank at most  $k$  on  $H$ . For  $(y_1, y_2) \in H$ , the representation matrix of  $d(\tilde{\varphi} \circ f_2^{-1})(y_1, y_2)$  in the canonical bases of  $\mathbb{R}^p$  and  $\mathbb{R}^q$  is under the form:

$$\begin{pmatrix} I_k & (0) \\ \frac{\partial \tilde{B}}{\partial y_1}(y_1, y_2) & \frac{\partial \tilde{B}}{\partial y_2}(y_1, y_2) \end{pmatrix}$$

As this matrix is of order  $k$ , then,  $\frac{\partial \tilde{B}}{\partial y_2}(y_1, y_2) = 0$ . Therefore, we see that  $\tilde{\varphi} \circ f_2^{-1}$  is independent of the second variable. Define then  $F$  on  $\pi_{\mathbb{R}^k}(H)$  by  $F(a) = \tilde{B}(a, b_a)$ , where for  $a \in \pi_{\mathbb{R}^k}(H)$ ,  $b_a$  is any element such that  $(a, b_a) \in H$ , and  $\pi_{\mathbb{R}^k} : \mathbb{R}^k \oplus \mathbb{R}^{p-k} \rightarrow \mathbb{R}^k$  is the projection in the first variable. We have then, for all  $(y_1, y_2) \in H$ :

$$\tilde{\varphi} \circ f_2^{-1}(y_1, y_2) = (y_1 + A(f_1(x_0)), F(y_1))$$

We end this proof by defining a final diffeomorphism. Define the open set:

$$\tilde{W} = \{(z_1, z_2) \in \mathbb{R}^q | z_1 - A(f_1(x_0)) \in \pi_{\mathbb{R}^k}(H)\}$$

Let  $g_2$  be the map defined on  $\tilde{W}$  by:

$$g_2(z_1, z_2) = (z_1 - A(f_1(x_0)), z_2 - F(z_1 - A(f_1(x_0))))$$

$dg_2$  is clearly  $\text{Lip} - (n + \varepsilon - 1)$  and  $g_2$  is a  $\text{Lip} - (n + \varepsilon)$  diffeomorphism when restricted to a bounded set of  $\tilde{W}$  or to  $f_1(U_0)$  and for  $(y_1, y_2) \in H$ :

$$g_2 \circ \tilde{\varphi} \circ f_2^{-1}(y_1, y_2) = (y_1, 0)$$

By defining  $f = f_2 \circ f_1$ ,  $g = g_2 \circ g_1$  and  $W = g_1^{-1}(\tilde{W})$ , we get the claimed result.

Assume now that  $k = p$ . Then  $g \circ \varphi \circ f^{-1} = i_p$ , where

$$\begin{aligned} i_p : \mathbb{R}^p &\rightarrow \mathbb{R}^p \oplus \mathbb{R}^{q-p} \\ x &\mapsto (x, 0) \end{aligned}$$

Let  $\hat{\varphi}$  be the map  $f^{-1} \circ \pi_p \circ g$  defined on  $W$ , where  $\pi_p : \mathbb{R}^p \oplus \mathbb{R}^{q-p} \rightarrow \mathbb{R}^p$  is the projection on the first  $p$  variables. Then  $\hat{\varphi}$  is a local inverse of  $\varphi$  on  $U_0$  and a  $\text{Lip} - (n + \varepsilon)$  map on  $\varphi(U_0)$ . Writing  $\hat{\varphi}$  as the composition of the two maps  $f^{-1}$  and  $\pi_p \circ g$ , one gets the control

$$\|d\hat{\varphi}\|_{\text{Lip}-(n+\varepsilon-1)} \leq C_{n+\varepsilon} \|df^{-1}\|_{\text{Lip}-(n+\varepsilon-1)} \|df(x_0)\|$$

Where  $C_{n+\varepsilon}$  is the constant specified in theorem (1.33). □

### 3. FLOWS OF LIPSCHITZ VECTOR FIELDS

We give first a definition of the flow of a vector field and state the fundamental theorem of ordinary differential equations (O.D.E.s) which ensures the existence and uniqueness of flows:



