

INTERNAL TURING MACHINES

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ABSTRACT. Using nonstandard analysis, we will extend the classical Turing machines into the internal Turing machines. The internal Turing machines have the capability to work with infinite ($*$ -finite) number of bits while keeping the finite combinatoric structures of the classical Turing machines. We will show the following. The internal deterministic Turing machines can do in $*$ -polynomial time what a classical deterministic Turing machine can do in an arbitrary finite amount of time. Given an element of $\langle *M; *x \rangle \in Halt$ (more precisely, the $*$ -embedding of $Halt$), there is an internal deterministic Turing machine which will take $\langle *M; *x \rangle$ as input and halt in the "yes" state, and for $\langle *M; *x \rangle \notin Halt$, the internal deterministic Turing machine will halt in the "no" state. The language $*HALT$ can not be decided by the internal deterministic Turing machines. The internal deterministic Turing machines can be viewed as the asymptotic behavior of finite precision approximation to real number computations. It is possible to use the internal probabilistic Turing machines to simulate finite state quantum mechanics with infinite precision. This simulation suggests that no information can be transmitted instantaneously and at the same time, the Turing machine model can simulate instantaneous collapse of the wave function. The internal deterministic Turing machines are powerful, but if $P \neq NP$, then there are internal problems which the internal deterministic Turing machines can solve but not in $*$ -polynomial time.

1. Introduction. Nonstandard analysis is well known for its ability to rigorously work with infinities and infinitesimals in finite discrete combinatoric manner. We will use this ability to extend the classical Turing machines to the internal Turing machines. The outcome of this extension can simply be summarized as follows. Allow the classical Turing machines to work with an infinite number of bits in infinite amount of time steps in such a way that it keeps the finite combinatoric structures of the classical machines.

The internal Turing machine model is a computational model that is much more powerful than the classical Turing machine model. As of today, we are not aware of any evidence that the internal Turing machine model is a reasonable model of computation for the universe that we live in, but for the following reasons, we believe that this model is worth studying. The classical Turing machines are embedded in the internal ones. Hence, from a purely mathematical point of view, the classical Turing machines live in a much bigger universe. In mathematics, results are often obtained by embedding objects into a bigger universe. The internal Turing machines are capable of performing computations that are infinitesimally close to real variable computations. This could be useful as a bridge between finite discrete computations, experimental science, and continuous variable physical modeling. The internal probabilistic Turing machine model can simulate finite state quantum mechanics in such a way that it can model instantaneous collapse

of the wave function and suggests the non-allowance of instantaneous transmission of information. This suggests that the classical definition of deterministic Turing machine might be more fundamental than previously thought (more on this later). The internal Turing machines are much more powerful than the classical ones, but at the same time, they have the same types of limitations as the classical ones. For the philosophers and theologians, this suggests that our computational power was created in the image of more godly computers which are infinitely more powerful than what we can do and at the same time, have the same types of limitations as we do.

The structure of this paper is as follows. We will assume that the reader is familiar with the basic notions of nonstandard analysis, complexity theory, and quantum physics but we will try to make the paper self contained. In section 2, we will give a overview of nonstandard analysis. The main goal is familiarize the reader with the $*$ -transform. For more details on nonstandard analysis, we refer the reader to [3], [5], [7], and [9]. In section 3, we will give a rigorous definition of the classical deterministic Turing machines in terms of sets and functions. This is needed to apply the superstructure methods of nonstandard analysis. After this rigorous definition, we will extend the classical definition to the internal Turing machines. For the rest of section 3, we will take a look at what the internal deterministic Turing machines can and can not do. Namely, we will show that the internal Turing machines can do in $*$ -polynomial time what the classical Turing machines can do in finite amount of time. There exists an internal deterministic Turing machines that will answer "yes" on input from ${}^{\sigma}HALT$, the $*$ -embedding of $HALT$. On the other hand, the internal deterministic Turing machines can not decide $*HALT$. In section 4, we will show that the internal deterministic Turing machines can be viewed as the asymptotic behavior of the real computation model proposed in [6]. In section 5, we will assume that nature uses internal Turing machines to compute physical evolutions, and we will simulate finite state quantum mechanics with the internal probabilistic Turing machines. We will see that the simulation suggests that the classical Turing machine model might be more fundamental than previously thought. In section 6, we extend the classical nondeterministic Turing machines to internal nondeterministic Turing machines. We will show that even though the internal Turing machines are much more powerful than the classical ones, they have the same types of limitations that the classical Turing machines have.

2. Nonstandard Analysis. We begin by giving a quick overview of nonstandard analysis. For more details on nonstandard analysis, see [3], [5], [7], and [9].

Theorem 1. *There exists a finite additive measure m over \mathbb{N} , the positive integers, such that the following three properties hold. For all $A \subset \mathbb{N}$, $m(A)$ is defined and it is either 0 or 1, $m(\mathbb{N}) = 1$, and $m(A) = 0$ for all finite A .*

Proof. See [3]. \square

We now use the measure to construct the nonstandard reals. Let

$$(2.1) \quad S = \{ \{a_n\} \mid a_n \in \mathbb{R}, n \in \mathbb{N} \},$$

be the collection of all infinite sequences of real numbers. Define an equivalent relations on S as follows. For any two sequences $\{a_n\}$ and $\{b_n\}$, $\{a_n\} \sim \{b_n\}$

iff $m\{n|a_n = b_n\} = 1$. The nonstandard reals is defined as ${}^*\mathbb{R} = S/\sim$. For any two equivalence classes $\langle a_n \rangle$ and $\langle b_n \rangle$ with representative sequences $\{a_n\}$ and $\{b_n\}$, numerical operations like addition and multiplication are defined by $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$, and $\langle a_n \rangle * \langle b_n \rangle = \langle a_n * b_n \rangle$. Orderings like \leq is defined by $\langle a_n \rangle \leq \langle b_n \rangle$ iff $m\{n|a_n \leq b_n\} = 1$. It can be shown that these operations are independent of the representatives. It can also be shown that ${}^*\mathbb{R}$ is an ordered field with zero element $0 = \langle 0, 0, \dots, 0, \dots \rangle$ and unit element $1 = \langle 1, 1, \dots, 1, \dots \rangle$. Finally, for $a \in \mathbb{R}$, the mapping $a \rightarrow \langle a, a, \dots, a, \dots \rangle$ is an order preserving homomorphism embedding \mathbb{R} into ${}^*\mathbb{R}$.

The nonstandard reals contain a copy of \mathbb{R} and elements that are not in \mathbb{R} . Namely, it contains the infinities, the infinitesimals, and the reals plus the infinitesimals. Consider the equivalence class $\omega = \langle 1, 2, 3, \dots \rangle$. For $a \in \mathbb{R}$, its embedding in ${}^*\mathbb{R}$ is given by $\langle a, a, \dots \rangle$. Further, $\{\lceil a \rceil, \lceil a \rceil + 1, \dots\} = \{n|a \leq n, n \in \mathbb{N}\}$ and $m\{n|a \leq n, n \in \mathbb{N}\} = 1$. The last result comes from

$$(2.2) \quad \begin{aligned} 1 &= m(\mathbb{N}) = m(\{\lceil a \rceil, \lceil a \rceil + 1, \dots\} \cup \{1, \dots, \lceil a \rceil - 1\}) = \\ &= m(\{\lceil a \rceil, \lceil a \rceil + 1, \dots\}) + m(\{1, \dots, \lceil a \rceil - 1\}) = m(\{\lceil a \rceil, \lceil a \rceil + 1, \dots\}) + 0. \end{aligned}$$

This shows that for any $a \in \mathbb{R}$, $\langle a, a, \dots \rangle$ is less than $\langle 1, 2, 3, \dots \rangle$. The nonstandard real numbers of this type are the infinite nonstandards. For our purpose, we will mainly be interested in the nonstandard infinite integers. These are nonstandard integers of the form $\langle b_n \rangle, b_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, and for all $a \in \mathbb{Z}$, $|\langle a, a, \dots \rangle| \leq |\langle b_n \rangle|$, where $|\langle a_n \rangle| \leq |\langle b_n \rangle|$ is defined by

$$(2.3) \quad |\langle a_n \rangle| \leq |\langle b_n \rangle| \quad \text{iff} \quad m(\{n| |a_n| \leq |b_n|\}) = 1.$$

At the other end of the spectrum are the infinitesimals. Consider the equivalence class $\delta = \langle 1, 1/2, 1/3, \dots, 1/n, \dots \rangle$. For any $a \in \mathbb{R} - 0$,

$$(2.4) \quad \langle 0, 0, \dots \rangle < \delta < |\langle a, a, \dots \rangle|.$$

Hence, δ is smaller than $|a|$ for all $a \in \mathbb{R} - 0$. The nonstandard number δ is an infinitesimal. Both the infinities and the infinitesimals can be positive or negative, and the nonstandard number 0 is the only infinitesimal in \mathbb{R} . An element $x \in {}^*\mathbb{R}$ is called finite if there exists a positive real number $a \in \mathbb{R}$ such that $-a < x < a$. It can be shown that for any finite $x \in {}^*\mathbb{R}$, there is a real $a \in \mathbb{R}$ and an infinitesimal $\delta \in {}^*\mathbb{R}$ such that $x = a + \delta$. For any $x, y \in {}^*\mathbb{R}$, we write $x \approx y$ to denote that $x = y + \delta$, where δ is an infinitesimal. In other-words, we say that x and y are infinitesimally close to each other. For any finite nonstandard $x = a + \delta$, we denote the standard part of x by $st(x) = a$. It can be shown that if the sequence $\{a_n\}$ converges to a , then $a \approx \langle a_n \rangle$ or $a = st(\langle a_n \rangle)$. Finally, the set of all x such that $st(x) = a$ is called the monad of a , i.e., the monad of $a \in \mathbb{R}$ is the set $a + \delta$ where δ is an infinitesimal.

