MEROMORPHIC BRAIDED CATEGORY ARISING IN QUANTUM AFFINE ALGEBRAS

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1 Introduction

Main result of this article consists in construction of what is called meromorphic braided (or tensor) category, see [So]. It arises in the representation theory of quantum affine algebras and lives on an elliptic curve.

There are many papers devoted to different aspects of finite-dimensional representations of quantum affine algebras. Surprisinly few of them consider categorical picture. Most of the authors study irreducible representations. Some fundamental questions remain unanswered. For example the fact that the universal R-matrix is meromorphic for *any* two finite-dimensional representations was not proved (to my knowledge) before [KS]. This fact is crucial for construction of meromorphic braided structure on the category of finite-dimensional representations. The latter contains an interesting subcategory with objects naturally "localized" on an elliptic curve.

In this article we consider the simplest example related to the quantum affine algebra $sl(2)$. Our constructions remind "chiral" objects from [BD]: we derive the braiding considering infinitesimal neighbourhood of the diagonal in the square of an elliptic curve.

The paper is organized as follows. Next two sections contain recollections from [So]. Last section contains construction and discussion. It can be read independently of the other part of the paper. In the last section we mainly concentrate on the case of quantum affine $sl(2)$. It is explained at the very end of the paper how main construction can be generalized to higher ranks if one uses results of [FR].

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2 Pseudo-braided categories

In this section we are going to recall (in a slightly revised form) main definitions from [So].

2.1

We denote by $\mathcal{T}$ the class of all planar trees, and by $\mathcal{T}(n)$ the subclass of trees having n tails (see [GK], [KM] about terminology). Then $\mathcal{T}(n)$ is a category (morphisms are identities and contraction of edges). For a given tree we orient edges and tails in such a way that there is a unique vertex with only one outgoing tail (this vertex is called the root of tree). We fix such an orientation for each tree. We also fix a numeration of tails : for each $T \in \mathcal{T}(n)$ they are numbered from 1 to n in such a way that the only outgoing tail is numbered by n.

An additional structure on $\mathcal{T}$ is given by the gluing operation: if $T \in \mathcal{T}(n+1), T_i \in \mathcal{T}(k_i+1), i = 1, ..., n+1$ then one can construct a new tree $T(T_1, ..., T_n) \in \mathcal{T}(k_1 + ... + k_n + 1)$ by gluing outgoing tail of T_i to the ith tail of T. The orientation of edges and numeration of tails for the new tree are defined in the natural way. The gluing operation is associative, and hence $\mathcal{T}$ becomes a strict monoidal 2-operad . The role of a unit object is played by the only tree $e \in \mathcal{T}(1)$.

Let $\mathcal{S} = (S_T, \mathcal{O}_{S_T})$ be a family of ringed spaces parametrized by trees from $\mathcal{T}$. We say that $\mathcal{S}$ is a monoidal operad of spaces if the following conditions are satisfied:

1) for any morphism $f : T' \to T \in \mathcal{T}(n)$ we are given a morphism of ringed spaces $l_f : S_{T'} \to S_T$.

2) For a gluing operation of trees $T \times T_1 \times ... \times T_n \to T(T_1, ..., T_n)$ we are given a morphism of ringed spaces (operadic composition) $\gamma : S_T \times S_{T_1} \times$

$... \times S_{T_n} \to S_{T(T_1,...,T_n)}$ which is strictly associative with respect to the gluing of trees. It is also assumed to be functorial with respect to the morphisms of trees.

In particular if we put $A = \Gamma(S_e, \mathcal{O}_{S_e})$, then all $\Gamma(S_T, \mathcal{O}_{S_T})$ become $A - A^{\otimes n}$ bimodules, where $T \in \mathcal{T}(n)$.

2.2

Let X be a set, $\mathcal{A}$ a class. Its elements are called objects. A family of objects of $\mathcal{A}$ parametrized by X (or simply X-family) is an element of $\prod_X \mathcal{A} = \mathcal{A}^X$. Suppose that $\mathcal{A}$ is a category. We keep the same notation for the class of objects of $\mathcal{A}$. Then X-families form a category $\mathcal{F}_X$ with $Hom_{\mathcal{F}_X}(M,N) = \prod_X Hom_{\mathcal{A}}(M_x, N_x)$. If $f : X \to Y$ is a map the there is a pull-back functor $f^* : \mathcal{F}_Y \to \mathcal{F}_X$.

If $(X, \mathcal{O}_X)$ is a ringed space then we understand an X-family as a family of objects which are $\mathcal{O}_{X,x}$-modules, where $O_{X,x}$ denotes the fiber at $x \in X$. The pull-back functors exist in the case of ringed spaces as well. We do not impose any continuity condition on the family at the moment.

