FOLIATIONS ON COMPLEX PROJECTIVE SURFACES

Marco Brunella

In this text we shall review the classification of foliations on complex projective surfaces according to their Kodaira dimension, following McQuillan's seminal paper [MQ1] with some complements and variations given by [Br1] and [Br2]. Most of the proofs will be only sketched, and the text should be considered as guidelines to the above works (and related ones), with no exhaustivity nor selfcontainedness pretention. There are no new results, but some old results are presented in a new way, by adopting systematically an orbifold point of view.

Contents

1. Basic definitions

Let X be a smooth complex surface (the smoothness assumption will be soon relaxed). A **foliation** $\mathcal{F}$ on X is given by an open covering $\{U_j\}$ of X and holomorphic vector fields $v_j \in H^0(U_j, \Theta_X)$ with isolated zeroes such that

$$v_i = g_{ij} v_j \qquad \text{on} \qquad U_i \cap U_j$$

for some nonvanishing holomorphic functions $g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}_X^*)$. This allows to glue together the local orbits of the vector fields $\{v_j\}$ to obtain the **leaves** of $\mathcal{F}$. The **singular set** $Sing(\mathcal{F})$ of $\mathcal{F}$ is the discrete subset of X defined by $Sing(\mathcal{F}) \cap U_j = \{$ zeroes of $v_j\}$.

The functions $\{g_{ij}\}$ form a multiplicative cocycle and define a holomorphic line bundle $K_{\mathcal{F}}$, called **canonical bundle** of $\mathcal{F}$. The relations $v_i = g_{ij} v_j$ allow to construct a global holomorphic section $s \in H^0(X, K_{\mathcal{F}} \otimes \Theta_X)$, vanishing only on $Sing(\mathcal{F})$. If $T_{\mathcal{F}}$ denotes the dual of $K_{\mathcal{F}}$ (the **tangent bundle** of $\mathcal{F}$) we therefore have an exact sequence of sheaves induced by s

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow \Theta_X \longrightarrow \mathcal{I}_Z \cdot N_{\mathcal{F}} \longrightarrow 0$$

for a suitable line bundle $N_{\mathcal{F}}$ (the **normal bundle** of $\mathcal{F}$) and a suitable ideal sheaf $\mathcal{I}_Z$ supported on $Sing(\mathcal{F})$ (see e.g. [Fri, Chapter 2]).

In a dual way, the foliation $\mathcal{F}$ can be also defined by (kernels of) holomorphic 1-forms with isolated zeroes $\omega_j \in H^0(U_j, \Omega_X^1)$ satisfying

$$\omega_i = f_{ij} \omega_j \qquad \text{on} \qquad U_i \cap U_j$$

for a suitable cocycle $\{f_{ij}\}$. This cocycle actually defines the normal bundle $N_{\mathcal{F}}$, and if $N_{\mathcal{F}}^*$ denotes its dual (the **conormal bundle** of $\mathcal{F}$) we obtain an exact sequence of sheaves

$$0 \longrightarrow N_{\mathcal{F}}^* \longrightarrow \Omega_X^1 \longrightarrow \mathcal{I}_Z \cdot K_{\mathcal{F}} \longrightarrow 0$$

which is the dual of the previous one.

The canonical and the conormal bundle of $\mathcal{F}$ are related by the formula

$$K_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^*$$

where K_X $(= \Omega_X^2)$ is the canonical bundle of the surface X. Indeed, the choice of a local nonvanishing holomorphic 2-form induces, by contraction, a local isomorphism between vector fields and 1-forms generating $\mathcal{F}$.

Let us consider now the case of a singular surface X. In fact, we shall need only the case where X has only *cyclic quotient singularities*, i.e. around each $p \in Sing(X)$ the surface is of the type $\mathbf{B}^2/\Gamma_{k,h}$ where $\mathbf{B}^2$ is the unit ball in $\mathbf{C}^2$ and $\Gamma_{k,h}$ is the cyclic group of order k generated by $(z,w) \mapsto (e^{\frac{2\pi i}{k}} z, e^{\frac{2\pi i}{k} h} w)$, for suitable coprime positive integers k, h with $0 < h < k$. Equivalently [BPV, pages 80-85], such a singularity arises from the contraction of a Hirzebruch–Jung string on a smooth surface (a chain of rational curves, each one of selfintersection ≤ -2).

Because cyclic quotient singularities are normal, a foliation $\mathcal{F}$ on such a surface can be simply defined as a foliation on $X \setminus Sing(X)$. If $p \in Sing(X)$ and $U \simeq \mathbf{B}^2/\Gamma$ is a neighbourhood of p, then $\mathcal{F}|_{U \setminus \{p\}}$ can be lifted to $\mathbf{B}^2 \setminus \{0\}$, and on that covering the foliation can be defined by a holomorphic vector field which holomorphically extends to 0. We shall say that $\mathcal{F}$ is **nonsingular** at p if that extension is nonvanishing at 0. In other words, if $\mathcal{F}$ is not singular at p then on $U \simeq \mathbf{B}^2/\Gamma$ the foliation is the quotient of the vertical or horizontal foliation on $\mathbf{B}^2$ (up to an equivariant biholomorphism). Even if many things can be done in greater generality, we shall always assume that $\mathcal{F}$ is not singular at singular points of X:

$$Sing(\mathcal{F}) \cap Sing(X) = \emptyset.$$

In this context, leaves of $\mathcal{F}$ are defined as in the smooth case, by glueing together local leaves through nonsingular points of $\mathcal{F}$. The only difference is that if $p \in Sing(X)$ has order k then the local leaf of $\mathcal{F}$ through p is an *orbifold* in which p is affected by the multiplicity k. Indeed, this local leaf on $\mathbf{B}^2/\Gamma$ is the quotient of a disc $\mathbf{D} \subset \mathbf{B}^2$ by a k-cyclic group. Thus, leaves of $\mathcal{F}$ are *orbifolds* injectively immersed (in orbifold's sense) in $X \setminus Sing(\mathcal{F})$, and giving a partition of $X \setminus Sing(\mathcal{F})$ into disjoint subsets.

Given a foliation $\mathcal{F}$ on a surface X with (at most) cyclic quotient singularities, we can still define its canonical sheaf $K_{\mathcal{F}}$, for instance by taking the direct image under the inclusion $X \setminus Sing(X) \to X$. However, it is no more a genuine line bundle (an element of $Pic(X)$) but only a $\mathbf{Q}$-bundle (an element of $Pic(X) \otimes \mathbf{Q}$). Indeed, if $p \in Sing(X)$ has order k then $K_{\mathcal{F}}$ is not locally free at p, but its k-power $K_{\mathcal{F}}^{\otimes k}$ is, and moreover k is the minimal positive integer with that property (the vector field $\frac{\partial}{\partial w}$ on $\mathbf{B}^2$ is not $\Gamma_{k,h}$-invariant, but its power $(\frac{\partial}{\partial w})^{\otimes k}$ is). Anyway, for many things $\mathbf{Q}$-bundles are as good as line bundles.

Similar considerations hold also for $T_{\mathcal{F}}$, $N_{\mathcal{F}}$, $N_{\mathcal{F}}^*$, and of course K_X. We still have the equality $K_X = K_{\mathcal{F}} \otimes N_{\mathcal{F}}^*$, as sheaves or $\mathbf{Q}$-bundles.

2. Basic formulae

We shall need many times some elementary formulae which compute the degree of $K_{\mathcal{F}}$ over compact curves in X.

As before, let X be a surface with at most cyclic quotient singularities, and let $\mathcal{F}$ be a foliation on X, with $Sing(\mathcal{F}) \cap Sing(X) = \emptyset$. Let $C \subset X$ be a compact connected (possibly singular) curve, and suppose that each irreducible component of C is not invariant by $\mathcal{F}$. For every $p \in C$ we can define an index $tang(\mathcal{F}, C, p)$ which measure the tangency order of $\mathcal{F}$ with C at p (and thus which is 0 for a generic $p \in C$, where we have transversality).

If $p \notin Sing(X)$ the definition is the following: we take a local equation f of C at p, a local holomorphic vector field v generating $\mathcal{F}$ around p, and we set

$$tang(\mathcal{F}, C, p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_p}{< f, v(f) >}$$

where $\mathcal{O}_p$ is the local algebra of X at p (germs of holomorphic functions), $v(f)$ is the Lie derivative of f along v, $< f, v(f) >$ is the ideal in $\mathcal{O}_p$ generated by f and $v(f)$. This index is finite, because C is not $\mathcal{F}$-invariant; it is a nonnegative integer; it is 0 iff $p \notin Sing(\mathcal{F})$ and $\mathcal{F}$ is transverse to C at p.

If $p \in Sing(X)$, we take a neighbourhood $U \simeq \mathbf{B}^2/\Gamma$ and we lift $\mathcal{F}|_U$ and $C \cap U$ on $\mathbf{B}^2$. We compute the index on $\mathbf{B}^2$ (at 0) and we divide the result by k, the order of the singularity. This is, by definition, $tang(\mathcal{F}, C, p)$. It is a nonnegative rational number, and it is 0 iff $\mathcal{F}$ is transverse to C at p (in the sense that the lift of $\mathcal{F}$ on $\mathbf{B}^2$ is transverse to the lift of C at 0).

We also have at our disposal the selfintersection $C \cdot C$ of C in X: it is the degree on C of the $\mathbf{Q}$-bundle $\mathcal{O}_X(C)$, and it is a rational number, possibly a noninteger one if C passes through $Sing(X)$ (see e.g. [Sak] for the intersection theory on normal surfaces). We then have the following formula for the degree of the $\mathbf{Q}$-bundle $K_{\mathcal{F}}$ on C:

$$K_{\mathcal{F}} \cdot C = -C \cdot C + tang(\mathcal{F}, C)$$

where $tang(\mathcal{F}, C) = \sum_{p \in C} tang(\mathcal{F}, C, p)$ (a finite sum). This is proved in [Br1, page 23] when X is smooth, but the general case can be handled along the same lines. The most important consequence of this formula is the inequality

$$(K_{\mathcal{F}} + C) \cdot C \geq 0$$

where the equality is realized if and only if $\mathcal{F}$ is everywhere transverse to C.

Let us consider now the case where each irreducible component of C is $\mathcal{F}$-invariant. Again, we shall define an index $Z(\mathcal{F}, C, p)$ for every $p \in C$. If $p \notin Sing(\mathcal{F})$ we simply set $Z(\mathcal{F}, C, p) = 0$. If $p \in Sing(\mathcal{F})$ (thus, in particular, $p \notin Sing(X)$) then we choose a local equation f of C and a local 1-form ω generating $\mathcal{F}$. We can factorize around p (see e.g. [Suw, Chapter V])

$$g\omega = hdf + f\eta \qquad (i.e.\ \frac{\omega}{f} = \frac{h}{g}\frac{df}{f} + \frac{1}{g}\eta)$$

where η is a holomorphic 1-form, g and h are holomorphic functions, and g and h do not vanish identically on each local branch of C at p. Thus h/g is a meromorphic function on each local branch of C at p (the residue of $\frac{\omega}{f}$ along the branch), not identically zero nor infinity. It has a well defined vanishing or polar order at p on each local branch of C at p, and the sum of these vanishing or polar orders is, by definition, the index $Z(\mathcal{F}, C, p)$.

