THE NEWTON POLYGON OF A PLANAR SINGULAR CURVE AND ITS SUBDIVISION

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Abstract. Let a planar algebraic curve C be defined over a valuation field by an equation $F(x,y)=0$. Valuations of the coefficients of F define a subdivision of the Newton polygon Δ of the curve C.

If a given point p is of multiplicity m for C, then the coefficients of F are subject to certain linear constraints. These constraints can be visualized on the above subdivision of Δ. Namely, we find a distinguished collection of faces of the above subdivision, with total area at least $\frac{3}{8}m^2$. In a sense, the union of these faces is "the region of influence" of the singular point p on the subdivision of Δ. Also, we discuss three different definitions of a tropical point of multiplicity m.

0. Introduction

Fix a non-empty finite subset $\mathcal{A}\subset\mathbb{Z}^2$ and any valuation field $\mathbb{K}$. We consider a curve C given by an equation $F(x,y)=0$, where

$$F(x,y)=\sum_{(i,j)\in\mathcal{A}} a_{ij}x^iy^j,\ a_{ij}\in\mathbb{K}^*. \tag{1}$$

Suppose that we know only the valuations of the coefficients of the polynomial $F(x,y)$. Is it possible to extract any meaningful information from this knowledge? Unexpectedly, many geometric properties of C are visible from such a viewpoint.

The *Newton polygon* $\Delta=\Delta(\mathcal{A})$ of the curve C is the convex hull of $\mathcal{A}$ in $\mathbb{R}^2$. The *extended Newton polyhedron* $\widetilde{\mathcal{A}}$ of the curve C is the convex hull of the set $\{((i,j),s)\in\mathbb{R}^2\times\mathbb{R}|(i,j)\in\mathcal{A}, s\leq \mathrm{val}(a_{ij})\}$. Projection of all the faces of $\widetilde{\mathcal{A}}$ along $\mathbb{R}$ induces a *subdivision* of Δ. Note, that the valuations of the coefficients of F completely determine $\widetilde{\mathcal{A}}$ and this subdivision of Δ.

A point p is *of multiplicity* m (or m-fold point) for C if the lowest term in the Taylor expansion of F at p has degree m. Nagata's conjecture proposes the estimate $d\geq m\sqrt{n}$ for the minimal degree d of a curve which has $n>9$ points of multiplicity m in general position. Motivated by this conjecture we study the following question: how do the points of multiplicity m on C influence the subdivision of Δ? This paper is devoted to the case of one m-fold point, whereas [15] concerns the case of several m-fold points.

By definition, the non-Archimedean amoeba of C is $\mathrm{Val}(C)=\{(\mathrm{val}(x),\mathrm{val}(y))|(x,y)\in C\}$. Also, we define the tropical curve $\mathrm{Trop}(C)$ as the set of non-smooth points of the function $\max\limits_{(i,j)\in\mathcal{A}}(iX+jY+\mathrm{val}(a_{ij}))$. It is known that $\mathrm{Val}(C)\subset\mathrm{Trop}(C)$. Furthermore, $\mathrm{Trop}(C)$ is a graph which is combinatorially dual to the subdivision of Δ (described above), in particular each vertex V of $\mathrm{Trop}(C)$ corresponds to a face $d(V)$ of this subdivision of Δ.

Fix a point $p=(p_1,p_2)\in(\mathbb{K}^*)^2$. Define $P=\mathrm{Val}(p)=(\mathrm{val}(p_1),\mathrm{val}(p_2))$. We consider a curve C given by (1) such that p is of multiplicity m for C. In such a case, the coefficients a_{ij} of C satisfy a certain set of $\frac{m(m+1)}{2}$ linear constraints. In their turn, the constraints for the numbers $\mathrm{val}(a_{ij})$ manifest themselves via the fact that the subdivision of Δ enjoys very special properties.

In particular, there is a certain collection $\mathfrak{I}(P)$ of vertices of $\mathrm{Trop}(C)$ (Figure 1, lower row). We estimate the total area of the faces in the subdivision of Δ, dual to the vertices in $\mathfrak{I}(P)$ (Figure 1,

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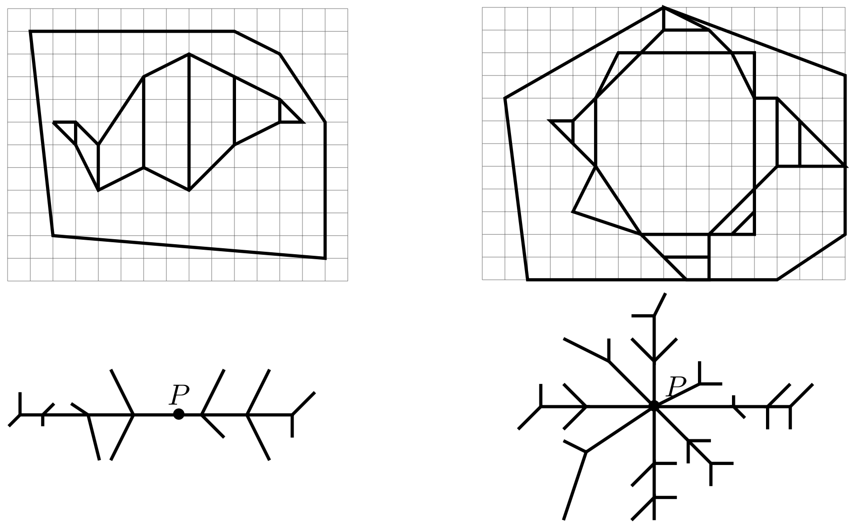

(A) if Val(p) is not a vertex (B) if Val(p) is a vertex

FIGURE 1. If P is not a vertex of Trop(C) (left column), then the collection $\mathfrak{I}(P)$ of vertices consists of all the vertices of Trop(C), lying on the prolongation of the edge through P. If P is a vertex of Trop(C) (right column), then we take the vertices on the prolongations of all the edges through P. In each case the corresponding set of faces of the subdivision of Δ, "the region of influence" of P, is drawn on the top. The sum of the areas of the faces in (2) is at least $\frac{1}{2}m^2$ in (A) and at least $\frac{3}{8}m^2$ in (B).

top row). Namely, if the minimal lattice width of Δ is at least m, then the following inequality holds

$$\sum_{V\in\mathfrak{I}(P)} \text{area}(d(V)) \geq cm^2. \tag{2}$$

If P is not a vertex of Trop(C), then (2) holds with $c = \frac{1}{2}$; if P is a vertex of Trop(C), then (2) holds with $c = \frac{3}{8}$, see Lemma 2.8, Theorems 1,2 in Section 2 for more derails.

Remark 0.1. Let us fix points $p_1, p_2, \dots, p_n$ in general position, suppose that C passes through them. In [15] we prove that in this case each vertex of Trop(C) belongs to at most two sets $\mathfrak{I}(P_i)$, i.e., for indices $i_1 < i_2 < i_3$ we have $\mathfrak{I}(P_{i_1}) \cap \mathfrak{I}(P_{i_2}) \cap \mathfrak{I}(P_{i_3}) = \varnothing$.

Definition 0.2. ([8],[17]) The multiplicity of a point P on a tropical curve H is at least m *in the* $\mathbb{K}$-*extrinsic sense* if there exists an algebraic curve $H' \subset (\mathbb{K}^*)^2$ and a point $p \in H'$ of multiplicity m such that $\text{Trop}(H') = H, \text{Val}(p) = P$.

This definition is extrinsic because it involves other objects besides H. We find new necessary intrinsic conditions (in terms of the subdivision of Δ) for the presence of an m-fold point on C. We give two other definitions (Def. 3.4, Def. 2.7) of a tropical singular point and compare them in Section 6.2.

The previous research in this direction has been done for $m = 2$ in [17, 18], for inflection points in [4], and for cusps in [12]. Lifting of tropical singular points to the usual singular points is discussed in [23]. In [6, 7, 8] the matroid M associated with the aforementioned linear constraints on a_{ij} is studied, see also Remark 6.4.

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1. Preliminaries

1.1. **Tropical geometry and valuation fields.** Let $\mathbb{T}$ denote $\mathbb{R} \cup \{-\infty\}$, $\mathbb{T}$ is usually called *the tropical semi-ring*. Let $\mathbb{K}$ be any valuation field, i.e., a field equipped with a *valuation map* $\mathrm{val} : \mathbb{K} \to \mathbb{T}$, where this map val possesses the following properties:

- $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,
- $\mathrm{val}(a + b) \le \max(\mathrm{val}(a), \mathrm{val}(b))$,
- $\mathrm{val}(0) = -\infty$.

Example 1.1. Let $\mathbb{F}$ be an arbitrary (possibly finite) field. An example of a valuation field is the field $\mathbb{F}\{\{t\}\}$ of generalized Puiseux series. Namely,

$$\mathbb{F}\{\{t\}\} = \{\sum_{\alpha \in I} c_\alpha t^\alpha | c_\alpha \in \mathbb{F}, I \subset \mathbb{R}\},$$

where t is a formal variable and I is a well-ordered set, i.e., each of its nonempty subsets has a least element. The valuation map $\mathrm{val} : \mathbb{K} \to \mathbb{T}$ is defined by the rule

$$\mathrm{val}(\sum_{\alpha \in I} c_\alpha t^\alpha) := -\min_{\alpha \in I}\{\alpha | c_\alpha \neq 0\}, \mathrm{val}(0) := -\infty.$$

Different constructions of Puiseux series and their properties are listed in [19, 22].

Remark 1.2. It follows from the axioms of the valuation map that if $a_1 + a_2 + \cdots + a_n = 0, a_i \in \mathbb{K}^*$, then the maximum among $\mathrm{val}(a_i), i = 1, \ldots, n$ is attained at least twice.

Example 1.3. Suppose that $\mathbb{K} = \mathbb{C}\{\{t\}\}$ and all the coefficients $a_{ij} \in \mathbb{K}^*$ in (1) are convergent series in t for t being close to zero. Then, specializing t to be $t_k \in \mathbb{C}$ close to zero, we obtain a family of complex curves C_{t_k} defined by the equations $\sum_{(i,j)\in\mathcal{A}} a_{ij}(t_k)x^i y^j = 0$. Note, that the valuation $\mathrm{val}(\sum_{\alpha \in I} c_\alpha t^\alpha) = -\min_{\alpha \in I}\{\alpha | c_\alpha \neq 0\}$ is a measure of the asymptotic behavior of a_{ij} as t_k tends to 0, i.e., $a_{ij}(t_k) \sim t_k^{-\mathrm{val}(a_{ij})}$.

The combinatorics of the extended Newton polyhedron reflects some asymptotically visible properties of a generic member of the family $\{C_{t_k}\}$. In such a way, real algebraic curves with a prescribed topology can be constructed, see Viro's patchworking method.

Definition 1.4. ([9]) *The non-Archimedean amoeba* $\mathrm{Val}(C) \subset \mathbb{T}^2$ of an algebraic curve $C \subset \mathbb{K}^2$ is the image of C under the map val applied coordinate-wise.

Now we recall some basic notions of tropical geometry.

Definition 1.5. For the given $F(x, y) = \sum_{(i,j)\in\mathcal{A}} a_{ij}x^i y^j$ we define

$$\mathrm{Trop}(F)(X, Y) = \max_{(i,j)\in\mathcal{A}} (iX + jY + \mathrm{val}(a_{ij})). \tag{3}$$

We use the letters x, y for variables in $\mathbb{K}$, and we use X, Y for the corresponding variables in $\mathbb{T}$. Fix a finite subset $\mathcal{A} \subset \mathbb{Z}^2$. Let us consider a curve C given by (1).

Definition 1.6. Let $\mathrm{Trop}(C) \subset \mathbb{T}^2$ be the set of points where $\mathrm{Trop}(F)$ is not smooth, that is, the set of points where the maximum in (3) is attained at least twice.

It is clear that $\mathrm{Trop}(C)$ is a planar graph, whose edges are straight.