We now look at the sets of ${}^*\mathbb{R}$. A sequence of subsets $\{A_n\}$ of \mathbb{R} defines a subset $\langle A_n \rangle$ of ${}^*\mathbb{R}$ by $\langle x_n \rangle \in \langle A_n \rangle$ iff $m(\{n|x_n \in A_n\}) = 1$. Any subset of ${}^*\mathbb{R}$ that can be obtained this way is called internal. For any subset A of \mathbb{R} , the internal subset ${}^*A = \langle A \rangle$ is called the nonstandard version of A . An internal set is called standard if it is of the form *A . For any subset A of \mathbb{R} , $A \subseteq {}^*A$ with equality

hold if and only if A is finite. The $*$ operation is called the $*$ -transform. Hence, the $*$ -transform of an infinite set is a proper extension of the set. An internal set $A = \langle A_n \rangle$ is called hyperfinite if almost all of the A_n 's are finite; the set A has internal cardinality $|A| = \langle |A_n| \rangle$ where $|A_n|$ is the number of elements in A_n . There are sets of ${}^*\mathbb{R}$ that are not internal. The set of infinitesimals of ${}^*\mathbb{R}$ is not an internal set. A set that is not internal is called external.

Example 1. Let $N = \langle N_n \rangle \in {}^*\mathbb{N}$ be an infinite element of nonstandard natural numbers. The set $T = \{0, 1/N, 2/N, \dots, (N-1)/N, 1\}$ should have internal cardinality $N+1$ (notice that N is an infinite integer). Since $N = \langle N_n \rangle$, $T = \langle T_n \rangle$, where $T_n = \{0, 1/N_n, 2/N_n, \dots, 1\}$. The cardinality of T is then $|T| = \langle |T_n| \rangle = \langle N_n + 1 \rangle = N + 1$. The point here is that the set T has infinite cardinality, but we can treat the combinatorics as if it were finite. Properties like this is better illustrated with the $*$ -transform theorem below.

After having defined sets, we now define functions. Let $f_n : A_n \rightarrow B_n$ be a sequence of functions. The internal function $\langle f_n \rangle : \langle A_n \rangle \rightarrow \langle B_n \rangle$ is defined by $\langle f_n \rangle \langle x_n \rangle = \langle f_n(x_n) \rangle$. Any function that can be obtained this way is called an internal function. If $f : A \rightarrow B$, then the function $*f = \langle f \rangle : \langle A \rangle \rightarrow \langle B \rangle$ is called the nonstandard version of f . An internal function which can be obtained this way is called standard. A characterization of the internal cardinality of an internal set A is that there exist an internal bijection $f : \{1, 2, \dots, N\} \rightarrow A$.

To make the process of working with nonstandard analysis less cumbersome, we introduce superstructures and the $*$ -transform on superstructures. This allows us to work with nonstandard analysis described above in one big scoop. For our purpose, we will mainly be interested in functions and relations. Functions and relations can be described in terms of set theory. Given a set S , an n -ary relation P on S is a subset of $S^n = S \times S \cdots \times S$. For $\langle a_1, a_2, \dots, a_n \rangle \in P \subseteq S^n$, The domain of P consists of those elements $\langle a_1, a_2, \dots, a_{n-1} \rangle$ of S^{n-1} such that there exists an a in S with $\langle a_1, a_2, \dots, a_{n-1}, a \rangle \in P \subseteq S^n$. The set of all such a 's is the range of P . A relation P is a function if whenever

$$(2.5) \quad \langle a_1, a_2, \dots, a_{n-1}, a \rangle \in P \text{ and } \langle a_1, a_2, \dots, a_{n-1}, b \rangle \in P,$$

then $a = b$. We now define superstructures. For a set X , $\mathcal{P}(X)$ is the power set or the set of all subsets (including \emptyset) of X . The n -th cumulative power set of X is defined recursively by

$$(2.6) \quad V_0(X) = X, V_{n+1} = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The superstructure over X is the set

$$(2.7) \quad V(X) = \bigcup_{n=0}^{\infty} V_n(X).$$

The symbols of the language for $V(X)$ consist of the following. The connective symbols \neg , \vee , \wedge , \rightarrow , and \leftrightarrow will be interpreted as "not", "or", "and", "implies", and "if and only if" (we will occasionally overload the symbol \rightarrow as function domain to range notation). The quantifier symbols \forall , and \exists will be interpreted as "for all" and "there exists". The symbols $[,]$, $(,)$, and \langle, \rangle will be used for bracketing. At

least one symbol a for each element $a \in V(X)$ (for notation convenience, we have used the same notation for both the element and its symbol). A countable collection of symbols like x, y, z, \dots to be used as variables. The symbol $=$ will be interpreted as "equal", and the symbol \in will be interpreted as "an element of".

A formula of the language is built up inductively as follows. If x_1, x_2, \dots, x_n, x , and y are either constants or variables, the expressions $x \in y$, $\langle x_1, \dots, x_n \rangle \in y$, $\langle x_1, \dots, x_n \rangle = y$, $\langle \langle x_1, \dots, x_n \rangle, x \rangle \in y$, and $\langle \langle x_1, \dots, x_n \rangle, x \rangle = y$ are formulas called atomic formulas. If Φ and Ψ are formulas, then so are $\neg\Phi$, $\Phi \vee \Psi$, $\Phi \wedge \Psi$, $\Phi \rightarrow \Psi$, and $\Phi \leftrightarrow \Psi$. If x is a variable symbol, and y is either a variable or a constant symbol, and Φ is a formula which does not already contain an expression of the form $(\forall x \in z)$ or $(\exists x \in z)$ (with the same variable x), then $(\forall x \in y)\Phi$ and $(\exists x \in y)\Phi$ are formulas. A variable occurs in the scope of a quantifier if whenever a variable x occurs in Φ , then x is contained in a formula Ψ which occurs in Φ in the form $(\forall x \in z)\Psi$ or $(\exists x \in z)\Psi$; it is then said to be bounded, and otherwise it is called free. A sentence is a formula in which all variables are bounded.

The decision of true or false of a sentence Φ in the language of $V(X)$ is as follows. The atomic sentences $a \in b$, $\langle a_1, \dots, a_n \rangle \in b$, $\langle \langle a_1, \dots, a_n \rangle, c \rangle \in b$, $a = b$, $\langle a_1, \dots, a_n \rangle = b$, and $\langle \langle a_1, \dots, a_n \rangle, c \rangle = b$ are true in $V(X)$ if the entity corresponding to its name is an element of or identical to b . If Φ and Ψ are sentences, then $\neg\Phi$ is true if Φ is not true; $\Phi \wedge \Psi$ is true if both Φ and Ψ are true; $\Phi \vee \Psi$ is true if at least one of Φ or Ψ is true; $\Psi \rightarrow \Phi$ is true if either Φ is true or Ψ is not true; $\Psi \leftrightarrow \Phi$ is true if Ψ and Φ are either both true or both not true. The expression $(\forall x \in b)\Phi$ is true if for all entities $a \in b$, when the symbol corresponding to the entity a is substituted for x in Φ , the resulting formula $\Phi(a)$ is true. The expression $(\exists x \in b)\Phi$ is true if there is an entity $a \in b$ such that $\Phi(a)$ is true.

Example 2. Let us denote the set of polynomials with coefficients in $\mathbb{N} \cup \{0\} \equiv \bar{\mathbb{N}}$ and domain $\bar{\mathbb{N}}$ by $\bar{\mathbb{N}}POLY$, then the following statements are true

$$(2.8) \quad \begin{aligned} & (\forall p \in \bar{\mathbb{N}}POLY)(\exists n \in \bar{\mathbb{N}})[(\exists a_0 \in \bar{\mathbb{N}})(\exists a_1 \in \bar{\mathbb{N}}) \dots (\exists a_n \in \bar{\mathbb{N}}) \\ & [(\langle x, y \rangle \in p \rightarrow x \in \bar{\mathbb{N}}) \wedge (\langle x, y \rangle \in p \rightarrow y = a_0x^0 + a_1x + \dots + a_nx^n)], \\ & (\forall n \in \bar{\mathbb{N}}) \left[(\forall a_0 \in \bar{\mathbb{N}})(\forall a_1 \in \bar{\mathbb{N}}) \dots (\forall a_n \in \bar{\mathbb{N}})(\exists p \in \bar{\mathbb{N}}POLY) \right. \\ & \left. [(z \in p \rightarrow z = \langle x, y \rangle) \wedge (\langle x, y \rangle \in p \rightarrow x \in \bar{\mathbb{N}}) \wedge \right. \\ & \left. (\langle x, y \rangle \in p \rightarrow y = a_0x^0 + a_1x + \dots + a_nx^n)] \right], \end{aligned}$$

where we have used the set theoretic notation for functions. This is just a fancy way of saying that p is a polynomial with coefficient in $\bar{\mathbb{N}}$ if and only if

$$(2.9) \quad p(x) = \sum_{i=0}^n a_i x^i : \bar{\mathbb{N}} \rightarrow \text{range}(p), a_i \in \bar{\mathbb{N}}.$$

The $*$ -transform of superstructures together with the superstructures' language and true or false assignment allow us to efficiently apply the methods of nonstandard analysis. The injection $*$: $V(\mathbb{R}) \rightarrow V(*\mathbb{R})$ has the following properties. It preserves basic set operations, $*\emptyset = \emptyset$, $*(A \cup B) = *A \cup *B$, $*(A - B) = *A - *B$, $*(A \cap B) =$

