On the Hausdorff Dimension of Piecewise Hyperbolic Attractors

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Abstract

We study non-invertible piecewise hyperbolic maps in the plane. The Hausdorff dimension of the attractor is calculated in terms of the Lyapunov exponents, provided that the map satisfies a transversality condition. Explicit examples of maps for which this condition holds are given.

1 Introduction

A general class of piecewise hyperbolic maps was studied by Pesin in [10]. Pesin proved the existence of SRB-measures and investigated their ergodic properties. Results from Pesin's article and Sataev's article [11] are described in Section 3. The assumptions in [10] and [11] did not allow overlaps of the images. Schmeling and Troubetzkoy extended in [12] the theory in [10] to allow maps with overlaping images.

Using the results of Pesin and techniques from Solomyak's paper [14], the author of this paper proved in [8] and [9] that for two classes of piecewise affine hyperbolic maps, there exists, for almost all parameters, an invariant measure that is absolutely continuous with respect to Lebesgue measure, provided that the map expands area. This result had previously been obtained for fat baker's transformations by Alexander and Yorke in [1]. The main difficulty that arises for the class of maps in [9] is that in difference from the fat baker's transformation the symbolic space associated to the systems, changes with the parameters, and also the SRB-measure changes in a way that is hard to control. By embedding all symbolic spaces into a larger space it was possible get sufficient control to prove the result.

Solomyak's proof in [14] uses a transversality property of power series. The proofs in [8] and [9] uses that iterates of points can be written as power series with such a transversality property. For the possibility of writing

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iterates as power series, it is important that the directions of contraction is maped onto each other throughout the manifold. The method in [8] and [9] is therefore not good for proving similar results for more general maps. It should also be noted that this method only gives results that holds for almost every map, with respect to some parameter.

Tsujii studied in [15] a class of area-expanding solenoidal attractors and proved that generically these systems has an invariant measure that is absolutely continuous with respect to Lebesgue measure. Tsujii also used a transversality condition, but in a different way. Instead of transversality of power series, Tsujii used transversality of intersections of iterates of curves. This technique makes it possible to show the existence of an absolutely continuous invariant measure for a fixed system, provided that the appropriate transversality condition is satisfied. Tsujii proved that this transversality condition is generically satisfied.

In this paper we will use this idea from Tsujii's article [15] to prove a formula for the dimension of the attractor for some piecewise hyperbolic maps in the plane, provided that a transversality condition is satisfied. This is done by an estimate of the dimension from below. This estimate coinsides with a previously known estimate from above (see [5] and [12]) and thereby provide the following formula for the dimension:

$$\dim_{\mathrm{H}} \Lambda = 1 - \frac{\chi_{\mathrm{u}}}{\chi_{\mathrm{s}}},$$

where Λ denotes the attractor and χ_{u} and χ_{s} denote the positive and the negative Lyapunov exponents. This formula has previously been proved by Falconer in [5] and by Simon in [13], but for a much smaller class of systems. Both Falconer and Simon considered maps that are scew-products with the underlying shift being a full shift on n symbols. These restrictions are not assumed in this paper. Hence, this paper generalises the results of Falconer and Simon.

We will also need the assumption that the multiplicity entropy is zero, which also is the case in Falconer's and Simon's results. This seems often to be the case, and we provide a condition which guaranties that the multiplicity entropy is zero.

2 Outline of the Paper

In Section 3 we present the general theory of piecewise hyperbolic maps, that will be used later in the paper. In Section 4 we introduce a transversality condition. Under the assumption that this transversality condition holds, a theorem that estimates the dimension from below is stated in Section 5. This estimate gives the dimension formula. The theorem is proved in Section 8 and Section 7 contains explicit examples of maps that satisfy the assumptions of this theorem. There are also examples that the dimension formula may fail if the transversality condition does not hold.

3 Piecewise Hyperbolic Maps

There is a study of general piecewise hyperbolic maps in Pesin's article [10]. He studied maps of the following form.

Let M be a smooth Riemannian manifold with metic d, let $K \subset M$ be an open, bounded and connected set and let $N \subset K$ be a closed set in K. The set N is called the discontinuity set. Let $f\colon K \setminus N \to K$.

Put

$$\begin{aligned} K^+ &= \{\, x \in K : f^n(x) \notin N \cup \partial K,\ n = 0, 1, 2, \dots \,\}, \\ D &= \bigcap_{n \in \mathbb{N}} f^n(K^+). \end{aligned}$$

The attractor of f is the set $\Lambda = \overline{D}$.

The maps studied in [10] were assumed to satisfy the following conditions.

$$f\colon K \setminus N \to f(K \setminus N) \text{ is a } C^2\text{-diffeomorphism.} \tag{A1}$$

Let $N^+ = N \cup \partial K$ and

$$N^- = \{\, y \in K : \exists z_n \in K \setminus N^+,\ z \in N^+ \text{ such that } y = f(z),\ z_n \to z,\ f(z_n) \to y \,\}.$$

One might want to think of N^- as the image of N^+ although f is not defined on N^+. We can now formulate the second and third assumption.

There exists $C > 0$ and $\alpha \geq 0$ such that (A2)

$$\begin{aligned} \|\mathrm{d}_x^2 f\| &\leq C d(x, N^+)^{-\alpha}, &\quad \forall x \in K \setminus N, \\ \|\mathrm{d}_x^2 (f^{-1})\| &\leq C d(x, N^-)^{-\alpha}, &\quad \forall x \in f(K \setminus N). \end{aligned}$$

For $\varepsilon > 0$ and $l = 1, 2, \dots$ let (A3)

$$\begin{aligned} D_{\varepsilon,l}^+ &= \{\, x \in K^+ : d(f^n(x), N^+) \geq l^{-1} e^{-\varepsilon n},\ n \in \mathbb{N} \,\}, \\ D_{\varepsilon,l}^- &= \{\, x \in \Lambda : d(f^{-n}(x), N^-) \geq l^{-1} e^{-\varepsilon n},\ n \in \mathbb{N} \,\}, \\ D_{\varepsilon,l}^0 &= D_{\varepsilon,l}^+ \cap D_{\varepsilon,l}^-, \\ D_\varepsilon^0 &= \bigcup_{l \geq 1} (D_{\varepsilon,l}^+ \cap D_{\varepsilon,l}^-). \end{aligned}$$

The set D_ε^0 is not empty for sufficiently small $\varepsilon > 0$. (Here sufficiently small means so small that there are local unstable manifolds.)

The attractor is called regular if (A3) is satisfied. For a given map, it is usually not apperent whether the condition (A3) is satisfied or not. There exist however conditions that implies (A3) and are such that it easily can be checked if they hold true. These conditions are given in the end of this section.