We refer to [Br1, page 24] for a more detailed discussion of this index. Here we just recall that: i) the index is an integer number, but it may be negative, when C is a so-called "dicritical separatrix at p"; ii) if C is smooth at p, the index equals the vanishing order of $v|_C$ at p, v being a local holomorphic vector field generating $\mathcal{F}$, and so it is a positive number; iii) if C has a normal crossing singularity at p, with local branches C_1 and C_2, then $Z(\mathcal{F}, C, p) = Z(\mathcal{F}, C_1, p) + Z(\mathcal{F}, C_2, p) - 2$, which is nonnegative.

As in [Br1, page 25] we then have the formula:

$$K_{\mathcal{F}} \cdot C = -\chi_{orb}(C) + Z(\mathcal{F}, C)$$

where $Z(\mathcal{F}, C) = \sum_{p \in C} Z(\mathcal{F}, C, p)$. Here $\chi_{orb}(C)$ denotes the orbifold-arithmetic Euler characteristic of C, which may be defined by the adjunction formula

$$\chi_{orb}(C) + C \cdot C = -K_X \cdot C$$

(each $p \in C \cap Sing(X)$ is affected by a multiplicity k_p, the order of the singularity; if C is smooth then $\chi_{orb}(C) = \chi_{top}(C) + \sum \frac{1-k_p}{k_p}$, the sum being over all $p \in C \cap Sing(X)$ and $\chi_{top}(C) = 2 - 2genus(C)$ being the topological Euler characteristic; if C has singularities then one has the usual correction terms arising from smoothing them [BPV, page 68]).

Another very useful formula is Camacho–Sad formula [C-S] [Suw] [Br1, Chapter 3]. Let us consider again a compact connected curve C invariant by $\mathcal{F}$. For each $p \in Sing(\mathcal{F}) \cap C$ we take a local factorization $g\omega = hdf + f\eta$ as above. On each local branch of C at p the meromorphic 1-form $-\frac{1}{h}\eta$ has a well defined residue at p. The sum of these residues, over all local branches of C at p, is by definition the Camacho–Sad index $CS(\mathcal{F}, C, p)$. Then we have [Br1, page 37]

$$C \cdot C = CS(\mathcal{F}, C) = \sum_{p \in Sing(\mathcal{F}) \cap C} CS(\mathcal{F}, C, p).$$

We refer to [Br1, Chapter 3] or [Suw, Chapter V] for a more detailed discussion of these Camacho–Sad index and formula.

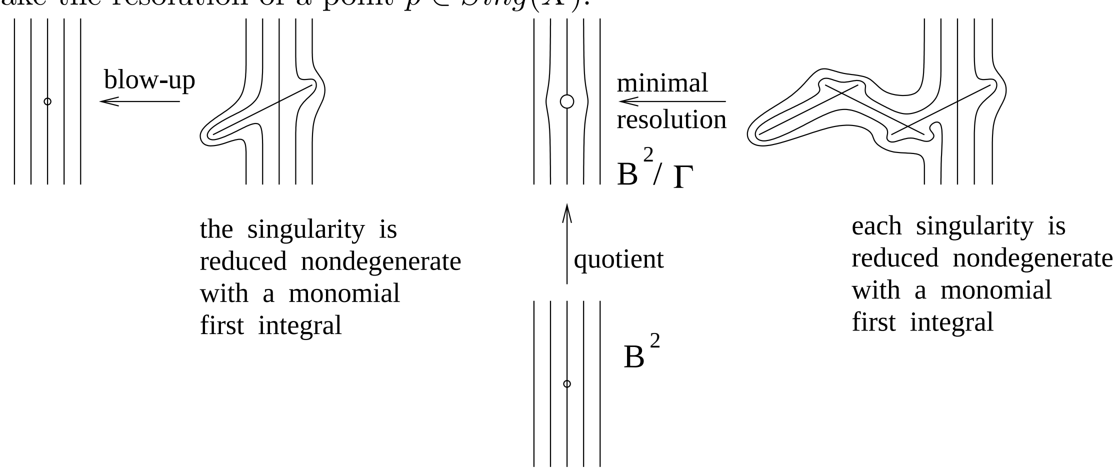
3. Singularities

In order to simplify some statements and some proofs, and also to get a more coherent theory, we will need some assumptions on the singularities of the foliation [Br1, Chapter 1] [C-S] [M-M] [MR1] [MR2].

A singularity $p \in Sing(\mathcal{F})$ is called **reduced** (in Seidenberg's sense) if $\mathcal{F}$ around p is generated by a vector field v whose linear part $(Dv)_p$ has eigenvalues λ_1, λ_2 such that either $\lambda_1 \neq 0 \neq \lambda_2$ and $\lambda_1/\lambda_2 \notin \mathbf{Q}^+$, or $\lambda_1 \neq 0 = \lambda_2$. In the former case p is called **nondegenerate**, in the latter case a **saddle-node**.

A fundamental result of Seidenberg [M-M] [C-S] [Br1, page 13] says that, given any $p \in Sing(\mathcal{F})$, we can perform a sequence of blow-ups based at p, $\widetilde{X} \xrightarrow{\pi} X$, such that the lifted foliation $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ has only reduced singularities on the exceptional divisor $\pi^{-1}(p)$. Thus, from a bimeromorphic point of view, we can work without loss of generality with **reduced foliations**, i.e. foliations all of whose singularities are reduced.

Note also that the class of reduced singularities is stable by blow-ups: the blow-up of a reduced singularity produces, on the exceptional divisor, only reduced singularities. This remains true even if we blow up a regular point $p \in X \setminus \{Sing(\mathcal{F}) \cup Sing(X)\}$, or if we take the resolution of a point $p \in Sing(X)$.



Let us compute the indices Z and CS of the previous section, in the case of reduced singularities. Suppose firstly that $p \in Sing(\mathcal{F})$ is reduced and nondegenerate. Then, in suitable local coordinates (z,w) centered at p, $\mathcal{F}$ is generated by a vector field of the form [M-M] [Br1]

$$v = z\frac{\partial}{\partial z} + \lambda w(1+...)\frac{\partial}{\partial w}$$

with $\lambda \neq 0$, $\lambda \notin \mathbf{Q}^+$. The axis $\{z=0\}$ and $\{w=0\}$ are $\mathcal{F}$-invariant, and they are the only local curves through p invariant by $\mathcal{F}$. We easily find

$$Z(\mathcal{F},\{z=0\},0) = Z(\mathcal{F},\{w=0\},0) = 1 \qquad Z(\mathcal{F},\{zw=0\},0) = 0$$

and

$$CS(\mathcal{F},\{w=0\},0) = \lambda \qquad CS(\mathcal{F},\{z=0\},0) = \frac{1}{\lambda} \qquad CS(\mathcal{F},\{zw=0\},0) = \lambda + \frac{1}{\lambda} + 2.$$

Suppose now that $p \in Sing(\mathcal{F})$ is a saddle-node. In suitable local coordinates (z,w), $\mathcal{F}$ is generated by a vector field of the type [M-M] [Br1]

$$v = [z(1+\nu w^k) + wF(z,w)]\frac{\partial}{\partial z} + w^{k+1}\frac{\partial}{\partial w}$$

with $k \in \mathbf{N}^+$, $\nu \in \mathbf{C}$, and F vanishes at $(0,0)$ up to order k. The axis $\{w=0\}$ (called **strong separatrix**) is $\mathcal{F}$-invariant, and we find

$$Z(\mathcal{F},\{w=0\},0) = 1$$

$$CS(\mathcal{F},\{w=0\},0) = 0.$$

There exists *at most* one more local curve through p and $\mathcal{F}$-invariant. If it exists, it is smooth and transverse to the strong separatrix, and it is called **weak separatrix**. Its indices are given by $Z = k+1$ and $CS = \nu$.

Let us observe the following important consequence of the above discussion: if $\mathcal{F}$ is a reduced foliation and C is a $\mathcal{F}$-invariant curve, then C has *at most normal crossings* as singularities.

We shall need also some information on the holonomy of the separatrices of reduced singularities [Br1, pages 10-12]. If $p \in Sing(\mathcal{F})$ is nondegenerate, generated by a vector field v as above, then the holonomy along a loop in $\{w=0\}$ has the form $h(w) = e^{2\pi i\lambda}w + ...$. By a result of Mattei and Moussu [M-M] [MR2], this holonomy characterizes the analytic conjugacy class of $\mathcal{F}$ around p. In particular, if h is periodic of period m (thus conjugate to $w \mapsto e^{\frac{2\pi i n}{m}}w$ for some n) then $\mathcal{F}$ around p is conjugate to the foliation generated by $mz\frac{\partial}{\partial z} + lw\frac{\partial}{\partial w}$, for some $l = n \bmod m < 0$ ($\lambda = l/m$). Such a foliation has a local holomorphic first integral, given by $z^l w^m$.

If $p \in Sing(\mathcal{F})$ is a saddle-node, generated by a vector field v as above, then the holonomy along a loop in $\{w = 0\}$ has the form $h(w) = w + w^{k+1} + ...$, in particular it is never periodic. By a result of Martinet and Ramis [MR1], this holonomy characterizes the analytic conjugacy class of $\mathcal{F}$ around p.

Let us finally remark that if $p \in Sing(X)$ and γ is a loop around p in the leaf L_p of $\mathcal{F}$ through p, then the holonomy of $\mathcal{F}$ along γ is *not* the identity, but it is a periodic diffeomorphism of period equal to the order k of the singularity. This is consistent with the fact that the local fundamental group of the orbifold L_p at p is the cyclic group of order k.

4. Nef models

As the title suggests, one of the purposes of [MQ1] is to develop a "Mori theory" for foliations on algebraic surfaces (or even on higher dimensional varieties). Therefore, as in the ordinary Mori theory (see e.g. [M-P]), one of the first steps should be the construction of birational models of foliations whose canonical bundles have suitable semipositivity properties.

In what follows, X will denote a complex projective surface with (at most) cyclic quotient singularities. Recall that a $\mathbf{Q}$-bundle L (or a $\mathbf{Q}$-divisor D) on X is called **pseudoeffective** if $L \cdot H \geq 0$ for every ample divisor H on X. This is equivalent to say that $L \cdot C \geq 0$ for every irreducible curve $C \subset X$ whose selfintersection $C \cdot C$ is nonnegative. If the same inequality holds for every irreducible curve C, regardless its selfintersection, then L is said to be **nef** (numerically eventually free). A useful result of Zariski and Fujita [Fuj] [Sak] states that any pseudoeffective $\mathbf{Q}$-bundle L has a *Zariski decomposition*: it can be uniquely written as

$$L = P + N$$

where P (the positive part) is a nef $\mathbf{Q}$-bundle and N (the negative part) is an effective $\mathbf{Q}$-bundle whose support is contractible and orthogonal to P (i.e. $N = \mathcal{O}_X(\sum_{j=1}^{n} a_j C_j)$, with $a_j \in \mathbf{Q}^+$, C_j irreducible curves with $(C_i \cdot C_j)_{i,j}$ negative definite, and $P \cdot C_j = 0$ for every j).

Let now $\mathcal{F}$ be a foliation on X, with as usual $Sing(\mathcal{F}) \cap Sing(X) = \emptyset$, and let $K_{\mathcal{F}} \in Pic(X) \otimes \mathbf{Q}$ be its canonical bundle. The following result has been proved (in a much more general context) by Miyaoka [Miy] and Shepherd-Barron [ShB] using positive characteristic arguments (see also [M-P, Lecture III]). We shall reproduce here a characteristic zero proof due to Bogomolov and McQuillan [B-M], which reduces the statement to a classical result of Arakelov on fibrations by curves [Ara] [Szp] [BPV, pages 107-110].