Remark 1.7. We have $\mathrm{Val}(C) \subset \mathrm{Trop}(C)$ because if $F(x, y) = 0$, then the maximum among $\mathrm{val}(a_{ij}x^i y^j)$ must be attained at least twice (Remark 1.2). If $\mathbb{K}$ is algebraically closed and the image of val contains $\mathbb{Q}$, then $\overline{\mathrm{Val}(C)} = \mathrm{Trop}(C)$ (c.f. [9], Theorem 2.1.1).

To the curve C we associate a subdivision of its Newton polygon $\Delta = \mathrm{ConvHull}(\mathcal{A})$ by the following procedure. Consider *the extended Newton polyhedron* ([9])

$$\widetilde{\mathcal{A}} = \mathrm{ConvHull}(\bigcup\{(i, j, x)|(i, j) \in \mathcal{A}, x \le \mathrm{val}(a_{ij})\}) \subset \mathbb{R}^3.$$

The projection of the edges of $\widetilde{\mathcal{A}}$ to the first two coordinates gives us a subdivision of Δ. Hence the curve C produces the tropical curve $\mathrm{Trop}(C)$ and the subdivision of Δ.

Proposition 1.8. *This subdivision is dual to* $\mathrm{Trop}(C)$ *in the following sense (see Example 1.10) :*

- *each vertex Q of* $\mathrm{Trop}(C)$ *corresponds to some face $d(Q)$ of the subdivision of Δ;*
- *each edge E of* $\mathrm{Trop}(C)$ *corresponds to some edge $d(E)$ in the subdivision of Δ, and the direction of the edge $d(E)$ is perpendicular to the direction of E;*
- *if a vertex $Q \in \mathrm{Trop}(C)$ is an end of an edge $E \subset \mathrm{Trop}(C)$, then $d(Q)$ contains $d(E)$;*
- *each vertex of $\widetilde{A}$ corresponds to a connected component of $\mathbb{T}^2 \setminus \mathrm{Trop}(C)$.*

Proof. This proposition follows from Definition 1.6. □

See Figure 2 for an example of the above duality. Also, parts of tropical curves and the corresponding parts of the dual subdivisions are shown in Figure 1.

Definition 1.9. Suppose that $\mathrm{Trop}(F)$ is equal to $i_1X + j_1Y + \mathrm{val}(a_{i_1j_1})$ on one side of an edge $E \subset \mathrm{Trop}(C)$ and to $i_2X + j_2Y + \mathrm{val}(a_{i_2j_2})$ on the other side of E. Therefore E is locally defined by the equation $(i_1 - i_2)X + (j_1 - j_2)Y + (\mathrm{val}(a_{i_1j_1}) - \mathrm{val}(a_{i_2j_2})) = 0$. In this case the endpoints of $d(E)$ are $(i_1, j_1), (i_2, j_2)$, and, by definition, the *weight* of E is equal to the lattice length of $d(E)$, which is $\gcd(i_1 - i_2, j_1 - j_2)$ by definition.

Example 1.10. Consider a curve C' defined by the equation $G(x, y) = 0$, where

$$G(x,y) = t^{-3}xy^3 - (3t^{-3} + t^{-2})xy^2 + (3t^{-3} + 2t^{-2} - 2t^{-1})xy - (t^{-3} + t^{-2} - 2t^{-1} - 3)x+$$
$$+t^{-2}x^2y^2 - (2t^{-2} - t^{-1})x^2y + (t^{-2} - t^{-1} - 3)x^2 + t^{-1}y - (t^{-1} + 1) + x^3.$$

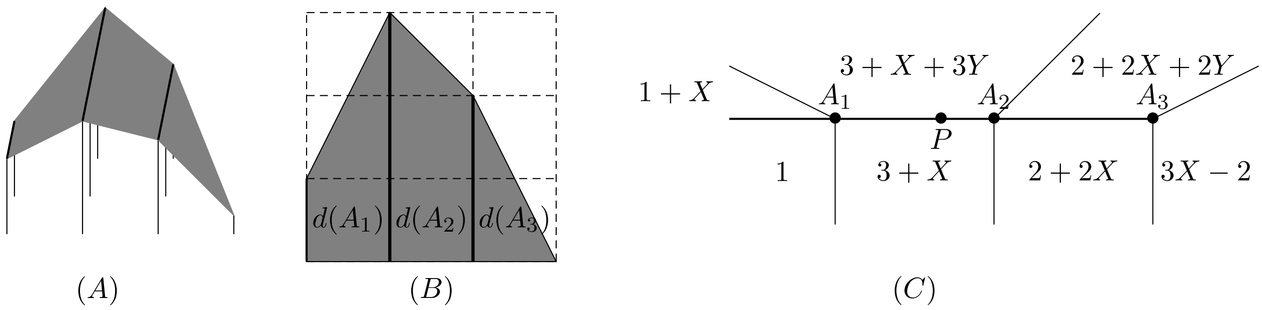

FIGURE 2. The extended Newton polyhedron $\widetilde{\mathcal{A}}$ of the curve C' (Example 1.10) is drawn in (A). The projection of its faces gives us the subdivision of the Newton polygon of C', see (B). The tropical curve $\mathrm{Trop}(C')$ is drawn in (C). The vertices A_1, A_2, A_3 have coordinates $(-2, 0), (1, 0), (4, 0)$. The edge A_1A_2 has weight 3, while the edge A_2A_3 has weight 2. The point P is $(0, 0) = \mathrm{Val}((1, 1))$.

The curve $\mathrm{Trop}(C')$ is equal to the set of non-smooth points of the function

$$\mathrm{Trop}(F) = \max(3+X+3Y, 3+X+2Y, 3+X+Y, 3+X, 2+2X+2Y, 2+2X+Y, 2+2X, Y+1, 1, 3X).$$

The plane is divided by $\mathrm{Trop}(C')$ into regions corresponding to the vertices of $\widetilde{\mathcal{A}}$. In Figure 2, the value of $\mathrm{Trop}(F)(X, Y)$ is written on each region, and, for example, $3X - 2$ corresponds to the vertex $(3, 0, -2)$ of $\widetilde{\mathcal{A}}$.

A tropical curve $H \subset \mathbb{T}^2$ is the non-smooth locus of a function (3) with finite $\mathcal{A} \subset \mathbb{Z}^2$.

Remark 1.11. The tropical curves defined by the equations $\max(x, y, 0)$ and $\max(2x, 2y, 0)$ coincide as sets, but the weights of the edges of the second curve are equal to 2, whereas for the first curve the weights of its edges are equal to 1.

Given a tropical curve H as a subset of $\mathbb{T}^2$ with weights on its edges (as we always assume in this paper), we can construct an equation, defining H. Then we construct the extended Newton polyhedron for H, using the same formula as for algebraic curves. Since the function defining H is not unique, the extended Newton polyhedron for H is defined up to a translation.

Remark 1.12. When we pass from the set $\{(i, j, \mathrm{val}(a_{ij}))\}$ to $\widetilde{\mathcal{A}}$, some information is lost. Nevertheless, we do not suppose that all the points $\{(i, j, \mathrm{val}(a_{ij}))\}$ belong to the boundary of $\widetilde{\mathcal{A}}$.

The reader should be familiar with the notions mentioned above, or is kindly requested to refer to [3, 13, 16].

1.2. **Change of coordinates and m-fold points.**

Definition 1.13. If the lowest term in the Taylor expansion of F at a point p has degree m, then $m = \mu_p(C)$ is called the *multiplicity* of p. The point p is called an m-fold point or a point of multiplicity m.

Another way to say the same thing is to define $\mu_p(C)$ for $p = (p_1, p_2)$ as the maximal m such that the polynomial F belongs to the m-th power of the ideal of the point p, i.e., $F \in \langle x - p_1, y - p_2\rangle^m$ in the local ring of the point p.

Example 1.14. The condition for a point p to be of multiplicity one for C means that $p \in C$. Multiplicity greater than one implies that p is a singular point of C.

Example 1.15. Consider a curve C' of degree d given by an equation

$$G(x, y) = \sum b_{ij} x^i y^j, 0 \le i, j, i + j \le d.$$

The point $(0, 0)$ is of multiplicity at least m for the curve C' if and only if $b_{ij} = 0$ for all i, j with $i + j < m$. As a consequence, for a given point $p \in (\mathbb{K}^*)^2$ the condition that $\mu_p(C') \ge m$ can be rewritten as a certain system of $\frac{m(m+1)}{2}$ linear equations in the coefficients $\{b_{ij}\}$ of G.

Example 1.16. Refer to Example 1.10. The point $p = (1, 1)$ is a point of multiplicity $m = 3$ for the curve C'. This affects the subdivision of the Newton polygon of C' in the following way:

- The point $P = (0, 0)$ belongs to an edge E of the weight $m = 3$.
- The sum of the areas of the faces, dual to the vertices of $\mathrm{Trop}(C')$ on the prolongation of E, is $2 + 5/2 + 1 = 11/2$ which is greater than $m^2/2 = 3^2/2$.

These two facts are particular incarnations of the Exertion Theorem for edges.

Lemma 1.17. *Suppose $ad-bc = 1$ where $a, b, c, d \in \mathbb{Z}$. The transformation $\Psi : (x, y) \mapsto (x^a y^b, x^c y^d)$ preserves multiplicity at the point $p = (1, 1)$, i.e., $\mu_{(1,1)}(C) = \mu_{(1,1)}(\Psi(C))$.*

Proof. Indeed, Ψ defines an isomorphism in the local ring of $p = (1, 1)$. One can prove this by verifying that $\langle x - 1, y - 1\rangle = \langle x^a y^b - 1, x^c y^d - 1\rangle$ in the local ring of p. $\square$

Definition 1.18. A map f *tropicalizes* to a map $\mathrm{Trop}(f)$ if the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{K}^2 & \xrightarrow{f} & \mathbb{K}^2 \\ \downarrow{\scriptstyle \mathrm{Val}} & & \downarrow{\scriptstyle \mathrm{Val}} \\ \mathbb{T}^2 & \xrightarrow{\mathrm{Trop}(f)} & \mathbb{T}^2 \end{array}$$

Proposition 1.19. *A map $\Psi : (x, y) \mapsto (x^a y^b, x^c y^d)$ tropicalizes to the integer affine map* $\mathrm{Trop}(\Psi)$: $(X, Y) \mapsto (aX + bY, cX + dY)$. $\square$

We define a new curve C' given by the equation $G(x, y) = 0$, where $G(x, y) = F(\Psi(x, y))$. Then, the Newton polygon of C' is the image of Δ under $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$, the same for the extended Newton polyhedron, and $\mathrm{Trop}(C') = \mathrm{Trop}(\Psi)(\mathrm{Trop}(C))$.

Proposition 1.20. *A map $\Psi_{r,q} : (x, y) \mapsto (rx, qy)$ with $r, q \in \mathbb{K}^*$ tropicalizes to the map* $\mathrm{Trop}(\Psi_{r,q})$: $(X, Y) \mapsto (X + \mathrm{val}(r), Y + \mathrm{val}(q))$. $\square$

For $G(x,y) = \sum a'_{ij}x^iy^j$ defined as $G(x,y) = F(\Psi_{r,q}(x,y))$, easy computation gives $\mathrm{val}(a'_{ij}) = \mathrm{val}(a_{ij}) + l(i,j)$, with $l(i,j) = i\cdot\mathrm{val}(r) + j\cdot\mathrm{val}(q)$. This adds the linear function $l(i,j)$ to the extended Newton polyhedron $\widetilde{\mathcal{A}}$, therefore the subdivision of the Newton polygon for G coincides with the subdivision for F. This is not surprising because of Proposition 1.8 and the fact that $\mathrm{Trop}(\Psi_{r,q})$ is a translation. Thus, $SL(2,\mathbb{Z})$-invariant properties of the subdivision of Δ for the curve C with $\mu_p(C) = m$ for a given point $p \in (\mathbb{K}^*)^2$ do not depend on the point p.

1.3. **Lattice width and m-thick sets.** *Lattice width* is the most frequent notion in our arguments, already proved to be a practical tool elsewhere, e.g., the article [5] uses it to estimate the gonality of a general curve with a given Newton polygon. The minimal genera of surfaces dual to a given 1-dimensional cohomology class in a three-manifold are related to the lattice width of its Alexander polynomial ([11],[20]). A good survey about lattice geometry and related problems can be found in [1].