There exists $C > 0$ and $0 < \lambda < 1$ such that for every $x \in K \setminus N^+$ there exists cones $C^{\mathrm{s}}(x), C^{\mathrm{u}}(x) \subset T_x M$ such that the angle between $C^{\mathrm{s}}(x)$ and $C^{\mathrm{u}}(x)$ is uniformly bounded away from zero, (A4)

$$\begin{aligned} &\mathrm{d}_x f(C^{\mathrm{u}}(x)) \subset C^{\mathrm{u}}(f(x)) && \forall x \in K \setminus N^+, \\ &\mathrm{d}_x (f^{-1})(C^{\mathrm{s}}(x)) \subset C^{\mathrm{s}}(f^{-1}(x)) && \forall x \in f(K \setminus N^+), \end{aligned}$$

and for any $n > 0$

$$\begin{aligned} &\|\mathrm{d}_x f^n(v)\| \geq C\lambda^{-n}\|v\|, && \forall x \in K^+,\ \forall v \in C^{\mathrm{u}}(x), \\ &\|\mathrm{d}_x f^{-n}(v)\| \geq C\lambda^{-n}\|v\|, && \forall x \in f^n(K^+),\ \forall v \in C^{\mathrm{s}}(x). \end{aligned}$$

The last assumption makes it possible to define stable and unstable manifolds, $W^{\mathrm{s}}(x)$ and $W^{\mathrm{u}}(x)$ as well as local ones for any $x \in D^0_\varepsilon$.

The condition

There exists a point $x \in D^0_\varepsilon$ and $C, t, \delta_0 > 0$ such that for any $0 < \delta < \delta_0$ and any $n \geq 0$ (A3′)

$$\nu^{\mathrm{u}}(f^{-n}(U(\delta, N^+))) < C\delta^t,$$

where ν^{u} is the measure on the local unstable manifold of x, induced by the Riemannian measure, and $U(\delta, N^+)$ is an open δ-neigbourhood of N^+.

implies condition (A3). Pesin proved the following theorem.

Theorem 1 (Pesin [10])**.** *Assume that f satisfies the assumptions* (A1)–(A4) *and* (A3′)*. Then there exists an f-invariant measure μ such that Λ can be decomposed $\Lambda = \bigcup_{i\in\mathbb{N}} \Lambda_i$ where*

- $\Lambda_i \cap \Lambda_j = \emptyset$*, if* $i \neq j$*,*
- $\mu(\Lambda_0) = 0$*,* $\mu(\Lambda_i) > 0$ *if* $i > 0$*,*
- $f(\Lambda_i) = \Lambda_i$*,* $f|_{\Lambda_i}$ *is ergodic,*
- *for* $i > 0$ *there exists* $n_i > 0$ *such that* $(f^{n_i}|_{\Lambda_i}, \mu)$ *is isomorphic to a Bernoulli shift.*

The metric entropy satisfy

$$h_\mu(f) = \int \sum \chi_i(x) \, \mathrm{d}\mu(x),$$

where the sum is over the positive Lyapunov exponents $\chi_i(x)$.

The measure μ in Theorem 1 is called SRB-measure (or Gibbs u-measure). For piecewise hyperbolic maps the SRB-measures are characterised by the property that their conditional measures on unstable manifolds are absolutely continuous with respect to Lebesgue measure and the set of typical points has positive Lebesgue measure.

For a somewhat smaller class of maps Sataev proved in [11] that the ergodic components of the SRB-measure (the sets Λ_i in Theorem 1) are finitely many.

3.1 Non-Invertible Piecewise Hyperbolic Maps

The maps studied by Pesin and Sataev are all invertible on their images. Schmeling and Troubetzkoy generalised in [12] the results of Pesin to non-invertible maps: If

> the set $K \setminus N$ can be decomposed into finitely many sets K_i such that $f \colon K_i \to f(K_i)$ can be extended to a diffeomorphism from $\overline{K_i}$ to $\overline{f(K_i)}$ (A5)

and f satisfies the assumptions (A2)–(A4) and (A3′), then the statement of Theorem 1 is still valid. Note that $f(K_i) \cap f(K_j)$ is allowed to be non-empty so that $f \colon K \setminus N \to f(K \setminus N)$ is not a diffeomorphism. Schmeling and Troubetzkoy proved their result by lifting the map and the set K to a higher dimension; Let $\hat{K} = K \times [0,1]$, $\hat{K}_i = K_i \times [0,1]$ and

$$\hat{f}|_{K_i} \colon (x,t) \mapsto (f(x), \tau t + i/p), \quad i = 0, 1, \ldots, p-1,$$

where $\tau < 1$ and p is the number of sets K_i. The map $\hat{f}$ is then invertible if τ is sufficiently small and then $\hat{f}$ satisfies the assumptions of Theorem 1, in particular there is an SRB-measure $\hat{\mu}$ on the lifted set $\hat{K}$. The projection of this measure to the set K was shown to be an SRB-measure of the original map f, in the sence that the set of typical points with respect to the projected measure has positive Lebesgue measure.

We will let $\hat{D}$, $\hat{D}^0_{\varepsilon,l}, \ldots$ denote the lifted variants of the corresponding sets D, $D^0_{\varepsilon,l}, \ldots$

It is often hard to check whether (A3′) holds. It is proved in [12] that if f satisfies (A2), (A4), (A5) and the assumptions (A6)–(A8) below, then f satisfies condition (A3′), and hence also (A3).

> The sets ∂K and N are unions of finitely many smooth curves such that the angle between these curves and the unstable cones are bounded away from zero. (A6)

The cone families $C^{\mathrm{u}}(x)$ and $C^{\mathrm{s}}(x)$ depends continuously on $x \in K_i$ and they can be extend continuously to the boundary. (A7)

There is a natural number q such that at most L singularity curves of f^q meet at any point, and $a^q > L+1$ where (A8)

$$a = \inf_{x \in K \setminus N} \inf_{v \in C^{\mathrm{u}}(x)} \frac{|\mathrm{d}_x f(v)|}{|v|}.$$

3.2 Multiplicity entropy

The assumption (A8) implies that the multiplicity entropy [2] is not larger than $\log(L+1)$. We will need a stronger assumption (assumption (A9) below) than (A8), namely that the multiplicity entropy is zero. We will show that this is satisfied under rather mild assumptions on the map. Let us start with defining the multiplicity entropy, and then give the assumption (A9).

Let $\mathcal{K} = \{K_1, \ldots, K_p\}$ be the partion of K into sets on which f is continuous, and let $\mathcal{K}_n$ be the corresponding partition for the map f^n. Let k_n be the maximal numbers of elements of $\mathcal{K}_n$ that meet in one point. The multiplcity entropy $h_{\mathrm{mult}}(f)$ is defined as

$$h_{\mathrm{mult}}(f) = \limsup_{n \to \infty} \frac{1}{n} \log k_n.$$

We can now give our next assumption.

$$h_{\mathrm{mult}}(f) = 0. \tag{A9}$$

One might wounder how general this condition is. The author of this paper knows of no example of a map in dimension two, satisfying (A1)–(A8), with one positive and one negative Lyapunov exponent and such that (A9) is not satisfied. In Section 6 we give sufficient conditions for the map to satisfy (A9), and hence also (A8). These conditions are for instance satisfied by Belykh maps. Hence Belykh maps have zero multiplicity entropy.