Theorem 1 [Miy] [ShB] [B-M]. *Let X be a complex projective surface with at most cyclic quotient singularities, and let $\mathcal{F}$ be a foliation on X with reduced singularities. Then the following two statements are equivalent:*
i) $\mathcal{F}$ is a rational fibration, i.e. a fibration whose generic fibre is a rational curve $\mathbf{C}P^1$;
ii) $K_{\mathcal{F}}$ is not pseudoeffective.

Proof.

i) $\Rightarrow$ ii). This is obvious: a generic fibre C satisfies $C \cdot C = 0$ and $K_{\mathcal{F}} \cdot C = -\chi(C) = -2 < 0$, thus $K_{\mathcal{F}}$ cannot be pseudoeffective.

ii) $\Rightarrow$ i). If $K_{\mathcal{F}}$ is not pseudoeffective then we can find an irreducible curve $C \subset X$ with $C \cdot C > 0$ and $K_{\mathcal{F}} \cdot C < 0$. We may also suppose that C is smooth, disjoint from $Sing(\mathcal{F}) \cup Sing(X)$, and that it is not $\mathcal{F}$-invariant (C is the zero set of a generic section of a very ample bundle).

On the projective threefold $Y = X \times C$ let us consider the 1-dimensional foliation $\mathcal{G}$ which is tangent to each stratum $X \times \{c\}$, $c \in C$, and which coincides there with $\mathcal{F}$. Let $D \subset Y$ be the diagonal curve $\{(c,c)|c \in C\} \subset X \times C$. On a neighbourhood U of D we can construct a smooth complex surface $Z \subset U \subset Y$ by glueing together the local leaves of $\mathcal{G}$ through points of D: note that at each $(c,c) \in D$ the foliation $\mathcal{G}$ is nonsingular, and its local leaf through (c,c) is a disk *not* tangent to D (even if the local leaf of $\mathcal{F}$ through c may be tangent to C).

This surface Z contains D, and we can compute the selfintersection of D in Z. Because $\mathcal{G}$ is tangent to Z and everywhere transverse to D in Z, we have $(D \cdot D)_Z = T_{\mathcal{G}} \cdot D$. But $T_{\mathcal{G}}$ is the pull-back of $T_{\mathcal{F}}$ by the projection $Y \to X$, and D projects to C under the same projection, thus we have

$$(D \cdot D)_Z = -K_{\mathcal{F}} \cdot C > 0$$

and in particular Z is a so-called *pseudoconcave* surface [And]. By a theorem of Andreotti [And] (which, in this special case, is reproved in [B-M] by an easier formal argument) the Zariski-closure of Z in Y has the same dimension as Z: there exists an *algebraic* surface $W \subset Y$ which contains Z.

Such a surface W is $\mathcal{G}$-invariant (for Z is), and the projection $W \xrightarrow{\pi} C$ to the second factor of $X \times C$ is a fibration which coincides with $\mathcal{G}|_W$ (for this is true on Z). The curve $D \subset W$ is a section of this fibration, and the positivity of its selfintersection implies, by Arakelov's theorem [Ara] [Szp], that π is a rational fibration. By projecting from W to

X, we see that $\mathcal{F}$ is a foliation by rational curves, and indeed a rational fibration for its singularities are reduced (each $p \in Sing(\mathcal{F})$ belongs to at most 2 $\mathcal{F}$-invariant curves, thus a generic rational fibre of π projects to a rational leaf of $\mathcal{F}$, free of singularities).

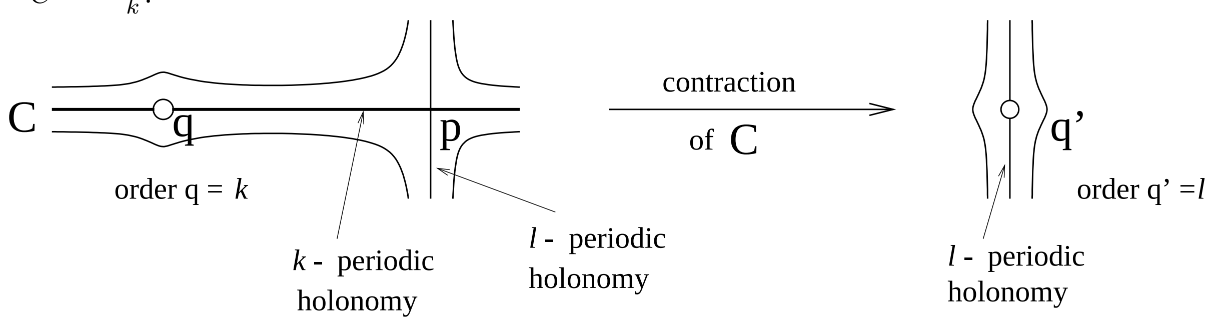
△

Suppose now that the reduced foliation $\mathcal{F}$ is not a rational fibration, and let us analyse the failure of nefness of its canonical bundle $K_{\mathcal{F}}$. Thus, take an irreducible curve $C \subset X$ with $K_{\mathcal{F}} \cdot C < 0$. Because $K_{\mathcal{F}}$ is pseudoeffective, we forcely have $C \cdot C < 0$, and this implies that C is $\mathcal{F}$-invariant: otherwise we should have $(K_{\mathcal{F}} + C) \cdot C \geq 0$, which is not the case. We can therefore apply the formula

$$K_{\mathcal{F}} \cdot C = -\chi_{orb}(C) + Z(\mathcal{F}, C).$$

From $K_{\mathcal{F}} \cdot C < 0$ we obtain $\chi_{orb}(C) > Z(\mathcal{F}, C) \geq 0$ (C has at most normal crossing singularities, because $\mathcal{F}$ is reduced). In particular, C is a smooth rational curve, possibly passing through $Sing(X)$. Observe now that C must intersect $Sing(\mathcal{F})$, otherwise by Camacho–Sad formula we would deduce $C \cdot C = 0$. Hence $Z(\mathcal{F}, C) \geq 1$, and consequently $\chi_{orb}(C) > 1$. On the other side, $\chi_{orb}(C) \leq 2$ and so $Z(\mathcal{F}, C) < 2$. This leaves open only the following possibility: C is a smooth rational curve which contains at most 1 point q from $Sing(X)$ (if q has order k then $\chi_{orb}(C) = 1 + \frac{1}{k}$), and exactly 1 point p from $Sing(\mathcal{F})$, with $Z(\mathcal{F}, C, p) = 1$.

The holonomy of $\mathcal{F}$ along a loop in C around p is of course equal to the holonomy along a loop around q, hence it is k-periodic ($k = 1$ if $C \cap Sing(X) = \emptyset$). Thus $\mathcal{F}$ around p is generated by a vector field of the type $kz\frac{\partial}{\partial z} - lw\frac{\partial}{\partial w}$, with $C = \{w = 0\}$, $l \in \mathbf{N}^+$ prime to k. By Camacho–Sad formula l can be computed from the selfintersection of C: we have $C \cdot C = -\frac{l}{k}$.



We can contract this curve C to a point, obtaining in this way a new surface X' with a point q' arising from C. It turns out that q' is still a cyclic quotient singularity, of order l (a regular point if $l = 1$), and X' is still a projective surface [Fri, pages 75-76]. Moreover, the foliation $\mathcal{F}$ induces a foliation $\mathcal{F}'$ on X' which is nonsingular at q' (and hence reduced, as $\mathcal{F}$ is).

In other words, by contracting a curve over which the canonical bundle of the foliation has negative degree we obtain a new surface and a new foliation which still satisfy our standing assumptions on $Sing(X)$ and $Sing(\mathcal{F})$. By iterating this procedure a finite number of steps we finally obtain the following result of McQuillan [MQ1] [Br1].

Corollary 1. *Let X be a complex projective surface with at most cyclic quotient singularities, and let $\mathcal{F}$ be a foliation on X with reduced singularities. Suppose that $\mathcal{F}$ is not a rational fibration. Then there exists a birational morphism $(X, \mathcal{F}) \to (X', \mathcal{F}')$ such that:*

i) X' is still projective with at most cyclic quotient singularities, and $\mathcal{F}'$ is still reduced;

ii) the canonical bundle of $\mathcal{F}'$ is nef.

It is here that the presence of cyclic quotient singularities becomes unavoidable: even if X is smooth, it may happen that X' is singular. Of course, the contraction $X \to X'$ is nothing but the contraction of the support of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$ [MQ1, §III.2] [Br1, page 113], thus the Corollary can be seen as a description of that negative part.

We shall say that a foliation $\mathcal{F}$ is a **nef foliation** if $K_{\mathcal{F}}$ is nef, so that the previous results (Seidenberg's resolution, Miyaoka's theorem, McQuillan's contraction) say that *any foliation, which is not birational to a rational fibration, has a birational model which is reduced and nef.* Such a model, in general, is not unique: for instance, the blow-up of a reduced nef foliation at a singular point is still reduced and nef (whereas nefness is lost if we blow up a regular point). In order to get uniqueness we need, at least, to contract more curves, and to obtain the so-called *minimal models*; but even after these additional contractions uniqueness may fail, and we refer to [MQ1] [Br1] [Br3] for a more complete discussion of this point and related ones.

From now on, we shall concentrate our attention to *reduced nef foliations*. Given such a foliation $\mathcal{F}$, with canonical bundle $K_{\mathcal{F}} \in Pic(X) \otimes \mathbf{Q}$, we define the **numerical Kodaira dimension** $\nu(\mathcal{F})$ of $\mathcal{F}$ as the numerical Kodaira dimension of $K_{\mathcal{F}}$, that is:

- $\nu(\mathcal{F}) = 2$ if $K_{\mathcal{F}} \cdot K_{\mathcal{F}} > 0$;
- $\nu(\mathcal{F}) = 1$ if $K_{\mathcal{F}} \cdot K_{\mathcal{F}} = 0$ but $K_{\mathcal{F}}$ is not numerically trivial (there exists C such that $K_{\mathcal{F}} \cdot C > 0$);
- $\nu(\mathcal{F}) = 0$ if $K_{\mathcal{F}}$ is numerically trivial ($K_{\mathcal{F}} \cdot C = 0$ for every C).

(Because $K_{\mathcal{F}}$ is nef, we have $K_{\mathcal{F}} \cdot K_{\mathcal{F}} \geq 0$).

If $\nu(\mathcal{F}) = 2$ then $\mathcal{F}$ is said to be of **general type**. In some sense, most foliations belong to this class. Our ultimate task will be the classification of foliations which are *not* of general type.

5. Numerically trivial foliations

A reduced nef foliation $\mathcal{F}$ on a projective surface X is said to be **numerically trivial** if $\nu(\mathcal{F}) = 0$, that is its canonical bundle $K_{\mathcal{F}}$ has zero degree on every curve $C \subset X$. The classification of these foliations is rather easy: not surprisingly, they are related to foliations whose canonical bundle is *holomorphically* trivial, i.e. foliations which are globally generated by a single global holomorphic vector field with isolated zeroes.

Theorem 2 [MQ1] [Br1]. *Let $\mathcal{F}$ be a reduced nef foliation with*

$$\nu(\mathcal{F}) = 0$$

on a projective surface X. Then there exists a finite regular covering $Y \stackrel{\pi}{\to} X$ such that the lifted foliation $\mathcal{G} = \pi^*(\mathcal{F})$ has a holomorphically trivial canonical bundle $K_{\mathcal{G}}$.

(Note: Y is smooth, and *regular covering* has to be understood in orbifold's sense, i.e. on a neighbourhood of a point $q \in Y$ sent to $p \in Sing(X)$ the map π looks like $\mathbf{B}^2 \to \mathbf{B}^2/\Gamma$).

Proof.