**Definition 3.1.** Let  $I$  be an open interval. Let  $M$  be a  $C^1$ -manifold and  $A$  be a vector field on  $M$ . A  $C^1$ -path  $\gamma : I \rightarrow M$  is said to be an integral curve of  $A$  if:

$$\forall t \in I : \quad \gamma'(t) = A(\gamma(t))$$

If  $0 \in I$  and  $x = \gamma(0)$ , we say that  $x$  is the starting point of  $\gamma$ . If furthermore  $U$  is a subset of  $M$  and  $\tilde{A} : I \times U \rightarrow M$  is such that, for every  $x \in U$ ,  $t \mapsto \tilde{A}(t, x)$  is an integral curve of  $A$  starting at  $x$ , we say then that  $\tilde{A}$  is a local flow (or global flow if  $I \times U = \mathbb{R} \times M$ ) of  $A$  on  $I \times U$ .

**Notation.** Under the assumption of existence, we will be denoting the flow of a vector field  $A$  by  $\tilde{A}$ .

**Theorem 3.2.** Let  $d \in \mathbb{N}^*$  and let  $A$  be a Lip-1 vector field on  $\mathbb{R}^d$ . Let  $\xi \in \mathbb{R}^d$ . Then there exists a unique global flow of  $A$  on  $\mathbb{R} \times \mathbb{R}^d$ .

**Remark 3.3.** Note that theorem 3.2 is a special case of Picard-Lindelöf's theorem (see for example [2]) dealing with differential equations driven by paths of bounded variation.

In the following, we will need the comparison lemma (which is also used to prove the previous theorem).

**Lemma 3.4** (Comparison lemma). Let  $I$  be an open interval. Let  $d \in \mathbb{N}^*$  and  $u : I \rightarrow \mathbb{R}^d$  be a differentiable map such that there exists  $a \geq 0$  and  $b \geq 0$  such that:

$$\forall t \in I : \|u'(t)\| \leq a\|u(t)\| + b$$

Then, if  $t_0 \in I$ , we have:

$$\forall t \in I : \|u(t)\| \leq e^{a|t-t_0|}\|u(t_0)\| + \frac{b}{a}(e^{a|t-t_0|} - 1)$$

We start by proving some basic properties of flows:

**Lemma 3.5.** Let  $d \in \mathbb{N}^*$ . Let  $A$  be a Lip-1 vector field on  $\mathbb{R}^d$  and  $\tilde{A}$  its global flow. Then:

- $\forall t \in \mathbb{R}, \forall y, \tilde{y} \in \mathbb{R}^d : \|\tilde{A}(t, y) - \tilde{A}(t, \tilde{y})\| \leq e^{t\|A\|_{Lip-1}}\|y - \tilde{y}\|.$
- $\tilde{A}$  is locally 1-Hölder: for all  $t, \tilde{t} \in \mathbb{R}$  and  $y, \tilde{y} \in \mathbb{R}^d$ :

$$\|\tilde{A}(t, y) - \tilde{A}(\tilde{t}, \tilde{y})\| \leq e^{(|t| \wedge |\tilde{t}|)\|A\|_{Lip-1}}\|y - \tilde{y}\| + \|A\|_{\infty}|t - \tilde{t}|$$

- $\forall T, r \in \mathbb{R}_+^*, \forall x_0 \in \mathbb{R}^d : \tilde{A}((-T, T) \times B(x_0, r)) \subseteq B(x_0, r + T\|A\|_{\infty}).$

*Proof.* Let  $y, \tilde{y} \in \mathbb{R}^d$ . Define  $u$  on  $\mathbb{R}$  by the identity:  $u(t) = \tilde{A}(t, y) - \tilde{A}(t, \tilde{y})$ . Note that  $u(0) = y - \tilde{y}$ .  $u$  is differentiable and:

$$\forall t \in \mathbb{R} : u'(t) = A(\tilde{A}(t, y)) - A(\tilde{A}(t, \tilde{y}))$$

Therefore, for all  $t \in \mathbb{R}$ ,  $\|u'(t)\| \leq \|A\|_{Lip-1}\|u(t)\|$ . Hence, by the comparison lemma 3.4:

$$\forall t \in \mathbb{R} : \|\tilde{A}(t, y) - \tilde{A}(t, \tilde{y})\| = \|u(t)\| \leq e^{t\|A\|_{Lip-1}}\|y - \tilde{y}\|$$

