Characterization of Driver Nodes of Anti-Stable Networks

Ram Niwash Mahia
Indian Institute of Technology Jodhpur
Rajasthan, INDIA-342011
Email: ramyashu@iitj.ac.in

Deepak Fulwani
Indian Institute of Technology Jodhpur
Rajasthan, INDIA-342011
Email: df@iitj.ac.in

Mahaveer Singh
Indian Institute of Technology Jodhpur
Rajasthan, INDIA-342011
Email: pg201373010@iitj.ac.in

Abstract—A controllable network can be driven from any initial state to any desired state using driver nodes. A set of driver nodes to control a network is not unique. It is important to characterize these driver nodes and select the right driver nodes. The work discusses theory and algorithms to select driver node such that largest region of attraction can be obtained considering limited capacity of driver node and with unstable eigenvalues of adjacency matrix. A network which can be controllable using one driver node is considered. Nonuniqueness of driver node poses a challenge to select right driver node when multiple possibilities exist. The work addresses this issue.

Index Terms—Complex network, Characterization of Driver Nodes, Region of Attraction.

I. Introduction

Twenty first century has been witnessing unprecedented growth in use of networked systems. Application domains of networked systems are as diverse as social systems to bilogical networks. A network can be defined as an entity comprising of nodes and edges where each node represents an entity such as genes in a biological network, sensors in a detection system, vehicles in a traffic system or persons/individuals in a social network; while the edges denote connection or interactions between the nodes [1]. The state of a node can be described as position coordinates of a sensor or a robot, an opinion of a person, expression of protein by a gene etc [2]. A complex network system is composed of several nodes, which can interact with each other as well as share information. Each node has a state variable to represent its state values. These nodes, together, form a system, which performs tasks collectively.

Recent control theoretic analysis of complex networks provides a great deal of insight about the behaviour of complex network [1]–[5]. Controllability property is investigated by modeling complex network as a linear time invariant system; adjacency matrix is used as system matrix. A recent contribution has been made by Liu et al. [1], who developed a minimum input theory to efficiently characterize the structural controllability of directed networks using a minimum set of driver nodes to control of the system. The complex network system can also be controlled by the driver node with minimum control energy of the input which is an unavoidable and important issue [6]. In [6], authors proposed the trade-off between the driver nodes and the control energy as a function of the network dynamics using the smallest eigenvalue of the

Controllability Gramian. One of the challenges in the control of complex network systems is to find a set of right nodes (physical systems), so that controlling these nodes eventually leads to control of the entire network in a desired manner. In recent years there have been quite a few work devoted to this problem [1]–[5], [7], [8].

While considering the problem of controlling a complex network, it is important to consider limitations of driver nodes. Driver nodes can not have infinite (unlimited) actuating capability i.e., maximum input a driver node can provide is limited. When actuator has limited capacity, control objectives may not be achieved if this limitation is not considered *a priori*. This problem becomes more complex when adjacency matrix has unstable eigenvalues; even stability can not be guaranteed in this situation. With unstable eigenvalues of adjacency matrix of complex network, region in state space, where stability is guaranteed, is finite when actuator (driver node) limitation is considered. We denote this region as region of attraction (ROA). This ROA depends on choice of driver node and control law. This work characterizes driver nodes and proposes a theory and algorithms such that the right driver node, among many, can be found to maximize region of attraction. In n dimensional state space, region is described by ellipsoid. In context of linear system with actuator saturation, many excellent work exist see [9]–[11] and the references therein. Number of driver nodes required to control the network is fixed i.e., set of driver nodes has fixed number of nodes, however, driver nodes are not unique. This raises a very important question about the choice of driver nodes when there are many nodes qualify for driver nodes. In this work we address two important issues the first issue is regarding selection of driver node and in second we consider driver node limitation. We propose a theory and algorithm to select driver node such that region of attraction corresponding to this driver node maximizes. In this work to obtain largest possible region with a given control law, the low and high gain (LHG) technique [9], [12]–[14] is used. Our emphasis on selection of right driver nodes rather than on control law. To the best of authors knowledge this work does not exist.