Definition 1.21. We denote by $P(\mathbb{Z}^2)$ the set of all directions in $\mathbb{Z}^2$. Each direction u has a representative $(u_1,u_2) \in \mathbb{Z}^2$ with $\gcd(u_1,u_2) = 1$, we will write $u \sim (u_1,u_2)$ in this case.

Let us consider a compact set $B \subset \mathbb{R}^2$.

Definition 1.22. *The lattice width* of B in a direction $u \in P(\mathbb{Z}^2)$ is $\omega_u(B) = \max\limits_{x,y\in B}(u_1,u_2)\cdot(x-y)$ where $u \sim (u_1,u_2)$. *The minimal lattice width* $\omega(B)$ is defined to be $\min\limits_{u\in P(\mathbb{Z}^2)} \omega_u(B)$.

Consider an interval I with a rational slope. Let $(p,q) \in \mathbb{Z}^2$ be a primitive vector in the direction of I. The *lattice length* of I is its Euclidean lengths divided by $\sqrt{p^2+q^2}$.

Definition 1.23. A set $B \subset \mathbb{R}^2$ is called *m-thick* in the following cases:

- B is empty,
- $\mathrm{ConvHull}(B)$ is 1-dimensional with a rational slope and its lattice length is at least m,
- $\mathrm{ConvHull}(B)$ is 2-dimensional and for each $u \in P(\mathbb{Z}^2)$, if $\omega_u(\mathrm{ConvHull}(B)) = m - a_u$ with $a_u > 0$, then $\mathrm{ConvHull}(B)$ has two sides of the lattice length at least a_u, and these sides are perpendicular to u.

The relation between m-thickness property and Euler derivatives is discussed in Remark 6.4.

Proposition 1.24. *If $B \subset \mathbb{Z}^2$ is m-thick and* $\mathrm{ConvHull}(B)$ *is a polygon with at most one vertical side, then $\omega_{(1,0)}(B) \geq m$. If B is m-thick and* $\mathrm{ConvHull}(B)$ *is a polygon without parallel sides, then $\omega(B) \geq m$.* □

Lemma 1.25. *If $\mu_{(1,1)}(C) = m$ and $\omega_u(\mathcal{A}) = m - a$ for some $a > 0, u \sim (u_1,u_2)$, then C contains a rational component parametrized as (s^{u_1}, s^{u_2}).*

Proof. By Lemma 1.17, it is enough to prove this lemma only for $u = (1,0)$. The degree of the polynomial $F(x,1)$ is $m-a$, $F(x,1)$ has a root of multiplicity m at 1, therefore F is identically zero on $y = 1$, hence F is divisible by $y-1$. Let b be the maximal number such that F is divisible by $(y-1)^b$. Clearly $b \geq a$, otherwise we can repeat the above argument. Therefore F is divisible by $(y-1)^a$, and both vertical sides of $\mathrm{ConvHull}(\mathcal{A})$ have lattice length at least a. □

Corollary 1.26. *If $\mu_{(1,1)}(C) = m$, then the Newton polygon Δ of C is m-thick.*

For a polynomial $G(x,y) = \sum b_{ij}x^iy^j$ we define its *support set* by $\mathrm{supp}(G) = \{(i,j) | b_{ij} \neq 0\}$.

Definition 1.27. For $\mu \in \mathbb{R}$, denote by $\mathcal{A}_\mu$ the set $\{(i,j) \in \mathcal{A} | \mathrm{val}(a_{ij}) \geq \mu\}$.

The following lemma describes the set of valuations of the coefficients a_{ij} of $F(x,y)$.

Lemma 1.28 (The M-thickness lemma)**.** *If $\mu_{(1,1)}(C) = m$, then for each real number μ the set $\mathcal{A}_\mu$ is m-thick (Def. 1.23).*

Proof. We will find a polynomial G with $\mathrm{supp}(G) = \mathcal{A}_\mu$, which defines a curve passing through $(1,1)$ with the multiplicity m. Then Corollary 1.26 concludes the proof. Let us consider the set of linear conditions in the coefficients a_{ij}, imposed by the fact that $\mu_{(1,1)}(C) = m$. If there is no required polynomial G, then by setting all the coefficients $a_{ij} = 0$ for $(i,j) \in \mathcal{A} \setminus \mathcal{A}_\mu$, we see that the above system of linear equations would imply that $a_{i'j'} = 0$ for some $(i',j') \in \mathcal{A}_\mu$. That would mean that there exists an equation $\sum \lambda_{ij} a_{ij} = a_{i'j'}, \lambda_{ij} \in \mathbb{Q}, (i,j) \in \mathcal{A} \setminus \mathcal{A}_\mu$ which is a consequence of the above system. The latter leads us to the contradiction, because for the polynomial F we have $\mathrm{val}(\lambda_{ij} a_{ij}) < \mu \leq \mathrm{val}(a_{i'j'})$ for $(i,j) \in \mathcal{A} \setminus \mathcal{A}_\mu$ (see Remark 1.2). The attentive reader can notice that $\mathcal{A}_\mu$ is a *flat* in the matroid corresponding to the above linear conditions. Indeed, no dependent set intersects $\mathcal{A}_\mu$ by exactly one element, because the valuation of this element would be strictly bigger than the valuations of the other elements in this dependent set. □

1.4. **One lemma about concave functions.** Suppose that $h : [a,b] \to \mathbb{R}$ is a concave and piecewise smooth function on the interval $[a,b]$. Define $\hat{h}_{[a,b]}(x)$ as the length of the subinterval of $[a,b]$ where the values of h are at least $h(x)$, i.e., $\hat{h}_{[a,b]}(x) = \mathrm{measure}\{y \in [a,b] | h(y) \geq h(x)\}$.

Lemma 1.29. *Suppose that h attains its maximal value at a unique point. Then $\int_a^b \hat{h}_{[a,b]}(x)dx = (b-a)^2/2$.*

Proof. Without loss of generality $h(a) \geq h(b) = 0$. Let q be the point where the maximum of h is attained. On the intervals $[a,q]$ and $[q,b]$ the function h is invertible. Call f_1, f_2 the respective inverses, that is $f_1(h(x)) = x$ for $x \in [a,q]$ and $f_2(h(x)) = x$ for $x \in [q,b]$. For $y \in [0, h(a)]$, we define $f_1(y) = a$. Hence, $f_1(h(q)) = f_2(h(q)) = q, f_1(0) = a, f_2(0) = b$. Let $H(y) = f_2(y) - f_1(y)$, note that $H(y) = \hat{h}(f_1(y)) = \hat{h}(f_2(y))$. Finally, we integrate $\hat{h}_{[a,b]}(x)$ along the y-axis. In between, we change the measure in the integration. The integral becomes

$$\int_a^b \hat{h}_{[a,b]}(x)dx = \int_{h(q)}^0 (h_2(y) - h_1(y))d(h_2(y) - h_1(y)) = \int_{h(q)}^0 H(y)d(H(y)) = \frac{H^2(0)}{2} = \frac{(b-a)^2}{2}.$$

□

Corollary 1.30. *If $h(a') = h(b')$ for some $a' < b'$ in $[a,b]$, then*

$$\int_a^{a'} \hat{h}_{[a,b]}(x)dx + \int_{b'}^b \hat{h}_{[a,b]}(x)dx = \frac{1}{2}((b-a)^2 - (b'-a')^2).$$

Proof. We proceed as in the proof of the lemma, and

$$\int_a^{a'} \hat{h}_{[a,b]}(x)dx + \int_{b'}^b \hat{h}_{[a,b]}(x)dx = \int_{h(a')}^0 H(y)d(H(y)) = \frac{1}{2}((b-a)^2 - (b'-a')^2).$$

□

Proposition 1.31. *If h is linear with non-zero slope on an interval $[a',b'] \subset [a,b]$, then the function $\hat{h}$ is concave on $[a',b']$.*

Proof. Without loss of generality we suppose that $a' < b', f(a') < f(b')$. It is it enough to check that $\hat{h}(\frac{1}{2}(x+y)) \geq \frac{1}{2}(\hat{h}(x) + \hat{h}(y))$ for $x, y \in [a',b']$. Since $h(\frac{1}{2}(x+y)) = \frac{1}{2}(h(x) + h(y))$ we have

$$\hat{h}(\frac{1}{2}(x+y)) = -h_1(h(\frac{1}{2}(x+y))) + h_2(h(\frac{1}{2}(x+y))) = -\frac{1}{2}(h_1(h(x)) + h_1(h(y))) + h_2(h(\frac{1}{2}(x+y)) \geq$$

$$\geq -\frac{1}{2}(h_1(h(x)) + h_1(h(y))) + \frac{1}{2}(h_2(h(x)) + h_2(h(y))) = \frac{1}{2}(\hat{h}(x) + \hat{h}(y)),$$

because $h_2 \circ h$ is concave on the interval $[x,y]$. Note, that linearity of h and h_1 on $[x,y]$ is crucial, since we have h_1 with a negative coefficient. □

2. Formulation of main theorems

In this section, we state the main results of this paper. For the terminology of faces, vertices, edges, and the duality among them, refer to Proposition 1.8.

2.1. **Influenced sets.** We consider a tropical curve $H \subset \mathbb{T}^2$ and a point $Q \in H$.

Definition 2.1. Let $l_Q(u)$ be the line through Q in the direction $u \in P(\mathbb{Z}^2)$. Take the connected component, containing Q, of the intersection $H \cap l_Q(u)$. We call this component *the long edge through Q in the direction u* and denote it by $E_Q(u)$.

Definition 2.2. For each $u \in P(\mathbb{Z}^2)$ we denote by $\mathfrak{I}_Q(u)$ the set of vertices of H which belong to the long edge $E_Q(u)$. Define $\mathfrak{I}(Q) = \bigcup_{u \in P(\mathbb{Z}^2)} \mathfrak{I}_Q(u)$.

Note, that $\mathfrak{I}(Q)$ is not a multiset, it contains only one copy of Q. Examples of $\mathfrak{I}(P)$ are presented in Figure 1. On the left part we see one long edge $E_P((1,0))$ and $\mathfrak{I}(P)$ consists of 7 vertices, and above we see 7 corresponding faces in the subdivision of Δ. On the right part, we see long edges $E_P((1,0)), E_P((0,1)), E_P((-1,1))$. Each of the long edges $E_P((1,2))$ and $E_P((-3,-2))$ consists of only one edge. In Example 1.10, $E_P((1,0))$ is the union of the horizontal edges of $\mathrm{Trop}(C')$, and $\mathfrak{I}(P)$ is the set of all vertices of $\mathrm{Trop}(C')$.

Definition 2.3. For a point $Q \in H$ we define $\mathfrak{Infl}(Q) = \bigcup\limits_{V \in \mathfrak{I}(Q)} d(V)$, the union of the faces of the Newton polygon of H, dual to the vertices in $\mathfrak{I}(Q)$.

Definition 2.4. For a point $Q \in H$ which is not a vertex of H we define

$$\mathrm{area}(\mathfrak{Infl}(Q)) = \sum_{F \in \mathfrak{Infl}(Q)} \mathrm{area}(F).$$

Note, that $\mathrm{area}(\mathfrak{Infl}(Q))$ depends only on H and does not depend on a particular choice of an equation defining H. Also, if Q belongs to an edge E of H, and Q is not a vertex of H, then $\mathfrak{I}_u(Q) = \mathfrak{I}(Q)$ where u is the direction of E. Indeed, for any other not collinear to u direction v the connected component of Q in the intersection $H \cap l_P(v)$ is just Q.

Recall, that if Q is a vertex of H, then $d(Q)$ is a face, dual to Q, in the subdivision of Δ.