For future use, we note that condition (A9) implies that the topological entropy is equal to the entropy of the SRB-measure. This follows by the result of Kruglikov and Rypdal in [7], that $h_{\mathrm{top}} \leq \chi_{\mathrm{u}} + h_{\mathrm{mult}}$ (in the case of a map on the plane with one positiv and one negative Lyaponov exponent; the statement in [7] is for any dimension).

4 A Transversality Condition

Let $\varepsilon > 0$ and $0 < \delta < 1$. We will say that an intersection of two smooth curves γ_1 and γ_2 is (ε, δ)-transversal if for any ball B_ε of radius ε intersecting both γ_1 and γ_2, there exist points $x_1 \in B_\varepsilon \cap \gamma_1$ and $x_2 \in B_\varepsilon \cap \gamma_2$ such that the following holds true. If d_1 and d_2 are the induced metrics on γ_1 and γ_2 respectively, then the intersection of the open sets

$$\bigcup_{y \in \gamma_i \cap B_\varepsilon} B(y, \delta d_i(x_i, y)), \quad i = 1, 2, \tag{1}$$

is empty. The symbols $B(x, r)$ denotes the open ball of radius r around x. Note that if γ_1 and γ_2 intersect (ε, δ)-transversal then the intersection $\gamma_1 \cap \gamma_2$ can be empty.

Definition 1. *We will say that a piecewise hyperbolic system $f\colon K \setminus N \to K$ satisfies condition* (T) *if*

> *there exists numbers $\varepsilon, \delta > 0$ such that if γ_1 and γ_2 are two smooth curves such that every tangent lies in the unstable cone, and γ_1 and γ_2 are in different K_i, then the curves $f(\gamma_1)$ and $f(\gamma_2)$ intersect (ε, δ)-transversal.* (T)

5 Dimension of the Attractor

Consider a map $f\colon K \setminus N \to K \subset \mathbb{R}^2$ that satisfies the conditions (A2), (A4) and (A5)–(A9). We denote by $\chi_{\mathrm{s}}(x) < 0 < \chi_{\mathrm{u}}(x)$ the two Lyapunov exponents at the point x if they exist. If Λ_1 is an ergodic component of the attractor, then the Lyapunov exponents are constant almost everywhere with respect to the SRB-measure on Λ_1, and we write $\chi_{\mathrm{s}}(x) = \chi_{\mathrm{s}}$ and $\chi_{\mathrm{u}}(x) = \chi_{\mathrm{u}}$ for almost every x.

Theorem 2. *Suppose that $f\colon K \setminus N \to K \subset \mathbb{R}^2$ is a piecewise hyperbolic map that satisfies the conditions* (T), (A2), (A4) *and* (A5)–(A9). *Let Λ_1 be an ergodic component of the attractor, whith one positive and one negative Lyapunov exponent. Then the Hausdorff dimension of Λ_1 satisfies*

$$\dim_{\mathrm{H}} \Lambda_1 \geq \min\left\{2,\, 1 - \frac{\chi_{\mathrm{u}}}{\chi_{\mathrm{s}}}\right\}.$$

Theorem 2 is proved in Section 8.

Note that in [12], it is proved that $\dim_{\mathrm{H}} \Lambda_1 \leq 1 - \chi_{\mathrm{u}}/\chi_{\mathrm{s}}$ with equality if and only if f restricted to Λ_1 is almost everywhere invertible, meaning that f is invertible on a set of full measure. Hence we get the following corollary.

Corollary 1. *If the assumptions of Theorem 2 are satisfied then*

$$\dim_{\mathrm{H}} \Lambda_1 = \min\left\{2,\, 1 - \frac{\chi_{\mathrm{u}}}{\chi_{\mathrm{s}}}\right\},$$

and f is invertible almost everywhere on Λ_1 if and only if $\chi_{\mathrm{u}} + \chi_{\mathrm{s}} \leq 0$.

Remark 1. *In case the transverality condition* (T) *is not satisfied we can only give the trivial estimate* $\dim_{\mathrm{H}} \Lambda_1 \geq 1$. *Indeed, the map* $f\colon [0,1]^2 \to [0,1]^2$ *defined by* $f\colon (x_1, x_2) \mapsto (x_1/2, 2x_2 \bmod 1)$ *has the attractor* $\Lambda_1 = \{\,(x_1, x_2) : x_1 = 0,\ 0 \leq x_1 < 1\,\}$, *and so* $\dim_{\mathrm{H}} \Lambda_1 = 1$. *Moreover, any map satisfying the conditions* (A2), (A4) *and* (A5)–(A8) *has an attractor* Λ_1 *that contains curves of unstable manifolds. This implies that* $\dim_{\mathrm{H}} \Lambda_1 \geq 1$. *So, unless one impose an additional condition, such as* (T), *one can not get a better estimate than* $\dim_{\mathrm{H}} \Lambda_1 \geq 1$.

Remark 2. *It should be noted that if* f *satisfies the conditions in Theorem 2, then so does any sufficiently small smooth perturbation of* f.

6 Vanishing multipicity entropy

In this section we give a condition which guaranties that the multiplicity entropy is zero.

Theorem 3. *Let* $K, N \subset \mathbb{R}^2$ *where* N *is a union of smooth curves, and let* $f\colon K \setminus N \to K$ *satisfy conditions* (A1)–(A7). *Assume that there is a family of cones* $C^{\mathrm{d}}(p, \gamma) \subset T_p\mathbb{R}^2$ *where* p *is a point on a smooth curve* $\gamma \subset N$, *such that*

$$C^{\mathrm{d}}(p, \gamma) \cap C^{\mathrm{u}}(p) = \{0\} \quad \text{and} \quad \mathrm{d}f(C^{\mathrm{d}}) \subset C^{\mathrm{u}}. \tag{2}$$

Then the multiplicity entropy of f *is zero.*

Remark 3. *The condition* $C^{\mathrm{d}}(p, \gamma) \cap C^{\mathrm{u}}(p) = \{0\}$ *in Theorem 3 is nonsence since* $C^{\mathrm{u}}(p)$ *is not defined for* $p \in N$. *But* $C^{\mathrm{u}}(p)$ *depends continuously on* $p \in K_i$ *so the condition should be understand as* $C^{\mathrm{u}}(p)$ *replaced by its limit for each* K_i *that meet* p.

Proof. For simplicity, let us start with the case that the curves of N do not intersect. Let $p \in N$. We will iterate p and see in how many pieces a small neigborhood U of p is cut by a curve in N that goes through $f^n(p)$. Of cource, $f^n(p)$ is not defined but we will use this notation for simplicity, for the collection of accumulation points of $f^n(q)$, when $q \to p$.