Rest of the paper is organized as follows. Starting with an introduction in section I, the section II describes the modeling of the networks with saturated actuator. Section III presents the theoretical results to characterize the maximum volume of the

invariant ellipsoids of anti-stable system and transformed anti-stable subsystem. The simulation results of the two networks and their discussion are included in section IV. Finally, the concluding remark is included in the section V.

II. MODELING OF NETWORK WITH SATURATED ACTUATOR

A network with n nodes can be represented by a graph $G := (V, E)$, where $V := \{1, 2, \ldots, n\}$ and $E \subseteq V \times V$ are the set of nodes and the set of edges respectively. The weighted adjacency matrix of the network G is described as A(G), where the elements are $a_{ij} \in \mathbb{R}$ if node i and j adjacent and 0 otherwise. Let us assume that network G is independently controllable by each of the nodes from the set $\bar{U} = \{u_1, u_2, \ldots, u_p\}$ i.e. each input u_i, *where* $i = 1, 2, \ldots, p$, acting alone can control the entire network.

Consider a complex network with input u_i, let us designate corresponding input matrix as B_i, with this, dynamics of network with bounded control can be written as

$$\frac{d\mathbf{x}(t)}{dt} = A(G)\mathbf{x}(t) + B_i sat(u_i(t)) \tag{1}$$

where $A(G) \in \mathbb{R}^{n \times n}$ adjacency matrix of the network G, $B_i \in \mathbb{R}^{n \times 1}$ input matrix corresponding to input u_i, $\mathbf{x} \in \mathbb{R}^{n \times 1}$ state vector (n-vector), $u_i \in \mathbb{R}$ control signal (scalar); It should be noted that any u_i can control the network. The standard saturation function 'sat' is defined as $sat: \mathbb{R} \to \mathbb{R}$, i.e., $sat(u_i(t)) = sign(u_i(t))min\{u_{i,max}, |u_i(t)|\}$ where $u_{i,max} > 0$, $\forall i = 1, 2, \ldots, p$ is the saturation limit control input u_i. Without loss of generality let us assume $u_{i,max} = 1$. The state vector is defined as $\mathbf{x} = [x_1 \ x_2 \ldots x_n]^T$.
Let us define control law for network (1) as

$$u_i := -K_i\mathbf{x} = -R^{-1}B_i^T P_i\mathbf{x} \tag{2}$$

Where $R > 0$ is a chosen matrix and the matrix $P_i > 0$ is obtained by solving the following Ricatti equation for some $Q > 0$

$$A^T P_i + P_i A - P_i B_i R^{-1} B_i^T P_i + Q = 0 \tag{3}$$

Now, the control law (2) is divided into two equal parts, low gain and high gain i.e. $u_{i,L} = -\frac{1}{2}R^{-1}B_i^T P_i\mathbf{x}$ and $u_{i,H} = -\frac{1}{2}R^{-1}B_i^T P_i\mathbf{x}$ respectively, where the low gain part is not allowed to saturate but the overall control law is allowed [12].

$$u_i := u_{i,L} + u_{i,H}$$

Under the control law (2), the network is asymptotically stable for all states lie in the invariant ellipsoid $\varepsilon(P_i, \delta_i)$, [9] given as

$$\varepsilon(P_i, \delta_i) = \{\mathbf{x} : \mathbf{x}^T P_i\mathbf{x} \le \delta_i\} \tag{4}$$

and $\delta_i > 0$ is defined as

$$\delta_i := \frac{4r_i^2}{B_i^T P_i B_i} \tag{5}$$

Where r_i is the ith diagonal entry of $R > 0$ and it is assumed $r_i = 1$. Equation (5) gives the radius of the invariant ellipsoid $\varepsilon(P_i, \delta_i)$ and it is obtained by using the low gain part of the

linear quadric (LQ) control (2). The proof of (5) is given in [9], [12], [15]. It is evident from (5), that with different input node (driver node) acting independently, the matrices P_i and B_i will change and this subsequently changes δ_i and corresponding region of attraction.

III. MAIN RESULTS

This section is divided into three parts, first, we explain the problem statement, second, we derive the conditions for ROA of the anti-stable network with different control inputs and in third, we derive the conditions for ROA of the anti-stable subsystem network with different control inputs of the network (1).