The main step consists in showing that $K_{\mathcal{F}}$ is a torsion $\mathbf{Q}$-bundle (i.e., some positive power of $K_{\mathcal{F}}$ is effective), then the conclusion will follow by the standard covering trick [MQ1, §IV.3] [Br1, pages 110-112].

If $H^1(X, \mathcal{O}_X) = 0$ then there is nothing to prove: on such a surface, any numerically trivial $\mathbf{Q}$-bundle is forcely a torsion $\mathbf{Q}$-bundle, because the Chern class map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$ is injective and the numerical triviality is equivalent to the vanishing of the rational Chern class in $H^2(X, \mathbf{Q})$.

Thus let us suppose $H^1(X, \mathcal{O}_X) \neq 0$, whence $H^0(X, \Omega^1_X) \neq 0$ by Hodge symmetry. If some global 1-form does not vanish identically when restricted to the leaves of $\mathcal{F}$, then this restriction defines a nontrivial section of $K_{\mathcal{F}}$, and we are done (in this case $K_{\mathcal{F}}$ is already trivial and $Y = X$). If every global 1-form vanishes on the leaves of $\mathcal{F}$, then we necessarily are in the following situation: the Albanese map $X \stackrel{alb}{\to} Alb(X)$ has a 1-dimensional image B, and $\mathcal{F}$ coincides with the fibration $X \stackrel{alb}{\to} B$. The generic fibres are elliptic curves and the fibration is isotrivial, for $K_{\mathcal{F}}$ is numerically trivial (Arakelov's theorem [Ara] [Ser]). Then it is still a consequence of results of Arakelov that $K_{\mathcal{F}}$ is a torsion $\mathbf{Q}$-bundle [MQ1, page 65] [Br1, page 111].

△

Recall now that global holomorphic vector fields on projective surfaces are well understood, see for instance [Br1, Chapter 6]. If v is such a vector field, with isolated singularities, on a surface Y, then we have one of the following four possibilities:

1) Y has an isotrivial elliptic fibration, all of whose fibres have smooth reduction, and v is tangent to the fibres, nowhere vanishing;
2) Y is a torus $\mathbf{C}^2/G$, and v is the quotient of a constant vector field on $\mathbf{C}^2$;
3) Y is a $\mathbf{C}P^1$-bundle over an elliptic curve E, and v is transverse to the fibres and projects on E to a constant vector field;
4) Y is a rational surface, and up to a birational map we have $Y = \mathbf{C}P^1 \times \mathbf{C}P^1$ and $v = v_1 \oplus v_2$, v_1 and v_2 being holomorphic vector fields on $\mathbf{C}P^1$.

These facts and Theorem 2 give a rather exhaustive description of reduced nef foliations with vanishing numerical Kodaira dimension. Let us just observe the following corollary to Theorem 2: if $\nu(\mathcal{F}) = 0$ then every leaf of $\mathcal{F}$ is *parabolic*, i.e. uniformized by $\mathbf{C}$. We shall see later another way to prove such a corollary, using the Poincaré metric on the leaves. Be careful, however, that there are foliations with $\nu(\mathcal{F}) = 1$ which also have all their leaves parabolic: see for instance [Br4] for some natural examples. This is quite unpleasant.

6. Kodaira dimension

The **Kodaira dimension** $kod(\mathcal{F})$ of a reduced nef foliation $\mathcal{F}$ on a projective surface X is defined as the Kodaira-Iitaka dimension of its canonical bundle $K_{\mathcal{F}} \in Pic(X) \otimes \mathbf{Q}$, that is

$$kod(\mathcal{F}) = \limsup_{n \to +\infty} \frac{\log \dim H^0(X, K_{\mathcal{F}}^{\otimes n})}{\log n} \in \{-\infty, 0, 1, 2\}.$$

Standard arguments give $kod(\mathcal{F}) \leq \nu(\mathcal{F})$, and moreover $kod(\mathcal{F}) = 2 \iff \nu(\mathcal{F}) = 2$. In the proof of Theorem 2 we showed $\nu(\mathcal{F}) = 0 \Longrightarrow kod(\mathcal{F}) = 0$. Here we shall prove the converse.

Theorem 3 [MQ1]. *Let $\mathcal{F}$ be a reduced nef foliation with $\nu(\mathcal{F}) = 1$, on a projective surface X. Then $kod(\mathcal{F})$ is either $-\infty$ or 1.*

Proof.

We shall give a proof following [Br1, pages 119-126], which is slightly different from [MQ1].

Suppose $kod(\mathcal{F}) \geq 0$, i.e. for some positive integer n the canonical bundle $K_{\mathcal{F}}^{\otimes n}$ has a nontrivial global section s vanishing on an effective divisor D. We have $D \neq \emptyset$ and $D \cdot D = 0$, for $\nu(\mathcal{F}) = 1$. For each irreducible component D_j of $Supp(D)$ we have $D \cdot D_j = 0$

and $D_j^2 \leq 0$, for D is nef.

1). If some component D_j is not $\mathcal{F}$-invariant then, from $0 = K_{\mathcal{F}} \cdot D_j = -D_j^2 + tang(\mathcal{F}, D_j)$, we find that D_j is everywhere transverse to $\mathcal{F}$ and $D_j^2 = 0$, in particular D_j is disjoint from the other components of D. The section s of $K_{\mathcal{F}}^{\otimes n}$ induces on each local leaf L_p of $\mathcal{F}$ through $p \in D_j$ a section of $\Omega^1(L_p)^{\otimes n}$, vanishing only at p. By integrating this section on each local leaf [Br1, page 121] we obtain a holomorphic function f on a neighbourhood of D_j, vanishing on D_j and only there. The level sets of f define a fibration which contains D_j as a (multiple) fibre, and standard compactness arguments show that such a fibration extends to the full X. Thus there exists a fibration $X \xrightarrow{\pi} B$ such that $D_j = \pi^*(Z)$ where Z is a positive $\mathbf{Q}$-divisor on B, supported on a single point (the image of D_j by π). It follows that $kod(\mathcal{O}_X(D_j)) = kod(\mathcal{O}_B(Z)) = 1$, whence $kod(\mathcal{F}) = kod(\mathcal{O}_X(D)) = 1$.

2). From now on we shall therefore assume that each irreducible component D_j is $\mathcal{F}$-invariant. From $0 = K_{\mathcal{F}} \cdot D_j = -\chi_{orb}(D_j) + Z(\mathcal{F}, D_j)$ we thus find

$$\chi_{orb}(D_j) = Z(\mathcal{F}, D_j).$$

Remark that each point of $D_j \cap D_k$, $k \neq j$, is a singularity of $\mathcal{F}$ and gives a positive contribution to $Z(\mathcal{F}, D_j)$. Hence, setting $D_{red} = \cup_i D_i$ we have

$$Z(\mathcal{F}, D_j) \geq D_{red} \cdot D_j - D_j^2$$

and consequently

$$K_X \cdot D_j = -D_j^2 - \chi_{orb}(D_j) \leq -D_{red} \cdot D_j$$

$$K_X \cdot D \leq -D_{red} \cdot D = 0.$$

If $K_X \cdot D < 0$ then by Riemann–Roch formula we deduce $kod(\mathcal{O}_X(D)) = 1$, as desired. Hence we shall assume from now on that

$$K_X \cdot D = 0.$$

Thus the previous inequalities become equalities, that is $Z(\mathcal{F}, D_j) = D_{red} \cdot D_j - D_j^2$ for every j. This means that $\mathcal{F}$ is singular on D_j only in correspondence of the intersections with the other components, or in correspondence of its selfintersections. Moreover all these singularities are nondegenerate ($Z(\mathcal{F}, D_j, p) = Z(\mathcal{F}, D_k, p) = 1$ if $p \in D_j \cap D_k$).

Let now D_0 be a connected component of $Supp(D)$, not necessarily irreducible. By the previous considerations, we have $Sing(\mathcal{F}) \cap D_0$ = nodal points of D_0. From this and $\chi_{orb}(D_j) = Z(\mathcal{F}, D_j)$ for every j it is then easy to find the structure of D_0 (see the discussion before Corollary 1). Here is a list of all the possibilities (compare with Kodaira's table of elliptic fibres [BPV, page 150] [Br1, page 67]: case (a) corresponds to I_0, case (b) to I_b, $b \geq 1$, case (c) to II, II^*, III, III^*, IV or IV^*, case (d) to I_0^*, case (e) to I_b^*, $b \geq 1$):

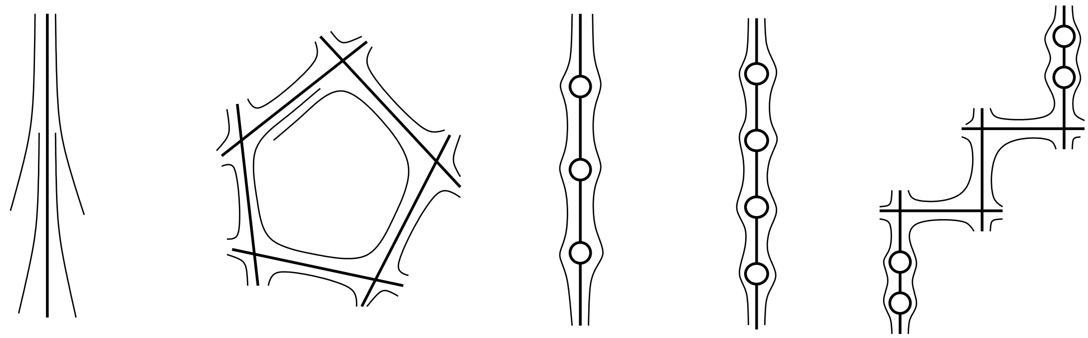
(a) D_0 is a smooth elliptic curve, disjoint from $Sing(\mathcal{F})$ and $Sing(X)$;

(b) D_0 is a cycle of smooth rational curves, or a rational curve with a node; $D_0 \cap Sing(X) = \emptyset$ and $D_0 \cap Sing(\mathcal{F}) =$ nodes of D_0;

(c) D_0 is a rational curve, disjoint from $Sing(\mathcal{F})$ and with $D_0 \cap Sing(X) = \{q_1, q_2, q_3\}$; the orders k_j of q_j satisfy $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1$;

(d) D_0 is a rational curve, disjoint from $Sing(\mathcal{F})$ and with $D_0 \cap Sing(X) = \{q_1, q_2, q_3, q_4\}$; each q_j has order 2;

(e) D_0 is a chain of rational curves, with $D_0 \cap Sing(\mathcal{F}) =$ nodes of D_0; each extreme curve in D_0 contains two points from $Sing(X)$, both of order 2; each interior curve in D_0 is disjoint from $Sing(X)$.



(a) (b) (c) (d) (e)

In case (b) there could be a curve of selfintersection -1 among the rational curves of the cycle, but then we may contract it and we obtain a similar and simpler situation. Thus, we may suppose that all the curves in (b) have selfintersection -2 (recall that D_0 is a connected component of $Supp(D)$ and D is nef with zero selfintersection). Same thing in case (e): we may suppose that the interior curves have selfintersection -2, the extreme ones -1. In both cases the singularities of $\mathcal{F}$ on D_0 can be computed via Camacho–Sad formula: the CS indices must be all equal to -1, hence the singularities are all of the type $z\frac{\partial}{\partial z} - w(1 + ...)\frac{\partial}{\partial w}$.

Remark also that in all the cases we have $D = mD_0$ (+ other components), i.e. all the irreducible components of D_0 appear in D with the same multiplicity m (this follows again from D nef, $D^2 = 0$). Hence $D_0^2 = 0$.