In the first iterate U is cut through p in at most two pieces, which we denote by U_1 and U_2 (or just U_1 if U is not cut). In the next iterate, each of the pieces U_1 and U_2 is cut throught $f(p)$ in at most two pieces. Denote by $U_{1,1}$ and $U_{1,2}$ the pieces of U_1 and similarly for U_2.

By the property (2), one of $U_{1,1}$ and $U_{1,2}$ lies in the cone $C^{\mathrm{u}}(f^2(p))$ and no iterate of this piece will be cut through $f^n(p)$ for any n. The same argument holds for the pieces $U_{2,1}$ and $U_{2,2}$. So we now have at most four pieces of which at most two can be cut in future iterations. There is a picture of this in Figure 1.

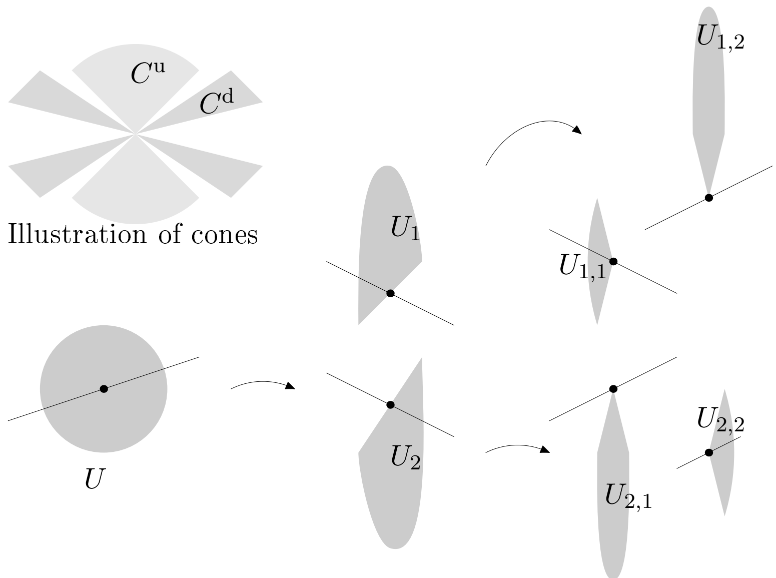

Figure 1: Illustration to the proof of Theorem 3. Note that $U_{1,2}$ and $U_{2,1}$ cannot be cut throught $f^2(p)$ since the slopes of the discontinuities are too small.

By induction we get that after n iterates $f^n(U)$ consists of at most $2n$ pieces. This shows that the multiplicity entropy is zero.

The case with N containing curves that cut each other is similar. If at most L curves meet in one point, we get that after n iterates, U consists of at most $2(L+1)n$ pieces of which at most two can be cut through $f^n(p)$. $\square$

7 An Example

In this section we give an example of maps satifying the assumptions of Theorem 2.

Let $K = (-1,1) \times (-1,1)$ be a square. Take $-1 < k < 1$ and let $N = \{ (x_1, x_2) \in K : x_2 = kx_1 \}$ be the singularity set. Take $\rho \neq 0$ and let ψ_1 and ψ_2 be two C^2 functions, such that $|\psi_1'|, |\psi_2'| < \rho_\psi < |\rho|/2$. We take parameters $\frac{1}{2} < \lambda < 1$, $1 < \gamma < 2$, a_1, a_2, b_1 and b_2 such that the map f defined by

$$f(x_1, x_2) = \begin{cases} (\lambda x_1 + a_1 + \rho x_2 + \psi_1(x_2), & \gamma x_2 + b_1) \quad \text{if } x_2 > kx_1 \\ (\lambda x_1 + a_2 + \psi_2(x_2), & \gamma x_2 + b_2) \quad \text{if } x_2 > kx_1 \end{cases} \tag{3}$$

maps $K \setminus N$ into K. There is a picture of f in Figure 2.

The case $\rho \neq 0$, $k = \psi_1 = \psi_2 = 0$ and $\gamma = 2$ is threated by Falconer in [5]. He proved that for almost all parameters γ and λ, the dimension satisfies

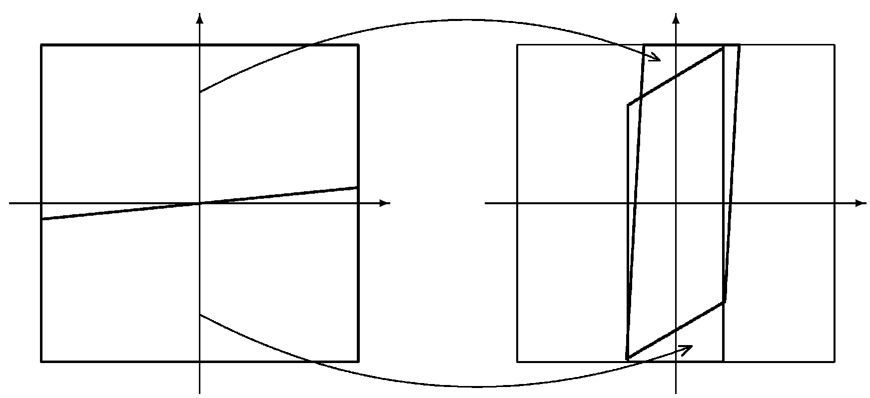

Figure 2: A picture of f with $\rho = 0.1$, $\psi_1 = \psi_2 = 0$, $\gamma = 1.8$, $\lambda = 0.3$, $k = 0.1$, $a_1 = a_2 = 0$ and $-b_1 = b_2 = 0.8$

$\dim_{\mathrm{H}} \Lambda = 1 - \log\gamma/\log\lambda$. The case $k = 0$ and $\gamma = 2$ is covered by Simon's paper [13]. He proved equality for all parameters. We prove that we have equality for all parameters in also when k, ψ_1 and ψ_2 are not nessesarily zero. More precisely, we use Theorem 2 to prove the following theorem.

Theorem 4. *If* a_1, a_2, $-b_1 = b_2 = (\gamma - 1)$ *and*

$$(\gamma, \lambda, k, \rho) \in \{\, (\gamma, \lambda, k, \rho) : \gamma > 2\lambda,\ \rho \neq 0 \,\}$$

are numbers such that $f\colon K \setminus N \to K$, *then* $f\colon K \setminus N \to K$ *defined by* (3) *has an attractor* Λ *with dimension*

$$\dim_{\mathrm{H}} \Lambda = \min\left\{2,\ 1 - \frac{\log\gamma}{\log\lambda}\right\}. \tag{4}$$

Let $\psi_1 = \psi_2 = 0$, $1 < \gamma < 2$, $0 < \lambda < 1$, $a_1 = a_2 = 0$ and $b_1 = -b_2 = 1 - \gamma$. Then if $\rho = 0$, the attractor is $\Lambda = \{\, (x_1, x_2) : x_1 = 0,\ |x_2| \leq \gamma - 1 \,\}$, and so $\dim_{\mathrm{H}} \Lambda = 1$. If $\rho \neq 0$ and $\gamma > 2\lambda$ then the dimension $\dim_{\mathrm{H}} \Lambda$ is given by (4). The dimension can be made arbitrarily close to 2 by choosing λ close to 1. Then the dimension is bounded away from 1 for any $\rho \neq 0$ but the dimension is 1 for $\rho = 0$.