A. Problem Statement

A complex network system of homogeneous nodes with a set of nodes, having capability to control the network acting alone, is considered. A set of driver nodes—each driver node can control the network independently—is considered. Furthermore, with different input, region where stability, with saturated input, is guaranteed may expand or shrink. This work presents theory and algorithms to find an input such that largest region of attraction can be achieved. The objective is to characterize an input such that ROA becomes largest in comparison to region obtained by each of the other driver nodes from $\bar{U}$ acting alone.

B. Region of Attraction and Selection of Driver Node of an Anti-stable Network With Saturated Driver Noder

This section discusses region of attraction and volume of invariant ellipsoid for the system defined in (1) with all eigenvalues of $A(G)$ are unstable. Consider an anti-stable network (all eigenvalues of $A(G)$ are unstable) (1) with driver node saturation. The contractively invariant ellipsoids corresponding to input matrix B_i of the network (1) depends on the positive definite matrix P_i.

Lemma 1. *For the system defined in (1) with control input u_i, control law (2) and radius of the invariant ellipsoid defined in (5):*
(i) The volume of the invariant ellipsoid is given by

$$Vol_i = \frac{S(0, \sqrt{\delta_i})}{\sqrt{det(P_i)}} \tag{6}$$

Where $S(0, \sqrt{\delta_i})$ is the volume of $n-$dimensional sphere $(n \ge 3)$ of radius $\sqrt{\delta_i}$ and center at origin, $det(P_i)$ is the determinant of the positive definite matrix P_i.
(ii) Largest region of attraction corresponds to i^{th} driver node for which volume becomes largest i.e.
Largest volume $= Max_i\{Vol_i\} \quad i = 1, 2, \ldots, p$

Proof: To prove the first part, the outer boundaries of the contractively invariant ellipsoids $\varepsilon(P_i, \delta_i) = \{\mathbf{x} : \mathbf{x}^T P_i\mathbf{x} \le \delta_i\}$ of the linear anti-stable system (1) are obtained from the following equation

$$\mathbf{x}^T P_i\mathbf{x} = \delta_i \tag{7}$$

and the volume of the contractively invariant ellipsoid is given by

$$Vol_i = \underbrace{\int \cdots \int}_{x_{j_{min}}, j=1,..,n}^{x_{j_{max}}} \underbrace{dx_1 dx_2 .... dx_n} \tag{8}$$

Where $x_{j_{min}}$ and $x_{j_{max}}$ are the minimum and maximum limits of state vector $\mathbf{x}$ in j^{th} direction.

Now assume $\mathbf{x} = T_i \mathbf{y}$, where $T_i > 0$ is a positive definite matrix and $\mathbf{y} \in \mathbb{R}^n$ state vector, then equation (7) can be rewritten as

$$\mathbf{y}^T T_i^T P_i T_i \mathbf{y} = \delta_i \tag{9}$$

The positive definite matrix T_i is chosen in such a way that $T_i^T P_i T_i = I_{n \times n}$. This implies that $T_i = ((M_i \sqrt{D_i})^{-1})^T$, where M_i is the eigenvector matrix of positive definite matrix P_i and matrix D_i is the diagonal matrix (elements are the eigenvalues of the matrix P_i).

Using change of variables $\mathbf{x} = T_i \mathbf{y}$, we can obtain $\underbrace{dx_1 \ dx_2 \ \ldots \ dx_n} = (\frac{1}{\sqrt{det(P_i)}}) \underbrace{dy_1 \ dy_2 \ \ldots \ dy_n}$. Equation (8) become

$$Vol_i = (\frac{1}{\sqrt{det(P_i)}}) \underbrace{\int \cdots \int}_{y_{j_{min}}, j=1,...,n}^{y_{j_{max}}} \underbrace{dy_1 \ dy_2 \ \ldots \ dy_n}. \tag{10}$$

Now, the volume of $n-$dimensional sphere $(n \geq 3)$ is defined as

$$S(0, \sqrt{\delta_i}) =: \underbrace{\int \cdots \int}_{y_{j_{min}}, j=1,...,n}^{y_{j_{max}}} \underbrace{dy_1 \ dy_2 \ \ldots \ dy_n}. \tag{11}$$

with center at origin and radius $\sqrt{\delta_i}$.