Let  $t, \tilde{t} \in \mathbb{R}$  and assume that  $|t| \leq |\tilde{t}|$ :

$$\begin{aligned} \|\tilde{A}(t, y) - \tilde{A}(\tilde{t}, \tilde{y})\| &\leq \|\tilde{A}(t, y) - \tilde{A}(t, \tilde{y})\| + \|\tilde{A}(t, \tilde{y}) - \tilde{A}(\tilde{t}, \tilde{y})\| \\ &\leq e^{t\|A\|_{Lip-1}}\|y - \tilde{y}\| + \left\| \int_{\tilde{t}}^t A(\tilde{A}(u, \tilde{y})) du \right\| \\ &\leq e^{t\|A\|_{Lip-1}}\|y - \tilde{y}\| + \|A\|_{\infty}|t - \tilde{t}| \end{aligned}$$

Therefore,  $\tilde{A}$  is locally 1-Hölder continuous and, by taking  $(t, y) = (0, x_0)$  in the previous inequality, we see that  $\tilde{A}((-T, T) \times B(x_0, r)) \subseteq B(x_0, r + T\|A\|_{\infty})$ .  $\square$

We show now that the flows of differentiable vector fields are differentiable too:

**Lemma 3.6.** *Let  $0 < \varepsilon \leq 1$  and  $d \in \mathbb{N}^*$ . Let  $A$  be a Lip  $-(1 + \varepsilon)$  vector field on  $\mathbb{R}^d$  and  $\tilde{A}$  be its global flow. Then  $\tilde{A}$  is continuously differentiable and, if  $(\vec{e}_1, \dots, \vec{e}_d)$  is a basis for  $\mathbb{R}^d$ , then for all  $(t, y) \in \mathbb{R} \times \mathbb{R}^d$ :*

$$\|\partial_t \tilde{A}(t, y)\| \leq \|A\|_{\text{Lip}-(1+\varepsilon)}$$

and

$$\|\partial_{x_i} \tilde{A}(t, y)\| \leq e^{|t|\|A\|_{\text{Lip}-1}} \|\vec{e}_i\|$$

*Proof.* Let  $y \in \mathbb{R}^d$ . By definition of the flow:

$$\forall t \in \mathbb{R} : \quad \tilde{A}(t, y) = y + \int_0^t A(\tilde{A}(u, y)) du$$

As both  $A$  and  $\tilde{A}(\cdot, y)$  are continuous,  $t \mapsto \tilde{A}(t, y)$  is then continuously differentiable and, for all  $t \in \mathbb{R} : \partial_t \tilde{A}(t, y) = A(\tilde{A}(t, y))$ . Moreover:

$$(4) \quad \forall (t, y) \in \mathbb{R} \times \mathbb{R}^d : \|\partial_t \tilde{A}(t, y)\| \leq \|A\|_\infty$$

We will prove now that  $\tilde{A}$  is continuously differentiable in space. Let  $(\vec{e}_1, \dots, \vec{e}_d)$  be a basis for  $\mathbb{R}^d$  and  $T > 0$ . Let  $i \in [1, d]$ . For  $h \in \mathbb{R}^*$ , we define the map  $\Delta_h^i$  on  $(-T, T) \times \mathbb{R}^d$  by the relation:

$$\Delta_h^i(t, y) = \frac{\tilde{A}(t, y + h\vec{e}_i) - \tilde{A}(t, y)}{h}$$

We are going to show that the sequence  $(\Delta_h^i)_{|h|>0}$  converges uniformly on  $(-T, T) \times \mathbb{R}^d$  (as  $h$  goes to zero). Let  $h \in \mathbb{R}^*$ . For  $(t, y) \in (-T, T) \times \mathbb{R}^d$ , lemma 3.5 gives the inequality:

$$(5) \quad \|\Delta_h^i(t, y)\| \leq e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\|$$

$A$  being Lip  $-(1 + \varepsilon)$ , let  $R$  be a map defined on  $\mathbb{R}^d \times \mathbb{R}^d$  with values in  $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$  such that, for all  $a, b \in \mathbb{R}^d$ :

$$A(a) = A(b) + dA(b)(a - b) + R(a, b)$$

and

$$\|R(a, b)\| \leq \|A\|_{\text{Lip}-(1+\varepsilon)} \|a - b\|^{1+\varepsilon}$$