Proof of Theorem 4. It is clear from Theorem 3 that f has zero multiplicity entropy if $k \neq 0$. If $k = 0$ then the multiplicity entropy is trivialy zero.

We claim that if $\gamma > 2\lambda$ and $\rho \neq 0$ then f satisfies condition (T). Let us prove this claim. It is clear that the cone spanned by the vectors

$$\left(\frac{-\rho_\psi}{\gamma - \lambda}, 1\right) \quad \text{and} \quad \left(\frac{\rho + \rho_\psi}{\gamma - \lambda}, 1\right)$$

defines an unstable cone family at any point of $K \setminus N$. Denote this cone by C^{u}.

If $\sigma_1 \subset K \cap \{x_2 > kx_1\}$ and $\sigma_2 \subset K \cap \{x_2 < kx_1\}$ are two curves such that if v_1 and v_2 are two tangent vectors of the curves, then $v_1, v_2 \in C^{\mathrm{u}}$. The vectors v_1 and v_2 are mapped by $\mathrm{d}_x f$ to

$$u_1 = \begin{bmatrix} \lambda & \rho + \psi_1(x_2) \\ 0 & \gamma \end{bmatrix} v_1 \quad \text{and} \quad u_2 = \begin{bmatrix} \lambda & \psi_2(x_2) \\ 0 & \gamma \end{bmatrix} v_2$$

respectively. One checks that u_1 is contained in the cone spanned by

$$\Big(-\rho_\psi \frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho - \rho_\psi}{\gamma}, 1\Big) \quad \text{and} \quad \Big((\rho+\rho_\psi)\frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho+\rho_\psi}{\gamma}, 1\Big)$$

and u_2 is contained in the cone spanned by

$$\Big(-\rho_\psi \frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{-\rho_\psi}{\gamma}, 1\Big) \quad \text{and} \quad \Big((\rho+\rho_\psi)\frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho_\psi}{\gamma}, 1\Big)$$

The intersection of these two cones is trivial if

$$-\rho_\psi \frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho-\rho_\psi}{\gamma} > (\rho+\rho_\psi)\frac{\lambda}{\gamma(\gamma-\lambda)} + \frac{\rho_\psi}{\gamma},$$

or equivalently, if $\gamma > 2\lambda$. This proves the claim.

By Corollary 1 it now follows that

$$\dim_{\mathrm{H}} \Lambda = 1 - \frac{\log \gamma}{\log \lambda},$$

unless $\log \gamma + \log \lambda > 0$, in which case $\dim_{\mathrm{H}} \Lambda = 2$. □

Let us end this section by considering the attractor of the map in Figure 2. The dimension of the attractor is

$$\dim_{\mathrm{H}} \Lambda = 1.488\ldots$$

There is a picture of the attractor Λ in Figure 3.

We may also consider the dimension of Λ when $\gamma = 1.8$, $\lambda = 0.5$ and $k = 0.1$. Then

$$\dim_{\mathrm{H}} \Lambda = 1.848\ldots$$

A picture of this attractor is in Figure 4. Both pictures where drawn by calculating the iterates of a small curve with tangents in the unstable cones.

8 Proof of Theorem 2

Assume that f satisfies condition (T) with (ε_0, δ)-intersections. Let $\varepsilon > 0$.

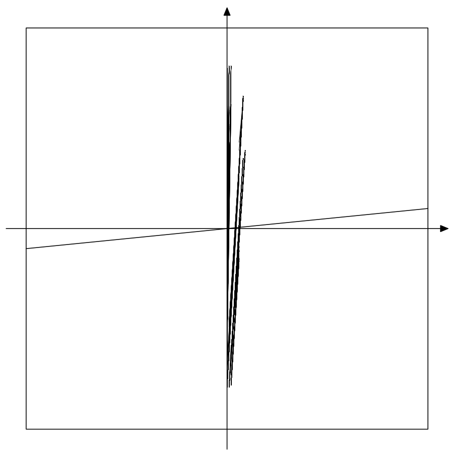

Figure 3: The attractor Λ of the map in Figure 2.

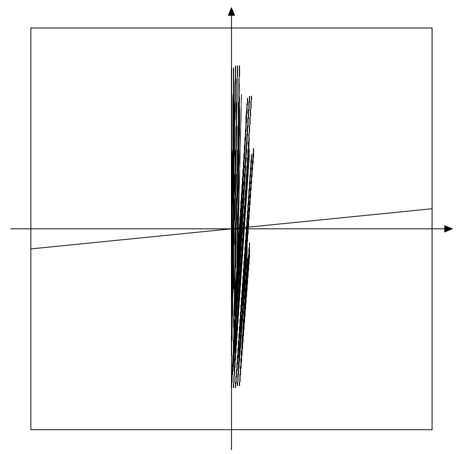

Figure 4: The attractor Λ of the map in Figure 2, but with $\lambda = 0.5$.

8.1 Coding of the system

Let $\hat{f}$ be the lift of f as described in Section 3.1 and let $\hat{\Lambda}$ denote the attractor of $\hat{f}$. We start by introducing a coding of the system $\hat{f} \colon \hat{\Lambda} \to \hat{\Lambda}$. If $\hat{x} \in \hat{\Lambda}$ then there is a sequence $\underline{\hat{s}}(\hat{x}) = \{i_k\}_{k \in \mathbb{Z}}$ such that $\hat{f}^k(\hat{x}) \in \hat{K}_{i_k}$ for every $k \in \mathbb{Z}$. We let $\Sigma = \Sigma(\hat{\Lambda})$ be the set of all such sequences, that is $\Sigma(\hat{\Lambda}) = \underline{\hat{s}}(\hat{\Lambda})$. Then there is an one-to-one correspondance $\rho \colon \Sigma \to \hat{D} \subset \hat{\Lambda}$, defined in the natural way. Let $\pi \colon \hat{K} \to K$ be the projection $\pi(x, y) = x$. Given a sequence $\underline{a} = \{a_i\}_{i \in \mathbb{Z}}$, we define the cylinder set ${}_k[\underline{a}]_l$ by

$$ {}_k[\underline{a}]_l := \{\, \underline{b} = \{b_i\}_{i \in \mathbb{Z}} \in \Sigma : b_i = a_i,\ \forall i = k, k+1, \ldots, l \,\}. $$

The sets $\rho({}_k[\underline{a}]_l)$ and $\pi(\rho({}_k[\underline{a}]_l))$ will also be called cylinders.