As in the first part of the proof (with one irreducible component not $\mathcal{F}$-invariant), we need just to prove that D_0 is a (multiple) fibre of some (elliptic) fibration $X \xrightarrow{\pi} B$.

3). If $H^1(X, \mathcal{O}_X) \neq 0$ we consider, as in Theorem 2, the Albanese map $X \xrightarrow{alb} Alb(X)$. Because $D_0^2 = 0$, we have the alternative: either alb is a fibration over a curve and D_0 is a fibre, or D_0 is not contracted by alb and so there exists a 1-form $\omega \in \Omega^1(X)$ with $\omega|_{D_0} \not\equiv 0$. In the former case we have obviously finished (with $\pi = alb$). In the latter case $\omega|_{\mathcal{F}}$ defines a section of $K_{\mathcal{F}}$ not vanishing on D_0, which easily gives $kod(\mathcal{F}) = 1$. Therefore we shall assume from now on that

$$H^1(X, \mathcal{O}_X) = 0.$$

We may also assume that $Supp(D)$ is connected, i.e.

$$D = mD_0.$$

Indeed, if there exists a second connected component D_0' then, by Hodge index theorem [BPV] and $H^1(X, \mathcal{O}_X) = 0$, we have $\mathcal{O}_X(aD_0) = \mathcal{O}_{\mathcal{X}}(bD_0')$ for some positive a and b, whence $kod(\mathcal{O}_X(D)) = 1$.

4). Let l be the minimal positive integer such that lD_0 is a Cartier divisor: $l = 1$ in cases (a) and (b), $l = 2$ in cases (d) and (e), $l = m.c.m.(k_1, k_2, k_3) \in \{3, 4, 6\}$ in case (c). By Riemann–Roch formula (and $D_0^2 = K_X \cdot D_0 = 0$, $\chi(X, \mathcal{O}_X) \geq 1$) we have, for $n \gg 0$,

$$h^0(X, \mathcal{O}_X(nlD_0)) \geq h^1(X, \mathcal{O}_X(nlD_0)) + 1.$$

If $h^1(X, \mathcal{O}_X(nlD_0)) \geq 1$ for some n then $h^0(X, \mathcal{O}_X(nlD_0)) \geq 2$ and so $kod(\mathcal{O}_X(D)) = 1$. If $h^1(X, \mathcal{O}_X(nlD_0)) = 0$ for every n then we consider the exact sequences

$$0 \to \mathcal{O}_X((n-1)lD_0) \to \mathcal{O}_X(nlD_0) \to \mathcal{O}_X(nlD_0)|_{lD_0} \to 0$$

from which we see that

$$h^0(X, \mathcal{O}_X(nlD_0)) \geq h^0(lD_0, \mathcal{O}_X(nlD_0)|_{lD_0}) + 1.$$

Hence we are reduced to prove that $\mathcal{O}_X(nlD_0)$ *has a nontrivial section over* lD_0, for some (large) positive n. This is the same as to prove that $\mathcal{O}_X(lD_0)|_{lD_0}$ is a torsion line bundle.

5). Recall that $D = mD_0$ is the zero set of a section s of $K_{\mathcal{F}}^{\otimes n}$ for some $n > 0$. However, by the branched covering trick [MQ1, page 83] we may assume that $n = 1$, i.e. s is a section of $K_{\mathcal{F}}$. Because $K_{\mathcal{F}}$ is not locally free at points of $Sing(X)$, the section s necessarily vanishes on $Sing(X)$, and so $Sing(X) \subset D_0$.

Let us consider now the logarithmic conormal bundle $N_{\mathcal{F}}^* \otimes \mathcal{O}_X(D_0)$, and observe that it is a genuine line bundle and not simply a **Q**-bundle (the logarithmic 1-form $\frac{dz}{z}$ is invariant by $(z, w) \mapsto (e^{\frac{2\pi i}{k}} z, e^{\frac{2\pi i}{k} h} w)$). By Riemann–Roch formula (and $D_0^2 = K_X \cdot D_0 = 0$, $N_{\mathcal{F}}^* = K_X - mD_0$) we have, for $n \gg 0$,

$$h^0(X, N_{\mathcal{F}}^* \otimes \mathcal{O}_X(D_0) \otimes \mathcal{O}_X(nlD_0)) \geq 1$$

that is $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_0) \otimes \mathcal{O}_X(nlD_0) = \mathcal{O}_X(rD_0 + R)$ for some effective Cartier divisor $rD_0 + R$, with $D_0 \not\subset Supp(R)$. In fact, we have $D_0 \cap Supp(R) = \emptyset$, for otherwise we would have $R \cdot D_0 > 0$ which contradicts $N^*_{\mathcal{F}} \cdot D_0 = 0$, $D_0^2 = 0$. Because $rD_0 + R$ is Cartier, we therefore have $r =$ multiple of l. Thus we finally obtain, for some $q \in \mathbf{Z}$:

$$N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_0) = \mathcal{O}_X(qlD_0 + R)$$

where R is a Cartier divisor disjoint from D_0. In other words, $\mathcal{F}$ is globally defined by a meromorphic 1-form ω with poles along D_0 of order $1 - ql$.

The case $q > 0$ is excluded: ω would be holomorphic, contradicting $H^0(X, \Omega^1_X) = H^1(X, \mathcal{O}_X) = 0$. Also the case $q = 0$ is excluded: ω would be a logarithmic 1-form with nontrivial residue $\sum \lambda_j D_j$, but this is impossible because the residue of a logarithmic 1-form must be cohomologous to zero (in the formalism of currents, $Res(\omega) = \frac{1}{2\pi i}\bar{\partial}\omega$), which is certainly not the case of $\sum \lambda_j D_j$ unless $\lambda_j = 0$ for every j. Thus we have $q < 0$.

6). If $l = 1$ (cases (a) and (b)) the proof is easily achieved. Indeed, we claim that $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_0)|_{D_0}$ is trivial. To see this, observe that each local section of $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_0)$ has a residue on D_0. If p is a smooth point of D_0 then the residue is, around p, a holomorphic function on D_0. If p is a nodal point of D_0 then the residue is, around p, a pair of holomorphic functions on the two local branches of D_0 through p; however, using the fact that the singularities of $\mathcal{F}$ are of the type $zdw + w(1 + ...)dz = 0$, we see that those two functions coincide at p, and so they can be identified with a single function on D_0. This means that we have a map

$$N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_0) \xrightarrow{Res} \mathcal{O}_{D_0}$$

and this map becomes an isomorphism when we restrict to D_0.

From the triviality of $N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D_0)|_{D_0}$ and the previous step it now follows that $\mathcal{O}_X(qD_0 + R)|_{D_0} = \mathcal{O}_X(qD_0)|_{D_0}$ is trivial too and has a nonvanishing section.

If $l > 1$ we need a more deep analysis of the structure of $\mathcal{F}$ around D_0, which takes into account higher order terms given by the holonomy of the foliation.

7). If $l > 1$ (cases (c), (d) and (e)) we may do, on a neighbourhood U of D_0, the semistable reduction [Br1, page 125] [BPV, page 155]: there exists a regular covering of order l, $V \xrightarrow{r} U$ (V smooth, r regular in orbifold's sense), such that $E_0 = r^{-1}(D_0)$ is a curve of type (a) or (b) and $\mathcal{G} = r^*(\mathcal{F})$ is tangent to E_0. By pulling back the meromorphic 1-form ω we see that

$$N^*_{\mathcal{G}} \otimes \mathcal{O}_V(E_0) = \mathcal{O}_V(qlE_0).$$

Let us firstly consider the case in which D_0 is of type (c) or (d), so that E_0 is of type (a). The holonomy group $Hol(\mathcal{F}) \subset Diff(\mathbf{C}, 0)$ of $\mathcal{F}$ along D_0 is a solvable group, generated by the finite order holonomies around points in $Sing(X)$. If this group is finite then $\mathcal{F}$, around D_0 and hence everywhere, is a fibration, and the proof is finished. If this group is not finite then [C-M] [L-M] its commutator is an abelian infinite group which embeds in a flow of a germ of vector field of the type $z^{p+1}\frac{\partial}{\partial z}$, where $p \in \mathbf{N}^+ \setminus l\mathbf{N}^+$. This commutator is in fact the holonomy group of $\mathcal{G}$ along E_0, and the previous property can be used to construct, around E_0, a (closed) meromorphic 1-form which defines $\mathcal{G}$ and has a pole of order $p+1$ along E_0 [Br1, page 126] [Pau]. This means, in particular, that

$$N^*_{\mathcal{G}} \otimes \mathcal{O}_V(E_0) = \mathcal{O}_V(-pE_0).$$

Hence

$$\mathcal{O}_V(qlE_0) = \mathcal{O}_V(-pE_0)$$

and from $p \neq -ql$ we infer that $\mathcal{O}_V(E_0)$ is a torsion line bundle. Then the same holds for $\mathcal{O}_U(lD_0)$, so that $\mathcal{O}_X(nlD_0)|_{lD_0}$ is trivial and has a nonvanishing section for some $n > 0$.

Let us finally consider the case in which D_0 is of the type (e), so that E_0 is of type (b). Recall that the singularities of $\mathcal{F}$ on D_0 are of the type $zdw + w(1+...)dz = 0$. If one singularity has a first integral, i.e. trivial holonomy, then it is easy to see that the same holds for any other singularity and $\mathcal{F}$, around D_0 and hence everywhere, is a fibration. Thus we are left with the case in which no singularity has a first integral. Each extreme curve of D_0 has then a solvable nonabelian holonomy, and as before we find [C-M] that the singularity of $\mathcal{F}$ on that extreme curve has holonomy embeddable (at least formally) in the flow of $z^{p+1}\frac{\partial}{\partial z}$ for some $p \in \mathbf{N}^+ \setminus 2\mathbf{N}^+$ (in fact, this is just a simple consequence of the fact that this holonomy is the product of two 2-periodic holonomies). This means [MR2] that, in suitable (formal) coordinates around that point, $\mathcal{F}$ is defined by the closed meromorphic 1-form $[\frac{1}{(zw)^p}+\lambda-1]\frac{dz}{z}+[\frac{1}{(zw)^p}+\lambda]\frac{dw}{w}$, for some $\lambda \in \mathbf{C}$ (in fact here λ is zero). This property propagates to the other singularities, with the same p (and λ integer). We now pass to the covering $\mathcal{G}$, where we find the same singularities, and then [Pau] we may construct (at least formally) a (closed) meromorphic 1-form which defines $\mathcal{G}$ and has a pole of order $p+1$ along E_0. As before, because $p \neq 2$ this gives the triviality of $\mathcal{O}_X(2nD_0)|_{2D_0}$ for some $n > 0$ (when we restrict to $2D_0$ all the formal problems disappear, of course).

$\triangle$

We shall see later that foliations with $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) = -\infty$ indeed exist. In the next section we address to the much easier case $kod(\mathcal{F}) = 1$.

7. Riccati and Turbulent foliations

A foliation $\mathcal{F}$ on X is a **Riccati foliation**, resp. a **Turbulent foliation**, if there exists a fibration $\pi : X \to B$ whose generic fibre is rational, resp. elliptic, and transverse to $\mathcal{F}$. Riccati and Turbulent foliations constitute a basic piece of the classification of nongeneral type foliations, as the next result shows.