Using (10) and (11), we get (6).

After obtaining region of attraction (volume) corresponding to each of the driver nodes, (ii) implies directly.

This completes the proof of lemma 1. ∎

Corollary 1. *For $n = 2$, we need to compute area. The area of ellipse (region) is given by*

$$Area_i = \frac{\pi \ \delta_i}{\sqrt{det(P_i)}} \tag{12}$$

$Area_i > Area_j \iff \delta_j^2 det(P_i) < \delta_i^2 det(P_j), \ i \neq j = 1, 2.$ *The ith node is chosen as the driver node of the system (1), corresponding to the largest area.*

We summarize computation of δ_i and $Area_{max}$ / Vol_{max} for a general n nodes network with p nodes being (each of the nodes can control the network) driver nodes.

Algorithm 1 Find maximum area/volume of the ellipsoid and driver node of a network with all eigenvalues in RHP

Input : Network $G := (V, E)$, Number of nodes n, driver nodes p (These driver nodes can control the network independently), $Q > 0$ and $R > 0$.

Output : Maximum Area/Volume of the ellipsoid and corresponding i^{th} driver node

1: **for** $i = 1 : p$ **do**
2:    Select $B_i = [0, 0, \ldots, b_i, \ldots, 0, 0]^T$
3:    Obtain the positive definite matrix $P_i > 0$ using (3)

$$A^T P_i + P_i A - P_i B_i R^{-1} B_i^T P_i + Q = 0$$

   where $i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, n.$
4:    Calculate the radius of the ellipsoid δ_i as
   $\delta_i = \frac{4}{\Sigma_i}$
   where $\Sigma_i = B_i^T P_i B_i = b_i^2 p_{jj}^{(i)}$
5: **end for**
6: **if** $n = 2$ **then**
7:    Calculate $Area_{max}$ as
   $Area_{max} = arg \ max_i \{Area_i \mid i = 1, 2, \ldots, p\}$
   where $Area_i = \frac{\pi \delta_i}{\sqrt{det(P_i)}}$
8: **else**
9:    Calculate Vol_{max} as
   $Vol_{max} = arg \ max_i \{Vol_i \mid i = 1, 2, \ldots, p\}$
   where $Vol_i = \frac{S(0, \sqrt{\delta_i})}{\sqrt{det(P_i)}}$
   where $S(0, \sqrt{\delta_i})$ is the volume of $n-$dimensional sphere $(n \geq 3)$ of radius $\sqrt{\delta_i}$ and center at origin, $det(P_i)$ is the determinant of the positive definite matrix P_i.
10: **end if**
11: **return** $Area_{max}$ / Vol_{max} and node i for which $Area_{max}$ / Vol_{max}

C. Region of Attraction and Selection of Driver Node of a Network with Stable and Unstable Eigenvalues

Now, we consider a more general network which has stable and unstable eigenvalues. We transform the original system (1) into two subsystems. The first subsystem has all unstable eigenvalues and the second subsystem has all stable eigenvalues. In what follows, we consider subsystem with unstable eigenvalues and obtain region of attraction. In this case, the system will be diagonalized using a transformation matrix V. Let $\mathbf{x} = V\mathbf{z}$, the transformation matrix V is the matrix which has eigenvectors of the matrix $A(G)$.

The transformed system has an anti-stable subsystem as well as a stable subsystem of the original system (1) defined as

$$\frac{d\mathbf{z}(t)}{dt} = \tilde{A}\mathbf{z}(t) + \tilde{B}_i sat(u_i(t)) \tag{13}$$

Where

$$\tilde{A} = V^{-1} A V = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \tag{14}$$

where $\tilde{A}_1 \in \mathbb{R}^{k \times k}$ anti-stable matrix and $\tilde{A}_2 \in \mathbb{R}^{n-k \times n-k}$ stable matrix, and the control input matrix $\tilde{B}_i$ can be partitioned as

$$\tilde{B}_i = V^{-1}B_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix} \quad (15)$$

where $\tilde{B}_{1i} \in \mathbb{R}^{k \times 1}$ and $\tilde{B}_{2i} \in \mathbb{R}^{n-k \times 1}$, n and k are the number of nodes (order) in system (1) and order of the anti-stable subsystem respectively.