$*A \cap *B$, $*(A \times B) = *A \times *B$, $* \langle a_1, a_2, \dots, a_n \rangle = \langle *a_1, *a_2, \dots, *a_n \rangle$. It preserves domain and range of relations, $dom(*\Phi) = *dom(\Phi)$, $rng(*\Phi) = *rng(\Phi)$. It preserves standard definition of sets, $* \{(x, y) : x \in y \in A\} = \{(z, w) : z \in w \in *A\}$. It produces a proper extension: ${}^\sigma A \subseteq *A$, where ${}^\sigma A = \{*x : x \in A\}$ is the $*$ -embedding of A , and equality holds if and only if A is a finite set. For all $a \in \mathbb{R}$ implies $*a \in *\mathbb{R}$, and $a \in \mathbb{R}$ implies $*a = a$. If $a \in V_{n+1}(\mathbb{R}) - V_n(\mathbb{R})$, $*a \in V_{n+1}(*\mathbb{R}) - V_n(*\mathbb{R})$. If $a \in *V_n(\mathbb{R})$, $n \geq 1$ and $b \in a$, then $b \in *V_{n-1}(\mathbb{R})$. For any sentence Φ in the language of $V(\mathbb{R})$, $*\Phi$ is the sentence in $V(*\mathbb{R})$ obtained by replacing all "constants" in Φ by $*$ "constants". For any sentence Φ in the language of $V(\mathbb{R})$, Φ is true in $V(\mathbb{R})$ if and only if $*\Phi$ is true in the language of $V(*\mathbb{R})$. Any entity A of the form $A = *B$ is called standard. Any element $a \in *B$ is called internal. An entity that is not internal (standard entities are internal) is called external. A sentence or formula Φ in the language of $V(*\mathbb{R})$ is called either internal or standard if the constants in Φ are names of internal or standard entities. A sentence which is not internal is called external.

Example 3. *The $*$ -transform of the sentences in example 2 are*

(2.10)

$$\begin{aligned} & (\forall p \in *\bar{\mathbb{N}}POLY)(\exists n \in *\bar{\mathbb{N}})[(\exists a_0 \in *\bar{\mathbb{N}})(\exists a_1 \in *\bar{\mathbb{N}}) \dots (\exists a_n \in *\bar{\mathbb{N}}) \\ & [(\langle x, y \rangle \in p \rightarrow x \in *\bar{\mathbb{N}}) \wedge (\langle x, y \rangle \in p \rightarrow y = a_0x^0 + a_1x + \dots + a_nx^n)]], \\ & (\forall n \in *\bar{\mathbb{N}}) \left[(\forall a_0 \in *\bar{\mathbb{N}})(\forall a_1 \in *\bar{\mathbb{N}}) \dots (\forall a_n \in *\bar{\mathbb{N}})(\exists p \in *\bar{\mathbb{N}}POLY) \right. \\ & \left. [(z \in p \rightarrow z = \langle x, y \rangle) \wedge (\langle x, y \rangle \in p \rightarrow x \in *\bar{\mathbb{N}}) \wedge \right. \\ & \left. (\langle x, y \rangle \in p \rightarrow y = a_0x^0 + a_1x + \dots + a_nx^n)] \right]. \end{aligned}$$

Since the original sentences are true, their $*$ -transforms are also true. This says that $*\bar{\mathbb{N}}POLY$ consists of elements of the form $a_0 + a_1x + \dots + a_nx^n$ where the coefficients and n are in $0 \cup *\bar{\mathbb{N}}$. Elements $p \in *\bar{\mathbb{N}}POLY$ are internal polynomials. If the coefficients and n are in ${}^\sigma\bar{\mathbb{N}} = \bar{\mathbb{N}}$, the $*$ -embedding of $\bar{\mathbb{N}}$, then the polynomial is the $*$ -embedding of a "classical" polynomial and it is a standard entity. The extension consists of those internal polynomials which some or all of the coefficients and n are infinite number(s) in $*\bar{\mathbb{N}} = 0 \cup *\bar{\mathbb{N}}$.

For our purposes, the nonstandard extension allows us to work with infinite integers and to keep the combinatoric structures of the standard finite integers. This is what gives life to the internal Turing machines.

3. Deterministic Turing Machines. We start this section with a semi-formally description of a k -tape deterministic Turing machine, this definition is taken from [8]. A k tape Turing machine, where $k \in \mathbb{N}$, is a quadruple

$$(3.1) \quad M = \langle K, \Sigma, \delta, s \rangle,$$

where K is a countable set of states; $s \in K$ is the initial state; Σ is a countable set of symbols; Σ contains the blank and the first symbol \sqcup and \triangleright ; δ is a transition function which maps $K \times \Sigma^k$ to $(K \cup \{h, \text{"yes"}, \text{"no"}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k$, where h (halting state), "yes" (accepting state), "no" (rejecting state), \leftarrow (left), \rightarrow (right),

and $-$ (stay) are not in $K \cup \Sigma$. Further, the transition function δ has the following property. If $\delta(q, \sigma_1, \dots, \sigma_k) = (p, \rho_1, D_1, \dots, \rho_k, D_k)$, and $\sigma_i = \triangleright$, then $\rho_i = \triangleright$ and $D_i = \rightarrow$. Intuitively, if the machine M is in state q and the cursor of the first tape reads σ_1 , the second tape reads σ_2 , and so on, then the next state will be p , the cursor of the first tape will write ρ_1 over σ_1 and then move in the direction D_1 , of the second tape will write ρ_2 over σ_2 , then move in the direction D_2 , and so on. Initially, all tapes start with a \triangleright and this symbol can not be overwritten. Further, the cursors can not move to the left of the starting point \triangleright of the corresponding tape. The k -string Turing machine starts its computation in the configuration $(s; \triangleright, x; \triangleright, \sqcup; \dots; \triangleright, \sqcup)$, where \sqcup is the blank symbol, and $x \in (\Sigma - \sqcup)^*$, where $(\Sigma - \sqcup)^*$ denotes the set of all finite strings from $\Sigma - \sqcup$. If the machine M reaches state h , "yes", or "no", then the machine is said to have halted, accepted the input, or rejected the input. In the case that the Turing machine computes functions, the output y is read from the last tape. This can be denoted by $M(x) = \langle \text{"yes"}, y \rangle$, $\langle \text{"no"}, y \rangle$, or $\langle \text{"h"}, y \rangle$. If the machine computes forever, then it is denoted by $M(x) = \nearrow$.

Let us divert for a moment to discuss the issue of time in the above definition. The definition of the k tape Turing machine allows the simultaneous reading or writing of the tapes at each time step. What is of interest is that the definition does not speak of the time required for the tape heads to communicate with the part of the machine that computes the transition function. The definition in fact allows instantaneous transmission of information at this communication junction, which violates Einstein's special theory of relativity. In section 5, we will use internal probabilistic Turing machines to simulate finite state quantum mechanics. The time issue just described allows the internal probabilistic Turing machines to model collapse of the wave function. On the other hand, section 5 also suggests that information can not be instantaneously transmitted because given an input, it takes time for the internal Turing machines to compute the output. This suggests that the k tape Turing machines could be more fundamental than previously thought.

Now back to internal Turing machines. In order to apply nonstandard analysis to the above, we must write down the definition of the Turing machine in terms of sets, functions and relations. Let x and δ be the input to the Turing machine and the transition function as previously defined. We recursively define a sequence of tape states as follows. For $n \in \bar{\mathbb{N}}$, where $\bar{\mathbb{N}}$ was defined in section 2, and for $1 \leq i \leq k$,

$$(3.2) \quad \begin{aligned} \text{Tape}_i^n &: \bar{\mathbb{N}} \rightarrow \Sigma, \\ \text{Cursor}_i &: \bar{\mathbb{N}} \rightarrow \bar{\mathbb{N}}, \\ \text{State} &: \bar{\mathbb{N}} \rightarrow K \cup \{\text{"yes"}, \text{"no"}, h\}. \end{aligned}$$

Here, $\text{Tape}_i^n(m)$ is the symbol of the m^{th} cell of the i^{th} tape at the n^{th} time step, $\text{Cursor}_i(n)$ is the position of the i^{th} cursor at time step n , and $\text{State}(n)$ is the state of the machine at time step n . Mathematically, this models all the cells of the tapes for all time steps. The functions will be defined as follows. At time $t = 0$,

$$(3.3) \quad \begin{aligned} \forall i, \text{Cursor}_i(0) &= 0, \quad \text{State}(0) = s, \\ \text{Tape}_1^0(0) &= \triangleright, \\ \text{Tape}_1^0(j) &= x_j, 1 \leq j \leq m, \text{ where } x = x_1 x_2 \dots x_m \\ \text{Tape}_1^0(j) &= \sqcup \text{ otherwise,} \end{aligned}$$

for $2 \leq i \leq k$,

$$(3.4) \quad \begin{aligned} \text{Tape}_i^0(0) &= \triangleright, \\ \text{Tape}_i^0(j) &= \sqcup \text{ otherwise.} \end{aligned}$$