Definition 2.5. If Q is a vertex of H, we define

$$\mathrm{area}(\mathfrak{Infl}(Q)) = \sum_{F \in \mathfrak{Infl}(Q)} \mathrm{area}(F) + \mathrm{area}(d(Q)),$$

$$\mathrm{area}^*(\mathfrak{Infl}(Q)) = \sum_{F \in \mathfrak{Infl}(Q)} \mathrm{area}(F).$$

From the point of view of combinatorics, studying of $\mathrm{area}^*(\mathfrak{Infl}(Q))$ is more natural, whereas $\mathrm{area}(\mathfrak{Infl}(Q))$ is motivated by Nagata's conjecture (see [15] for details). The name $\mathfrak{Infl}(P)$ is chosen because the linear constraints, imposed by the fact $\mu_p(C) = m$, asymptotically *influence* (see Remark 0.1) the coefficients a_{ij} where $(i,j) \in \mathfrak{Infl}(P), P = \mathrm{Val}(p)$.

2.2. **Multiplicity of a tropical point in the intermediate sense.** Consider a tropical curve H, given by a tropical polynomial $\mathrm{Trop}(F)$. Using $\mathrm{Trop}(F)$ we construct the extended Newton polyhedron $\widetilde{\mathcal{A}}$ for H.

Definition 2.6. We denote by $\widetilde{\mathcal{A}}_\mu$ the xy-projection of the section of $\widetilde{\mathcal{A}}$ by the plane $z = \mu$.

Note, that $\mathcal{A}_\mu$ (Def. 1.27) is contained in $\widetilde{\mathcal{A}}_\mu$. On Figure 5 (below) the set $\widetilde{\mathcal{A}}_\mu$ is colored in gray.

Definition 2.7. A point $P = (0,0)$ on the tropical curve H is of multiplicity at least m *in the intermediate sense* (we write $\mu_P^{trop}(H) \geq m$), if for each $\mu \in \mathbb{R}$ the set $\widetilde{\mathcal{A}}_\mu$ is m-thick (Def. 1.23).

Using Proposition 1.20, we can use this definition for any other point of $P \in \mathrm{Val}((\mathbb{K}^*)^2)$, after an appropriate change of coordinates.

Lemma 2.8. *If $\mu_p(C) = m$ and $P = \mathrm{Val}(p)$, then $\mu_P^{trop}(\mathrm{Trop}(C)) \geq m$.*

Corollary 2.9. *If a point P on a tropical curve $H \subset \mathbb{T}^2$ is of multiplicity at least m in the $\mathbb{K}$-extrinsic sense (Def. 0.2), then P is of multiplicity at least m for H in the intermediate sense.*

Unfortunately, this lemma does not immediately follow from Lemma 1.28.

2.3. **Exertion Theorems.** If $\omega(\mathcal{A}) < m$, i.e., $\omega_u(\mathcal{A}) < m$ for some $u \sim (u_1, u_2)$ (Def. 1.21), and $\mu_p(C) = m$, then Lemma 1.25 asserts that C contains a rational component with parameterization $(p_1 s^{u_1}, p_2 s^{u_2})$. We also prohibit such cases on the tropical side of the story.

Definition 2.10. A tropical curve is *admissible* if the minimal lattice width (Def. 1.22) of its Newton polygon is at least m.

The following theorems estimate the total area of the region of influence of p in Δ. The point p *exerts* its influence on the faces whose area is counted in the theorem, whence the name.

Theorem 1 (Exertion Theorem for edges). *If H is admissible, $\mu_P^{trop}(H) = m$ (Def. 2.7), and P is not a vertex of H, then* $\text{area}(\mathfrak{Infl}(P)) \geq \frac{1}{2}m^2$ *(Def. 2.4). Furthermore, if P belongs to an edge $E \subset H$, then the lattice length of $d(E)$ is at least m.*

In this case we see a collection of faces with parallel sides in the subdivision of Δ, see Figure 1(A).

Theorem 2 (Exertion Theorem for vertices). *If H is admissible, $\mu_P^{trop}(H) = m$, and the point P is a vertex of H, then* $\text{area}^*(\mathfrak{Infl})(P) \geq \frac{3}{8}m^2$ *and* $\text{area}(\mathfrak{Infl}(P)) \geq \frac{1}{2}m^2$ *(Def. 2.5).*

Here we will see a collection of faces like in Figure 1(B). Exertion theorems are valid only for admissible curves. The following example illustrates this problem.

Example 2.11. Consider a curve C' defined by the polynomial $F_k(x, y) = (x-1)^k (y-1)^{m-k}$. Clearly, $\mu_{(1,1)}(C') = m$ but the curve $\text{Trop}(C')$ is not admissible. The Newton polygon of F_k is the rectangle with vertices $(0,0), (k,0), (0, m-k), (k, m-k)$, it is m-thick and its area is $k(m-k)$ which is always less than $\frac{3}{8}m^2$. The curve C' consists of the line $x = 1$ with multiplicity k and the line $y = 1$ with multiplicity $m - k$. The tropical curve $\text{Trop}(C')$ consists of the vertical line of the weight k and the horizontal line of the weight $m - k$. Note, that Lemma 2.8 holds in this example.

3. Intrinsic definition of a tropical m-fold point

The *multiplicity* $m(P)$ of the point P of the intersection of two lines in directions $u, v \in P(\mathbb{Z}^2)$ is $|u_1 v_2 - u_2 v_1|$ where $u \sim (u_1, u_2), v \sim (v_1, v_2)$ (Def. 1.21).

Given two tropical curves $A, B \subset \mathbb{T}^2$ we define their *stable intersection* as follows. Let us choose a generic vector v. Then we consider the curves $T_{tv}A$ where $t \in \mathbb{R}, t \to 0$ and T_{tv} is the translation by the vector tv. For a generic small positive t the intersection $T_{tv}A \cap B$ is transversal and consists of points $P_i^t, i = 1, \dots, k$ with multiplicities $m(P_i^t)$.

Definition 3.1. ([21]) For each connected component X of $A \cap B$ we define *the local stable intersection of A and B along X* as $A \cdot_X B = \sum_i m(P_i^t)$ for t being close to zero, where the sum runs over $\{i | \lim_{t \to 0} P_i^t \in X\}$. For a point $Q \in A$ we define $A \cdot_Q B$ as $A \cdot_X B$ where X is the connected component of Q in the intersection $A \cap B$.

Definition 3.2. A generalized tropical line is the non-smooth locus of a function (3) with $\mathcal{A} \subset \mathbb{Z}^2$ such that $\mathcal{A}$ is an interval of the lattice length 1 or $|\mathcal{A}| = 3, \text{area}(\text{ConvHull}(\mathcal{A})) = \frac{1}{2}$.

Proposition 3.3. *Let Q be a vertex of a tropical curve H. If the face $d(Q)$ has no vertical sides, and L is the usual horizontal line through Q, then $H \cdot_Q L = \omega_{(1,0)}(d(Q))$.*

Proof. This follows from a direct computation and Proposition 1.8. □

Definition 3.4. A point P on a tropical curve H is of multiplicity at least m *in the intrinsic sense* if for each generalized tropical line L through P we have $L \cdot_P H \geq m$.

Remark 3.5. Given $Q \in H$, we call *the tangent cone $TC(Q)$ at Q* the connected component of Q of the intersection $H \cap \bigcup_{u \in P(\mathbb{Z}^2)} \{l_Q(u)\}$ (Def.2.1). Note, that only the vertices of H in $TC(Q)$ contribute to the multiplicity (in the intrinsic and intermediate senses) of Q. Also, the set of these vertices is exactly $\mathfrak{I}(Q)$.

Proposition 3.6. *Let P be of multiplicity m in the intrinsic sense. If P is a vertex of H, then $d(P)$ is m-thick (Def.1.23). If P is not a vertex of H, then the edge of H, containing P, is of weight at least m.*

Proof. For each $u \in P(\mathbb{Z}^2)$ we can find a generalized tropical line L such that P is the vertex of L, and L has an edge in the direction u. Like in Proposition 3.3, a direct calculation of $L \cdot_P H$ finishes the proof. □

Consider an edge of H through P. Without loss of generality we can suppose that this edge is horizontal. Let A_1 (resp. A_2) be the leftmost (resp. rightmost) vertex of H on the horizontal long edge $E_P((1,0))$.

Look at Example 2. The edge with P has the weight 3, therefore the stable intersection with each non-horizontal line is at least 3. The stable intersection of H with the horizontal line through P is exactly the width of the curve's Newton polygon in the direction $(1,0)$.

Proposition 3.7. *(c.f. Lemma 5.16.) If P is of multiplicity m in the intrinsic sense, then the difference between x-coordinates of the leftmost vertex of $d(A_1)$ and rightmost vertex of $d(A_2)$ is at least m.*

Proof. Let L be the usual line containing E. A direct calculation of $L \cdot_P H$ concludes the proof. □

Proposition 3.8. *Suppose that $P \in H$ is not a vertex of H, let P belongs to an edge E of H with endpoints A_1, A_2. Let P be of multiplicity m for H in the intrinsic sense. Suppose that $E_P((1,0)) = E$. Then* $\text{area}(d(A_1)) + \text{area}(d(A_2)) \geq \frac{1}{2}m^2$.

Proof. The lattice length of $d(E)$ is at least m and the sum of heights of $d(A_1)$ and $d(A_2)$ is at least m by Proposition 3.7. Therefore

$$\text{area}(d(A_1)) + \text{area}(d(A_2)) \geq m \cdot m/2.$$

□

4. Two combinatorial lemmata

Definition 4.1. The *defect* of $B \subset \mathbb{Z}^2$ in a direction $u \in P(\mathbb{Z}^2)$ is $\text{def}_u(B) = \max(m - \omega_u(B), 0)$.

This section is devoted to the proofs of the following statements.

Lemma 4.2. *For an m-thick (Def.1.23) polygon B with lattice vertices we have*

$$\text{area}(\text{ConvHull}(B)) + \frac{1}{2}\sum_{u \in P(\mathbb{Z}^2)} \text{def}_u(B)^2 \geq \frac{3}{8}m^2.$$

Lemma 4.3. *For an m-thick polygon $B \subset \mathbb{Z}^2$ with lattice vertices we have*

$$2 \cdot \text{area}(\text{ConvHull}(B)) + \frac{1}{2}\sum_{u \in P(\mathbb{Z}^2)} \text{def}_u(B)^2 \geq \frac{1}{2}m^2.$$

Unfortunately, though the proofs use only standard combinatorial arguments, they are cumbersome and rather tedious. So, the reader is recommended to skip this section while reading this paper the first time.

4.1. Using the direction $(0,1)$ or the direction $(1,1)$.

Suppose that B is not $(m+1)$-thick and the minimal lattice width $a \leq m$ of B is attained in the horizontal direction. Using m-thickness property, we can find two points M, L on the left vertical side of B and two points N, K on the right vertical side in such a way (Figure 3(A)) that the distances ML and NK are equal to $m - a$; so $MNKL$ is a parallelogram. Let us call it the *initial* parallelogram. Note that in the case $a = m$ we have a degenerate initial parallelogram with $M = L, N = K$.

Definition 4.4. Denote by $x(A)$ (resp. $y(A)$) the x-coordinate (resp. y-coordinate) of a point $A \in \mathbb{Z}^2$.

Let $b = y(M) - y(N)$. Applying a suitable coordinate change in $SL(2,\mathbb{Z})$ we may assume that $0 \leq b < a$, see Figure 3 (A).

Proposition 4.5. *The width $\omega_{(0,1)}(MNKL)$ of the initial parallelogram $MNKL$ in the direction $(0,1)$ is equal to $m - a + b$. The width $\omega_{(1,1)}(MNKL)$ is equal to $m - b$.* □

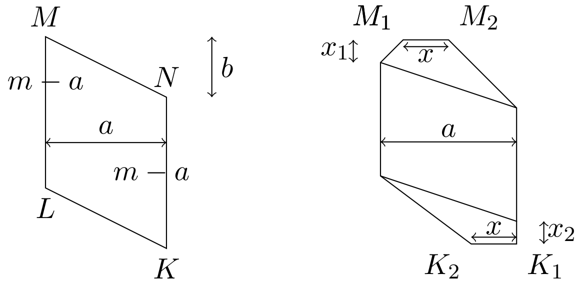

FIGURE 3. The initial parallelogram $MNKL$ is depicted on the left. The set B is m-thick. Therefore, by taking into consideration $\omega_{(0,1)}(B)$, we find a polygon $MM_1M_2NKK_1K_2L$, which is a subset of B.