$\Delta_h^i$  is obviously continuously differentiable in time. Let  $(t, y) \in (-T, T) \times \mathbb{R}^d$ :

$$\begin{aligned} \partial_t \Delta_h^i(t, y) &= \frac{1}{h} (A(\tilde{A}(t, y + h\vec{e}_i)) - A(\tilde{A}(t, y))) \\ &= \frac{1}{h} (dA(\tilde{A}(t, y))(\tilde{A}(t, y + h\vec{e}_i) - \tilde{A}(t, y)) + \\ &\quad R(\tilde{A}(t, y + h\vec{e}_i), \tilde{A}(t, y))) \\ &= dA(\tilde{A}(t, y))(\Delta_h^i(t, y)) + \frac{1}{h} R(\tilde{A}(t, y + h\vec{e}_i), \tilde{A}(t, y)) \end{aligned}$$

Let  $\tilde{h} \in \mathbb{R}^*$ . From the calculation above and the inequality (5), we get that:

$$\begin{aligned} \|\partial_t \Delta_h^i(t, y) - \partial_t \Delta_{\tilde{h}}^i(t, y)\| &\leq \|dA(\tilde{A}(t, y))(\Delta_h^i(t, y) - \Delta_{\tilde{h}}^i(t, y))\| + \\ &\quad \left\| \frac{1}{h} R(\tilde{A}(t, y + h\vec{e}_i), \tilde{A}(t, y)) \right\| + \\ &\quad \left\| \frac{1}{\tilde{h}} R(\tilde{A}(t, y + \tilde{h}\vec{e}_i), \tilde{A}(t, y)) \right\| \\ &\leq \|A\|_{\text{Lip}-(1+\varepsilon)} (\|\Delta_h^i(t, y) - \Delta_{\tilde{h}}^i(t, y)\| + \\ &\quad |h|^\varepsilon \|\Delta_h^i(t, y)\|^{1+\varepsilon} + |\tilde{h}|^\varepsilon \|\Delta_{\tilde{h}}^i(t, y)\|^{1+\varepsilon}) \\ &\leq \|A\|_{\text{Lip}-(1+\varepsilon)} (\|\Delta_h^i(t, y) - \Delta_{\tilde{h}}^i(t, y)\| + \\ &\quad 2(|h| \vee |\tilde{h}|)^\varepsilon (e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\|)^{1+\varepsilon}) \end{aligned}$$

Therefore, using the comparison lemma and the fact that  $\Delta_h^i(0, y) = \Delta_{\tilde{h}}^i(0, y) = \vec{e}_i$ , we get the following inequality:

$$\|\Delta_h^i - \Delta_{\tilde{h}}^i\|_{\infty, (-T, T) \times \mathbb{R}^d} \leq 2(|h| \vee |\tilde{h}|)^\varepsilon (e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\|)^{1+\varepsilon} (e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1)$$

We therefore see that  $(\Delta_h^i)_{|h|>0}$  converges uniformly on  $(-T, T) \times \mathbb{R}^d$  and that  $\partial_{x_i} \tilde{A}$  exists (as its limit). As for every  $h$ ,  $\Delta_h^i$  is continuous,  $\partial_{x_i} \tilde{A}$  is also continuous. We also get the following inequality from (5):

$$\forall (t, y) \in (-T, T) \times \mathbb{R}^d : \quad \|\partial_{x_i} \tilde{A}(t, y)\| \leq e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\|$$

□

**Lemma 3.7.** *Let  $0 < \varepsilon \leq 1$ . Let  $d \in \mathbb{N}^*$ ,  $T > 0$ ,  $r > 0$  and  $x_0 \in \mathbb{R}^d$ . Let and  $A$  be a  $\text{Lip}-(1+\varepsilon)$  vector field on  $\mathbb{R}^d$  and  $\tilde{A}$  its global flow. Then  $d\tilde{A}$  is  $\varepsilon$ -Hölder on  $(-T, T) \times B(x_0, r)$  and there exists a constant  $C$  depending on  $\|A\|_{\text{Lip}-(1+\varepsilon)}$ ,  $\varepsilon$ ,  $T$  and  $r$  such that  $\|d\tilde{A}\|_{\text{Lip}-\varepsilon} \leq C$ .*

*Proof.* The lemma is trivial in the case where  $A = 0$ . Assume then that  $A \neq 0$ . In the following, we endow  $\mathbb{R} \times \mathbb{R}^d$  with the  $l^\infty$  norm that we will denote by  $N$ .