For time $t \geq 1$, if $\text{State}(t-1) \neq \text{"yes"}, \text{"no"}, \text{ or } h$, then first apply the transition function to the corresponding cells and state of the tapes at time $t-1$, and then update the tapes, cursors, and state. Mathematically, it is as follows. Let

$$(3.5) \quad \begin{aligned} \delta(\text{State}(t-1), \text{Tape}_1^{t-1}(\text{Cursor}_1(t-1)), \dots, \text{Tape}_k^{t-1}(\text{Cursor}_k(t-1))) = \\ (q; \sigma_1, D_1; \dots; \sigma_k, D_k), \end{aligned}$$

then for all i ,

$$(3.6) \quad \begin{aligned} \text{Cursor}_i(t) &= \text{Cursor}_i(t-1) \text{ if } D_i = - \\ &= \text{Cursor}_i(t-1) + 1 \text{ if } D_i = \rightarrow \\ &= \text{Cursor}_i(t-1) - 1 \text{ if } D_i = \leftarrow, \end{aligned}$$

$$(3.7) \quad \begin{aligned} \text{Tape}_i^t(\text{Cursor}_i(t-1)) &= \sigma_i, \\ \text{Tape}_i^t(j) &= \text{Tape}_i^{t-1}(j), \quad \forall j \neq \text{Cursor}_i(t-1), \end{aligned}$$

and $\text{State}(t) = q$. If $\text{State}(t-1) = \text{"yes"}, \text{"no"}, \text{ or } h$, then

$$(3.8) \quad \begin{aligned} \text{Tape}_i^t(n) &= \text{Tape}_i^{t-1}(n), \quad \forall i, n, \\ \text{Cursor}_i(t) &= \text{Cursor}_i(t-1), \\ \text{State}(t) &= \text{State}(t-1). \end{aligned}$$

Equation 3.8 says if the machine goes into the "yes", "no", or h state at time $t-1$, then nothing changes from then on. Now define the function $MTime : \{\text{State}\} \rightarrow \mathbb{N} \cup \infty$ as follows. If there is an $n \in \mathbb{N}$ such that $\text{State}(n) = \text{"yes"}, \text{"no"}, \text{ or } h$, then

$$(3.9) \quad MTime = \min \{n \mid \text{State}(n) = \text{"yes"}, \text{"no"}, \text{ or } h\},$$

else, $MTime = \infty$. Define $Mstate$ to be "yes", "no", h if there exist some $n \in \mathbb{N}$ such that $\text{State}(n) = \text{"yes"}, \text{"no"}, \text{ or } h$ respectively. Otherwise, $Mstate = \nearrow$ Lastly, define $Mout$ to be the content of the last tape (not including the infinite continual \sqcup strings) if the machine at some point enters the "yes", "no", or h state and define it to be \emptyset otherwise (the model does not allow an output if the machine computes forever). Thus, given an input x , $M(x)$ produces

$$(3.10) \quad \begin{aligned} \{\text{Tape}_1^0, \text{Tape}_2^0, \dots, \text{Tape}_k^0\}, \{\text{Tape}_1^1, \text{Tape}_2^1, \dots, \text{Tape}_k^1\}, \dots \\ \{\text{Cursor}_1, \text{Cursor}_2, \dots, \text{Cursor}_k\} \\ \text{State}, Mtime, Mout, Mstate. \end{aligned}$$

At this point, a comment is in order. The superstructure we are working with is defined over \mathbb{R} . Some of the symbols that we used are not elements of R . Since everything is countable, we can map each symbol to an element of \mathbb{R} . Technically, this has to be done in order to work with the superstructure.

More generally, let

$$(3.11) \quad \begin{aligned} TAPE = & \\ & \left\{ \left\{ \{Tape_1^0, Tape_2^0, \dots, Tape_k^0\}, \{Tape_1^1, Tape_2^1, \dots, Tape_k^1\}, \dots \right\} \mid k \in \mathbb{N}, \right. \\ & \left. \Sigma \in \mathcal{P}(\mathbb{N}) \right\}, \end{aligned}$$

where $\mathcal{P}(\mathbb{N})$ is the set of all subsets of \mathbb{N} , and for all i, j , $Tape_i^j : \bar{\mathbb{N}} \rightarrow \Sigma$. Now let $FUNC$ be the set of all functions from $\bar{\mathbb{N}}$ to elements of $\mathcal{P}(\mathbb{N})$, then in terms of the superstructures' language, the sets $FUNC$ and $TAPE$ can be characterized by

$$(3.12) \quad \begin{aligned} & (\forall x \in FUNC) (\exists \Sigma \in \mathcal{P}(\mathbb{N})) \left[(|\Sigma| \in \mathbb{N} \vee |\Sigma| = |\mathbb{N}|) \wedge (x : \bar{\mathbb{N}} \rightarrow \Sigma) \right], \\ & (\forall x \in TAPE) (\exists \Sigma \in \mathcal{P}(\mathbb{N})) (\exists k \in \mathbb{N}) \\ & (\forall i \in \{1, \dots, k\}) (\forall j \in \bar{\mathbb{N}}) \left(\exists Tape_i^j \in FUNC \right) \\ & \left[(x = \{ \{Tape_1^0, Tape_2^0, \dots, Tape_k^0\}, \{Tape_1^1, Tape_2^1, \dots, Tape_k^1\}, \dots \}) \wedge \right. \\ & \left. (Tape_i^j : \bar{\mathbb{N}} \rightarrow \Sigma) \right], \end{aligned}$$

where we use the notation $|\Sigma| = |\mathbb{N}|$ to mean Σ is countable infinite. Similarly, define

$$(3.13) \quad \begin{aligned} & (\forall x \in FUNC_{\bar{\mathbb{N}}}) \left[x : \bar{\mathbb{N}} \rightarrow \bar{\mathbb{N}} \right], \\ & (\forall x \in CURSOR) (\exists k \in \mathbb{N}) (\forall i \in \{1, \dots, k\}) (\exists Cursor_i \in FUNC_{\bar{\mathbb{N}}}) \\ & \left[x = \{Cursor_1, Cursor_2, \dots, Cursor_k\} \right], \\ & (\forall x \in STATE) (\exists K \in \mathcal{P}(\mathbb{N})) \left[(|K| \in \mathbb{N} \vee |K| = |\mathbb{N}|) \wedge (x : \bar{\mathbb{N}} \rightarrow K) \right], \\ & Mtime : (\Sigma - \sqcup)^* \rightarrow \mathbb{N} \cup \infty, \\ & Mout : (\Sigma - \sqcup)^* \rightarrow (\Sigma)^* \cup \emptyset, \\ & Mstate : (\Sigma - \sqcup)^* \rightarrow \{ "yes", "no", h, \nearrow \}. \end{aligned}$$

The reader should note that the sets defined above are sets that contains "all" functions of the particular type. For sanity of notations, the descriptions in (3.12) and (3.13) do not indicate "all".

Definition 3.1 Deterministic Turing Machine. *Given δ, K, Σ , and k , a deterministic Turing machine is a function*

$$(3.14) \quad \begin{aligned} M_{\delta, K, \Sigma, k} : (\Sigma - \sqcup)^* \rightarrow \\ TAPE \times CURSOR \times STATE \times \mathbb{N} \cup \infty \times (\Sigma)^* \cup \emptyset \times \{ "yes", "no", h, \nearrow \}, \end{aligned}$$

where the sets are defined above. Denote the set of all deterministic Turing machines by

$$(3.15) \quad DTM = \{ M_{\delta, K, \Sigma, k}, \forall \delta, K, \Sigma, k \}.$$

We will write

$$(3.16) \quad \begin{aligned} M_{\delta, K, \Sigma, k}(x) = \\ < MTAPES, MCURSORS, MSTATES, Mtime(x), Mout(x), Mstate(x) >. \end{aligned}$$

We can now apply nonstandard analysis and obtain the internal deterministic Turing machines. For any finite or *-finite set S , we will use the notation $|S|$ to denote the cardinality of S . Notice that the following are true.

$$(3.17) \quad \begin{aligned} (\forall M_{\delta, K, \Sigma, k} \in DTM) [k \in \mathbb{N}], \\ (\forall M_{\delta, K, \Sigma, k} \in DTM) (\exists \Sigma \in \mathcal{P}(\mathbb{N})) [(\sqcup \in \Sigma) \wedge (\triangleright \in \Sigma) \wedge (|\Sigma| \in \mathbb{N} \vee |\Sigma| = |\mathbb{N}|)], \\ (\forall M_{\delta, K, \Sigma, k} \in DTM) (\exists K \in \mathcal{P}(\mathbb{N})) [(s \in K) \wedge (|K| \in \mathbb{N} \vee |K| = |\mathbb{N}|)], \\ (\forall M_{\delta, K, \Sigma, k} \in DTM) \left[\delta : K \times \Sigma^k \rightarrow (K \cup \{h, "yes", "no"\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k \right]. \end{aligned}$$