The transformed system (13) can be rewritten as

$$\begin{bmatrix} \frac{d\mathbf{z}_1(t)}{dt} \\ \frac{d\mathbf{z}_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix} sat(u_i(t)) \quad (16)$$

Now we describe the anti-stable subsystem of (16) as follows

$$\frac{d\mathbf{z}_1(t)}{dt} = \tilde{A}_1\mathbf{z}_1(t) + \tilde{B}_{1i}sat(u_i(t)) \quad (17)$$

Remark 1. *To obtain region of attraction, only anti-stable subsystem of (17) needs to be considered. For open-loop stable network, global stabilization is possible with saturated driver nodes.*

The control law $u_i(t)$ can be defined as

$$u_i(t) = -H\mathbf{z}_1 \quad (18)$$

where $H = R_1^{-1}\tilde{B}_{1i}^T\tilde{P}_{1i}$ is a feedback gain matrix.
The positive definite matrix $\tilde{P}_{1i} > 0$ is the solution to the following Algebraic Riccati Equation

$$\tilde{A}_1^T\tilde{P}_{1i} + \tilde{P}_{1i}\tilde{A}_1 - \tilde{P}_{1i}\tilde{B}_{1i}R_1^{-1}\tilde{B}_{1i}^T\tilde{P}_{1i} + Q_1 = 0 \quad (19)$$

where $Q_1 > 0$ and $R_1 > 0$ are chosen positive definite and diagonal matrices.
The radius $(\tilde{\delta}_{1i})$ of the invariant ellipsoid $(\varepsilon_1(\tilde{P}_{1i}, \tilde{\delta}_{1i}) = \{\mathbf{z}_1 : \mathbf{z}_1^T\tilde{P}_{1i}\mathbf{z}_1 \leq \tilde{\delta}_{1i}\})$ of the anti-stable network (17) is defined as

$$\tilde{\delta}_{1i} = \frac{4}{\Sigma_{1i}} \quad (20)$$

where $\Sigma_{1i} := \tilde{B}_{1i}^T\tilde{P}_{1i}\tilde{B}_{1i}, \quad i = 1,2,\ldots,p$.

Lemma 2. *For the anti-stable subsystem defined in (17) with control law (18) and radius of invariant ellipsoid (20)*
(i) The volume of the invariant ellipsoid is given by

$$Vol_{1i} = \frac{S(0, \sqrt{\tilde{\delta}_{1i}})}{\sqrt{det(\tilde{P}_{1i})}} \quad (21)$$

Where $S(0, \sqrt{\tilde{\delta}_{1i}})$ is the volume of $k-$dimensional sphere $(k \geq 3)$ of radius $\sqrt{\tilde{\delta}_{1i}}$ and center at origin, $det(\tilde{P}_{1i})$ is the determinant of the positive definite matrix $\tilde{P}_{1i}$.
(ii) Largest region of attraction corresponds to i^{th} input for which volume becomes largest i.e.
Largest volume $= Max_i\{Vol_{1i}\} \quad i = 1,2,\ldots,p$

   Proof: The proof of lemma 2 is similar to given in lemma 1. ∎

   We summarize computation of $\tilde{\delta}_{1i}$ and $Area_{1,max}$ / $Vol_{1,max}$ for a general network of order n with dimension of anti-stable subsystem k, $k < n$ in the algorithm 2.

Algorithm 2 Find maximum area/volume of the invariant ellipsoid and driver nodes of a network system (1)

Input : Network $G := (V,E)$, Number of nodes n, Number of anti-stable nodes k, driver nodes p, $Q_1 > 0$ and $R_1 > 0$.
Output : Maximum area/volume of the ellipsoid and corresponding i^{th} driver node

1: Partition the matrix $\tilde{A}$ using (14)