Suppose that $\mathcal{S}$ is a monoidal operad of spaces as in the previous subsection. We suppose that we are given a field k , and that all sheaves of rings are in fact k-algebras and all morphisms of sheaves of rings are morphisms of k-algebras.

Definition 1 *A pseudo-monoidal category $\mathcal{C}$ of ringed spaces is given by the following data:*

a) a class $\mathcal{C}$ called class of objects;

b) for every $T \in \mathcal{T}(n)$, a sequence $\{X_i\}, 1 \leq i \leq n$ of objects of $\mathcal{C}$ and an object $Y \in \mathcal{C}$ a family of k-vector spaces $P_T(\{X_i\}, Y)$ over S_T (operations from $\{X_i\}$ to Y);

c) for every morphism $f : T' \to T$ a morphism $\phi_f : P_{T'}(\{X_i\}, Y) \to (l_f)^* P_T(\{X_i\}, Y)$ of families over $S_{T'}$;

d) for an operadic composition of spaces

$\gamma : S_T \times \prod_i S_{T_i} \to S_{T(T_1,...,T_n)}$ *we are given a morphism of families on $S_T \times \prod_i S_{T_i}$ (composition of operations):*

$\Phi_\gamma : P_T(\{X_i\}, Y) \times \prod_i P_{T_i}(\{K_{j_i}\}, X_i) \to \gamma^* P_{T(T_1,...,T_n)}(\{K_j\}, Y)$,

$\Phi_\gamma(\phi, (\psi_i)) = \phi(\psi_i)$.

Here $\{X_i\}$ are parametrized by the tails of T, $\{K_j\}$ are parametrized by the tails of $T(T_1, ..., T_n)$ and $\{K_{j_i}\}$ corresponds to the subsequence of objects parametrized by the tails of T_i;

e) for every object $X \in \mathcal{C}$ there exists a family $\mathbf{1}_X \in P_e(\{X\}, X)$ on S_e such that $\phi(\mathbf{1}_{X_i}) = \gamma^*(\phi)$ and $\mathbf{1}_X(\phi) = \eta^*(\phi)$.

Here $\phi \in P_T(\{X_i\}, X)$ and we use the notations γ and η for the natural operadic morphisms of spaces (see d)).

The composition morphisms Φ_γ from d) are required to be associative in the following sense:

$\Phi_\gamma(id \times \Phi_\delta) = \Phi_\delta(\Phi_\gamma \times id)$

(when applied to the corresponding families).

Sometimes we will call $\mathcal{C}$ a pseudo-monoidal category over $\mathcal{S}$.

Suppose our spaces are k-schemes. We call a pseudo-monoidal category *algebraic* if all families $P_T(\{X_i\}, Y)$ are quasi-coherent sheaves and the corresponding morphisms are morphisms of quasi-coherent sheaves.

Suppose that $k = \mathbf{C}$ and our spaces are complex analytic. We call a pseudo-monoidal category *analytic* if the families $P_T(\{X_i\}, Y)$ are complex analytic sheaves and the corresponding morphisms are morphisms of complex analytic sheaves.

Similarly, if these complex analytic spaces are irreducible, we can speak about *meromorphic* pseudo-monoidal categories (in this case P_T are analytic families on dense subsets and morphisms can be extended meromorphically to S_T).

In the case of schemes we obtain *rational* pseudo-monoidal categories.

We are going to use this terminology without further discussion.

2.3

Here we recall what is it a representable pseudo-monoidal structure. Let S be a topological space, $\mathcal{C}$ a category, $n \geq 1$ an integer.

We denote by $Funct_S(\mathcal{C}, n)$ the sheaf of categories on S such that for an open U in S we have $Funct_S(\mathcal{C}, n)(U)$= category of families of functors $\{F_x\}_{x \in U}, F_x : \mathcal{C}^n \to \mathcal{C}$. We will denote this sheaf of categories by $Funct_S(n)$ if it will not lead to a confusion.

Suppose that in addition we have a sequence of topological spaces $\{S_i\}, 1 \leq i \leq n$, and a sequence of positive integers $k_1, ..., k_n$. Then we have a morphism of sheaves of categories $Funct_S(n) \times \prod_i Funct_{S_i} \to Funct_{S\times\prod_i S_i}(k_1 + ... + k_n)$ (sheaves on $S \times \prod_i S_i$) such that $\{F_s\} \times \prod_i F^i_{s_i} \to F_s(F^1_{s_1}, ..., F^n_{s_n})$.