Definition 3.2 Internal Turing Machines. *Let $*DTM$ denote the set of internal Turing machines, then $*DTM$ is characterized by the following.*

$$(3.18) \quad \begin{aligned} (\forall M_{\delta, K, \Sigma, k} \in *DTM) [k \in *N], \\ (\forall M_{\delta, K, \Sigma, k} \in *DTM) (\exists \Sigma \in *\mathcal{P}(\mathbb{N})) [(\sqcup \in \Sigma) \wedge (\triangleright \in \Sigma) \wedge (|\Sigma| \in *N \vee |\Sigma| = |*\mathbb{N}|)], \\ (\forall M_{\delta, K, \Sigma, k} \in *DTM) (\exists K \in *\mathcal{P}(\mathbb{N})) [(s \in K) \wedge (|K| \in *N \vee |K| = |*\mathbb{N}|)], \\ (\forall M_{\delta, K, \Sigma, k} \in *DTM) \\ \left[\delta : K \times \Sigma^k \rightarrow (K \cup \{h, "yes", "no"\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k \right], \\ (\forall x \in *FUNC) (\exists \Sigma \in *\mathcal{P}(\mathbb{N})) \left[(|\Sigma| \in *\mathbb{N} \vee |\Sigma| = |*\mathbb{N}|) \wedge (x : *\bar{\mathbb{N}} \rightarrow \Sigma) \right], \\ (\forall x \in *TAPE) (\exists \Sigma \in *\mathcal{P}(\mathbb{N})) (\exists k \in *N) \\ (\forall i \in \{1, \dots, k\}) (\forall j \in *\bar{\mathbb{N}}) \left(\exists Tape_i^j \in *FUNC \right) \\ \left[(x = \{ \{Tape_1^0, Tape_2^0, \dots, Tape_k^0\}, \{Tape_1^1, Tape_2^1, \dots, Tape_k^1\}, \dots \}) \wedge \right. \\ \left. (Tape_i^j : *\bar{\mathbb{N}} \rightarrow \Sigma) \right], \\ (\forall x \in *FUNC_{\bar{\mathbb{N}}}) \left[x : *\bar{\mathbb{N}} \rightarrow *\bar{\mathbb{N}} \right], \end{aligned}$$

$$\begin{aligned}
& (\forall x \in {}^*CURSOR) (\exists k \in {}^*\mathbb{N}) (\forall i \in \{1, \dots, k\}) (\exists Cursor_i \in {}^*FUNC_{\bar{\mathbb{N}}}) \\
& \left[x = \{Cursor_1, Cursor_2, \dots, Cursor_k\} \right], \\
& (\forall x \in {}^*STATE) (\exists K \in {}^*\mathcal{P}(\mathbb{N})) \left[(|K| \in {}^*\mathbb{N} \vee |K| = |{}^*\mathbb{N}|) \wedge (x : {}^*\bar{\mathbb{N}} \rightarrow K) \right], \\
& {}^*Mtime : (\Sigma - \sqcup)^* \rightarrow {}^*\mathbb{N} \cup \infty, \\
& {}^*Mout : (\Sigma - \sqcup)^* \rightarrow (\Sigma)^* \cup \emptyset, \\
& {}^*Mstate : (\Sigma - \sqcup)^* \rightarrow \{ "yes", "no", h, \nearrow \}.
\end{aligned}$$

Finally,

(3.19)

$$\begin{aligned}
& (\forall M_{\delta, K, \Sigma, k} \in {}^*DTM) \left[M_{\delta, K, \Sigma, k} : (\Sigma - \sqcup)^* \rightarrow \right. \\
& \left. {}^*TAPE \times {}^*CURSOR \times {}^*STATE \times {}^*\mathbb{N} \cup \infty \times (\Sigma)^* \cup \emptyset \times \{ "yes", "no", h, \nearrow \} \right].
\end{aligned}$$

Basically, the internal Turing machines are allowed to work with *-finite quantities. This allows the internal Turing machine to become infinitely more powerful than the classical Turing machines.

A language $L \subset (\Sigma - \sqcup)^*$ is said to be decided by a deterministic Turing machine M if and only if $\forall x \in L, Mstate(x) = "yes"$. The complexity class P is the set of all languages that are decidable in polynomial time. In our set theoretic notation, the definition is as follows.

Definition 3.3 *P*. *The complexity class P consists of all the languages that can be decided by deterministic Turing machines in polynomial time in the length of the input. In other-words,*

(3.20)

$$\begin{aligned}
& (\forall L \in P) (\forall x \in L) (\exists p \in \bar{\mathbb{N}}POLY) (\exists M_{\delta, K, \Sigma, k}^L \in DTM) \\
& \left[Mstate(x) = "yes" \wedge Mtime(x) \leq p(|x|) \right], \\
& (\forall L \in P) (\exists p \in \bar{\mathbb{N}}POLY) (\exists M_{\delta, K, \Sigma, k}^L \in DTM) (\forall y \in (\Sigma - \sqcup)^*) \\
& \left[(Mstate(y) = "yes" \wedge Mtime(y) \leq p(|y|)) \rightarrow y \in L \right],
\end{aligned}$$

where the Turing machine $M_{\delta, K, \Sigma, k}^L$ depends on L and $|x|$ denotes the length of the input string x .

The internal class *P allows internal Turing machines to decide internal languages in *-polynomial run time.

Definition 3.4 *P . *The internal complexity class *P consists of all the internal languages that can be decided by internal deterministic Turing machines in*

**-polynomial time in the length of the input. In other-words,*

$$(3.21) \quad \begin{aligned} & (\forall L \in {}^*P) (\forall x \in L) (\exists p \in {}^*\bar{N}POLY) (\exists M_{\delta, K, \Sigma, k}^L \in {}^*DTM) \\ & \left[Mstate(x) = \text{"yes"} \wedge Mtime(x) \leq p(|x|) \right], \\ & (\forall L \in {}^*P) (\exists p \in {}^*\bar{N}POLY) (\exists M_{\delta, K, \Sigma, k}^L \in {}^*DTM) (\forall y \in (\Sigma - \sqcup)^*) \\ & \left[(Mstate(y) = \text{"yes"} \wedge Mtime(y) \leq p(|y|)) \rightarrow y \in L \right], \end{aligned}$$

We now show that whatever the classical Turing machines can do, the internal Turing machines can do in **-polynomial time*.

Theorem 3.1. *Let $M \in DTM$ be a classical deterministic Turing machine and let L be a language for M such that for all $x \in L$, $Mout(x) \neq \nearrow$. In other-words, M eventually halts on input x . Let ${}^\sigma L$ be the **-embedding* of L ,*

$$(3.22) \quad {}^\sigma L = \{ {}^*x \mid x \in L \}.$$

*Then, on input $x \in {}^\sigma L$, the internal Turing machine *M will halt in **-polynomial time*. Further, the internal Turing machine will halt in the same state as M and output the same output as M .*

proof. Let $\omega \in {}^*\bar{N}$ be an infinite integer, then the polynomial $\omega n : {}^*\bar{N} \rightarrow {}^*\bar{N}$ is an internal polynomial. Let $Mtime(x)$ be the runtime of M on input $x \in L$, this number is standard finite (an element of \bar{N}) since M halts on input x .

Since M halts on x , ${}^*M({}^*x)$ will halt and the time for it to halt is $Mtime(x) < \omega|x|$. Further, on input *x , *M will halt in the same state as M and output the same output as M . \square

Remark. In other words, as far as *M is concerned, on input from L (or equivalently, the **-embedding* of L), *M will halt in **-polynomial time* regardless of the time for M to halt on input from L . This says that whatever a classical deterministic Turing machine does and halts, there is an internal deterministic Turing machine that can do the same thing in **-polynomial time*. We have to be very careful in using and interpreting nonstandard analysis. This does not mean that we can use the **-transform principle* and conclude that the classical deterministic Turing machines can do anything in polynomial time since the set ${}^\sigma L$ is an external set.

One way to interpret theorem 3.1 is as follows. We can think of the internal Turing machines as digital machines that can work with an infinite number of bits and can compute infinitesimally close to continuous variables. For example, let $r \in [0, 1]$ and let

$$(3.23) \quad r = \lim_{k \rightarrow \infty} \sum_{i=0}^k a_i 2^{-i} = \lim_{k \rightarrow \infty} s_k,$$

be its base 2 expansion. Let $\omega \in {}^*\bar{N}$ be a nonstandard infinite integer and $s_\omega \in {}^*\{s_k\}$, then s_ω is represented by ω number of bits and it is infinitesimally close to r . Suppose it takes the internal Turing machine *M k units of time to write

ω number of bits. Theorem 3.1 says that on input $x \in {}^\sigma L$, ${}^*M(x)$ will halt in less than $k|x|$ units of time. We can think of this as exploiting the internal Turing machines' ability to efficiently compute quantities that are infinitesimally close to real variables. For more thoughts on time issues, see section 5.

Now we show that the internal deterministic Turing machines can do things which the classical deterministic Turing machines can not do. In particular, the internal deterministic Turing machines can decide ${}^\sigma Halt$, the $*$ -embedding of the classical $Halt$ language. It is well known that $Halt$ can not be decided by the classical deterministic Turing machines. As with the previous theorem, we must be careful how we interpret this because ${}^\sigma Halt$ is an external set. This does not imply that the internal deterministic Turing machines can decide *Halt because this would imply that the classical deterministic Turing machine can decide $Halt$ by the transfer principle. In fact, *Halt can not be decided by the internal deterministic Turing machines. Thus, the internal Turing machines are infinitely more powerful than the classical Turing machines but at the same time, the internal machines have the same type of limitations as the classical machines.

Proposition 3.2 Universal Turing machines. *There exists a universal deterministic Turing machine which can simulate any other deterministic Turing machine in polynomial time. In other words,*

$$(3.24) \quad \left(\exists U_{\delta, \mathbb{N}, \mathbb{N}, k} \in DTM \right) \left(\forall M_{\delta', K, \Sigma, k'} \in DTM \right) \left(\forall x \in (\Sigma - \sqcup)^* \right) \left(\exists p \in \bar{\mathbb{N}}POLY \right) \\ \left[Uout(M_{\delta', K, \Sigma, k'}; x) = Mout(x) \wedge Ustate(M_{\delta', K, \Sigma, k'}; x) = Mstate(x) \wedge \right. \\ \left(Mtime(x) \neq \infty \rightarrow Utime(M_{\delta', K, \Sigma, k'}; x) \leq p(Mtime(x)) \right) \wedge \\ \left. \left(Mtime(x) = \infty \rightarrow Utime(M_{\delta', K, \Sigma, k'}; x) = \infty \right) \right].$$

proof. See [8]. \square

Remark 3.1. The notation $U(M; x)$ implies an encoding of the deterministic Turing machine M and input x (for M) as input for the universal Turing machine U . Further, the number of symbols and the number of states for U is allow to be countable infinite to compensate for arbitrary $M_{\delta', K, \Sigma, k'}$

For the rest of this section, we will fix a universal Turing machine U .