Suppose that $\omega_{(0,1)}(B) = m - x$. Thus, by Proposition 4.5, $x \leq a - b$, and B must have two horizontal sides M_1M_2, K_1K_2, whose lengths are at least x. Note, that it is possible that $x = 0$; in that case we choose $M_1 = M_2 \in B, K_1 = K_2 \in B$. So, B contains a polygon $MM_1M_2NKK_1K_2L$. A particular example of such a polygon is shown in Figure 3, right side. Let $x_1 = x(M_1) - x(M), x_2 = x(K) - x(K_1)$. The inequality $x \geq m - (m - a + b + x_1 + x_2)$ holds because B is m-thick. All the notation is presented in Figure 3 and this picture serves as the main illustration tool for the following computations.

Note, that

$$\text{area}(MM_1M_2NKK_1K_2L \setminus MNKL) \geq a(x_1 + x_2)/2 + x(b + x_1 + b + x_2)/2, \tag{4}$$

and the minimum is attained if the bottom horizontal edge is in the extremal right position (like at the bottom in Figure 3(B)), and the top edge is in the extremal left position. Look at the top of Figure 3(B)): we minimize the area of $MM_1M_2NKK_1K_2L$, preserving $MNKL$ and x_1, x_2. For that, we should move the interval M_1M_2 to the left as much as possible, while preserving convexity of $MM_1M_2NKK_1K_2L$.

Definition 4.6. Define $S_{(0,1)} = \frac{1}{2}\text{def}_{(0,1)}(B)^2 + 2 \cdot \text{area}(B \setminus (MNKL))$.

Using (4) we see that

$$S_{(0,1)} \geq x^2/2 + 2(a(x_1 + x_2)/2 + x(b + x_1 + b + x_2)/2) \geq a(a - b) + xb - x^2/2. \tag{5}$$

Remark 4.7. If $c_2 < 0$, then a function $f(x) = c_2x^2 + c_1x + c_0$ defined on an interval $[c_3, c_4]$ always attains its minimum at c_3 or c_4.

We will extensively use this fact below. In particular, we have

$$S_{(0,1)} \geq \min(a(a - b), (a - b)a + (a - b)(b - \frac{a - b}{2}),$$

because $x \in [0, a - b]$. Moreover, if $b \geq a/3$, then $S_{(0,1)} \geq a(a - b)$. If $b \leq a/3$, then

$$S_{(0,1)} \geq a(a - b) + (3b - a)(a - b)/2.$$

We repeat the above procedure for the direction $(1, 1)$. We define $y = m - \omega_{(1,1)}(B)$. Then, let N_1N_2, L_1L_2 be the vertices of two sides of B, perpendicular to the direction $(1, 1)$ and y_1, y_2 be the increments of $\omega_{(1,1)}$ obtained by adding N_1, N_2, L_1, L_2 to $MNKL$. On Figure 4 we have $y_1 = 0, y = 1$; note that $y_2 = 2$ because $\omega_{(1,1)}(\{(0, 0), (1, 1)\}) = 2$.

As above, it follows from the consideration of $\omega_{(1,1)}(B)$ that the inequality $y_1 + y_2 + y \geq b$ holds.

Definition 4.8. We denote $S_{(1,1)} = \frac{1}{2}\text{def}_{(1,1)}(B)^2 + 2 \cdot \text{area}(B \setminus (MNKL))$.

By direct calculation of the areas of the triangles $L_1L_2K, LL_1K, MN_1N_2, MN_2N$ we obtain

$$S_{(1,1)} \geq y^2 + a(y_1 + y_2) + y(a - b + y_1 + a - b + y_2) \geq y^2 + a(b - y) + y(2a - b - y). \tag{6}$$

Proposition 4.9. *The following inequalities hold: 1) if $b \leq 2a/3$, then $S_{(1,1)} \geq ab$, 2) if $b \geq 2a/3$, then $S_{(1,1)} \geq ab + b(2a-3b)/2$.*

Proof. It follows from (6) and Remark 4.7. □

Lemma 4.10. *The following inequality holds:*

$$2 \cdot \mathrm{area}(B \setminus (MNKL)) + \frac{1}{2} \sum_{u \in P(\mathbb{Z}^2), u \neq (1,0)} \mathrm{def}_u(B)^2 \geq \frac{a^2}{2}.$$

Proof. Indeed, if $a/3 \leq b \leq 2a/3$, then $S_{(0,1)} + S_{(1,1)} = a^2$ and we are done. If $b \leq a/3$, then it follows from Remark 4.7 that i) $b = 0$ and $S_{(0,1)} = a^2/2$, or ii) $b = a/3$, and we get $S_{(0,1)} = 2a^2/3$. If $b \geq 2a/3$, then i) $b = a$ and we obtain $S_{(1,1)} = a^2/2$, or ii) $b = 2a/3$ and we get $S_{(1,1)} = 2a^2/3$. □

Proof of Lemma 4.3. It follows from the previous lemma that

$$2 \cdot \mathrm{area}(\mathrm{ConvHull}(B)) + \frac{1}{2} \sum_{u \in P(\mathbb{Z}^2)} \mathrm{def}_u(B)^2 \geq 2 \cdot \mathrm{area}(MNKL) + \frac{1}{2}a^2 + \frac{1}{2}\mathrm{def}_{(1,0)}(B)^2 \geq$$

$$\geq \frac{1}{2}(m-a)^2 + 2(a(m-a)) + \frac{1}{2}a^2 \geq \frac{1}{2}m^2 + a(m-a),$$

and $a(m-a) \geq 0$. □

4.2. **Using both directions** $(0,1)$ **and** $(1,1)$**.** Now we will use the widths of B in the directions $(0,1), (1,1), (1,0)$ at the same time. Consider the directions $(0,1), (1,1)$, and define x, y, x_1, y_1, x_2, y_2 as in the previous subsection. Now, B contains the polygon $s(B) = MM_1M_2NN_1N_2KK_1K_2LL_1L_2$. Some of its vertices are allowed to coincide. Refer to Figure 4. We assume that the polygon $s(B)$ satisfies the condition of m-thickness in the directions $(0,1), (1,0), (1,1)$. Our goal is to find an estimate for the area of $s(B)$ in terms of m, a, x, y. We can suppose that $s(B)$ is the minimal by area polygon with the above requirements.

Lemma 4.11. *The pairs of intervals M_1M_2, N_1N_2 and K_1K_2, L_1L_2 either share common vertex (like K_1K_2 and L_1L_2 in the bottom of Figure 4), or are maximally far from each other (like M_1M_2 and N_1N_2 at the top of the picture).*

Proof. This lemma follows from the fact that the area changes linearly when we move the sides $K_1K_2, L_1L_2, M_1, M_2, N_1, N_2$, preserving the distances x, y, x_1, y_1, x_2, y_2. □

Let A_1 denote the minimal area of the top augmented piece ($MM_1M_2N_1N_2N$) when N_1N_2 and M_1M_2 are maximally far from each other (Figure 4, top). Let A_2 denote the minimal area of the bottom augmented piece ($LKK_1K_2L_1L_2$) when L_1L_2 and K_1K_2 are maximally far from each other. Let A_3 denote the minimal area of the top augmented piece when $N_1 = M_2$. Let A_4 denote the minimal area of the bottom augmented piece when $L_1 = K_2$ (Figure 4, bottom).

Lemma 4.12. *For A_1, A_2, A_3, A_4 defined above we have $A_1 - A_3 = A_2 - A_4$.*

Proof. Computing $\omega_{0,1}(B), \omega_{1,1}(B)$, we get relations $x_1 + x_2 = a - b - x, y_1 + y_2 = b - y$. Now, by direct computations we obtain

$$A_1 = \frac{1}{2}(ax_1 - yx_1 - yb + yy_1 + ay_1 + ay + xx_1 + xb - xy_1 - xy).$$

Substituting x_1 by x_2 and y_1 by y_2 we obtain A_2:

$$A_2 = \frac{1}{2}(a^2 - ax_1 + yx_1 + by - yy_1 - y^2 - ay_1 - ay - xx_1 - x^2 - xb + xy_1 + xy).$$

For A_3, A_4 we get

$$A_3 = \frac{1}{2}(yy_1 + xx_1 + ax_1 + ab - b^2 - bx_1 + by_1),$$

$$A_4 = \frac{1}{2}(-yy_1 - y^2 - xx_1 - x^2 + a^2 - ab - ax_1 + bx_1 - by_1 + b^2).$$

It is straightforward to see that $A_1 - A_3 = A_4 - A_2$. □

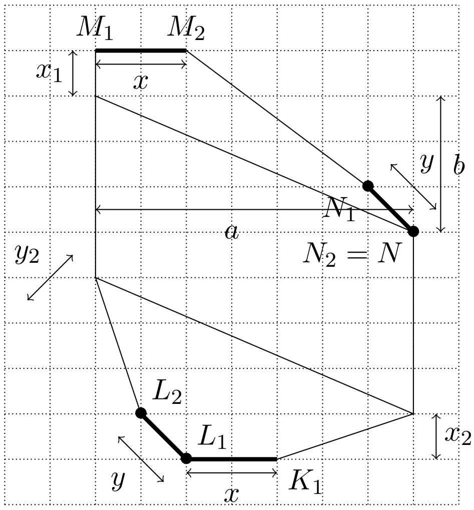

FIGURE 4. In this example $m = 11, a = 7$, and $y_1 = 0$ (therefore $N = N_2$). The vertices $MNKL$ are as in Figure 3(A), the vertices M_1M_2, K_1, K_2 are as in Figure 3(B), L_1, K_2 are coincide. We are looking for the minimum of the sum of the area of this polygon and $\frac{1}{2}(x^2 + y^2)$.

If $A_1 < A_3$, then $A_4 < A_2$. Therefore, the minimal total sum of the areas of the augmented pieces is $A_1 + A_4$ or $A_2 + A_3$. Suppose that the minimum is attained in the case $A_1 + A_4$.

Lemma 4.13. $\mathrm{area}(s(B) \setminus MNKL) + \frac{1}{2}(\mathrm{def}_{(0,1)}(B)^2 + \mathrm{def}_{(1,1)}(B)^2) \geq \frac{3}{8}a^2$.

Proof. The area of $s(B) \setminus MNKL$ is at least $A_1 + A_4$, $\mathrm{def}_{(0,1)}(B) = x, \mathrm{def}_{(1,1)}(B) = y$, and

$$A_1 + A_4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 = \frac{1}{2}(a^2 - ab + b^2 + xb + y(a - b - x) + x_1(b - y) + y_1(a - b - x)).$$

Minimizing, we get $x_1 = y_1 = 0$. Next, $y = 0, x = 0$. Finally, minimizing $\frac{1}{2}(a^2 - ab + b^2)$ by b we obtain $\frac{3}{8}a^2$.

□

Proof of Lemma 4.2. Using the previous Lemma, we get

$$\mathrm{area}(\mathrm{ConvHull}(B)) + \frac{1}{2}\sum_{u \in P(\mathbb{Z}^2)} \mathrm{def}_u(B)^2 \geq \frac{1}{2}(m - a)^2 + a(m - a) + \frac{3}{8}a^2 \geq \frac{3}{8}m^2,$$

and the equality is attained if $a = m$. □

Remark 4.14. As a side effect, for the special case $a = m$ we also proved

Theorem([10], based on [2], p. 716, formula II_3, p. 715 formula I). Let $B \subset \mathbb{Z}^2$ be a finite set. Then $\mathrm{area}(\mathrm{ConvHull}(B)) \geq \frac{3}{8}\omega(B)^2$.

In fact, from the above proofs it is easy to extract the extremal cases and exact bounds: if $\omega(B) = 2k$, then $\mathrm{area}(\mathrm{ConvHull}(B)) \geq \frac{3}{2}k^2$; and if $\omega(B) = 2k+1$, then $\mathrm{area}(\mathrm{ConvHull}(B)) \geq \frac{1}{2}(3k^2+3k+1)$.