Let  $(t, y), (\tilde{t}, \tilde{y}) \in (-T, T) \times B(x_0, r)$ . Using the definition of a  $\text{Lip} - 1$  map and our preliminary study of  $\tilde{A}$ , we get the following inequality:

$$\begin{aligned} \|\partial_t \tilde{A}(t, y) - \partial_t \tilde{A}(\tilde{t}, \tilde{y})\| &= \|A(\tilde{A}(t, y)) - A(\tilde{A}(\tilde{t}, \tilde{y}))\| \\ &\leq \|A\|_{\text{Lip}-1} \|\tilde{A}(t, y) - \tilde{A}(\tilde{t}, \tilde{y})\| \\ &\leq \|A\|_{\text{Lip}-1} (e^{T\|A\|_{\text{Lip}-1}} \|y - \tilde{y}\| + \|A\|_\infty |t - \tilde{t}|) \\ &\leq \|A\|_{\text{Lip}-1} (e^{T\|A\|_{\text{Lip}-1}} (2r)^{1-\varepsilon} + \|A\|_\infty (2T)^{1-\varepsilon}) \\ &\quad N((t, y) - (\tilde{t}, \tilde{y}))^\varepsilon \end{aligned}$$

Hence,  $\partial_t \tilde{A}$  is  $\varepsilon$ -Hölder on  $(-T, T) \times B(x_0, r)$ .

Let  $(\vec{e}_1, \dots, \vec{e}_d)$  is a basis for  $\mathbb{R}^d$  and let  $i \in \llbracket 1, d \rrbracket$  and let  $(y, \tilde{y}) \in B(x_0, r)^2$ . Define the map  $v$  on  $(-T, T)$  by the identity  $v(t) = \partial_{x_i} \tilde{A}(t, y) - \partial_{x_i} \tilde{A}(t, \tilde{y})$ . Since  $\partial_{x_i} \tilde{A}$  satisfies the following differential equation:

$$\forall t \in (-T, T) : \quad \partial_{x_i} \tilde{A}(t, y) = \vec{e}_i + \int_0^t dA(\tilde{A}(u, y)) \partial_{x_i} \tilde{A}(u, y) du$$

$v$  is then continuously differentiable. Using the fact that  $A$  is  $\text{Lip} - (1 + \varepsilon)$ , the controls obtained in lemma 3.6, we get the following inequality, for all  $t \in (-T, T)$ :

$$\|v'(t)\| \leq \|A\|_{\text{Lip}-(1+\varepsilon)} e^{(1+\varepsilon)T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\| \|y - \tilde{y}\|^\varepsilon + \|A\|_{\text{Lip}-(1+\varepsilon)} \|v(t)\|$$

Using the comparison lemma, we then get, for all  $t \in (-T, T)$ :

$$\|\partial_{x_i} \tilde{A}(t, y) - \partial_{x_i} \tilde{A}(t, \tilde{y})\| \leq e^{(1+\varepsilon)T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\| \|y - \tilde{y}\|^\varepsilon (e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1)$$

Let  $(s, t) \in (-T, T)^2$ . Using, successively, the differential equation satisfied by  $\partial_{x_i} \tilde{A}(\cdot, \tilde{y})$ , the fact that  $A$  is  $\text{Lip} - (1 + \varepsilon)$  and finally the lemma 3.6, one gets:

$$\begin{aligned} \|\partial_{x_i} \tilde{A}(t, \tilde{y}) - \partial_{x_i} \tilde{A}(s, \tilde{y})\| &= \left\| \int_s^t (dA(\tilde{A}(u, \tilde{y})) \partial_{x_i} \tilde{A}(u, \tilde{y})) du \right\| \\ &\leq |t - s| \|A\|_{\text{Lip}-(1+\varepsilon)} \|\partial_{x_i} \tilde{A}(\cdot, \tilde{y})\|_{\infty, [s, t]} \\ &\leq |t - s| \|A\|_{\text{Lip}-(1+\varepsilon)} e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\| \end{aligned}$$