Theorem 4 [MQ1] [Men]. *Let $\mathcal{F}$ be a reduced nef foliation with*

$$\nu(\mathcal{F}) = kod(\mathcal{F}) = 1$$

on a projective surface X. Then one of the following situations occurs:
i) $\mathcal{F}$ is a Riccati foliation;
ii) $\mathcal{F}$ is a Turbulent foliation;
iii) $\mathcal{F}$ is a nonisotrivial elliptic fibration;
iv) $\mathcal{F}$ is an isotrivial fibration of genus ≥ 2.

Proof.

Let $\pi : X \to B$ be the Iitaka fibration [M-P] associated to $K_{\mathcal{F}}$. A generic fibre C of π is a curve over which $K_{\mathcal{F}}$ has zero degree. If $\mathcal{F}$ coincides with the Iitaka fibration then $0 = K_{\mathcal{F}} \cdot C = -\chi(C)$, thus C is elliptic. The nonisotriviality of π follows from Arakelov's theorem [Ara] [Ser], and we are in case iii).

If $\mathcal{F}$ does not coincides with the Iitaka fibration, then $0 = K_{\mathcal{F}} \cdot C = -C^2 + tang(\mathcal{F}, C) = tang(\mathcal{F}, C)$, thus C is transverse to $\mathcal{F}$. If C is rational, resp. elliptic, then $\mathcal{F}$ is Riccati, resp. Turbulent.

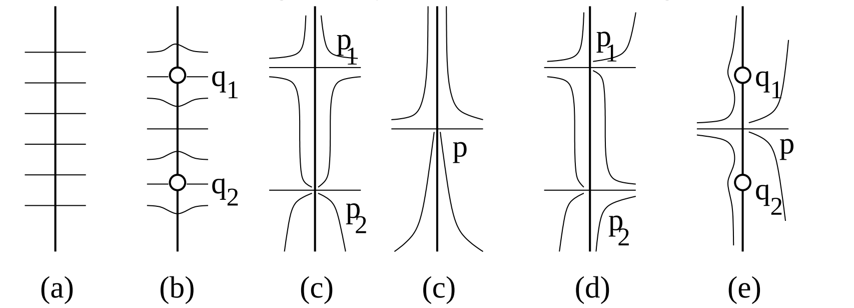
Suppose now that C has genus ≥ 2. The transversality of $\mathcal{F}$ to the generic fibres of π shows that the fibration π is isotrivial, and so it is a locally trivial fibre bundle up to ramified coverings and birational maps [BPV, Chapter III]. Standard hyperbolic arguments (finiteness of Aut(fibre), Picard's theorem) imply that $\mathcal{F}$, after those ramified coverings and birational maps, becomes a second fibration everywhere transverse to π. In other words, the universal covering of the surface is $\mathbf{D} \times \Sigma$, with $\Sigma = \mathbf{D}$ or $\mathbf{C}$ or $\mathbf{C}P^1$, and on that universal covering the foliation is the vertical one, with leaves $\{*\} \times \Sigma$. It follows easily that $\mathcal{F}$ itself is an isotrivial fibration, the isotriviality being given by the fibres of π. Finally, $K_{\mathcal{F}}$ has strictly positive degree on a fibre of $\mathcal{F}$, because that fibre is transverse to the Iitaka fibration of $K_{\mathcal{F}}$. Thus $\mathcal{F}$ has genus ≥ 2, and we are in case iv).

$\triangle$

A partial converse to this theorem is also true: nonisotrivial elliptic fibrations and isotrivial fibrations of genus ≥ 2 have $\nu(\mathcal{F}) = kod(\mathcal{F}) = 1$ [Ser], Riccati and Turbulent foliations have $\nu(\mathcal{F}) = kod(\mathcal{F}) \leq 1$ (see below).

Let us now rapidly describe the structure of Riccati and Turbulent foliations, following [Br1, Chapter 4] [Br4] [MQ1, §IV.4].

Let $\mathcal{F}$ be a reduced Riccati foliation on X. Then up to a birational morphism $X \to X'$ (contraction of curves in fibres) we have that around each fibre F of the rational fibration the foliation $\mathcal{F}$ has the structure given by one of the following models:



transverse | dicritical order = k | nondegenerate | semidegenerate multiplicity $=m$ | nilpotent multiplicity $=l$

In (b), q_1 and q_2 are quotient singularities of the same order k. In (c), p_1 and p_2 are nondegenerate, p is a saddle-node of multiplicity 2, with strong separatrix transverse to the fibre. In (d), p_1 and p_2 are saddle-nodes of the same multiplicity m, with strong separatrices inside the fibre. In (e), q_1 and q_2 are quotient singularities of order 2, p is a saddle-node of multiplicity l, with strong separatrix inside the fibre.

By pulling back sections of K_B under the fibration $X \stackrel{\pi}{\to} B$, we obtain sections of $K_{\mathcal{F}}$. We can in this way compute $K_{\mathcal{F}}$, which is the pull-back of its direct image $\pi_*(K_{\mathcal{F}}) \in Pic(B) \otimes \mathbf{Q}$. We finally obtain

$$deg(\pi_* K_{\mathcal{F}}) = -\chi_{orb}(B) + \sum_{(c)} 1 + \sum_{(d)} m_j + \sum_{(e)} \frac{l_j}{2}$$

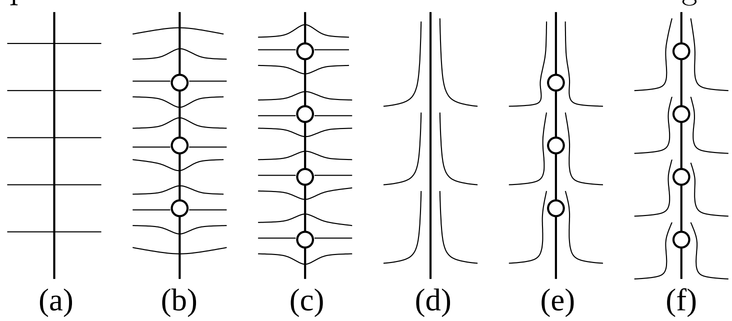
where the sums are over fibres of class (c), (d) and (e), and the orbifold structure of B is given by assigning the multiplicity k, resp. 2, to a point over which the fibre is of class (b), resp. (e). Hence

$$\chi_{orb}(B) = \chi_{top}(B) - \sum_{(b)} \frac{k_j - 1}{k_j} - \sum_{(e)} \frac{1}{2}.$$

We have $kod(\mathcal{F}) = 1$ if $deg(\pi_* K_{\mathcal{F}}) > 0$, $kod(\mathcal{F}) = 0$ if $deg(\pi_* K_{\mathcal{F}}) = 0$, $kod(\mathcal{F}) = -\infty$ if $deg(\pi_* K_{\mathcal{F}}) < 0$. It is a good exercise to check that if $deg(\pi_* K_{\mathcal{F}}) < 0$ then $\mathcal{F}$ is a rational fibration, and if $deg(\pi_* K_{\mathcal{F}}) = 0$ then, up to a covering, $\mathcal{F}$ is defined by a global vector field. Note also that it is well possible that $deg(\pi_* K_{\mathcal{F}}) > 0$ but all the leaves of $\mathcal{F}$ are parabolic.

If we set $B_0 = B \setminus \{$points over which the fibre is of class (c) (d) $(e)\}$ then the foliation induces a monodromy representation $\pi_1^{orb}(B_0) \to Aut(\mathbf{C}P^1)$ (recall that points over which the fibre is of class (b) have a local fundamental group of order k). For each leaf L outside the fibres of class (c), (d), (e) the map $L \xrightarrow{\pi} B_0$ is a regular covering (in orbifold's sense), and so the orbifold universal covering of L is equal to the one of B_0.

Consider now the case of a reduced Turbulent foliation. Here the main remark is that the elliptic fibration $X \xrightarrow{\pi} B$ is isotrivial, the isotriviality being defined by the transverse foliation. Hence fibres of type I_b and I_b^*, $b \geq 1$, cannot appear [BPV, pages 150-159]. Up to a birational morphism $X \to X'$ we then have one of the following models:



In cases (a) and (d) the fibre is smooth elliptic, and may be multiple. In cases (b) and (e) the fibre is rational with three quotient singularities of orders k_1, k_2, k_3, with $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = 1$; its multiplicity is $m.c.m.(k_1, k_2, k_3) \in \{3, 4, 6\}$. In cases (c) and (f) the fibre is rational with four quotient singularities of order 2; its multiplicity is 2. In cases (d), (e) and (f) the foliation may be tangent to the fibre at any order. One can easily find an explicit formula for $deg(\pi_* K_{\mathcal{F}})$, similar to the one for the Riccati case. This allows to compute $kod(\mathcal{F})$. We have a natural orbifold structure on the base B, and a monodromy representation $\pi_1^{orb}(B_0) \to Aut(E)$, where E is an elliptic curve.

By stable reduction [BPV, page 95] we may further simplify the situation: on a regular covering $Y \to X$ the elliptic fibration becomes a locally trivial elliptic bundle, so that only models (a) and (d) appear. A similar reduction can be done in the Riccati case, so that only models (a), (c) and (d) appear.

8. Poincaré metric

In this section we introduce some analytic tools which will be used to complete the classification of nongeneral type foliations.

Let $\mathcal{F}$ be a reduced nef foliation on X. Recall that each leaf of $\mathcal{F}$ is an orbifold, injectively immersed in $X \setminus Sing(\mathcal{F})$. The universal covering of such a leaf has no multiple point (indeed, this is true already for the holonomy covering), hence it is isomorphic to $\mathbf{D}$ or $\mathbf{C}$ or $\mathbf{C}P^1$. In fact, $\mathbf{C}P^1$ is not allowed, for the leaf would be a (quotient of a) rational curve C disjoint from $Sing(\mathcal{F})$ and we would have $K_{\mathcal{F}} \cdot C = -\chi_{orb}(C) < 0$. On each leaf of $\mathcal{F}$ we may therefore put its **Poincaré metric**: the unique complete metric of curvature -1 if the leaf is uniformised by $\mathbf{D}$ (hyperbolic leaf), and the identically zero "metric" if the leaf is uniformised by $\mathbf{C}$ (parabolic leaf).

The following result has been proved in [Br2], and it can be thougth as a metricised counterpart to Theorem 1. We refer to [Dem] for the basic theory of singular hermitian metrics.

Theorem 5 [Br2]. *Suppose that at least one leaf of $\mathcal{F}$ is hyperbolic. Then the Poincaré metric on the leaves of $\mathcal{F}$ induces on the canonical bundle $K_{\mathcal{F}}$ a singular hermitian metric whose curvature is a closed positive current.*

More explicitely, this means the following. Let U be an open subset of $X \setminus Sing(X)$, where $\mathcal{F}$ is generated by a holomorphic vector field v with isolated zeroes. This vector field induces a local trivialisation of $K_{\mathcal{F}}$ over U. On $U \setminus Sing(\mathcal{F})$ set

$$F = \log \|v\|_{Poin}$$

where, for each $p \in U \setminus Sing(\mathcal{F})$, $\|v(p)\|_{Poin}$ is the Poincaré norm of $v(p)$ with respect to the Poincaré metric on the leaf L_p through p. Thus F is a function with values into $[-\infty, +\infty)$, and $F(p) = -\infty$ iff L_p is parabolic. Moreover, it is not difficult to see that F is upper semicontinuous. Now, Theorem 5 says that F is a *plurisubharmonic* function. Such a function F is the local weight of a singular hermitian metric on $K_{\mathcal{F}}$, in the local trivialisation induced by v. The curvature Ω of this metric is locally expressed by

$$\Omega = \frac{i}{2\pi}\partial\bar{\partial}F$$

and the plurisubharmonicity of F is equivalent to the fact that Ω is a *closed positive current*. All of this can be done, of course, even if U cuts $Sing(X)$: replace $K_{\mathcal{F}}$ by a locally free power of it, etc. Finally, the fact that F above is plurisubharmonic permits to extend it, in a plurisubharmonic way, to $Sing(\mathcal{F})$; it turns out that $F = -\infty$ on $Sing(\mathcal{F})$. Thus the above singular metric on $K_{\mathcal{F}}$ is everywhere defined. The real Chern class of $K_{\mathcal{F}}$ is then equal to the De Rham cohomology class of Ω.