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}$$

   where $\tilde{A}_1 \in \mathbb{R}^{k \times k}$ and $\tilde{A}_2 \in \mathbb{R}^{n-k \times n-k}$.
2: **for** $i = 1 : p$ **do**
3:   Select $B_i = [0,0,\ldots,b_i,\ldots,0,0]^T$ of the original system (1)
4:   Partition $\tilde{B}_i$ using (15), given as

$$\tilde{B}_i = V^{-1}B_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix}$$

   where $\tilde{B}_{1i} \in \mathbb{R}^{k \times 1}$ and $\tilde{B}_{2i} \in \mathbb{R}^{n-k \times 1}$.
5:   Calculate matrix $\tilde{P}_{1i}$ using (19), given as

$$\tilde{A}_1^T\tilde{P}_{1i} + \tilde{P}_{1i}\tilde{A}_1 - \tilde{P}_{1i}\tilde{B}_{1i}R_1^{-1}\tilde{B}_{1i}^T\tilde{P}_{1i} + \tilde{Q}_1 = 0$$

6:   Calculate the radius of the invariant ellipsoid $\tilde{\delta}_{1i}$ as
   $\tilde{\delta}_{1i} = \frac{4}{\Sigma_{1i}}$
   where $\Sigma_{1i} = \tilde{B}_{1i}^T\tilde{P}_{1i}\tilde{B}_{1i}$
7: **end for**
8: **if** $k = 2$ **then**
9:   Calculate $Area_{1,max}$ as
   $Area_{1,max} = arg\ max_i\{Area_{1i} \mid i = 1,2,\ldots,p\}$
   where $Area_{1i} = \frac{\pi\tilde{\delta}_i}{\sqrt{det(\tilde{P}_i)}}$
10: **else**
11:   Calculate $Vol_{1,max}$ as
   $Vol_{1,max} = arg\ max_i\{Vol_{1i} \mid i = 1,2,\ldots,p\}$
   where $Vol_{1i} = \frac{S(0, \sqrt{\tilde{\delta}_{1i}})}{\sqrt{det(\tilde{P}_{1i})}}$
   where $S(0, \sqrt{\tilde{\delta}_{1i}})$ is the volume of $k-$dimensional sphere $(k \geq 3)$ of radius $\sqrt{\tilde{\delta}_{1i}}$ and center at origin, $det(\tilde{P}_{1i})$ is the determinant of the positive definite matrix $\tilde{P}_{1i}$.
12: **end if**
13: **return** $Area_{1,max}$ / $Vol_{1,max}$ and node i for which $Area_{1,max}$ / $Vol_{1,max}$

Corollary 2. *For $k = 2$, the region becomes area. The area of invariant ellipse (region) of system defined in (17) is given by*

$$Area_{1i} = \frac{\pi\ \tilde{\delta}_{1i}}{\sqrt{det(\tilde{P}_{1i})}} \quad (22)$$

$Area_{1i} > Area_{1j} \iff \tilde{\delta}_{1j}^2 det(\tilde{P}_{1i}) < \tilde{\delta}_{1i}^2 det(\tilde{P}_{1j})$, $i \neq j = 1,2$, *then i^{th} node will be chosen as the driver node of the system (1) corresponding to largest area.*

IV. SIMULATION RESULTS AND DISCUSSION

In this section, We consider two numerical examples to verify the theoretical results. First example considers anti-stable network which has all positive eigenvalues and other example has stable and unstable eigenvalues.

Example 1. *We consider the weighted directed network of two nodes system as shown in Fig.* 1.
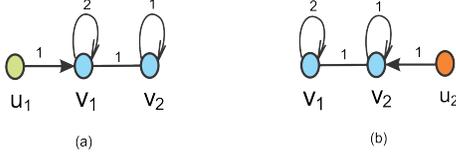

Fig. 1: Two nodes network system (a) with input u_1, (b) with input u_2.

The adjacency and control input matrices are given as:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (23)$$

Let us assume that B_i is a variable input matrix which depends on i^{th} driver node in order to control the system. The eigenvalues of the system that can be denoted as $\lambda(A) = \{0.3820, 2.6180\}$. Let $Q = I_2$, $R = 1$ and solve the Algebraic Riccati Equation (3) for different input matrices B_i. We obtain the radius of the invariant ellipsoids $\delta_1 = 0.6264$ and $\delta_2 = 0.6068$ using (5), for driver node 1 and 2 respectively. We get the values of $\sqrt{det(P1)} = 7.9030$, $\sqrt{det(P2)} = 9.4257$, and also obtain the area of the two ellipsoids using (12), they are 0.2473 Sq. units and 0.2014 Sq. units respectively. Now we obtain the maximum area of the invariant ellipsoid using lemma 1 is 0.2473 Sq. units. The region of attractions of the network considered in example 1 for different driver node, are plotted in the Fig. 3. From Fig. 3, we get the area of first ellipsoid (blue color) which is larger than a second ellipsoid (black color). We conclude here, choosing the first node as the driver node is advantageous because we get the maximum area of the invariant ellipsoid. According to table 1, results clearly validate the effectiveness of the theoritical results.