Finally, we get the inequality:

$$\begin{aligned}
\|\partial_{x_i} \tilde{A}(t, y) - \partial_{x_i} \tilde{A}(s, \tilde{y})\| &\leq \|\partial_{x_i} \tilde{A}(t, y) - \partial_{x_i} \tilde{A}(t, \tilde{y})\| + \|\partial_{x_i} \tilde{A}(t, \tilde{y}) - \partial_{x_i} \tilde{A}(s, \tilde{y})\| \\
&\leq e^{(1+\varepsilon)T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\| \|y - \tilde{y}\|^\varepsilon (e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1) + \\
&\quad |t - s| \|A\|_{\text{Lip}-(1+\varepsilon)} e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\| \\
&\leq e^{T\|A\|_{\text{Lip}-1}} \|\vec{e}_i\| (e^{\varepsilon T\|A\|_{\text{Lip}-1}} (e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1) + \\
&\quad (2T)^{1-\varepsilon} \|A\|_{\text{Lip}-(1+\varepsilon)}) N((t, y) - (s, \tilde{y}))^\varepsilon
\end{aligned}$$

Therefore,  $\partial_{x_i} \tilde{A}$  is  $\varepsilon$ -Hölder, which ends this part of the proof. Define the following constants:

$$\begin{cases} m_1 &= \|A\|_{\text{Lip}-1} (e^{T\|A\|_{\text{Lip}-1}} (2r)^{1-\varepsilon} + \|A\|_{\text{Lip}-(1+\varepsilon)} (2T)^{1-\varepsilon}) \\ m_2 &= e^{T\|A\|_{\text{Lip}-1}} (e^{\varepsilon T\|A\|_{\text{Lip}-1}} (e^{T\|A\|_{\text{Lip}-(1+\varepsilon)}} - 1) + (2T)^{1-\varepsilon} \|A\|_{\text{Lip}-(1+\varepsilon)}) \max_{1 \leq i \leq d} \|\vec{e}_i\| \\ m_3 &= \|A\|_{\text{Lip}-(1+\varepsilon)} \\ m_4 &= e^{T\|A\|_{\text{Lip}-1}} \max_{1 \leq i \leq d} \|\vec{e}_i\| \end{cases}$$

Then  $\|\text{d}\tilde{A}\|_{\text{Lip}-\varepsilon} \leq \max_{1 \leq i \leq 4} m_i$ . □

Finally, we show that flows of Lipschitz vector fields are also Lipschitz on bounded sets and have a well-controlled Lipschitz norm:

**Theorem 3.8.** *Let  $n, d \in \mathbb{N}^*$  and  $0 < \varepsilon \leq 1$ . Let  $A$  be a  $\text{Lip}-(n+\varepsilon)$  vector field on  $\mathbb{R}^d$ . Let  $x_0 \in \mathbb{R}^d$ ,  $T > 0$  and  $r > 0$  and  $\tilde{A}$  the local flow of  $A$  defined on  $(-T, T) \times B(x_0, r)$ . Then  $\tilde{A}$  is  $\text{Lip}-(n+\varepsilon)$  on  $(-T, T) \times B(x_0, r)$  and there exists a constant  $C$  depending only on  $T$ ,  $r$ ,  $\|x_0\|$ ,  $n$ ,  $\varepsilon$  and  $\|A\|_{\text{Lip}-(n+\varepsilon)}$  such that  $\|\tilde{A}\|_{\text{Lip}-(n+\varepsilon)} \leq C$ .*

*Proof.* We will prove the theorem by induction. The previous lemma states that if that  $A$  is  $\text{Lip}-(1+\varepsilon)$  then  $\tilde{A}$  is  $\text{Lip}-(1+\varepsilon)$  on  $(-T, T) \times B(x_0, r)$  and that the  $\text{Lip}-\varepsilon$  norm of its derivative can be controlled by a constant depending only on  $T$ ,  $r$ ,  $\varepsilon$  and  $\|A\|_{\text{Lip}-(1+\varepsilon)}$ .  $\|\tilde{A}\|_\infty$  is itself controlled by a constant depending on the same aforementioned variables and  $\|x_0\|$ .