Remark 4.15. The best constant c_n in the inequality $\mathrm{volume}(\mathrm{ConvHull}(B)) \geq c_n\omega(B)^n$ for $B \subset \mathbb{Z}^n$ is not known for $n > 2$. The above theorem says that $c_2 = \frac{3}{8}$.

5. The proofs of the Exertion Theorems

Firstly, we introduce the notation which we use throughout the remainder of this paper. Then we prove Lemma 2.8 and the Exertion Theorems.

5.1. **Notation.** Let H be a tropical curve, given by (3). The extended Newton polyhedron of H is $\widetilde{\mathcal{A}}$. We suppose that the point $P \in H$ is not a vertex of H. We assume that $P = (0,0)$ and the edge E containing P is horizontal. We consider the long edge $\mathfrak{E} = E_P((1,0))$.

Call the vertices on $\mathfrak{E}$ from left to right $A_1, A_2, A_3, \dots, A_n$. Clearly, we have $\mathfrak{I}_P((1,0)) = \bigcup_{i=1}^n \{A_i\}$ (Def. 2.2). We denote by E_i the edge of H such that $E_i \subset \mathfrak{E}$ and the left end of E_i, if exists, is the point A_i. If $\mathfrak{E}$ contains an infinite edge of H without the left end, as in Example 1.10, we call it E_0. Let P belongs to E_ℓ. Refer to Figure 5 for this notation.

If A_1 is the left end of $\mathfrak{E}$, then for the consistency of notation we add a "fictive" edge E_0 which has length zero, $d(E_0)$ will denote the leftmost vertex of the face $d(A_1)$. We say that $d(E_0)$ is a vertical edge of zero length. Similarly, if A_n is the right end of $\mathfrak{E}$, then we add a "fictive" edge E_n which has length zero, $d(E_n)$ will denote the rightmost vertex of the face $d(A_n)$. Now, regardless of finiteness of $\mathfrak{E}$, we always have edges $E_0, E_1, \dots, E_n$. Since $\mathfrak{E}$ is horizontal, it follows from Proposition 1.8 that for each $i = 0, \dots, n$ the edge $d(E_i)$ is vertical.

Definition 5.1. Refer to Figure 6(A). Let x_i be the x-coordinate of the edge $d(E_i)$. By $y_i \le y^i$ we denote the y-coordinates of the endpoints of $d(E_i)$, and by $m_i = y^i - y_i$ the lattice length of $d(E_i)$.

Note, that we have $y_i = y^i$ if and only if $i = 0$ (resp. $i = n$) and the long edge $\mathfrak{E}$ is finite from the left (resp. right) side.

Proposition 5.2. *For each $i = 1, ..., n$ we have*

$$\operatorname{area}(d(A_i)) \ge \frac{1}{2}(x_i - x_{i-1})(m_i + m_{i-1}). \tag{7}$$

Proof. Since $d(A_i)$ has two vertical sides of lengths m_i, m_{i-1}, the inequality follows from the convexity of $d(A_i)$. □

Definition 5.3. Recall that for each edge E' of H, there is the dual edge $d(E')$ in the subdivision of Δ. Also, all the edges in the subdivision of Δ come as the projections of the edges of $\widetilde{\mathcal{A}}$. We denote by $L(d(E'))$ the lifting of an edge $d(E')$ in the boundary of $\widetilde{\mathcal{A}}$.

If $d(E_0)$ is a point, then we denote by $L(d(E_0))$ the corresponding vertex of $\widetilde{\mathcal{A}}$, we apply the same rule for $d(E_n)$, look at the point $d(E_4)$ in Figure 5.

Proposition 5.4. *For each $i = 1, \dots, n$ the face of $\widetilde{\mathcal{A}}$ spanned by $L(d(E_{i-1}))$ and $L(d(E_i))$ projects to the face $d(A_i)$.*

Proof. This follows from Proposition 1.8. Refer to Figure 5. □

The edge E_ℓ is horizontal and passes through $(0,0)$, that implies the following lemma. Nevertheless, we give more details to illustrate the notation.

Lemma 5.5. *The direction of the edge $L(d(E_\ell))$ is $(0,1,0)$ and $L(d(E_\ell))$ is higher than all other points of $\widetilde{\mathcal{A}}$.*

Proof. Refer to Figure 2. The top end $(x_l, y^l) \in \mathcal{A}$ of $d(E_l)$ represents the tropical monomial $M_1 = \operatorname{val}(a_{x_l y^l}) + x_l X + y^l Y$ of $\operatorname{Trop}(F)$; M_1 dominates other monomials in the region above the edge E_l. The bottom end $(x_l, y_l) \in \mathcal{A}$ of $d(E_l)$ represents the monomial $M_2 = \operatorname{val}(a_{x_l y_l}) + x_l X + y_l Y$ which dominates other monomials in the region below the edge E_l. Therefore M_1 and M_2 are equal on the edge E_l, in particular at the point $(0,0)$; therefore $\operatorname{val}(a_{x_l y^l}) = \operatorname{val}(a_{x_l y_l})$, hence $L(d(E))$ is horizontal. Furthermore, $\max_{(i,j)\in\mathcal{A}}(\operatorname{val}(a_{ij}) + iX + jY) = \operatorname{val}(a_{x_l y^l}) = \operatorname{val}(a_{x_l y_l})$ at the point $(0,0)$.

If for some i, j we have $\operatorname{val}(a_{ij}) = \operatorname{val}(a_{x_l y^l})$, then $i = x_l$, otherwise $P = (0,0)$ is a vertex of H. It follows from the maximality of $\operatorname{val}(a_{x_l y^l}) + x_l X + y^l Y$ in the region above E_l that $j \le y^l$; then $y_l \le j$ by symmetric reasoning. □

Refer to Figure 5: the height of each bold edge $d(E_k)$ on the left side of the picture is greater then the heights $\operatorname{val}(a_{ij})$ of the points $(i, j, \operatorname{val}(a_{ij}))$ such that (i,j) lies to the left of E_k. In other words, the projections of bold edges on the xz-plane lie on the boundary of the xz-projection of $\widetilde{\mathcal{A}}$.

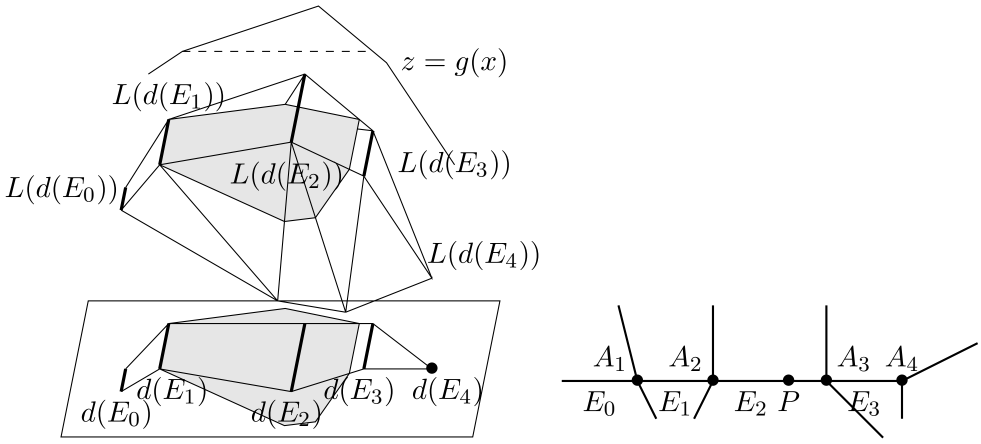

FIGURE 5. On the left we see a part of the extended Newton polyhedron, which corresponds to a horizontal long edge on the right. The long edge $E_P((1,0))$ consists of the edges E_0, E_1, E_2, E_3, $l = 2$, and $\mathfrak{I}(P) = \{A_1, A_2, A_3, A_4\}$. The edges $L(d(E_i))$ of $\widetilde{\mathcal{A}}$ are depicted as thick black horizontal intervals, while a section of the extended Newton polyhedron by a horizontal plane is marked in gray, as well as its projection onto the xy-plane. In this example $l = 2$. The projection of $\widetilde{\mathcal{A}}$ onto the xz-plane is also depicted, the projection of the section is dashed. Note, that we added a fictive edge E_4, and $d(E_4)$ is the rightmost vertex of $d(A_4)$.

Lemma 5.6. *Consider an edge E_q with $q < l$. Therefore, for each $(i,j) \in \mathcal{A}$ with the property 1) $i < x_q$ or 2) $i = x_q, j < y_q$, or 3) $i = x_q, j > y^q$, the number* $\mathrm{val}(a_{ij})$ *is less than* $\mathrm{val}(a_{x_q y_q}) = \mathrm{val}(a_{x_q y^q})$. *The symmetric statement holds for $q > l$.*

Proof. Refer to Figure 5. Each two consecutive edges $d(E_i), d(E_{i+1})$ bound the face $d(A_{i+1})$, therefore the edged $L(d(E_i)), L(d(E_{i+1}))$ (bold in Figure 5) also bound a face of the polyhedron $\widetilde{\mathcal{A}}$. The edges $d(E_i)$ are all parallel to $d(E_l)$, therefore all the edges $L(d(E_i))$ are parallel to each other as well. Provided $\widetilde{\mathcal{A}}$ is a convex polytope, all the points $(i, j, \mathrm{val}(a_{ij}))$ lie under each plane passing through a face of $\widetilde{\mathcal{A}}$. The part with $q > l$ can be proven by a word-by-word repetition of the above arguments. $\square$

Definition 5.7. Let us define $v_q := \mathrm{val}(a_{x_q y_q}) = \mathrm{val}(a_{x_q y^q})$, the height of the edge $L(d(E_q))$.

Lemma 5.6 implies that $v_0 < v_1 < \cdots < v_\ell > v_{l+1} > \cdots > v_n$.

Let us project the boundary of $\widetilde{\mathcal{A}}$ to the xz-plane. Each edge $L(d(E_i))$ is projected to the point (x_i, v_i) (Figure 5(A) and Figure 6(B) show examples of the result of such a projection).

Definition 5.8. Let $g(x)$ equal $\max\{z | (x, y, z) \in \widetilde{\mathcal{A}}\}$.

The xz-projection of the face of $\widetilde{\mathcal{A}}$ stretched on the edges $L(d(E_i)), L(d(E_{i+1}))$ coincides with the graph of g on the interval $[x_i, x_{i+1}]$, i.e., with the interval $(x_i, v_i), (x_{i+1}, v_{i+1})$ (compare Figures 5 and 6).

For $x' \in [x_0, x_n]$ let $\hat{g}(x')$ be the length of the interval excised out of the line $z = g(x')$ by the graph of g (see Figure 6, and the definition before Lemma 1.29).

Remark 5.9. If P is a vertex of H, then we can repeat all the above steps for each long edge through P.

5.2. **The proof of Lemma 2.8.** In Example 1.10, G can be written as

$$t^{-3} \cdot x(y-1)^3 + t^{-2} \cdot x(x-1)(y-1)^2 + t^{-1} \cdot (x-1)^2(y-1) + 1 \cdot (x-1)^3.$$

Therefore, in that example the extended Newton polyhedron is made of layers of m-thick sets, namely $\mathrm{supp}(x(y-1)^3), \mathrm{supp}(x(x-1)(y-1)^2), \mathrm{supp}((x-1)^2(y-1)), \mathrm{supp}((x-1)^3)$.

Let $H = \mathrm{Trop}(C)$ and $\mu_{(1,1)}(C) \geq m$. We will prove that the horizontal sections of $\widetilde{\mathcal{A}}$ passing through edges $L(d(E_i)), i = 0, \dots, n$ are m-thick, and then we extend this result to all the horizontal sections by Proposition 1.31.