8.2 Images of curves

In this section we make use of condition (A9), that the multiplicity entropy is zero, to get some estimates.

For $r \in \mathbb{N}$, we let $\mathcal{D}_r(\varepsilon)$ be the set of r-cylinders ${}_0[\underline{a}]_{r-1}$ such that there exists a point $p \in {}_0[\underline{a}]_{r-1}$ with

$$ \begin{aligned} e^{(\chi_{\mathrm{u}} - \varepsilon) r} &\leq \|\mathrm{d}_p(f^r)\| \leq e^{(\chi_{\mathrm{u}} + \varepsilon) r}, & \forall v \in C^{\mathrm{u}}(p), \\ e^{(\chi_{\mathrm{s}} - \varepsilon) r} &\leq \|\mathrm{d}_p(f^r)\| \leq e^{(\chi_{\mathrm{s}} + \varepsilon) r}, & \forall v \in C^{\mathrm{s}}(p). \end{aligned} $$

Let $q, r \in \mathbb{N}$, $l > 0$ and let γ be a curve of length l with tangents in the unstable cones. Let $\mathcal{W}_0 = \{\gamma\}$. We define $\mathcal{W}_n$ inductively. If $\mathcal{W}_{n-1}$ is a collection of curves, then we let $\mathcal{W}_n$ be the set of curves that are connected pieces of length between l and $2l$, contained in the union of $\mathcal{D}_r$ and in some $f^q(\sigma)$, $\sigma \in \mathcal{W}_{n-1}$.

Since we require that the length of the curves in $\mathcal{W}_n$ are between l and $2l$, the set $\mathcal{W}_n$ might not be uniquely defined, since there are several ways to divide a curve of length larger than $2l$ into pieces of length between l and $2l$. It is however not important how this is done, so we will not give a precise definition of $\mathcal{W}_n$.

Lemma 1. *Let $f\colon K \setminus N \to K$ satisfy the conditions* (A2), (A4), (A5)–(A9). *For any $\varepsilon > 0$, there exist constants C, q, r and $l > 0$, and a curve γ with tangents in the unstable cones, such that if $N(n)$ denotes the number of curves in $\mathcal{W}_n$, then*

$$C^{-1} e^{(\chi_{\mathrm{u}}-\varepsilon)q(n-k)} \leq \frac{N(n)}{N(k)} \leq C e^{(\chi_{\mathrm{u}}+\varepsilon)q(n-k)},$$

holds for all $n \geq k \geq 1$, and the derivatives of f^q at a point $p \in W \in \mathcal{W}_n$ satisfies

$$C^{-1} e^{(\chi_{\mathrm{u}}-\varepsilon)qk} \|v\| \leq \|\mathrm{d}_p(f^{qk})(v)\| \leq C e^{(\chi_{\mathrm{u}}+\varepsilon)qk} \|v\|, \qquad \forall v \in C^{\mathrm{u}}(p), \tag{5}$$

$$C^{-1} e^{(\chi_{\mathrm{s}}-\varepsilon)qk} \|v\| \leq \|\mathrm{d}_p(f^{qk})(v)\| \leq C e^{(\chi_{\mathrm{s}}+\varepsilon)qk} \|v\|, \qquad \forall v \in C^{\mathrm{s}}(p). \tag{6}$$

Proof. Since the multiplicity entropy is zero, we can take q large and $l > 0$ small, so that any curve of length l with tangents in the unstable cone is cut in at most $e^{\varepsilon q}$ pieces when mapped by f^q.

Since the Lebesgue measure of the complement of the union of $\mathcal{D}_r$ vanishes as $r \to \infty$, we can choose r large so that the Lebesgue measure of the union of $\mathcal{D}_r$ is as close to that of K as we like. Using property (A6), we see that it is even possible to choose r so large that the intersection of the complement of the union of $\mathcal{D}_r$ with any curve of length at least l with tangents in the unstable cone has as small one dimensional Lebesgue measure as we like.

Hence by first choosing q and l, and then r depending on l, it is possible to achieve that the sums of the lengths of the curves in $\mathcal{W}_n$ satisfies

$$C_0 e^{(\chi_{\mathrm{u}}-\varepsilon)qn} \leq \sum_{\sigma \in \mathcal{W}_n} \mathrm{length}\,(\sigma) \leq C_0 e^{(\chi_{\mathrm{u}}+\varepsilon)qn},$$

where C_0 is a constant depending on f, q, l and r. This implies that the number of curves in $\mathcal{W}_n$ satisfies the statement in the lemma. □

8.3 Frostman's lemma

We define a probability measure μ_n with support on $\cup \mathcal{W}_n$ by

$$\mu_n = \frac{1}{N(n)} \sum_{W \in \mathcal{W}_n} \nu_W,$$

where ν_W denotes the normalised Lebesgue measure on the curve W, and $N(n)$ denotes the number of elements in $\mathcal{W}_n$ as in Lemma 1.

By taking a subsequence we can achieve that μ_n converges weakly to a probability measure μ with support in Λ. This measure will not be invariant, but its conditional measures on unstable manifold will be absolutely continuous with respect to the corresponding conditional measures of the SRB-measure, almost surely.

We will use the following method, originating from Frostman [6], to estimate the dimension of Λ. If

$$\iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^s} < \infty,$$

then $\dim_{\mathrm{H}} \Lambda \geq \dim_{\mathrm{H}} \operatorname{supp} \mu \geq s$. For a proof of this, see Falconer's book [4].

Let M be a number. Then

$$\iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \to \iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\mu(x)\mathrm{d}\mu(y), \quad \text{as } n \to \infty,$$

and

$$\iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\mu(x)\mathrm{d}\mu(y) \to \iint \frac{1}{|x-y|^s} \,\mathrm{d}\mu(x)\mathrm{d}\mu(y), \quad \text{as } M \to \infty.$$

We will therefore estimate

$$E_s(n, M) = \iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y).$$

It is clear that $E_s(n, M) \leq M$. By the definition of the measure μ_n we immediately get that

$$E_s(n, M) = \sum_{W,V \in \mathcal{W}_n} \frac{1}{N(n)^2} \iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y). \tag{7}$$

We rewrite (7) as

$$E_s(n) = J_1 + J_2,$$

with

$$J_1 = \sum_{W \in \mathcal{W}_n} \frac{1}{N(n)^2} \iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\nu_W(x)\mathrm{d}\nu_W(y),$$

$$J_2 = \sum_{\substack{W,V \in \mathcal{W}_n, \\ V \neq W}} \frac{1}{N(n)^2} \iint \min\Big\{M, \frac{1}{|x-y|^s}\Big\} \,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y).$$

To estimate J_1 we note that

$$\iint \min\Big\{M,\ \frac{1}{|x-y|^s}\Big\}\,\mathrm{d}\nu_W(x)\mathrm{d}\nu_W(y) \leq M.$$

Hence

$$J_1 \leq \sum_{W\in\mathcal{W}_n} \frac{M}{N(n)^2} = \frac{M}{N(n)},$$

and so $J_1 \to 0$ as $n \to \infty$.