If $\mathcal{C}$ is a k-linear category then for a family $\{F_x\} \in Funct_S(n)(S)$ and a sequence $\{X_i\}, 1 \leq i \leq n$ of objects of $\mathcal{C}$ and an object $Y \in \mathcal{C}$ we have a family of vectors spaces $Hom_{\mathcal{C}}(F_x(\{X_i\}), Y)$ on S.

Suppose now that we have a monoidal operad of spaces $\mathcal{S} = (S_T)$ as before. Suppose that for every $T \in \mathcal{T}(n)$ we are given a family of functors $\{F^T_x\}_{x\in S_T} \in Funct_{S_T}(n)$. We remark that there is a natural operadic composition $Funct_{S_T}(n) \times \prod_i Funct_{S_{T_i}}(k_i) \to Funct_{S_T\times\prod_i S_{T_i}}(k_1 + ... + k_n) \to \gamma^*(Funct_{S_{T(T_1,...,T_n)}}(k_1 + ... + k_n)$ of sheaves of categories.

The last arrow corresponds to the operadic composition γ on the spaces which induces the obvious pull-back on the families. Suppose that we are given a pseudo-monoidal category on $\mathcal{S}$ as before.

Definition 2 *We say that it is representable if for every $T \in \mathcal{T}(n)$ there exists a family $F^T_x \in Funct_{S_T}(n)(S_T)$ and an isomorphism of families of k-vector spaces $P_T(\{X_i\}, Y) \to Hom_{\mathcal{C}}(F^T_x(\{X_i\}), Y)$ which is compatible with the operadic composition on both families.*

It is also required that for the only tree $e \in \mathcal{T}(1)$ we have: $Hom_{\mathcal{C}}(F^e_x(Y), Y)$ corresponds to $\mathbf{1}_Y$ under this isomorphism.

If $\mathcal{C}$ is an algebraic or analytic pseudo-monoidal category then representability is understood in the corresponding category. In particular it is assumed that families of functors define families of Hom's in the corresponding category, and the isomorphisms of families must be isomorphisms of quasi-coherent sheaves or analytic sheaves. Similarly one defines representable rational and meromorphic pseudo-monoidal categories. We skip "pseudo" in the case when all morphisms in the Definition 1c) are isomorphisms. In this way we obtain for example the notion of meromorphic monoidal category discussed in [So].

It is easy to define the notion of a functor between two pseudo-monoidal categories $\mathcal{A}$ and $\mathcal{B}$ which live over different operads of spaces,say, $\mathcal{S} = (S_T)$ and $\mathcal{R} = (R_T)$ respectively. It consists of morphisms of ringed spaces $h_T : S_T \to R_T$ which produce a morphism of the monoidal operads of spaces, of the mapping of objects $F : \mathcal{A} \to \mathcal{B}$, and of the morphisms of families $l_T : P^{\mathcal{A}}_T(\{X_i\}, Y) \to h^*_T P^{\mathcal{B}}_T(\{F(X_i)\}, F(Y))$ which are compatible with the compositions and the unit family. It is clear how to specify this definition for the case of schemes or analytic spaces as well as to the rational or meromorphic case.

2.4

Let $\mathcal{C}$ be a pseudo-monoidal category over $\mathcal{S}$. It is called $pseudo - braided$ if for every element σ of the braid group B_n we are given a morphism of families $\mu_\sigma P_T(\{X_i\}, Y) \to P_T(\{X_{\sigma(i)}\}, Y)$ which is identical on S_T. Here σ acts on i as the corresponding permutation from the permutation group S_n.

These morphisms are required to satisfy various natural properties. Details can be found in [So]. Here we list them shortly:

a) $\mu_{\sigma\tau} = \mu_\sigma\mu_\tau$, $\mu_1 = id$ where 1 denotes the unit of the group;

b) for any $\sigma \in B_n$ the morphism μ_σ commutes with morphisms in $\mathcal{T}(n)$, and μ_1 preserves $\mathbf{1}_X$ for any object X;

c) compatibility with the composition maps (this means natural commutative diagram, see [So]).

Now we can specify an operad $\mathcal{S}$ of spaces (schemes, analytic spaces, manifolds, etc.). Then we require that all μ_σ are morphisms in the corresponding category and arrive to various versions of pseudo-braded categories (algebraic, analytic, etc.). If the our spaces are irreducible (in the corresponding category) we can require that all μ_σ are defined in the generic point only. In this way we obtain the notions ot *rational* (in the case of schemes) and *meromorphic* (in the case of analytic spaces or complex manifolds) pseudo-braided category (see [So] for the details). The notion of a functor between such categories is defined in the natural way. The definitions in the representable case are also clear. Then we also have the notion of braided category, meromorphic braided category, etc. Sometimes we will simply call them *tensor*, *meromorphic tensor*, etc.