A first consequence of the theorem is that the existence of a hyperbolic leaf implies that "most" leaves are hyperbolic: parabolic leaves constitute a pluripolar subset of $X \setminus Sing(\mathcal{F})$. Of course, when all the leaves are parabolic (we shall say that $\mathcal{F}$ is a **parabolic foliation**) then the Poincaré metric is not very helpful. We shall see later how to handle with these parabolic foliations [MQ1] [MQ2].

Another consequence is that the existence of a hyperbolic leaf implies that $\nu(\mathcal{F}) \geq 1$: the current Ω is strictly positive along hyperbolic leaves, and so $K_{\mathcal{F}}$ is certainly not numerically trivial.

Let us now discuss the principle of the proof of Theorem 5. Take, in $X \setminus \{Sing(\mathcal{F}) \cup Sing(X)\}$, a disc T transverse to the foliation. For each $t \in T$, let L_t be the leaf through t, with universal covering $\widetilde{L}_t \simeq \mathbf{D}$ or $\mathbf{C}$. Thanks to the nefness of $K_{\mathcal{F}}$ (and also the projectivity, or better the Kählerianity, of X) it turns out that these universal coverings, for $t \in T$, glue together to give a smooth complex surface U_T [Br2, Proposition 1], called **covering tube**. See also [Ily]. Such a surface is equipped with a fibration over T, $U_T \xrightarrow{P} T$, a section $T \xrightarrow{s} U_T$, and an immersion $U_T \xrightarrow{\pi} X$. For each $t \in T$ the pointed fibre $(P^{-1}(t), s(t))$ is isomorphic to $(\widetilde{L}_t, t)$, the universal covering of L_t with basepoint t, and π sends $P^{-1}(t)$ onto L_t and $s(t)$ to t, as a regular (universal) covering. By construction, the plurisubharmonicity of the leafwise Poincaré metric on X is equivalent to the plurisubharmonicity of the fibrewise Poincaré metric on U_T, for every T. By results of Yamaguchi [Yam], this last one follows from a sort of "holomorphic convexity" of U_T. Thus, the core of [Br2] consists exactly in proving that, for each T, the covering tube U_T possesses this sort of "holomorphic convexity".

Another very important property of the leafwise Poincaré metric is its continuity (in the sense that the functions e^F are continuous). This is one of the major results of [MQ1], and it is based on some ideas previously introduced in [MQ2].

Theorem 6 [MQ1] [MQ2]. *The Poincaré metric on the leaves of $\mathcal{F}$ is continuous. Moreover, the polar set of this metric (i.e. $Sing(\mathcal{F}) \cup \{parabolic\ leaves\}$) is either the full X or a proper algebraic subset of X.*

As for Theorem 5, we discuss here only some of the ideas appearing in the proof of Theorem 6.

We may assume that $\mathcal{F}$ is not a parabolic foliation (otherwise the result is trivial) nor a fibration (otherwise the result is easy). By a theorem of Jouanolou [Br1, page 84], $\mathcal{F}$ has a finite number of algebraic leaves. Take $p \in X \setminus Sing(\mathcal{F})$ such that the leaf L_p is transcendental. We shall prove that L_p is hyperbolic and that the Poincaré metric is continuous at p.

Suppose, by contradiction, that L_p is parabolic or that the metric is discontinuous at p. In both cases, we may find a sequence $p_n \to p$ and a sequence of holomorphic maps $\mathbf{D} \xrightarrow{f_n} X$ with $f_n(0) = p_n$ and $f_n(\mathbf{D}) \subset L_{p_n}$ such that $\{f_n\}$ does *not* converge (up to subsequences) to a map $\mathbf{D} \xrightarrow{f} X$ with $f(0) = p$ and $f(\mathbf{D}) \subset L_p$ (see [Br2, Proposition 2]; if L_p is parabolic we may just choose $p_n = p$ for every n and $f_n(z) = g(nz)$ for some nonconstant entire $g : \mathbf{C} \to L_p$, $g(0) = p$). More precisely [Br2, Proposition 3], there exists some $r \in (0,1)$ such that $area(f_n(\mathbf{D}_r)) \to +\infty$, where $\mathbf{D}_r \subset \mathbf{D}$ is the disc of radius r and $area(\,\cdot\,)$ is computed with respect to any hermitian metric ω on X. This allows to construct a nontrivial closed positive current $\Phi \in A^{1,1}(X)'$, by a variation on Ahlfors' lemma [MQ1, Lemma V.2.5]: if $\eta \in A^2(X)$, we set

$$\Phi(\eta) = \lim_{n_k \to +\infty} \frac{\int_0^R \frac{dt}{t} \int_{\mathbf{D}_t} f_{n_k}^*(\eta)}{\int_0^R \frac{dt}{t} \int_{\mathbf{D}_t} f_{n_k}^*(\omega)}$$

and it turns out that for some $R \in (r,1)$ and some subsequence $n_k \to +\infty$ this indeed defines a *closed* positive current.

The current Φ is $\mathcal{F}$-invariant, and using the fact that L_p is transcendental (which implies, among other things, that the cohomology class $[\Phi]$ of Φ is nef) one can prove the following two inequalities:

i) $K_{\mathcal{F}} \cdot [\Phi] \leq 0$ [MQ1] [MQ2]

ii) $N_{\mathcal{F}}^* \cdot [\Phi] \leq 0$ [Br5].

By Hodge index theorem, the first inequality implies $\nu(\mathcal{F}) \leq 1$. In fact we must have $\nu(\mathcal{F}) = 1$, otherwise $\mathcal{F}$ would be parabolic by Theorem 2 (or Theorem 5). Still by Hodge theorem, we therefore have $c_1(K_{\mathcal{F}}) = \lambda[\Phi]$ for some $\lambda > 0$, hence the second inequality becomes $N_{\mathcal{F}}^* \cdot K_{\mathcal{F}} \leq 0$, i.e. $K_X \cdot K_{\mathcal{F}} \leq 0$. Now Riemann–Roch formula gives $h^0(K_{\mathcal{F}}^{\otimes n}) \geq 1 - h^1(\mathcal{O}_X)$ for n large. Hence if $h^1(\mathcal{O}_X) = 0$ then $kod(\mathcal{F}) \geq 0$, but the same holds even if $h^1(\mathcal{O}_X) > 0$, by the usual play with the Albanese map (as in Theorems 2 and 3). By Theorem 3 we therefore have $kod(\mathcal{F}) = 1$. The classification given by Theorem 4 says that $\mathcal{F}$ is Riccati or Turbulent. However, in both cases the Poincaré metric on the leaves arises from the pull-back of a metric on the base B under the rational or elliptic fibration $X \xrightarrow{\pi} B$, and in both cases the validity of Theorem 6 is easily checked.

Remarks that the same reasoning gives the classification of parabolic foliations: they have Kodaira dimension 0 or 1.

We conclude this section by noting that this way to prove the continuity of the leafwise Poincaré metric is not totally satisfying, because it is based on previous classification results (Theorems 2, 3 and 4). It would be better to find an *a priori* argument. When there are no parabolic leaves at all then this is indeed possible, using Brody's lemma [Br2, Proposition 2] in place of the above current Φ.

9. Hilbert modular foliations

With a negligible abuse of terminology, we shall say that a complex projective surface X (with cyclic quotient singularities) is a **Hilbert modular surface** [BPV, page 177] if there exists a (possibly empty) curve $C \subset X \setminus Sing(X)$ such that:
i) each connected component of C is a cycle of smooth rational curves, contractible to a normal singularity (a "cusp");
ii) $X \setminus C$ is uniformised (in orbifold's sense) by the bidisc $\mathbf{D} \times \mathbf{D}$, more precisely $X = \mathbf{D} \times \mathbf{D}/\Gamma$ where Γ is a discrete subgroup of $Aut_0(\mathbf{D} \times \mathbf{D}) = PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R})$;
iii) Γ is irreducible, in the sense that it does not contain a finite index subgroup of the form $\Gamma_1 \times \Gamma_2$, with $\Gamma_j \subset PSL(2, \mathbf{R})$, $j = 1, 2$.

Such a surface is naturally equipped with two natural foliations $\mathcal{F}$ and $\mathcal{G}$, arising from the horizontal and the vertical foliations on $\mathbf{D} \times \mathbf{D}$, preserved by Γ. These are called **Hilbert modular foliations**. They are both tangent to C, and they are singular only in correspondence of the normal crossings of C; these singularities are reduced and nondegenerate (eigenvalues can by computed by Camacho–Sad formula).

We have $K_{\mathcal{F}} = N^*_{\mathcal{G}} \otimes \mathcal{O}_X(C)$, the logarithmic conormal bundle of $\mathcal{G}$: indeed, C is the tangency locus between $\mathcal{F}$ and $\mathcal{G}$, and along C the two foliations have a first order tangency. The results of section 4 show that $K_{\mathcal{F}}$ is nef. By the irreducibility hypothesis on Γ the foliations are not fibrations, therefore by the logarithmic Castelnuovo–De Franchis–Bogomolov lemma [Br1, Chapter 6] we have $kod(\mathcal{F}) = kod(N^*_{\mathcal{G}} \otimes \mathcal{O}_X(C)) \leq 0$. The case $kod(\mathcal{F}) = 0$ is however excluded because $\mathcal{F}$ is not a parabolic foliation, so we finally obtain: $\mathcal{F}$ is a reduced nef foliation with $\nu(\mathcal{F}) = 1$ and $kod(\mathcal{F}) = -\infty$. The same of course holds for $\mathcal{G}$.

The following result completes the classification of nongeneral type foliations.

Theorem 7 [Br2] [MQ1]. *Let $\mathcal{F}$ be a reduced nef foliation with*

$$\nu(\mathcal{F}) = 1 \qquad kod(\mathcal{F}) = -\infty$$

on a projective surface X. Then $\mathcal{F}$ is a Hilbert modular foliation.

Proof.

We give just a sketch, refering to [Br2] for more details.

The arguments explained around the proof of Theorem 6 show that $\mathcal{F}$ is not a parabolic foliation, and so the leafwise Poincaré metric is nontrivial. By Theorem 5, the curvature Ω of this metric is a closed positive current. Using $kod(\mathcal{F}) = -\infty$ one finds that Ω has zero Lelong number everywhere, and even more [Br2, Proposition 5]: Ω is absolutely continuous, i.e. its coefficients (measures) are L^1_{loc} functions. This permits to define the wedge product $\Omega \wedge \Omega$ (in a punctual way, as a $(2,2)$-form with $L^{\frac{1}{2}}_{loc}$ coefficients), and from $c_1(K_{\mathcal{F}}) = [\Omega]$, $c_1^2(K_{\mathcal{F}}) = 0$, it follows [Dem] that

$$\Omega \wedge \Omega \equiv 0.$$

Roughly speaking, this suggests that Ω has a nontrivial Kernel, and we want to prove that this Kernel integrates to a *holomorphic* foliation $\mathcal{G}$, which will be the "second" Hilbert modular foliation on X.