TABLE I: Area of the Contractively Invariant Ellipsoids of different driver nodes

Driver node	Area of the Contractively Invariant Ellipsoid	$\frac{Area_i}{Area_{max}}$
1	0.2490	1
2	0.2023	0.8124

Example 2. *We consider the weighted directed network with four nodes system as shown in Fig.* 2.

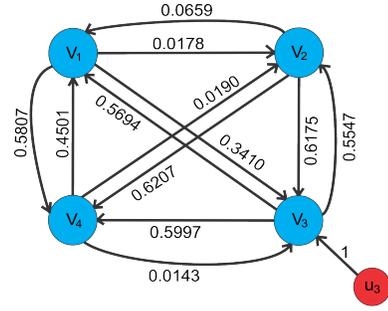The adjacency matrix and control input matrices are given

Fig. 2: Four nodes network system with control input u_3.

as:

$$A = \begin{bmatrix} 0 & 0.0178 & 0.3410 & 0.5807 \\ 0.0659 & 0 & 0.6175 & 0.6207 \\ 0.5694 & 0.5547 & 0 & 0.5997 \\ 0.4501 & 0.0190 & 0.0143 & 0 \end{bmatrix} \quad (24)$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (25)$$

Depending on choice of a particular node as a driver node, input matrix B_i has four possibilities. The eigenvalues of the network shown in Fig.2, obtained as $\lambda(A) = \{0.9613, 0.1318, -0.7706, -0.3225\}$. Here two eigenvalues are positive so the network will be partitioned into two subsystem according to (16). Now we analyse anti-stable subsystem (17). Let $Q = I_4$, $R_1 = 1$ and solve the Algebraic Riccati Equation (19) for different input matrices B_i corresponding to different driver nodes. We obtain the radius of the invariant ellipsoids $\tilde{\delta}_{11} = 1.2289$, $\tilde{\delta}_{12} = 1.4606$, $\tilde{\delta}_{13} = 1.6647$ and $\tilde{\delta}_{14} = 1.3311$ using (20). We get the values of $\sqrt{det(\tilde{P}_{11})} = 4.8641$, $\sqrt{det(\tilde{P}_{12})} = 12.0531$, $\sqrt{det(\tilde{P}_{13})} = 13.3164$ and $\sqrt{det(\tilde{P}_{14})} = 4.4125$ with B_i respectively.

Now from the table 1, we obtain the maximum area of the invariant ellipsoid 0.9449 Sq. units, corresponding to driver node 4. We observe that, the importance of driver node from network stability point of view depends upon the area of the ellipsoids. The region of attractions of the network considered in example 2 for different driver nodes are plotted in the Fig. 4.

From Fig. 4, we get the largest area when fourth node is selected as input (magenta color) vis-a-vis region corresponding to node 1, 2, 3 (blue,black and red colors). We conclude here, choosing the fourth node as the driver node is advantageous because we get the maximum area of the invariant ellipsoid. According to table 2, results clearly validate the effectiveness of the theoretical results.

V. CONCLUSIONS

The paper has presented a method to select driver nodes. A practical limitation on actuator magnitude has been considered.