3 Examples

We recall here main examples.

Example 1

Let G be a complex analytic group (the condition on G can be weaker of course). We can define the following operad of spaces. For $T \in \mathcal{T}(n)$ we put

$S_T = G^n, G^0 = id$. Then the morphisms from the Definition 1c) are identities. We also have an operadic composition $\gamma : G^n \times G^{k_1} \times ... \times G^{k_n} \to G^{k_1+...+k_n}$ such that $\gamma((g_i) \times \prod_i (g_{ij})) = (g_i g_{ij})$.

Let $\mathcal{C}$ be a category equipped with the action of G on objects: an object M is transformed by $g \in G$ into the object called $M(g)$.

Then we say that $\mathcal{C}$ is a meromorphic monoidal G-category if for any $T \in \mathcal{T}(n)$ we are given a functor $\otimes_T : \mathcal{C}^n \to \mathcal{C}$ such that the families $Hom_{\mathcal{C}}(\otimes_T X_i(g_i), Y)$ define a representable meromorphic pseudo-monoidal structure on $\mathcal{C}$ such that all morphisms from Definition 1c) are meromorphic isomorphisms. Subsequently we have the notions of G-braided category or meromorphic G-braided category, etc.

Example 2

This is a specialization of the Example 1.

We take as G the group $\mathbf{C}^*$ of non-zero complex numbers. We take as $\mathcal{C}$ the category of finite-dimensional $U_q(g)$- modules where $U_q(g)$ is the Drinfeld-Jimbo quantized enveloping algebra of an affine Kac-Moody Lie algebra g. We call it quantum affine algebra. The well-known fact explained in [So] is that $\mathcal{C}$ is a meromorphic $\mathbf{C}^*$-tensor category. We will discuss this one and related category later in the text.

Example 3

The usual notions of monoidal and braided (=tensor) categories are special cases. They can be described as representable pseudo-monoidal or pseudo-braided structures. One can take either the trivial operad of spaces with all spaces being just one point. Or one can take the operad of moduli of stable punctured complex curves with tangent vector attached to the last point. Taking a connected stratum of the real points we get monoidal categories. In complex case we note that the fundamental groups of the punctured curves are pure braid groups. Thus we arrive to the description of tensor categories in terms of local systems on punctured curves due to Deligne ([De]). See [So] for details.

4 Quantum affine algebras

4.1

Let X be a complex manifold, A_X be a bundle of Hopf algebras on X which is equipped with a flat connection ∇. The latter means that ∇ has zero curvature and equalities $\nabla(ab) = \nabla(a)b + a\nabla(b)$ and $(\nabla \otimes 1 + 1 \otimes \nabla)(\Delta(a)) = \Delta(\nabla(a))$ hold locally (here Δ is the comultiplication morphism for A_X).

We define a category $\mathcal{C}_X$ as a category of holomorphic vector bundles V on X of finite rank, equipped with a flat connection ∇_V which are (A_X, ∇)-modules. This means that V is a locally free sheaf of A_X-modules and the equality $\nabla_V(av) = \nabla_X(a)v + a\nabla_V(v)$ holds locally for all sections a of A_X and v of V. Morphisms in $\mathcal{C}_X$ are morphisms of vector bundles compatible with the structures.

It is clear that naturally defined kernel and cokernel of a morphism $(M, \nabla_M) \to (N, \nabla_N)$ belong to $\mathcal{C}_X$. The tensor product is defined fiber-wise, and the unit object is defined as a trivial line bundle over X equipped with the trivial connection. This implies the following lemma.

Lemma 1 *The category $\mathcal{C}_X$ is an abelian monoidal category.*

Let (M_i, ∇_i) be a sequence of objects of $\mathcal{C}_X$, $1 \leq i \leq n$. We can make the tensor product $M = \boxtimes_{i=1}^{i=n} M_i$ which is a holomorphic vector bundle on X^n equipped with the flat connection induced from tensor factors. The fiber over $(x_1, ..., x_n)$ carries a structure of $\otimes_i A_{X,x_i}$-module. Let Z_n denotes infinitesimal neigbourhood of the diagonal $\{x_1 = x_2 = ... = x_n\}$ with the union of all diagonals $\{x_i = x_j\}$ being removed. Then we have functors j_{n*} and j_n^* in the category of D-modules on X^n, where $j_n = j_{Z_n}$ is the canonical embedding of Z_n into X^n.

Let us consider $j_{n*}j_n^*(M)$. Using the flat connection on M we can identify the fiber $M_{x_1,...,x_n}$ with the fiber $M_{x_1,...,x_1}$. The latter carries the natural structure of an A_{X,x_1}-module.