Let  $n \in \mathbb{N}^*$ . Assume that the assertion is true for  $\text{Lip}-(n+\varepsilon)$  vector fields and let us prove it when  $A$  is  $\text{Lip}-(n+1+\varepsilon)$ . By the induction hypothesis, we know that  $\tilde{A}$  is  $\text{Lip}-(n+\varepsilon)$ . For  $(t, y) \in (-T, T) \times B(x_0, r)$ , we know that:

$$\partial_t \tilde{A}(t, y) = A(\tilde{A}(t, y))$$

Hence,  $\partial_t \tilde{A}$  is  $\text{Lip}-(n+\varepsilon)$  and  $\|\partial_t \tilde{A}\|_{\text{Lip}-(n+\varepsilon)}$  can be upper-bounded by a constant depending only on  $n$ ,  $\varepsilon$ ,  $\|A\|_{\text{Lip}-(n+1+\varepsilon)}$  and  $\|\tilde{A}\|_{\text{Lip}-(n+\varepsilon)}$  (the latter being less than  $\|\tilde{A}\|_\infty \vee \|\text{d}\tilde{A}\|_{\text{Lip}-(n+\varepsilon-1)}$ ).

Let  $Y$  be the vector field on  $\mathbb{R}^d \times \mathbb{R}^d$  defined by:

$$\forall a, b \in \mathbb{R}^d : \quad Y(a, b) = \begin{pmatrix} A(a) \\ \text{d}A(a)(b) \end{pmatrix}$$

It is then a simple exercise to show that, for all  $M > 0$ ,  $Y$  is a  $\text{Lip}-(n+\varepsilon)$  vector field on  $\mathbb{R}^d \times B(0, M)$  and to bound  $\|Y\|_{\text{Lip}-(n+\varepsilon)}$  with a constant depending only on  $n$ ,  $\varepsilon$  and  $\|A\|_{\text{Lip}-(n+1+\varepsilon)}$  and  $M$ . Moreover:

$$\|Y\|_{\infty, \mathbb{R}^d \times B(0, M)} \leq \|A\|_{\text{Lip}-(n+1+\varepsilon)} (1 \vee M)$$

Define  $m = \max_{1 \leq i \leq d} \|\vec{e}_i\|$  and take:

$$M = (1 \vee r \vee m \vee \|x_0\|)(1 + T\|A\|_{\text{Lip}-(n+1+\varepsilon)})$$

Let  $(\vec{e}_1, \dots, \vec{e}_d)$  be a basis for  $\mathbb{R}^d$ . Let  $i \in \llbracket 1, d \rrbracket$  and let  $\alpha$  be the local flow of  $Y$  on  $(-T, T) \times B((x_0, \vec{e}_i), r \vee m)$ . Then (lemma 3.5):

$$\begin{aligned} \alpha((-T, T) \times B((x_0, \vec{e}_i), r \vee m)) &\subseteq B((x_0, \vec{e}_i), r \vee m + T\|Y\|_{\infty, B((x_0, \vec{e}_i), r \vee m)}) \\ &\subseteq B(0, M) \end{aligned}$$

Using the induction hypothesis and proposition 1.10,  $\alpha$  is Lip  $-(n + \varepsilon)$  and  $\|\alpha\|_{\text{Lip}-(n+\varepsilon)}$  can be controlled with a constant depending only on  $\|A\|_{\text{Lip}-(n+\varepsilon+1)}$ ,  $T$ ,  $r$ ,  $\|x_0\|$ ,  $n$  and  $\varepsilon$ .

Using the uniqueness of solutions to differential equations with Lipschitz vector fields, we get, for every  $(t, y) \in (-T, T) \times B(x_0, r)$ :

$$\alpha(t, (y, \vec{e}_i)) = \begin{pmatrix} \tilde{A}(t, y) \\ \partial_{x_i} \tilde{A}(t, y) \end{pmatrix}$$

Therefore,  $\partial_{x_i} \tilde{A}$  is Lip  $-(n + \varepsilon)$  and that  $\|\partial_{x_i} \tilde{A}\|_{\text{Lip}-(n+\varepsilon)}$  can be upper-bounded by a constant depending only on  $\|A\|_{\text{Lip}-(n+\varepsilon+1)}$ ,  $n$ ,  $\varepsilon$ ,  $T$ ,  $r$  and  $\|x_0\|$ , which ends the proof.  $\square$

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