Proposition 5.10. *If P is not a vertex of H, then the edge $d(E_l)$ (see Section 5.1 for the notation) has the lattice length at least m.*

Proof. Let $\mu' = \max\{\mu \in \mathbb{R} | \mathcal{A}_\mu \neq \varnothing\}$. Clearly, $d(E_l) = \mathrm{ConvHull}(\mathcal{A}_{\mu'})$. By M-thickness Lemma, $d(E_l)$ is m-thick, which finishes the proof. □

Remark 5.11. If P is a vertex of $\mathrm{Trop}(C)$, then the same reasoning shows that $\widetilde{\mathcal{A}}_{\mu'} = d(P)$ is m-thick. Furthermore, $\widetilde{\mathcal{A}}_\mu$ (Def. 2.6) always contains $\mathcal{A}_{\mu'}$ for each $\mu < \mu'$.

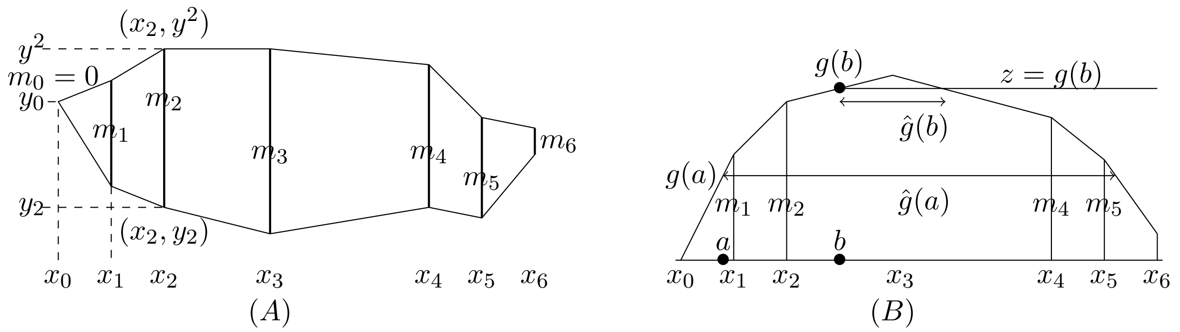

FIGURE 6. Projections of $\widetilde{\mathcal{A}}$ to the xy-plane (A) and to the xz-plane (B) are depicted. The number x_i is the x-coordinate of the edge $d(E_i)$ in (A). In this example the long edge $E_P((1,0))$ is finite from the left side (therefore $m_0 = 0$) and infinite from the right side (therefore $m_n = m_6 > 0$). By definition $g(x_i) = v_i$ in (B). Also $\hat{g}(a)$ and $\hat{g}(b)$ are presented in (B), and $\hat{g}(x_3) = 0, l = 3$. The key observation is that $\hat{g}(x_i) + m_i \geq m$ (Lemma 5.12). Furthermore, $\hat{g}$ is concave on $[x_i, x_{i+1}]$ for each i, see Proposition 1.31 for details.

By M-thickness Lemma, for each $i = 0, \dots, n$ the set $\mathcal{A}_{v_i}$ is m-thick. The following Lemma estimates the length of $d(E_i)$ via the width $\hat{g}(x_i)$ of the horizontal section through $L(d(E_i))$.

Lemma 5.12. *For each $i = 0, 1, \dots, n$ the length m_i of the edge $d(E_i)$ is at least $m - \hat{g}(x_i)$.*

Proof. We draw the horizontal section $\{z = v_i\}$ through the bold edge $L(d(E_i))$, refer to Figure 5 where $i = l - 1$. Consider the line $z = g(x_i)$ in the xz-plane. Suppose that the projection of the interval, excised on this line by the graph of g, onto the x-axis is $[x_i, x_i'], x_i' > x_i$. In fact, the length $\hat{g}(x_i)$ of the dashed line in Figure 5 satisfies $\hat{g}(x_i) = x_i' - x_i = \omega_{(1,0)}(\widetilde{\mathcal{A}} \cap \{z = v_i\}) = \omega_{(1,0)}(\widetilde{\mathcal{A}}_{v_i})$. The set $\mathcal{A}_{v_i}$ is inside the strip $\{(x, y)|\ x_i \leq x \leq x_i'\}$, and $\mathcal{A}_{v_i}$ is m-thick by M-thickness Lemma. Since $\mathrm{ConvHull}(\mathcal{A}_{v_i}) \cap \{x = x_i\}$ is $d(E_i)$, this lemma follows from the definition of m-thickness. □

Remark 5.13. In fact, $\mathcal{A}_{v_i}$ is contained in the xy-projection of $\{z = v_i\} \cap \widetilde{\mathcal{A}}$, but does not necessary coincide with it.

Consider the following piecewise linear function f on the interval $[x_0, x_n]$: let $f(x_i) = m_i$ for $i = 0, \dots, n$, then extend f to be linear on each interval $[x_i, x_{i+1}]$.

Proposition 5.14. *The length of the left vertical side of* $\mathrm{ConvHull}(\widetilde{\mathcal{A}}_{g(x)}), x \leq x_l$ *is at least* $f(x)$.

Proof. It follows from the fact that the face of $\widetilde{\mathcal{A}}$ stretched on $L(d(E_i)), L(d(E_{i+1}))$ contains the trapezoid stretched on $L(d(E_i)), L(d(E_{i+1}))$, and f calculates the lengths of its intersection with horizontal sections. □

Lemma 5.15. *The inequality $f(x)+\hat{g}(x)\geq m$ holds on the interval $[x_0,x_n]$.*

Proof. For each $i=0,\dots,n$ the inequality $f(x_i)+\hat{g}(x_i)\geq m$ is satisfied by Lemma 5.12. Consider an interval $[x_i,x_{i+1}]$. Since f is linear and $\hat{g}$ is concave on $[x_i,x_{i+1}]$ (Proposition 1.31), we have $f(x)+\hat{g}(x)\geq m$ for each $x\in[x_1,x_{i+1}]$. □

Proof of Lemma 2.8. Suppose that P is not a vertex of $\mathrm{Trop}(C)$, P belongs to a horizontal edge of $\mathrm{Trop}(C)$. It follows from Remark 5.11 that it is enough to check m-thickness of $\widetilde{\mathcal{A}}_\mu$ only in the direction $(1,0)$. The latter follows from Lemma 5.15 and Proposition 5.14. If P is a vertex of $\mathrm{Trop}(C)$, then, again, Remark 5.11 implies that we need to check m-thickness $\widetilde{\mathcal{A}}_\mu$ only in the directions of the edges through P. For each edge through P we use Propositions 1.19, 1.20 for making this edge horizontal. Then we repeat the above arguments. □

5.3. **Proof of the Exertion theorem for edges.** The second part of the Exertion Theorem for edges is proved in Proposition 5.10.

Lemma 5.16. *Refer to Figure 6(A) for the notation. If H is admissible (Def. 2.10) and $\mu_P^{trop}(H)\geq m$, then $x_n-x_0\geq m$.*

Proof. Let us suppose that $x_n-x_0<m$. If $m_0,m_n>0$, then $\omega_{(1,0)}(\mathcal{A})=x_n-x_0<m$, and the curve H is not admissible. If $m_0=0$ and $m_n>0$, then $\omega_{(1,0)}(\mathcal{A}_{v_0})<m$ and $\mathcal{A}_{v_0}$ does not have two vertical sides, which contradicts the fact that $\mathcal{A}_{v_0}$ is m-thick (Proposition 1.24). If both $m_0=m_n=0$, then we apply the above argument for $\mathcal{A}_{\max(v_0,v_n)}$. □

Proposition 5.17. *If a point P is of multiplicity at least m in the intermediate sense, then P is of multiplicity at least m in the intrinsic sense (Def. 3.4).*

Proof. Indeed, let us take a generalized tropical line L. We will verify Definition 3.4. If P is the vertex of L or $TC(P)$ does no contain the vertex of L, then the fact that $\widetilde{\mathcal{A}}_{\mu'}$ is m-thick (Remark 5.11) implies that $L\cdot_P H\geq m$. If the vertex V of L belongs to a long edge through P, then we use the notation in Section 5.1. Let V belongs to E_k. Recall that L passes through P, therefore L has a horizontal component. Draw the horizontal section through $L(d(E_k))$. A direct calculation and Lemma 5.12 show that m-thickness of $\widetilde{\mathcal{A}}_{v_k}$ implies that $L\cdot_P H\geq m$. □

Proposition 5.18. *By Lemma 5.16, there are points $b,c\in[x_0,x_n]$ such that $c-b=m$ and one of the following statements holds*

- $g(b)=g(c)$,
- $g(b)\leq g(c); c=x_n$,
- $g(b)\geq g(c), b=x_0$. □

The points b,c are chosen in such a way that $\hat{g}_{[b,c]}(x)=(\hat{g}_{[x_0,x_n]})|_{[b,c]}(x)$ for $x\in[b,c]$. By $h|_{[b,c]}$ we mean the restriction of h to $[b,c]$. The definition of $f(x)$ is given before Lemma 5.15.

Proof of Theorem 1. We complete the proof, applying Lemma 1.29 on the interval $[b,c]$ of the length m.

$$\mathrm{area}(\mathfrak{Infl}(P))\geq\int_{x_0}^{x_n}f(x)dx\geq\int_b^c f(x)dx\geq\int_b^c(m-\hat{g}(x))dx\geq m(c-b)-\frac{(c-b)^2}{2}=\frac{m^2}{2}.$$ □

Proposition 5.19. *If $E_P((1,0))$ coincides with the interval $[A_1,A_n]$ and $x_n-x_0=m$, then only one point $P\in[A_1,A_n]$ can be a point of multiplicity m in the intermediate sense.*

Proof. Indeed, using the m-thickness property of $\mathcal{A}_{\max(v_0,v_n)}$ we conclude that $v_0=v_n$ (cf. Lemma 5.16). This is equivalent to the fact that $\mathrm{val}(a_{x_0y_0})=\mathrm{val}(a_{x_ny_n})$ where (x_0,y_0) is the leftmost vertex of $d(A_1)$ and (x_n,y_n) is the rightmost vertex of $d(A_n)$, see Figure 6. All this notation (Section 5.1) was developed for the case $P=(0,0)$. Then, using Proposition 1.20 we see that the choice of other point $P'\in[A_1,A_n]$ and subsequent change of the coordinates in order to have $P'=(0,0)$ will destroy the equality $v_0=v_n$. □

For example, we can prove that in Example 1.10 if P is of multiplicity 3 in the extrinsic sense, then P must divide the edge it in the ratio $1:2$. Also, in the hypothesis of the above proposition it is possible to determine the position of the singular point via tropical modifications ([14]).

5.4. **Proof of the Exertion theorem for vertices.** Now we are in the hypothesis of the Exertion Theorem for vertices, i.e. $\mu_P^{trop}(H) \geq m$, P is a vertex of H, and the Newton polygon Δ of H has the minimal lattice width at least m. For each direction $u \in P(\mathbb{Z}^2)$, such that the face $d(P)$ has at most one side perpendicular to u, the width $\omega_u(d(P))$ is at least m. This follows from Lemma 1.25, since $d(P)$ is m-thick.

Suppose that the point P belongs to an edge $E \subset H$ of the direction u. If $\omega_u(d(P)) < m$, then the face $d(P)$ has two sides of lattice length at least $\mathrm{def}_u(d(P))$ (Def. 4.1), and these sides are perpendicular to the vector u, see Figure 7.

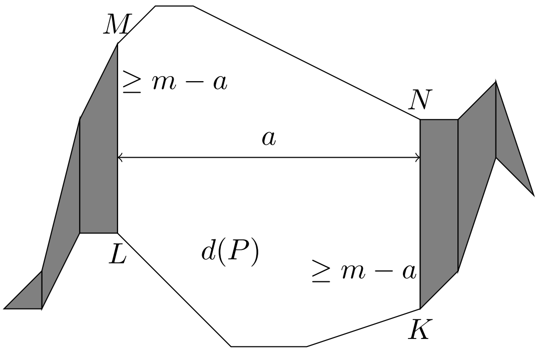

FIGURE 7. An example of the dual picture to a horizontal long edge through P, if P is a vertex of H. We have $\omega_{(1,0)}(d(P)) = a$ and $\mu_P^{trop}(H) \geq m$, therefore the lengths of LM and NK are at least $m-a$. The set $\bigcup d(Q)$ for $Q \in \mathfrak{I}_{(1,0)}(P), Q \neq P$ is colored. Lemma 5.20 states that the sum of the areas of the colored faces is at least $\frac{1}{2}(m-a)^2$.