Let us assume that for every such a fiber and every permutation $\sigma \in S_n$ we are given an isomorphism of A_{X,x_1}- modules $c_\sigma : j_{n*}j_n^*(\boxtimes_i M_i) \to j_{n*}j_n^*(\boxtimes_i M_{\sigma(i)})$ such that:

$c_1 = id, c_{\sigma\tau} = c_\sigma c_\tau$.

Definition 3 *We say that we are given an infinitesimal chiral braiding on $\mathcal{C}_X$ if*

a) the above-mentioned isomorphisms are functorial with respect to M_i;

b) they are compatible with the natural embeddings $X^n \to X^m, n \leq m$ in the sense that if $\sigma \in S_m$ permutes a subsequence of $\{1, ..., m\}$ consisting of n elements then $c_\sigma, \sigma \in S_m$ acts as $c_{\bar{\sigma}} \otimes id$ where $\bar{\sigma}$ is the corresponding element of S_n.

If the above mentioned structure exists globally on the complement to each main diagonal $\{x_1 = x_2 = ... = x_n\}$ then we say that $\mathcal{C}_X$ carries a chiral braiding.

Remark 1 *This definition can be restated in such a way that it admits generalization to the case of families of objects of a braided category which are parametrized by X and equipped with automorphisms of infinitesimally close fibers. Then to any planar tree with tails numbered from 1 to n, and to a sequence of families $M_i, 1 \leq i \leq n$ on X, one assigns a tensor product $M_T = \boxtimes_T M_i$. Although it actually depends on n, not on T, one needs to use trees to identify the fiber $(M_T)_{x_1,...,x_n}$ with the tensor product $\otimes_T M_{i,x_1}$ in the infinitesimal neighbourhood of the diagonal. Again, we get a family on X. We can now define chiral braiding as before (it is also similar to the definition of meromorphic braiding from 2.4). Notice that in this case we need additional compatibilies between trees with the same number of tails (chiral associativity) as well as compatibilities with the gluing operation on trees. We leave these details to the reader.*

If $\mathcal{C}_X$ carries a chiral braided structure, then one can define meromorphic braided structure on global sections of the objects from $\mathcal{C}_X$.

4.2

Let U be the quantized enveloping algebra of the affine Kac-Moody algebra $\widehat{sl(2)}$ corresponding to the parameter q such that $|q| < 1$. We denote by $\overline{U}$ the quantized subalgebra $U_q(sl(2))$. We denote by $\mathcal{A}$ the category of finite-dimensional U-modules of type 1 (see for example [ChP]). It follows from the results of [KS] that $\mathcal{A}$ is a meromorphic braided category (see discussion in [So]).

To fix the notation, we recall that U is a Hopf algebra generated by $X_i^\pm, K_i, K_i^{-1}, i = 0, 1$ subject to relations

$$K_i X_j^\pm K_i^{-1} = q^{\pm a_{ij}} X_j,$$

where (a_{ij}) is the Cartan matrix of $\widehat{sl(2)}$,

$$K_iK_i^{-1} = 1, K_iK_j = K_jK_i,$$

$$[X_i^+, X_i^-] = (K - K^{-1})/(q - q^{-1}),$$

as well as quantized Serre relations for X_i^+ and X_i^-, $i = 0, 1$. The coproduct $\Delta : U \to U \otimes U$ is defined by

$$\Delta(K) = K \otimes K, \Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i + K_i^{-1} \otimes X_i^{\pm}, i = 0, 1.$$

Therefore the antipode is defined on generators by $S(K) = K^{-1}, S(X_i^{\pm}) = -q^{\pm 2}X_i^{\pm}, i = 0, 1$.

The group $\mathbf{C}^*$ acts on U via $\phi_z(X_i^{\pm}) = z^{\pm 1}X_i^{\pm}, \phi_z(K_i) = K_i, z \in \mathbf{C}^*$.

4.3

One of our goals will be to construct a subcategory $\mathcal{C}$ of $\mathcal{A}$ and the extend it to a category $\mathcal{C}_X$ of the previous subsection, taking X to be the elliptic curve $\mathcal{E} = \mathbf{C}^*/q^{2\mathbf{Z}}$.

One way to do it is to construct a category $\mathcal{C}$ with the following properties:

1. $\mathcal{C}$ is a full monoidal rigid subcategory of $\mathcal{A}$;

2. $\mathcal{C}$ is closed with respect to taking submodules and factor modules;

3. If $V, W \in \mathcal{C}$ then meromorphic brading in $\mathcal{A}$ gives rise to an isomorphism $V(x) \otimes W(y) \to W(y) \otimes V(x)$ as long as x/y does not belong to the set $q^{2\mathbf{Z}}$.