Definition 3.5 Halting. *Let M be a deterministic Turing machine and x be an input for M . Define the language $Halt$ over the alphabet of U , the universal Turing machine as*

$$(3.25) \quad (\forall \langle M; x \rangle \in Halt) \left[Mstate(x) \neq \nearrow \right]$$

In otherwords, $Halt$ consists of the encoding of Turing machines M and input x for the universal Turing machine U such that M eventually halts on input x .

We now modify the universal Turing machine U into another Turing machine. Let $n \in \mathbb{N}$, define the Turing machine U_n as follows. On input $M; x$, U_n simulates

M with input x . If $U_n(M; x)$ halts in time less than n , then U_n computes as U , otherwise, $U_n(M; x)$ goes into the "no" state at step n . In terms of our set theoretic notation, this can be written as (with an abuse of notation)

$$(3.26) \quad \begin{aligned} & (\forall n \in N) [U_n \in DTM]; \\ & (\forall M_{\delta, K, \Sigma, k} \in DTM) (\forall x \in (\Sigma - \sqcup)^*) \\ & [(Mstate(x) = \text{"yes"}, \text{"no"}, h \wedge Mtime(x) < n) \rightarrow \\ & (U_nstate(M; x) = \text{"yes"} \wedge U_n time(M; x) < n) \wedge \\ & (else \rightarrow U_nstate(M; x) = \text{"no"} \wedge U_n time(M; x) = n)]. \end{aligned}$$

Theorem 3.3. *Let*

$$(3.27) \quad \sigma Halt = \{ \langle *M; *x \rangle \mid \langle M; x \rangle \in Halt \},$$

be the $*$ -embedding of $Halt$, then there is an internal deterministic Turing machine U_ω which on input $\langle *M, *x \rangle \in \sigma Halt$ will halt in the state "yes". Further, if $\langle *M, *x \rangle \notin \sigma Halt$, U_ω will halt in the "no" state.

Proof. The $*$ -transform of (3.26) is

$$(3.28) \quad \begin{aligned} & (\forall n \in *N) [U_n \in *DTM]; \\ & (\forall M_{\delta, K, \Sigma, k} \in *DTM) (\forall x \in (\Sigma - \sqcup)^*) \\ & [(Mstate(x) = \text{"yes"}, \text{"no"}, h \wedge Mtime(x) < n) \rightarrow \\ & (U_nstate(M; x) = \text{"yes"} \wedge U_n time(M; x) \leq n) \wedge \\ & (else \rightarrow U_nstate(M; x) = \text{"no"} \wedge U_n time(M; x) = n)]. \end{aligned}$$

Now let $n = \omega \in *N$ be a nonstandard infinite integer, then U_ω is an internal Turing machine. Suppose $\langle M; x \rangle \in Halt$, $*M(*x)$ produces $*Mstate(x) = \text{"yes"}, \text{"no"},$ or h , and $*Mtime(x) = t$ for some standard finite t . Since t is standard finite, we have $t < \omega$, which implies that $U_\omega(M; x)$ halts with a "yes" at time t . Further, if $M \in DTM$ and $M(x)$ does not halt, then $*M(*x)$ will not halt and $U_\omega(M; x)$ will halt with a "no" at time ω . \square

Theorem 3.4. *The language $Halt$ is not decidable by the classical deterministic Turing machines DTM .*

Proof. See [8] \square

Theorem 3.5. *The internal set $*Halt$ is not decidable by the internal deterministic Turing machines $*DTM$.*

Proof. This is a property of the $*$ -transform \square

4. Real computations and Asymptotic Behaviors. The internal Turing machines could be useful in determining asymptotic behaviors of real number computations. We will use the real number computation model proposed in [6]. The classical Turing machines are modeled with discrete mathematics. It would be unnatural to use limits to obtain asymptotic behaviors. The internal Turing machine model is quite natural for this purpose since it keeps all the combinatoric structures of the classical model. This could be a bridge for the gap between the discrete model

of computational complexity theory and the continuous variable model of physical theories. We will illustrate the latter idea in the next section.

We now set the notations and foundations for the real number computation model proposed in [6]. A dyadic rational number d is a number of the form $d = \frac{m}{2^n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N} \cup 0$. Denote by

$$(4.1) \quad D_n = \{m * 2^{-n} | m \in \mathbb{Z}\},$$

the set of dyadic rational numbers with precision n and $D = \cup_{n=1}^{\infty} D_n$ the set of all dyadic rational numbers.

Definition 4.1 Classical Computable Real Numbers. *Let $x \in \mathbb{R}$, x is said to be computable in polynomial time if the following is true. There exists a function $\Phi : \mathbb{N} \rightarrow D$ such that for all $n \in \mathbb{N}$, the precision $\text{prec}(\Phi(n)) = n$, $|\Phi(n) - x| \leq 2^{-n}$, and there exists a Turing machine that computes $\Phi(n)$ in polynomial time. In other words,*

$$(4.2) \quad \begin{aligned} &(\forall n \in \bar{\mathbb{N}})[\Phi(n) \in D_n], \\ &(\forall n \in \bar{\mathbb{N}})[|\Phi(n) - x| \leq 2^{-n}], \\ &(\exists M \in DTM) (\exists p \in \bar{\mathbb{N}}POLY) (\forall n \in \bar{\mathbb{N}}) [Mout(n) = \Phi(n) \wedge Mtime(n) \leq p(n)]. \end{aligned}$$

Theorem 4.2. *Let $x \in \mathbb{R}$ be computable in polynomial time in the sense of definition 4.1, then there exists an internal Turing machine which can compute an internal dyadic number $d \in {}^*D$ that is infinitesimally close to x . Further, this computation is done in $*$ -polynomial time.*

Proof. Applying the $*$ -transform to (4.2) yields

$$(4.3) \quad \begin{aligned} &(\forall n \in {}^*\bar{\mathbb{N}})[{}^*\Phi(n) \in {}^*D_n], \\ &(\forall n \in {}^*\bar{\mathbb{N}})[|{}^*\Phi(n) - x| \leq 2^{-n}], \\ &(\exists M \in {}^*DTM) (\exists p \in {}^*\bar{\mathbb{N}}POLY) (\forall n \in {}^*\bar{\mathbb{N}}) \\ &[Mout(n) = {}^*\Phi(n) \wedge Mtime(n) \leq p(n)]. \end{aligned}$$

In particular, let $n = \omega$ be an infinite integer, then $|{}^*\Phi(\omega) - x| \leq 2^{-\omega}$ and $2^{-\omega}$ is an infinitesimal. \square

For any real number $x \in \mathbb{R}$, the internal Turing machines might be able to output a $*$ -finite number of bits which could represent an element (in ${}^*\mathbb{R}$) that is infinitesimally close to x . However, unless x has a standard finite numerical representation, to output *x (the $*$ -embedding of x) would require the machine to write a ${}^*\infty$ number of bits on one of its tapes. This is the same limitation that the classical Turing machines have. Thus, for the internal Turing machines, in general it is not possible to compute the standard part function since $st(x)$ could requires an infinite amount of time to output. This is as expected since the standard part function is an external function. As far as outputting the exact value of *x is concerned, the best we could hope for is that an observer looks at the output of the internal machine and then apply the standard part operation to the machine's output. As far as humans are concerned, assuming that we can only

measure a standard finite number of bits, we will never be able to observe the exact value of x . If the internal Turing machines were a reasonable model for nature's computation structure, then the fact that it can never output the exact value of x indicates that it might be more suitable to model physical phenomena with *-finite nonstandard analysis rather than continuous variables in \mathbb{R} .

5. Probabilistic Turing Machines and Simulating Finite State Quantum Mechanics. In this section, we take on the idea that the internal Turing machine model is a reasonable model for nature's behind the scene physical computations. We assume that the working tapes of the internal Turing machines are hidden from us but we can observe the machines' output tapes when we perform a measurement. We will use an internal probabilistic Turing machine to simulate time independent finite state quantum mechanics in *-polynomial time. As far as we know, there is no evidence that nature uses internal Turing machines to compute its physical processes. Hence, we will think of this section as a thought experiment.

We believe that there are a few reasons why it is interesting to do this thought experiment. The first is that the internal Turing machines are extensions of the classical ones, and the internal ones are capable of doing computations that are infinitesimally close to continuous variables. This could be a digital bridge between the Church-Turing thesis, experimental science, and continuous variable modeling. The second is that the algorithm described below (using internal probabilistic Turing machine model) is not at odds with instantaneous collapse of the wave function and no information can travel faster than the speed of light. Further, it suggests that the classical definition of Turing machines could be much more fundamental than previously thought; the definition of the classical Turing machines might be more capable of dealing with quantum phenomena than previously thought. Recently, there has been much research activity in quantum computing and quantum Turing machines (see [1],[2], [10], [11], and references within). It is now widely believed that the quantum Turing machines are more powerful than the probabilistic Turing machines. The ideas in section could be of interest for quantum computing research.