TABLE II: Area of the Contractively Invariant Ellipsoids of different driver nodes

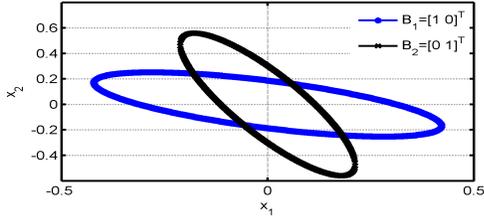| Driver node	Area of the Contractively Invariant Ellipsoid	$\frac{Area_i}{Area_{max}}$
4	0.9449	1
1	0.7917	0.8378
3	0.3910	0.4138
2	0.3796	0.4017

Fig. 3: Region of attractions of the network considered in example 1 for different control input matrices B_i.
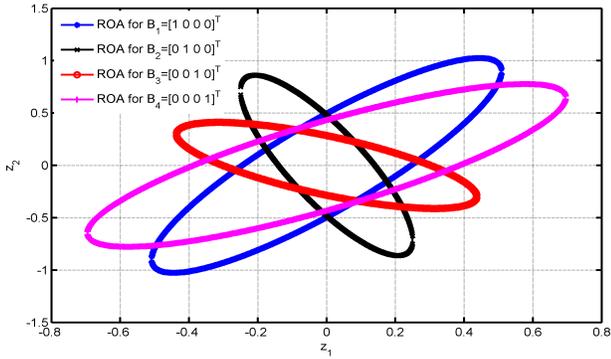

Fig. 4: Region of attractions of the network considered in example 2 for different control input matrices B_i.

The proposed theory and algorithms give most suitable driver nodes so that stability with saturated driver node can be ensured in the largest possible region. It has been shown, using numerical example, 40% reduction in stability region occurs if driver node is not selected using the proposed algorithm. A close agreement between theoretical and simulation has been obtained. The determinant of positive definite matrix P_i and the radius of the invariant ellipsoids decides which node should be considered as a driver node to control the entire system.

REFERENCES

[1] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, pp. 167–173, 2011.
[2] G. Yan, J. Ren, Y.-C. Lai, C.-H. Lai, and B. Li, "Controlling complex networks: How much energy is needed?" *Physical review letters*, vol. 108, no. 21, p. 218703, 2012.
[3] A. Lombardi and M. Hörnquist, "Controllability analysis of networks," *Phys. Rev. E*, vol. 75, p. 056110, May 2007.
[4] J. Sun and A. E. Motter, "Controllability transition and nonlocality in network control," *Phys. Rev. Lett.*, vol. 110, p. 208701, May 2013.
[5] G. Menichetti, L. Dall'Asta, and G. Bianconi, "Network controllability is determined by the density of low in-degree and out-degree nodes," *Phys. Rev. Lett.*, vol. 113, p. 078701, Aug 2014.
[6] F. Pasqualetti, S. Zampieri, and F. Bullo, "Controllability metrics and algorithms for complex networks," *arXiv preprint arXiv:1308.1201*, 2013.
[7] A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt, "Controllability of multi-agent systems from a graph-theoretic perspective," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 162–186, 2009.
[8] W.-X. Wang, X. Ni, Y.-C. Lai, and C. Grebogi, "Optimizing controllability of complex networks by minimum structural perturbations," *Phys. Rev. E*, vol. 85, p. 026115, Feb 2012.
[9] P.-O. Gutman and P. Hagander, "A new design of constrained controllers for linear systems," in *Decision and Control, 1982 21st IEEE Conference on*, vol. 21. IEEE, 1982, pp. 1007–1013.
[10] T. Hu and Z. Lin, *Control Systems With Actuator Saturation: Analysis and Design*. Boston, MA: Birkhauser, 2001.
[11] T. Lauvdal and T. I. Fossen, "Stabilization of linear unstable systems with control constraints," in *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, vol. 5. IEEE, 1997, pp. 4504–4509.
[12] M. Turner and I. Postlethwaite, "Guaranteed stability regions of linear systems with actuator saturation using the low-and-high gain technique," *International Journal of Control*, vol. 74, no. 14, pp. 1425–1434, 2001.
[13] Z. Lin, "Low gain and low-and-high gain feedback: A review and some recent results," in *Control and Decision Conference, 2009. CCDC '09. Chinese*, June 2009, pp. lii–lxi.
[14] A. Saberi, Z. Lin, and A. R. Teel, "Control of linear systems with saturating actuators," *Automatic Control, IEEE Transactions on*, vol. 41, no. 3, pp. 368–378, 1996.
[15] D. Henrion, G. Garcia, and S. Tarbouriech, "Piecewise-linear robust control of systems with input constraints," *European Journal of Control*, vol. 5, no. 1, pp. 157 – 166, 1999.