Let $\mathcal{C}$ be a monoidal subcategory of $\mathcal{A}$ which satisfies Property 2 only.

Lemma 2 *Property 3 holds for any two objects $V, W \in \mathcal{C}$ as long as it holds for simple V and W.*

Proof. One case use induction by $n = dimV \cdot dimW$. The Lemma holds for $n = 1$. Suppose that it holds for all $k < n$. Let us take V and W such that $dimV \cdot dimW = n$. If both objects are simple we are done. Suppose that V is not simple. Then there exists a non-trivial simple submodule $V_1 \in V$. Then we have an exact sequence

$$0 \to V_1(x) \to V(x) \to V(x)/V_1(x) \to 0.$$

We can tensor it with $W(y)$ from the left and from the right. Then using induction assumption, functoriality of meromorphic braiding and five-lemma we get the result. Q.E.D.

Suppose that we are given a category $\mathcal{C}$ which satisfies the properties 1-3 above. Every object $V \in \mathcal{C}$ gives rise to a trivial vector bundle $V_{\mathbf{C}^*}$ on $\mathbf{C}^*$ with the trivial connection.

We can assign to U a trivial bundle $U_{\mathbf{C}^*}$ of Hopf algebras on $\mathbf{C}^*$ equipped with the connection defined by the action ϕ_z of $\mathbf{C}^*$. Actually $U_{\mathbf{C}^*}$ is an equivariant bundle. Then $V_{\mathbf{C}^*}$ is a bundle of $U_{\mathbf{C}^*}$-modules such that the fiber $U_{\mathbf{C}^*,z} = U$ acts on the fiber $V_{\mathbf{C}^*,z} = V$ via automorphism ϕ_z.

To descent these data to $\mathcal{E}$ we need to define automorphisms between fibers at z and zq^2 compatible with module structures. We define them to be id_V for $V_{\mathbf{C}^*}$ (all fibers are canonically identified with V). We define an isomorphism $\gamma_z : U = U_{\mathbf{C}^*,z} \to U_{\mathbf{C}^*,zq^2} = U$ as $\gamma_z(a) = \phi_{q^{-2}}(a), a \in U$. Since $\gamma_z(\phi_z(a)v) = \phi_{zq^{-2}}(a)v = \phi_{zq^2}(\gamma_z(a))v$ for any $v \in V, a \in U$ we see that indeed all the structures are compatible and we obtain a bundle $U_{\mathcal{E}}$ of Hopf algebras on $\mathcal{E}$ equipped with a connection (in fact a structure of equivariant sheaf) as well as a bundle $V_{\mathcal{E}}$ of $U_{\mathcal{E}}$-modules.

Let $V_{\mathcal{E}}$ and $W_{\mathcal{E}}$ be two $U_{\mathcal{E}}$-modules. Then $V_{\mathcal{E},x} \boxtimes W_{\mathcal{E},y}$ carries a structure of $U_{\mathcal{E},x} \otimes U_{\mathcal{E},y}$-module. For any x and y there is an isomorphism of the fiber $U_{\mathcal{E},y}$ and the fiber $U_{\mathcal{E},x}$ given by $\phi_{xy^{-1}}$ (to be more precise this is the formula on $\mathbf{C}^*$ but it is compatible with all the structures so it descents to the elliptic curve). This makes $V_{\mathcal{E},x} \boxtimes W_{\mathcal{E},y}$ into $U_{\mathcal{E},x} \otimes U_{\mathcal{E},x}-$ module and via coproduct Δ into $U_{\mathcal{E},x}$-module.

If $(x, y) \in \mathcal{E} \times \mathcal{E} \setminus \{diag\}$ then the meromorphic braiding in $\mathcal{A}$ descents to an isomorphism of $U_{\mathcal{E},x}-$ modules $c_{x,y} : V_{\mathcal{E},x} \boxtimes W_{\mathcal{E},y} \to W_{\mathcal{E},y} \boxtimes V_{\mathcal{E},x}$.

Therefore we get a chiral braiding $c_{V,W} : j_*j^*(V_{\mathcal{E}} \boxtimes W_{\mathcal{E}}) \simeq j_*j^*(W_{\mathcal{E}} \boxtimes V_{\mathcal{E}})$.

Here j is the embedding of the complement of the diagonal to $\mathcal{E} \times \mathcal{E}$.

It is easy to check that in this way we have obtained a chiral braided category (associativity constraint is trivial). Taking sections of the bundles we get a meromorphic braided category.

4.4

We are going to construct a subcategory $\mathcal{C}$ which satisfies Properties 1-3 .