Lemma 5.20. *If $\mu_P^{trop}(H) \geq m$, P is a vertex of H, and $u \in P(\mathbb{Z}^2)$, then*

$$\sum_{V \in \mathfrak{I}_P(u), V \neq P} \mathrm{area}(d(V)) \geq \frac{1}{2}\mathrm{def}_u(d(P))^2. \tag{8}$$

Proof. Applying a change of coordinates (Proposition 1.19) we may assume that $u = (1,0)$. Let $\omega_u(d(P)) = a$. The faces of the subdivision, contributing to (8), are colored in Figure 7. Look at the set $\{(i,j) \in \mathbb{Z}^2\}$ where $\mathrm{val}(a_{ij})$ is maximal. It contains the vertices of $d(P)$ and maybe some integer points inside $d(P)$. As in the proof of the Exertion Theorem for edges, we consider the sets A_μ for different μ, and repeat all the other steps. In the final step of the proof, instead of the integral $\int_b^c (m-\hat{g})dx$ we consider the integral $\int_b^{x_i}(m-\hat{g}) + \int_{x_{i+1}}^c (m-\hat{g})$ where x_i, x_{i+1} are the x-coordinates of the vertical sides of $d(P)$. Finally,

$$\sum_{Q \in \mathfrak{I}_u(P), Q \neq P} \mathrm{area}(d(Q)) \geq \int_b^{x_i} (m-\hat{g})dx + \int_{x_{i+1}}^c (m-\hat{g})dx =$$

$$= m(x_i - b) + m(c - x_{i+1}) - \left(\int_b^{x_i} \hat{g}dx + \int_{x_i}^c \hat{g}dx\right) =$$

$$= m(m-a) - \left(\frac{1}{2}(c-b)^2 - \frac{1}{2}(x_{i+1}-x_i)^2\right) = \frac{1}{2}(m-a)^2 = \frac{1}{2}\mathrm{def}_u(d(P))^2,$$

by Corollary 1.30. □

Proof of Theorem 2. Indeed, it follows from Lemma 5.20 that

$$\mathrm{area}^*(\mathfrak{Infl}(P)) = \sum_{\substack{u \in P(\mathbb{Z}^2),\\ V \in \mathfrak{I}_u(P), V \neq P}} \mathrm{area}(d(V)) + \mathrm{area}(d(P)) \geq \mathrm{area}(d(P)) + \frac{1}{2}\sum_{u \in P(\mathbb{Z}^2)} \mathrm{def}_u(d(P))^2,$$

and the latter expression is at least $\frac{3}{8}m^2$ by Lemma 4.2.

Similarly, by Lemma 4.3 we get

$$\mathrm{area}(\mathfrak{Infl}(P)) \geq 2 \cdot \mathrm{area}(d(P)) + \frac{1}{2} \sum_{u \in P(\mathbb{Z}^2)} \mathrm{def}_u(d(P))^2 \geq \frac{1}{2} m^2.$$

□

6. Discussion

"The forceps of our minds are clumsy forceps,
and crush the truth a little in taking hold of it."
H. G. Wells

In this section we show that a point of multiplicity m can impose less than $\frac{m(m+1)}{2}$ linearly independent conditions on the coefficients of a curve's equation. Also, we summarize what is known about tropical points of multiplicity m.

6.1. **Examples and the Euler derivative.**

Example 6.1. Fix $k \in \mathbb{N}$. The polygon T_k of the minimal area with $\omega(T_k) = 2k$ is the triangle with the vertices $(0,0), (k,2k), (2k,k)$ (see Remark 4.14). The triangle T_k comes as the support set of the polynomial $(1 - 3xy + xy^2 + x^2y)^k = 0$ which defines a curve C with $\mu_{(1,1)}(C) = 2k$. The area of T_k is $\frac{3}{8}(2k)^2$ which shows that the estimate in the Exertion Theorem for vertices is sharp.

If $char(\mathbb{K}) = 0$, then $\mu_{(1,1)}(C) \geq 2k$ is equivalent to the set of linear equations $\frac{\partial^{q+r}}{\partial^q x \partial^r y} F(x,y) = 0, q + r < 2k$ in the coefficients of the polynomial $F = \sum\limits_{(i,j) \in T_k} a_{ij} x^i y^j$. Note that among these equations there are at least

$$\frac{2k(2k+1) - (3k^2 + 3k + 2)}{2} = \frac{k^2 - k - 2}{2}$$

linearly dependent ones. Here $\frac{2k(2k+1)}{2}$ is the number of equations and $\frac{3k^2+3k+2}{2}$ is the number of variables, i.e., the number of integer points in T.

Example 6.2. To see one more phenomenon we consider the set

$$\mathcal{A} = \mathrm{ConvHull}((0,0),(3,1),(6,3),(6,4),(3,6),(1,3)) = T_3 \cup \{(1,3),(3,1),(6,4)\}.$$

The only curve C with support in $\mathcal{A}$ and $\mu_{(1,1)}(C) = 6$ is given by the equation $(1 - 3xy + xy^2 + x^2y)^3 = 0$. Hence adding three new monomials $a_{13}xy^3 + a_{31}x^3y + a_{64}x^6y^4$ does not add new degrees of freedom and a_{13}, a_{31}, a_{64} are always 0.

We give the following explanation. Consider the constraint on a_{ij} imposed by the fact that $F_{xx}(1,1) = 0$. That is $\sum i(i-1)a_{ij} = 0$. Note, that the set of a_{ij} with non-zero coefficients in this equation is parametrized by $\mathcal{A} \setminus \{(i,j) | i(i-1) = 0\}$. So, we say that $i(i-1)$ corresponds to F_{xx}.

In a similar way, given $\mu_{(1,1)} = 6$, by considering linear combinations of $F, F_x, F_{xy}, \ldots, F_{yyyyy}$ we can obtain all the polynomials in i,j of degree at most five. Next, $(6,4)$ is the only point in $\mathcal{A}$ where $f(i,j) = (j-3)(i-j)(i-3)(i^2 + j^2 - ij - 3j - 3i + 6)$ is not zero. The corresponding to $f(i,j)$ linear equation

$$(9) \quad (F_{xxxxy} - 2F_{xxxyy} + 2F_{xxyyy} - F_{xyyyy} - 3F_{xxxx} + 4F_{xxxy} - \\ - 4F_{xyyy} + 3F_{yyyy} - 12F_{xx} + 12F_{yy} + 24F_x - 24F_y)_{(1,1)} = 0,$$

written in terms of a_{ij}, is just $a_{64} = 0$. Similar combinations of derivatives can be found for a_{13} and a_{31}.

Let char $\mathbb{K} = 0$. In this case, [6] contains the complete description of the matroid M associated with the linear conditions imposed by the m-fold point at $(1,1)$. Namely, all the dependent sets of M, minimal by inclusion, are the sets of the type $\mathcal{A} \setminus \{(i,j) | G(i,j) = 0\}$, where $G \in \mathbb{K}[i,j]$ is a polynomial of degree at most $m-1$.

Let $\mathcal{A}_G$ be $\mathcal{A} \setminus \{(t,w)|G(t,w)=0\}$. We call the operation

$$\partial_G : \sum_{(i,j)\in\mathcal{A}} a_{ij}x^iy^j \to \sum_{(i,j)\in\mathcal{A}_G} a_{ij}x^iy^j$$

the Euler derivative with respect to G. Suppose that a tropical curve H is given by $\mathrm{Trop}(F)$ where F is in (1).

Proposition 6.3. *([6]) A point $P \in H$ is a point of multiplicity at least m in the $\mathbb{K}$-extrinsic sense (Def. 0.2) if and only if for each polynomial $G \in \mathbb{K}[i,j]$ of degree no more than $m-1$ the tropical curve given by* $\mathrm{Trop}(\partial_G F)$ *passes through P.*

Remark 6.4. If char $\mathbb{K}=0$, then the above proposition implies the m-thickness property for $\mathcal{A}$ if $\mu_{(1,1)}(C)=m$ (cf. Corollary 1.26). Indeed, if the set $\mathcal{A}$ is not m-thick, then there exists a collection of $m-1$ lines $l_1,\dots,l_{m-1}$ such that $\mathcal{A}\setminus\bigcup\{l_i\}=(i',j')\in\mathbb{Z}^2$. Let the polynomial G be the product of the equations of the lines l_i. Clearly, $\deg(G)=m-1$. Then, $\partial_G F = a_{i'j'}x^{i'}y^{j'}$, and $\mathrm{Trop}(\partial_G F)$ is smooth at P. This contradicts to Proposition 6.3.

One can argue that in Examples 6.1, 6.2 we have lesser degree of freedom because the curves were reducible, so, look at the following example.

Example 6.5. Consider the curve C' given by the equation $(x^2y+xy^2-3xy+1)^8+xy^4(x-1)^8=0$. It is irreducible, $\mu_{(1,1)}(C')=8$ and the number of integer points in the Newton polygon of C' is 35 which is less than the number of linear condition, namely, 36.

6.2. **Tropical points of multiplicity** m**.** The aim of the present work was to improve understanding of the combinatorics of tropical singular points. Application of the Exertion Theorems for Nagata's conjecture can be found in [15].

For a tropical curve H, if a point P is of multiplicity at least m in the $\mathbb{K}$-extrinsic sense (Def. 0.2), then P is of multiplicity at least m in the intermediate sense (Def. 2.7), see Lemma 2.8.

We say that a tropical curve H can be lifted over a field $\mathbb{K}$ if there exists a curve C' over $\mathbb{K}$ such that $\mathrm{Trop}(C')=H$.

Question: is it true that for each m-thick (Def. 1.23) set $B\subset\mathbb{Z}^2$ there exists a polynomial $G\in\mathbb{Q}[x,y]$, defining the curve C', such that $\mu_{(1,1)}(C')\geq m$ and $\mathrm{ConvHull}(\mathrm{supp}(G))=\mathrm{ConvHull}(B)$? As shown in Example 6.2, the answer is "no".

Let a point $P\in H$ be of multiplicity m in the $\mathbb{K}$-extrinsic sense for some valuation field $\mathbb{K}$. Suppose that H can be lifted over another field $\mathbb{K}'$ of the same characteristic.

Question: is it true that the point P is of multiplicity m in the $\mathbb{K}'$-extrinsic sense? As far as the author knows, this question is an open problem.

For a tropical curve H, if a point $P\in H$ is of multiplicity at least m in the intermediate sense, then P is of multiplicity at least m in the intrinsic sense (Def. 3.4), see Proposition 5.17.

Note, that the method in Proposition 6.3, which allows us to verify the definition in the extrinsic sense, requires the information about all the valuations of the coefficients of the equation of the tropical curve H. Therefore, we have to know even those coefficients which can be perturbed without changing H. Hence, given only a tropical curve H, the verification of Definition 0.2 is not straightforward.

On the opposite side, it is enough to know only the dual subdivision of the Newton polygon for H in order to verify the definition in the intrinsic sense (Def. 3.4). The multiplicity in the intrinsic sense of a point $P\in H$ remains the same if we change the lengths of the edges of H. Quite the contrary, for Definition 2.7 of multiplicity in the intermediate sense the lengths of the edges of H are important because we operate with the extended Newton polyhedron $\widetilde{\mathcal{A}}$, see also Remark 1.12.

So, if a point P is a point of multiplicity m in the extrinsic sense, then P satisfies some necessary conditions, for example, estimates in the Exertion Theorems hold and can be easily verified. Nevertheless an ambiguity remains: it is possible that a lot of the points on an edge E are of multiplicity m in the extrinsic sense, but we can not realize them as tropicalizations of m-fold points simultaneously, see examples in [17, 18]. See also Proposition 5.19 for the case where we can prove that the position of P is unique.

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