Classically, a probabilistic Turing machines consists of two elements $\langle M, \Psi \rangle$, where M is a deterministic Turing machine and Ψ is a random coin flip. The operation of the machine is roughly described as follows (see [4] for a full description). At each step of the computation, Ψ flips its coin and with probability 1/2 outputs a 0 or 1 on a special random bit tape, then the machine M makes its next move according to all its tapes including the random bit tape. Let PTM denote the set of all probabilistic Turing machines, without spelling out the superstructure set theoretic definition of the probabilistic Turing machines, we can characterize PTM as follows.

$$(5.1) \quad (\forall \langle M, \Psi \rangle \in PTM)[M \in DTM \wedge prob(\Psi = 1) = .5 \wedge prob(\Psi = 0) = .5].$$

Its *-transform is

$$(5.2) \quad (\forall \langle M, \Psi \rangle \in *PTM)[M \in *DTM \wedge prob(\Psi = 1) = .5 \wedge prob(\Psi = 0) = .5].$$

Thus,

Definition 5.1 Internal Probabilistic Turing Machines. *An internal probabilistic Turing machines consists of two elements $\langle M, \Psi \rangle$ where $M \in {}^*DTM$ is an internal deterministic Turing machine and Ψ is a 0, 1 coin flip with probability 1/2. The machine operates as follows. At each step, Ψ flips a coin and outputs a 0 or 1 on its random bit tape, then the machine M makes its next move according to all its tapes including the random bit tape.*

We now proceed to simulate finite state quantum mechanics. This will be done with two algorithms, the evolution and measurement algorithm. We first describe the evolution. Let $U = U_R + iU_I$ be an n by n time independent unitary matrix where U_R and U_I are the real and imaginary parts of U . Let $v^{in} = \sum_{k=1}^n a^k e_k$ be a quantum state where the e_k 's form a basis for the underlying Hilbert space. Let $\omega \in {}^*\mathbb{N}$ be an infinite nonstandard integer. The algorithm is independent of which infinite nonstandard integer is picked. For $1 \leq k, j \leq n$, let $(U_{R\omega})_{k,j}, (U_{I\omega})_{k,j} \in D_\omega$ be the approximation of $(U_R)_{k,j}$ and $(U_I)_{k,j}$ by elements of D_ω (the internal set of dyadic rational numbers with precision ω as defined in the previous section) such that the approximation is infinitesimally close. Denote this by

$$(5.3) \quad U_R \approx U_{R\omega}, U_I \approx U_{I\omega},$$

where $U_{R\omega}$ and $U_{I\omega}$ denote the matrices of the corresponding ω precision approximations. Similarly, let v_ω^{in} be an ω precision approximation to v^{in} and denote it by $v^{in} \approx v_\omega^{in}$. Finally, let $U_\omega = U_{R\omega} + iU_{I\omega}$, and

$$(5.4) \quad Uv^{in} = v^{out} = \sum_{k=1}^n b^k e_k$$

We now describe the input to the internal probabilistic Turing machine which will simulate the evolution of finite state quantum mechanics. Recall that the internal Turing machines are allowed to have a $*$ -finite number of tapes, this allows for enough memory to deal with n number of quantum states for arbitrary standard finite n . We will take U_ω, v_ω^{in} , and 1^ω (a string consisting of an ω number of 1's) as the input to the machine. This input could be computed in $*$ -polynomial time by another internal machine as described in the previous section or it could be obtained from an internal oracle machine. We might think of v^{in} as the initial state of a quantum experiment, U as the evolution operator corresponding to the experiment, the environment dictates 1^ω (or ω is a universal constant, or ω is an intrinsic property of the probabilistic Turing machine), and the internal probabilistic Turing machine reads in v^{in}, U as v_ω^{in}, U_ω and then computes. In any case, we will assume that the machine obtains U_ω, v_ω^{in} , and 1^ω as input. This assumption is justified by the thesis that nature takes an initial state and the environment as input and computes the physical evolution. The size of the above input is linear ω .

Upon receiving the input, the internal machine computes as follows. First, it computes $\bar{\omega} = \lfloor \log_2 \omega \rfloor$, which is an infinite nonstandard integer. This computation can be done in $*$ -polynomial time in ω since its classical equivalent can be done in polynomial time, i.e.,

$$(5.5) \quad (\exists M \in DTM)(\exists p \in \bar{N}POLY)(\forall k \in \mathbb{N})[Mout(k) = \lfloor \log_2 k \rfloor \wedge Mtime(k) \leq p(k)].$$

Next, the internal machine computes $U_\omega v_\omega^{in}$. Let us denote the output of this stage by $\sum_{k=1}^n b_\omega^k e_k$. This computation can also be computed in $*$ -polynomial time in ω since its classical equivalent can be computed in polynomial time, i.e.,

$$(5.6) \quad (\exists M \in DTM)(\exists p \in \bar{N}POLY)(\forall k \in \mathbb{N}) \\ [Mout(U_k, v_k^{in}) = U_k v_k^{in} \wedge Mtime(U_k, v_k^{in}) \leq p(k)].$$

If the quantum experiment or environment does not perform a measurement, the algorithm writes the out state computed above to the output tape and halts, otherwise, the algorithm proceeds to the second part, the measurement algorithm. Notice that if it halts, then the output state vector will be entry-wise infinitesimally close to v^{out} , and we can interpret this as nature putting the quantum system into a state that is infinitesimally close to v^{out} . Now suppose a measurement is performed. After computing the out state, the machine then computes $|b_\omega^k|^2, 1 \leq k \leq n$ with $\bar{\omega}$ precision. This too can be done in $*$ -polynomial time since its classical equivalent can be done in polynomial time. Let us denote the output of this stage by $pr_k, 1 \leq k \leq n$. Notice that

$$(5.7) \quad pr_k \approx |b^k|^2, |b^k|^2 \leq 1, \sum_{k=1}^n pr_k \approx \sum_{k=1}^n |b^k|^2 = 1,$$

since $\bar{\omega}$ is nonstandard infinite (the error in the $\bar{\omega}$ roundoff is less than $2^{-\bar{\omega}}$). Further, pr_k having precision $\bar{\omega}$ means that it is of the form $\frac{m_k}{2^{\bar{\omega}}}$ for $m_k \in {}^*\bar{\mathbb{N}}$, or more generally, $pr_k \in {}^*D_{\bar{\omega}}$ where ${}^*D_{\bar{\omega}}$ is defined in the previous section.

The machine now uses its coin flip ability and performs the last stage of the computation. If there is a k such that $1 \leq pr_k$, then the machine outputs k and halts. This can be done in $*$ -polynomial time since its classical equivalent can be done in polynomial time. If for all $k, pr_k < 1$, then the machine proceeds as follows. Recall that $pr_k = \frac{m_k}{2^{\bar{\omega}}}$, and $\sum_{k=1}^n pr_k \approx 1$. Let $\sum_{k=1}^n pr_k = 1 + \epsilon$, where ϵ is an infinitesimal. The machine computes the internal number $2^{\bar{\omega}}$, then computes m_k for all k , and then computes $\sum_{k=1}^n m_k = T$. This can be done in $*$ -polynomial time in ω since $\bar{\omega} = \lfloor \log_2 \omega \rfloor$. The machine now flips its coin T number of times, outputs a state k according to the probability distribution $\{\frac{m_j}{T}\}$, and then halts. The last step can be done in $*$ -polynomial time in its input size since

$$(5.8) \quad T = \sum_{k=1}^n m_k = 2^{\bar{\omega}} \sum_{k=1}^n pr_k < 22^{\bar{\omega}} \leq 2\omega.$$

We now need to show that the probability distribution $\{\frac{m_j}{T}\}$ is infinitesimally close to the distribution $\{\frac{m_k}{2^{\bar{\omega}}}\}$. This is true because

$$(5.9) \quad \frac{m_k}{\sum_j m_j} = \frac{\frac{m_k}{2^{\bar{\omega}}}}{\frac{\sum_j m_j}{2^{\bar{\omega}}}} = \frac{\frac{m_k}{2^{\bar{\omega}}}}{1 + \epsilon},$$

which implies

$$(5.10) \quad \frac{m_k}{\sum_j m_j} \approx \frac{m_k}{\sum_j m_j} (1 + \epsilon) = \frac{m_k}{2^{\bar{\omega}}}.$$

Equations (5.7) and (5.10) imply that for all k , $\frac{m_k}{T}$ is infinitesimally close to $|b^k|^2$. Finally, since each stage of the computation can be done in $*$ -polynomial time, the complete algorithm halts in $*$ -polynomial time. Notice that if a partial measurement is performed, then the Turing machine must compute the output state after the partial measurement. For our purpose, we will not need to do this.

The result of the above algorithm is that it outputs a state k with probability infinitesimally close to $|b^k|^2$, which is the probability dictated by the theory of quantum mechanics. Further, the algorithm halts in $*$ -polynomial time. As an observer, a human can look at the output tape of the internal Turing machine and measures a standard finite state, namely a state e_k where $1 \leq k \leq n$. Further, as an observer, a human can only perform the experiment or run the above algorithm a standard finite number of times. This would imply that the observer will never be able to detect the fact that the statistics of the output of the above algorithm is only infinitesimally close to the result dictated by theory of quantum mechanics.