To do this we recall that for any non-negative integer n and for any non-zero complex number a one has a simple object $V_n(a)$ of $\mathcal{A}$. It is constructed as evaluation representation of U corresponding to the point a and finite-dimensional simple $(n+1)$-dimensional $U_q(sl(2))$-module V_n. We call $\otimes_i V_{n_i}(a_i)$ a *standard* module corresponding to $(a_1, a_2, ...)$. It is known (see [ChP]) that any simple object of $\mathcal{A}$ is a standard one with $a_1, a_2, ...$ satisfy certain properties. Namely to every $V_{n_i}(a_i)$ one assigns a finite set $S_{n_i}(a_i)$ called $q-string$. It is a subset of $a_i q^{2\mathbf{Z}}$. The condition mentioned above says that any two q-strings $S_{n_i}(a_i)$ and $S_{n_j}(a_j)$ are in a *generic position* (see [ChP] for precise definitions). If all a_i/a_j do not belong to $q^{2\mathbf{Z}}$ then the above-mentioned strings are in generic position.

We define $\mathcal{C}$ as a full rigid monoidal subcategory of $\mathcal{A}$ which is generated by $V_n(q^m), m \in 2\mathbf{Z}$ and closed under taking submodules and quotients. Clearly the trivial module $\mathbf{1}$ belongs to $\mathcal{C}$. Since $V_n(q^m)^* \simeq V_n(q^{2+m})$ our category is rigid monoidal. Therefore it satisfies the Properties 1 and 2.

Theorem 1 *Property 3 holds for the category $\mathcal{C}$.*

Proof. We will split the proof into several steps.

Step 1. Let $V = \otimes_{i+1}^{i=n} V_{k_i}(q^{l_i})$ and $W = \otimes_{j=1}^{j=m} V_{r_j}(q^{s_j})$. If x/y does not belong to $q^{2\mathbf{Z}}$ then two strings $S_{k_i}(xq^{l_i})$ and $S_{r_j}(yq^{s_j})$ are in generic position for any i and j. Consider the tensor product $V(x) \otimes W(y) = \otimes_i V_{k_i}(xq^{l_i}) \otimes \otimes_j V_{r_j}(yq^{s_j})$. We can use meromorphic bradings to interchange every module from the first group of tensor factors with every module from the second group of tensor factors. It is easy to deduce from [ChP], sections 4 , 5 and [KS], section 4, that meromorphic braidings do not have singularities and hence we obtain an isomorphism of U-modules $V(x) \otimes W(y) \to W(y) \otimes V(x)$.

Step 2. Let V and W be as on the Step 1. If M and N are either both submodules or factor modules of V and W then the Property 3 holds for M and N. This is clear from the Step 1 and the following functoriality of meromorphic braiding proven in [KS]: if meromorphic braiding $c_{A(x),B(y)}$: $A(x) \otimes B(Y) \to B(y) \otimes A(x)$ is well-defined and $f : A \to A', g : B \to B'$ are morphisms of U-modules then $(f \otimes g)(c_{A(x),B(y)})$ is a well-defined isomorphism $A'(x) \otimes B'(y) \to B'(y) \otimes A'(x)$.

Similarly one can prove the Property 3 in case if M is a submodule of V and N is a factor module of W. Using functoriality of meromorphic braiding once again, we can prove the Property 3 for any object of $\mathcal{C}$. Q.E.D.

4.5

It is natural to ask for a description of simple objects of $\mathcal{C}$. We start with the following elementary result.

Theorem 2 *Any simple submodule or factor module of* $V = \otimes_{i=1}^{i=n} V_{k_i}(q^{l_i})$ *is of the form* $M = \otimes_{i=1}^{i=n} V_{r_i}(q^{s_i})$.

Proof. Let us use induction by $k = \sum_i k_i$. For $k = 0$ the result is obvious. Suppose it holds for all $k < m$. Let us prove it for $k = m$. We prove it for submodules only. The case of factor modules easily follows if we take duals.

If V is simple we have nothing to prove. Let $M \subset V$ be a non-trivial submodule. Since V is not simple there are two q-strings $S_{k_i}(q^{l_i})$ and $S_{k_j}(q^{l_j})$ which are not in generic position. We may assume that $i = 1, j = 2$ (otherwise use the bradings to get two consequitive strings not in generic position). Then according to [ChP], Section 4.9, there is a unique simple submodule of $X = V_{k_1}(q^{l_1}) \otimes V_{k_2}(q^{l_2})$ of the type $A = V_d(q^e) \otimes V_f(q^h)$ with the factor module B of similar type. Then we have an exact sequence

$$0 \to A \otimes_{i \geq 3} V_{k_i}(q^{l_i}) \to V \to B \otimes_{i \geq 3} V_{k_i}(q^{l_i}) \to 0$$

If simple module M intersects $A \otimes_{i \geq 3} V_{k_i}(q^{l_i})$ then it belongs to it and the result follows from the induction assumption. If M does not intersect this submodule then it is projected isomorphically to $B \otimes_{i \geq 3} V_{k_i}(q^{l_i})$ and again the result follows by induction. Q.E.D.