We will now estimate J_2 and show that J_2 is bounded as $n \to \infty$, provided that s is sufficiently small.

Let $m < n$ and $W \in \mathcal{W}_n$. Then there is a unique $\alpha \in \mathcal{W}_{n-m}$ such that $W \subset f^{qm}(\alpha)$. Let W_{-m} denote the set $W_{-m} \subset \alpha$ such that $W = f^{qm}(W_{-m})$.

Fix $m < n$ and take two different α and β in $\mathcal{W}_{n-m}$ such that α_{-1} and β_{-1} are in different cylinders. By condition (T) this implies that α and β intersect (ε_0, δ)-transversal. We will consider all manifolds W and V in $\mathcal{W}_n$ such that $W_{-m} \subset \alpha$, $V_{-m} \subset \beta$, and W_{-m} and V_{-m} are in the same qm-cylinder, which we denote by $S_m(W_{-m})$. There is a picture of this in Figure 5.

Note that W and V intersect if and only if W_{-m} and V_{-m} intersect, since W_{-m} and V_{-m} are in the same qm-cylinder. If $W_{-m} \subset \alpha$ intersect β, then we estimate that

$$\sum_{\substack{V\in\mathcal{W}_n \\ V_{-m}\subset\beta\cap S_m(W_{-m})}} \iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \leq C_1 e^{(\chi_\mathrm{u}-\chi_\mathrm{s}+2\varepsilon)(s-1)m}, \qquad (8)$$

where C_1 does not depend on W, α and β. Indeed, if m is large, then we may assume that W_{-m} and $V_{-m} \subset \beta \cap S_m(W_{-m})$ are contained in a ball of radius ε_0, and so the manifolds $f^{qm}(\beta)$ and W intersect $(\varepsilon_0, C^2 e^{(\chi_\mathrm{u}-\chi_\mathrm{s}+2\varepsilon)m}\delta)$-transversal and we can estimate that

$$\sum_{\substack{V\in\mathcal{W}_n \\ V_{-m}\subset\beta\cap S_m(W_{-m})}} \iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \leq C_0 \int_{\gamma_1}\int_{\gamma_2} \frac{1}{|x-y|^s}\,\mathrm{d}x\mathrm{d}y,$$

where γ_1 and γ_2 are the curves

$$\gamma_1 = \{\, (x_1,x_2) : x_1 = 0, |x_2| < \operatorname{diam} K \,\},$$
$$\gamma_2 = \{\, (x_1,x_2) : |x_2| < l, x_2 = C^2 e^{(\chi_\mathrm{u}-\chi_\mathrm{s}+2\varepsilon)m}\delta x_1 \,\},$$

and C_0 is a constant, that depends only on the second derivative of the map and the constants $\operatorname{diam} K$ and l. To prove (8), one easily checks that there exists a constant C_1 such that

$$C_0 \int_{\gamma_1}\int_{\gamma_2} \frac{1}{|x-y|^s}\,\mathrm{d}x\mathrm{d}y \leq C_1 e^{(\chi_\mathrm{u}-\chi_\mathrm{s}+2\varepsilon)(s-1)m}.$$

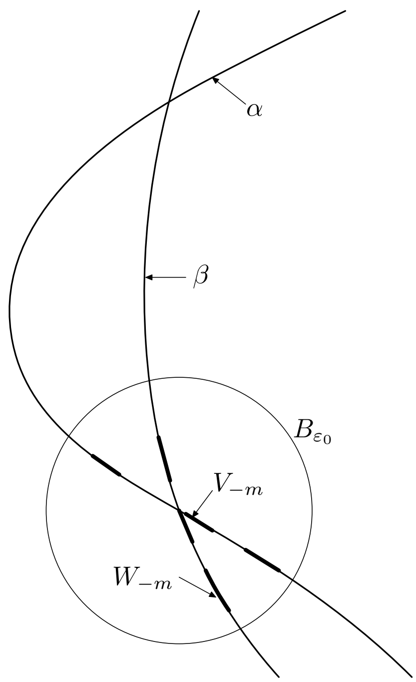

Figure 5: A picture of intersections of unstable manifolds. The pre-images W_{-m} and V_{-m} are the thicker segments.

We consider now those manifolds W, such that $W_{-m} \subset \alpha$ does not intersect β. First, we consider those V such that W_{-m} and V_{-m} lies in some ball B_{ε_0} in the spirit of (T). If the distance between W_{-m} and β is $d(W_{-m}, \beta)$, then the distance between W and $V \subset f^{qm}(\beta)$ is larger than $C^{-1} e^{(\chi_s - \varepsilon)m} d(W_{-m}, \beta)$ by (6). If we choose the length l in the construction of $\mathcal{W}_n$ sufficiently small, then we can approximate the integral by

$$\sum_{\substack{V \in \mathcal{W}_n \\ V_{-m} \subset \beta \cap S_m(W_{-m})}} \iint \frac{1}{|x-y|^s} \,\mathrm{d}\nu_V(x) \mathrm{d}\nu_W(y) \leq l^{-2} \int_{\gamma_1} \int_{\gamma_2} \frac{1}{|x-y|^s} \,\mathrm{d}x \mathrm{d}y,$$

where γ_1 and γ_2 are two parallell line segments of length l and $\operatorname{diam} K$, and with distance $d(W_{-m}, \beta)/2$. The last integral is estimated by

$$\begin{aligned} \int_{\gamma_1} \int_{\gamma_2} \frac{1}{|x-y|^s} \,\mathrm{d}x \mathrm{d}y &\leq \int_{-\infty}^{\infty} \frac{1}{\left(\sqrt{x^2 + (d(W_{-m}, \beta)/2)^2}\right)^s} \,\mathrm{d}x \\ &= 2^s e^{(\chi_s - \varepsilon)(1-s)m} d(W_{-m}, \beta)^{1-s} \int_0^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\frac{s}{2}}}, \end{aligned}$$

and so

$$\begin{aligned} \sum_{\substack{V \in \mathcal{W}_n \\ V_{-m} \subset \beta \cap S_m(W_{-m})}} \iint \frac{1}{|x-y|^s} \,\mathrm{d}\nu_V(x) \mathrm{d}\nu_W(y) & \\ \leq C_2 e^{(\chi_s - \varepsilon)(1-s)m} d(W_{-m}, \beta)^{1-s}, \end{aligned} \quad (9)$$

for some constant C_2, provided that $s > 1$. In fact, one easily shows that

$$\int_0^{\infty} \frac{\mathrm{d}x}{(1+x^2)^{\frac{s}{2}}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})},$$

by the change of variable $t = \frac{1}{1+x^2}$ and the observation that

$$\frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})} = \frac{1}{2} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{s-1}{2})}{\Gamma(\frac{s}{2})} = \frac{1}{2} B(\tfrac{1}{2}, \tfrac{s-1}{2}),$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^{\infty} t^{x-1} (1-t)^{y-1} \,\mathrm{d}t$ is the beta function.