Take a local chart $\mathbf{D} \times \mathbf{D} \subset X \setminus \{Sing(\mathcal{F}) \cup Sing(X)\}$, disjoint from the parabolic leaves (which are algebraic by Theorem 6), and in which $\mathcal{F}$ is expressed by the vector field $\frac{\partial}{\partial w}$. In this local trivialisation we have

$$\Omega = \frac{i}{2\pi} \partial \bar{\partial} F$$

where F is finite, continuous and plurisubharmonic, by Theorems 5 and 6. Moreover, the second derivatives $F_{\alpha\bar{\beta}}$ ($\alpha, \beta \in \{z, w\}$) are L^1_{loc} (Ω absolutely continuous), $F_{z\bar{z}}F_{w\bar{w}} - F_{z\bar{w}}F_{w\bar{z}}$ is identically zero ($\Omega \wedge \Omega \equiv 0$), and $F_{w\bar{w}} = e^F$ (curvature -1 along the leaves). The Kernel of Ω is expressed by the differential equation

$$\frac{dw}{dz} = -\frac{F_{z\bar{w}}}{F_{w\bar{w}}}$$

and we shall see below (Proposition 1, [Br2, Proposition 6]) that the function $-F_{z\bar{w}}/F_{w\bar{w}}$ is *holomorphic*. By integrating the differential equation, we obtain, at least outside the parabolic leaves and the singularities, a holomorphic foliation $\mathcal{G}$, transverse to $\mathcal{F}$. Using again the absolute continuity of Ω, one proves that $\mathcal{G}$ extends to the full X, as a holomorphic foliation with a first order tangency with $\mathcal{F}$ along the parabolic leaves: the above function $-F_{z\bar{w}}/F_{w\bar{w}}$ extends to a meromorphic function with a first order pole along $\{F = -\infty\}$. We thus have $N^*_{\mathcal{G}} \otimes \mathcal{O}_X(C) = K_{\mathcal{F}}$, where C is the closure of the parabolic leaves.

The next step consists in analysing the structure of C. By numerical arguments similar to those employed in the step 2) in the proof of Theorem 3, we obtain that each connected component of C is a contractible cycle of rational curves, and $\mathcal{F}$ and $\mathcal{G}$ are singular only in correspondence of its normal crossings. The singularities of $\mathcal{G}$ are nondegenerate, and using Camacho–Sad formula we see also that they are reduced. Finally, by arguments close to those before the statement of Theorem 7 ($K_{\mathcal{G}} = N^*_{\mathcal{F}} \otimes \mathcal{O}_X(C)$) we obtain that $\mathcal{G}$ is nef and its Kodaira dimension is $-\infty$.

Therefore we may repeat the previous steps, for $\mathcal{G}$ instead of $\mathcal{F}$, and we find a third holomorphic foliation $\mathcal{H}$, generated by the Kernel of the curvature of the Poincaré metric on the leaves of $\mathcal{G}$. Without surprise, one proves that $\mathcal{H}$ coincides with $\mathcal{F}$.

Thus, outside the parabolic leaves of $\mathcal{F}$ (which coincide with those of $\mathcal{G}$, and which form cycles C of rational curves) we have a pair of transverse foliations such that the leaves of one are in the Kernel of the curvature of the Poincaré metric on the leaves of the other. This means that the holonomy maps between pieces of leaves of $\mathcal{F}$ (resp. $\mathcal{G}$) induced by $\mathcal{G}$ (resp. $\mathcal{F}$) are *isometries* in the Poincaré metric (because they are holomorphic and they preserve the Poincaré area form). As a consequence of this, we can define a complete hermitian metric on $X \setminus C$ which is locally isometric to the Poincaré metric on $\mathbf{D} \times \mathbf{D}$, by taking the orthogonal sum of the metrics along $\mathcal{F}$ and along $\mathcal{G}$. It is then easy to conclude, by classical arguments, that $X \setminus C$ is uniformised by $\mathbf{D} \times \mathbf{D}$, and then that $\mathcal{F}$ and $\mathcal{G}$ are Hilbert modular foliations.

△

In the course of the proof we have used the following quite miraculous fact concerning Monge–Ampère foliations.

Proposition 1. *Let $F : \mathbf{D} \times \mathbf{D} \to \mathbf{R}$ be a continuous plurisubharmonic function such that:*
i) $\Omega = \frac{i}{2\pi}\partial\bar{\partial}F$ is absolutely continuous;
ii) $\Omega \wedge \Omega \equiv 0$;
iii) $F_{w\bar{w}} = e^F$.
Then $-F_{z\bar{w}}/F_{w\bar{w}}$ (the slope of the Kernel of Ω) is a holomorphic function.

Proof.

A full proof is given in [Br2, Proposition 6], here we shall explain a different, more geometric, proof, which however requires some additional regularity. More precisely, we shall suppose here that F is *smooth* (using standard elliptic regularity theory one can prove that, under the hypotheses of the proposition, F Lipschitz implies F smooth; however, the step from continuous to Lipschitz is more problematic).

The Kernel of Ω being smooth and Ω being closed, we certainly have a smooth foliation $\mathcal{G}$ with complex leaves obtained by integrating that Kernel, a so-called Monge–Ampère foliation [Kli]. Generally speaking, such a foliation is far from being holomorphic, and we need to prove that indeed it is, thanks to hypothesis iii).

Let L_0 be a leaf of $\mathcal{G}$, say the one through $(0,0)$; we may suppose $L_0 = \{w = 0\}$. The leaf L_t through $(0,t)$, t close to 0, has equation

$$w = a(z)t + b(z)\bar{t} + o(|t|)$$

where a, b are holomorphic functions and $a(0) = 1$, $b(0) = 0$. Let

$$\phi_s : \{z = 0\} \to \{z = s\}$$

be the holonomy map defined by $\mathcal{G}$, on a neighbourhood of $(0,0) \in \{z=0\}$. We have

$$\phi_s^*(dw \wedge d\bar{w}) = (|a(s)|^2 - |b(s)|^2) dw \wedge d\bar{w} + O(|w|)$$

and so, by hypothesis iii),

$$\phi_s^*(\Omega|_{\{z=s\}}) = e^{F(s,0)-F(0,0)}(|a(s)|^2 - |b(s)|^2)\Omega|_{\{z=0\}} + O(|w|).$$

But $\mathcal{G}$ is in the Kernel of Ω and Ω is closed, hence $\phi_s^*(\Omega|_{\{z=s\}})$ must be equal to $\Omega|_{\{z=0\}}$ by Stokes theorem. In particular:

$$\log(|a(s)|^2 - |b(s)|^2) = F(s,0) - F(0,0).$$

The function $F(s,0)$ is harmonic, and not only subharmonic, because $\Omega|_{L_0} \equiv 0$. Thus $\log(|a(s)|^2 - |b(s)|^2)$ is harmonic too, and the maximum principle plus the initial conditions $a(0) = 1$, $b(0) = 0$ give $b \equiv 0$. In other words, the linear part at 0 of the holonomy diffeomorphism ϕ_s is a complex linear map, for every s, and so the foliation $\mathcal{G}$ is holomorphic along L_0. This leaf being an arbitrary leaf, we obtain that $\mathcal{G}$ is holomorphic *tout court*.

$\triangle$

10. Kähler surfaces

Up to now we have mostly considered the case of foliations on projective surfaces. Let us explain in this last section how to extend the previous results to the slightly more general case of compact Kähler surfaces. More precisely, we shall draw a complete list of *all* foliations on compact Kähler nonprojective surfaces: this is consistent with the fact that on such a surface a foliation cannot be of general type, because a line bundle cannot have Kodaira dimension 2.

Instead of working with a special model of the foliation (reduced, nef), it is simpler here to work with a special model of the surface. Thus, let X be a *minimal* smooth compact Kähler nonprojective surface. According to Kodaira classification [BPV], the algebraic dimension $a(X)$ is either 1, in which case X has a unique elliptic fibration, or 0, in which case X is a torus or a K3 surface. Let $\mathcal{F}$ be any foliation on X. We distinguish three cases:

$\underline{a(X) = 1}$. Let $\pi : X \to B$ be the elliptic fibration. Any compact curve in X is contained in some fibre of π. In particular, if $\mathcal{F}$ is different from the fibration then the tangency locus between $\mathcal{F}$ and π is contained in some fibres. In other words, $\mathcal{F}$ is transverse to a generic fibre of π, and so $\mathcal{F}$ is a **Turbulent foliation**.

$\underline{X \text{ is a torus with } a(X) = 0}$. We may take on $X = \mathbf{C}^2/G$ a foliation $\mathcal{G}$ generated by a constant vector field and tangent to $\mathcal{F}$ at some point. The tangency locus between $\mathcal{F}$ and $\mathcal{G}$ is not empty and it cannot be a curve, because such an X contains no curve. Thus it remains only the possibility $\mathcal{F} = \mathcal{G}$, i.e. $\mathcal{F}$ is a **Kronecker foliation**, generated by a constant vector field.

$\underline{X \text{ is a K3 surface with } a(X) = 0}$. Such a surface admits, by Yau's theorem [BPV, page 40], a Kähler–Einstein metric in each Kähler cohomology class, which implies the semistability of its tangent bundle [Fri, Chapter 4]. That is, if $\omega \in A^{1,1}(X)$ is any Kähler form on X, then we have the semistability inequality

$$c_1(T_{\mathcal{F}}) \cdot [\omega] \leq \frac{1}{2} c_1(X) \cdot [\omega] = 0$$

i.e.

$$c_1(K_{\mathcal{F}}) \cdot [\omega] \geq 0.$$

This inequality plays here the same rôle as Theorem 1 in the projective case: it says that $K_{\mathcal{F}}$ is pseudoeffective [Dem] [Lam]. Looking at the proof of [Fuj], one realizes that such a line bundle has a Zariski decomposition: as a $\mathbf{Q}$-bundle, it can be written as

$$K_{\mathcal{F}} = P + N$$

where N is an effective $\mathbf{Q}$-bundle with contractible support and P is a $\mathbf{Q}$-bundle which satisfies: i) $P \cdot [\omega] \geq 0$ for every Kähler form ω; ii) $P \cdot C \geq 0$ for every compact curve $C \subset X$; iii) $P \cdot D = 0$ for every compact curve $D \subset Supp(N)$. From i) and ii) it follows [Lam] that $P \cdot P \geq 0$. But on a K3 surface with zero algebraic dimension there exists only one line bundle L with $L \cdot L \geq 0$: the trivial one (by Riemann–Roch we have $h^0(L) + h^0(L^*) \geq 2$ and by $a(X) = 0$ we have $h^0(L) \leq 1$ and $h^0(L^*) \leq 1$). It follows that P is numerically trivial, or more precisely $K_{\mathcal{F}}^{\otimes n} = \mathcal{O}_X(E)$ for some $n > 0$ and some effective divisor E supported on $Supp(N)$.

We can now repeat the arguments of sections 4 and 5 (with some care, because $\mathcal{F}$ could be nonreduced). After contraction of $Supp(N)$ we obtain a foliation $\mathcal{F}'$ on a surface X' with cyclic quotient singularities, whose canonical bundle $K_{\mathcal{F}'}$ is a torsion $\mathbf{Q}$-bundle. After a regular covering $Y \xrightarrow{\pi} X'$ this canonical bundle becomes holomorphically trivial.

Of course Y has still algebraic dimension 0, and $Y \neq X'$, because a K3 surface has no global holomorphic vector field. It follows that Y is a torus and $\pi^*(\mathcal{F}')$ is a Kronecker foliation. Thus $\mathcal{F}'$ is a **Kummer foliation**, quotient of a Kronecker foliation by a n-cyclic group.

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