On ther other hand one can try to use the notion of q-character introduced in [FR] in order to describe the Grothendieck ring of the category $\mathcal{C}$. For example it is natural to expect an affirmative answer to the following

Question Let χ_q be the q-character (notation from [FR]). Is it true that the Grothendieck ring K_0 of our category $\mathcal{C}$ is isomorphic to the subring of the Grothendieck ring of U generated by t_{q^n}, where t_{q^n} is the class of $V_1(q^n), n \in 2\mathbf{Z}$?

Clearly $K_0(\mathcal{C})$ contain the subring $\mathbf{Z}[t_{q^n}], n \in 2\mathbf{Z}$. Thus the question is whether the q-character of an object from $\mathcal{C}$ belongs to this ring.

As was pointed to me by Ed Frenkel (private communication) the answer to the Question is positive, and it can be generalized to the higher rank case. We sketch his arguments below in the case of quantum affine algebra of $sl(n)$. We denote by Γ_n the set $2\mathbf{Z}$ if $n = 2$ and the set $\mathbf{Z}$ if $n > 2$.

Let us consider the monoidal category $\mathcal{C}_n$ generated by evaluation representations $V_{\omega_i}(q^l)$ of $U_q(\widehat{sl(n)})$ where $l \in \Gamma_n$ and ω_i is the fundamental weight of $U_q(sl(n))$ $1 \leq i \leq n-1$. Then the q-character of the tensor product of such representations belongs to $A_n = \mathbf{Z}[t_{i,q^l}]$ where t_{i,q^l} is the class of $V_{\omega_i}(q^l)$ in the representation ring of $U_q(\widehat{sl(n)})$, and $l \in \Gamma_n$. Thus for $n = 2$ we have $\mathcal{C}_2 = \mathcal{C}$ and $A := A_2$ is expected to be isomorphic to $K_0(\mathcal{C})$.

Theorem 3 *A simple object in $\mathcal{C}_n$ is isomorphic to a subquotient of a tensor product $\otimes_i V_{\omega_i}(q^{l_i}), l_i \in \Gamma_n$.*

In the case $n = 2$ it is isomorphic to a tensor product $\otimes_i V_{n_i}(q^{l_i})$ where $V_{n_i}(q^{l_i})$ is a standard module, $l_i \in \mathbf{Z}$.

Proof. According to Chari and Pressley a simple $U_q(\widehat{sl(n)})$-module is isomorphic to a subquotient of $\otimes_i V_{\omega_i}(a_i)$ (we have an isomorphism to a tensor product of standard modules for $n = 2$). Then one uses the fact that the q-character of the simple module contains the dominant term (terminology and notation from [FR], Section 4) equal to $\prod_i Y_{i,a_i}$, and is the sum with positive coefficients of monomilas in $Y^{\pm 1}_{i,a_i q^{n_i}}$. This implies that all $a_i \in q^{\Gamma_n}$. Q.E.D.

Let $V = \otimes_i V_{\omega_i}(q^{l_i})$.

Theorem 4 *The q-character of every subquotient of V belongs to A_n.*

Proof. Let $L_j, 1 \leq j \leq m$ be the set of simple objects which appears in the composition series of V. Then $\chi_q(V) = \sum_j \chi_q(L_j)$.

Every L_j is the highest weight module over $U_q(\widehat{sl(n)})$. The q-character of such a module was computed in [FR]. It is equal to a sum of monomials in $t_{i,a}$ with positive coefficients. It follows from the previous theorem that if $t_{i,a}$ appears in such monomials we have $a \in q^{\Gamma_n}$. Hence $\chi_q(L_j) \in A_n$ for every j. This implies the theorem. Q.E.D.

Corollary 1 *The Grothendieck ring $K_0(\mathcal{C}_n)$ is isomorphic to A_n.*

Proof. It is easy to see that $K_0(\mathcal{C}_n)$ is generated by the isomorphism classes of subquotients of tensor products of the type $V = \otimes_i V_{\omega_i}(q^{l_i})$ and then apply the previous theorem. Q.E.D.

There is little doubt about positive answer to the Question for an arbitary quantum affine algebra. One can try to use this fact in order to construct the corresponding meromorphic braided category on an elliptic curve.

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