There are a few interesting things that we can conclude by taking on the idea that nature behaves this way. The first is that given an input, nature requires time to compute the output in both the evolution and measurement algorithm. In the most loose interpretation, this might be related to no information can travel faster than speed of light. In which case, the time required to perform one step of the above computation is related to the speed of light. On the other hand, collapse of the wave function says otherwise. It says that measurements on spatially separated quantum systems can instantaneously influence one another. At first thought, the above algorithm is at odds with collapse of the wave function, but in fact, that need not be the case. This is because the definition of a multi-tape deterministic Turing machine allows at each time step the simultaneous reading and writing of one cell on each of its tapes. For example, suppose Alice and Bob each has a qubit. They run the the above evolution algorithm with the appropriate unitary operator and obtain a state that is infinitesimally close to a Bell state, i.e., the internal Turing machine computes

$$(5.11) \quad b_\omega^1|0\rangle|0\rangle + b_\omega^2|1\rangle|1\rangle \approx \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle,$$

and writes the state $b_\omega^1|0\rangle|0\rangle + b_\omega^2|1\rangle|1\rangle$ on a working tape (or puts the qubits into the Bell state) and then temporarily halts until further notice to perform the measurement algorithm. At this point, we can think of nature putting Alice and Bob's qubits into the Bell state, and we assume that the machine has two output tapes, one for each qubit. Alice and Bob now each takes their output tape (their qubits) and they separate light years apart. After the spatial separation, Bob (or Alice) gives the internal Turing machine the go to perform the measurement stage of the computation. Suppose the machine uses its coin flip ability and outputs $|0\rangle|0\rangle$. At output, the machine writes the state $|0\rangle$ on Bob's tape and "simultaneously" writes $|0\rangle$ on Alice's tape. The definition of Turing machine does not prevent this from happening even though the tapes are separated light years apart. Hence, the Turing machine model has the ability to model instantaneous transmission of information during the read and write operation at each time step. This suggests that the Turing machine model might be more fundamental than previously thought.

As a final note for this section, notice that the standard part operation is not needed since the output of the algorithm is standard finite. Further, the infinite

number of bits of computations are completely oblivious to the the human observer since the working tapes of the Turing machines are hidden from the observer. Thus, the infinite number of bits of computation is a black box for the observer. Finally, the working tapes are hidden but no hidden variables are introduced in this thought experiment. In other words, at no time before the measurement is performed (more precisely, the coin flips) there exist variables which if know will completely determine the outcome of the output state.

6. Nondeterministic Turing Machines. In this last section, we show that if $P \neq NP$, then there exists problems which the internal Turing machines can solve but not in $*$ -polynomial time. This is basically a property of the $*$ -transform. In the previous sections, we mainly dealt with problems that the internal Turing machines can solve in $*$ -polynomial time (except deciding $*Halt$, which can not be decided). Thus, the internal Turing machines are very powerful but they also have limitations similar to the classical Turing machines. In terms of the physical, if the internal Turing machines properly model nature's computational power and if nature views polynomial time in a sense similar to ours, then nature would favor physical processes that are computable in $*$ -polynomial time.

The nondeterministic Turing machines are similar to the deterministic ones except that the transition function δ is allowed to be a transition relation. For notation convenience, we will just sketch the set theoretic definition and then apply the $*$ -transform and obtain the internal nondeterministic Turing machines. For a k tape nondeterministic Turing machine, we have

$$(6.1) \quad \delta \subset K \times \Sigma^k \times (K \cup \{h, \text{"yes"}, \text{"no"}\}) \times (\Sigma \times \{\leftarrow, \rightarrow, -\})^k.$$

For example, at time $t = 0$, the tape configuration would be given by

$$(6.2) \quad \{Tape_1^0, Tape_2^0, \dots, Tape_k^0\}.$$

The transition relation will then take the Turing machine into a computational tree. If the relation takes the machine into three configurations, then at time $t = 1$, we would get three configurations

$$(6.2) \quad \left\{ \begin{aligned} &\{Tape_1^{1,1}, Tape_2^{1,1}, \dots, Tape_k^{1,1}\}, \\ &\{Tape_1^{1,2}, Tape_2^{1,2}, \dots, Tape_k^{1,2}\}, \\ &\{Tape_1^{1,3}, Tape_2^{1,3}, \dots, Tape_k^{1,3}\} \end{aligned} \right\},$$

and similarly for the cursors, states, etc. At each time step, the machine would branch off from the configurations of the previous time step and continues with the computational tree. The machine halts if one of the configurations (computational branches) halts, otherwise, it computes forever. If the machine halts and one of the halting branches halts in "yes" state, then the machine is said to accept the input x . In which case, we will write $Mstate(x) = \text{"yes"}$. Notice that there could be different halting states for different computational paths. The time of the computation is somewhat vague since there are many computational paths where some of which might halt or some of which might compute forever. To make the

definition of computational time complete, we define three computational times, the time for it to halt into the "yes", "no" and h state. They are defined as the minimum amount of time over all configurations that goes into the "yes", "no" and h state respectively; if the machine never goes into any of three states, we will define the time to be ∞ for that particular state. We will denote them by $Mtime_y(x)$, $Mtime_n(x)$ and $Mtime_h(x)$. The output of the machine also has this ambiguity. One way of defining the output is to restrict the output to y if all computational branches that halts in the "yes" state outputs y (see [4]). For our purpose, we will not be needing the output of the machine. We will leave the definition of the output open. We will mainly be interested in the machine going into the "yes" state.

The set of all nondeterministic Turing machines can be cast into set theoretic languages as in the previous section for deterministic Turing machines. For notation sanity, we will not spell this out. The set of all nondeterministic Turing machines will be denoted by $NDTM$. Its $*$ -transform, the set of internal nondeterministic Turing machines will be denoted $*NDTM$.

Definition 6.1 *NP.* The complexity class NP consists of all the languages that can be decided by nondeterministic Turing machines in polynomial time in the length of the input. In other-words,

$$(6.3) \quad \begin{aligned} & (\forall L \in NP) (\forall x \in L) (\exists p \in \bar{NPOLY}) (\exists M_{\delta,K,\Sigma,k}^L \in NDTM) \\ & \left[Mstate(x) = \text{"yes"} \wedge Mtime_y(x) \leq p(|x|) \right], \\ & (\forall L \in NP) (\exists p \in \bar{NPOLY}) (\exists M_{\delta,K,\Sigma,k}^L \in DTM) (\forall y \in (\Sigma - \sqcup)^*) \\ & \left[(Mstate(y) = \text{"yes"} \wedge Mtime_y(y) \leq p(|y|)) \rightarrow y \in L \right]. \end{aligned}$$

Definition 6.2 **NP.* The internal complexity class $*NP$ consists of all the internal languages that can be decided by internal non-deterministic Turing machines in $*$ -polynomial time in the length of the input. In other-words,

$$(6.4) \quad \begin{aligned} & (\forall L \in *NP) (\forall x \in L) (\exists p \in *\bar{NPOLY}) (\exists M_{\delta,K,\Sigma,k}^L \in *NDTM) \\ & \left[Mstate(x) = \text{"yes"} \wedge Mtime_y(x) \leq p(|x|) \right], \\ & (\forall L \in *NP) (\exists p \in *\bar{NPOLY}) (\exists M_{\delta,K,\Sigma,k}^L \in *NDTM) (\forall y \in (\Sigma - \sqcup)^*) \\ & \left[(Mstate(y) = \text{"yes"} \wedge Mtime_y(y) \leq p(|y|)) \rightarrow y \in L \right]. \end{aligned}$$

The $*$ -transform property again shows that the internal Turing machines has the same type of limitations as the classical Turing machines.

Theorem 6.1. $P = NP$ if and only if $*P = *NP$ and $P \neq NP$ if and only if $*P \neq *NP$.

Proof. This comes from the $*$ -transform. \square

Corollary 6.2. *Suppose $P \neq NP$, then there are internal languages which can not be decided in $*$ -polynomial time by internal deterministic Turing machines.*

Proof. This follows from theorem 6.1. \square

While the internal Turing machines has the same types of limitations as the classical ones, the internal ones are much more powerful.

Theorem 6.2. *Let $L \in NP$ and*

$$(6.5) \quad {}^\sigma L = \{ *x \mid x \in L \},$$

be the $$ -embedding of L . Then, there exists an internal deterministic Turing machine $*M$ such that for all $x \in {}^\sigma L$, on input x , $*M$ outputs "yes" in $*$ -polynomial time.*

Proof. Any language L in NP that is decided by a nondeterministic Turing machine N in polynomial time $p(n)$ can be decided by a deterministic Turing machine M in time $\mathcal{O}(c^{p(n)})$, where $c > 1$ is a constant depending on N (see [4] and [8]). The theorem follows from Theorem 3.1 \square

REFERENCES

1. D. Aharonov, *Quantum Computation - A Review*, Annual Review of Computational Physics, World Scientific, volume VI, ed. Dietrich Stauffer (1998).
2. E. Bernstein and U. Vazirani, *Quantum Complexity Theory*, SIAM J. Comput. 26(5): 1411-1473 (1997).
3. N. Cutland, *NonStandard Analysis and its Applications*, Cambridge University Press, 1988.
4. D. Du and K. Ko, *Theory of Computational Complexity*, John Wiley and Son, 2000.
5. A. Hurd and P. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, 1985.
6. K. Ko, *Complexity Theory of Real Functions*, Birkhauser, 1991.
7. E. Nelson, *Internal Set Theory: A New Approach to Nonstandard Analysis*, Bulletin of American Mathematical Society, 83, 1165-1198 (1977).
8. C. Papadimitriou, *Computational Complexity*, Addison-Wesley, 1995.
9. A. Robinson, *Nonstandard Analysis*, Princeton Univ Pr; Revised edition, 1996.
10. P. Shor, *Polynomial-Time Algorithms For Prime Factorization and Discrete Logarithms on a Quantum Computer*, SIAM J. Comput. 26(5): 1484-1509 (1997).
11. K. Stroyan and J. Luxemburg, *Introduction to the Theory of Infinitesimals*, Academic Press, 1976.
12. M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.

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