We cover the intersections of α and β by balls B_{ε_0}. Since K is a bounded set the number of such balls will always be less than some number N_B. Given a manifold $W_{-m} \subset \alpha$, either W_{-m} intersect one of these balls or lies a distance of at least ε_0 from each of the intersections of α and β.

If W_{-m} lies in B_{ε_0}, with distance $d_{W_{-m}}$ from the center of the ball, then by property (T), the distance between W_{-m} and $V_{-m} \subset \beta$ is at least $\delta d_{W_{-m}}$. The manifolds W_{-m} and V_{-m} are subsets of the two larger manifolds α and β (see Figure 5). On each side of the intersection of these larger manifolds (or the closest point in case they do not intersect) we can enumerate the pairs W_{-m} and V_{-m}, such that the distance from the center of the ball B_{ε_0} to the ith manifold W_{-m} is increasing. Since two different W_{-m} do not intersect, the distance from the center of B_{ε_0} to the ith manifold W_{-m} is at least $i\frac{l}{Ce^{(\chi_u+\varepsilon)m}}$, since the length of each W_{-m} is at least $\frac{l}{Ce^{(\chi_u+\varepsilon)m}}$ by (5). (We measure the distance along the large manifold containing all the W_{-m}.) This implies that the distance between the ith W_{-m} and $V_{-m} \subset \beta$ is at least $\delta i\frac{l}{Ce^{(\chi_u+\varepsilon)m}}$ and so the distance between the corresponding W and V is at least $C^{-2}e^{(\chi_s-\varepsilon)m}\delta i\frac{l}{e^{(\chi_u+\varepsilon)m}}$. There are at most $C_3M(m)$ different W_{-m} in B_{ε_0}, where C_3 is a constant that dependds on l and $M(m)$ denotes the number of qm-cylinders. By (8) and (9) we estimate that

$$\begin{aligned}\sum_{\substack{W_{-m}\subset\alpha\cap B_{\varepsilon_0},\\ V_{-m}\subset\beta\cap S_m(W_{-m})\cap B_{\varepsilon_0}}} \iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \\ < C_1e^{(\chi_u-\chi_s+2\varepsilon)(s-1)m} + 2\sum_{i=1}^{C_3M(m)} C_2\Big(C^{-2}e^{(\chi_s-\varepsilon)m}\delta i\frac{l_0}{e^{(\chi_u+\varepsilon)m}}\Big)^{1-s} \\ < C_4e^{(\chi_u-\chi_s+2\varepsilon)(s-1)m}M(m)^{2-s}. \qquad (10)\end{aligned}$$

If we sum over the balls B_{ε_0} needed to cover the intersection of α and β, we get

$$\begin{aligned}\sum_{B_{\varepsilon_0}}\sum_{\substack{W_{-m}\subset\alpha\cap B_{\varepsilon_0},\\ V_{-m}\subset\beta\cap S_m(W_{-m})\cap B_{\varepsilon_0}}} \iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \\ < C_5e^{(\chi_u-\chi_s+2\varepsilon)(s-1)m}M(m)^{2-s}. \qquad (11)\end{aligned}$$

For those W and V such that W_{-m} and V_{-m} are not inside a ball B_{ε_0} we have $d(W,V) > C^{-1}e^{(\chi_s-\varepsilon)m}\varepsilon_0$, and estimate by (9) that

$$\begin{aligned}\iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \\ \leq C_2e^{(\chi_s-\varepsilon)(1-s)m}C^{s-1}\varepsilon_0^{1-s} = C_6e^{(\chi_s-\varepsilon)(1-s)m}. \qquad (12)\end{aligned}$$

The number of such pairs W and V are at most some constant C_7 times the number of qm-cylinders, denoted by $M(m)$. We get by (11) and (12) that

$$\begin{aligned}\sum_{\substack{W,V\in\mathcal{W}_n,\\ W_{-m}\subset\alpha,\\ V_{-m}\subset\beta}} \iint \frac{1}{|x-y|^s}\,\mathrm{d}\nu_V(x)\mathrm{d}\nu_W(y) \\ < C_5e^{(\chi_u-\chi_s+2\varepsilon)(s-1)m}M(m)^{2-s} + C_6C_7e^{(\chi_u-\chi_s(s-1)+s\varepsilon)m}M(m). \qquad (13)\end{aligned}$$

We will now sum over all m, α and β, and write J_2 as $J_2 = J_3 + J_4$, with

$$J_3 = \sum_{m=0}^{n-1} \sum_{\substack{\alpha,\beta \in \mathcal{W}_{n-m} \\ \alpha \neq \beta}} \sum_{\substack{W,V \in \mathcal{W}_n, \\ W_{-m} \subset \alpha, \\ V_{-m} \subset \beta}} \frac{\iint \min\left\{M,\ \frac{1}{|x-y|^s}\right\} \mathrm{d}\nu_V(x) \mathrm{d}\nu_W(y)}{N(n)^2},$$

$$J_4 = \sum_{m=0}^{n-1} \sum_{\alpha \in \mathcal{W}_{n-m}} \sum_{\substack{W,V \in \mathcal{W}_n, \\ W_{-m}, V_{-m} \subset \alpha, \\ W_{-m} \neq V_{-m}}} \frac{\iint \min\left\{M,\ \frac{1}{|x-y|^s}\right\} \mathrm{d}\nu_V(x) \mathrm{d}\nu_W(y)}{N(n)^2}.$$

Similarly as for J_1 we obtain that $J_4 \to 0$ as $n \to \infty$. It remains to estimate J_3.

Using that there are $N(n-m)$ different α and β, we get by (10) and (13) that

$$J_3 \leq \sum_{m=0}^{n-1} N(n-m)^2 \frac{C_5 e^{(\chi_\mathrm{u} - \chi_\mathrm{s} + 2\varepsilon)(s-1)qm} M(m)^{2-s}}{N(n)^2} + \sum_{m=0}^{n-1} N(n-m)^2 \frac{C_6 C_7 e^{(\chi_\mathrm{u} - \chi_\mathrm{s}(s-1) + s\varepsilon)qm} M(m)}{N(n)^2}.$$

We now use that the topological entropy is χ_u, and thus the number of cylinders satisfy $M(m) \leq C_6 e^{(\chi_\mathrm{u} + \varepsilon)qm}$, for some constant C_6. This yields

$$J_3 \leq C_8 \sum_{m=0}^{n-1} \frac{N(n-m)^2 e^{(\chi_\mathrm{u} - \chi_\mathrm{s}(s-1) + s\varepsilon)qm}}{N(n)^2},$$

for some constant C_8 that does not depend on n. By Lemma 1 we have $N(n-m)/N(n) \leq C e^{-(\chi_\mathrm{u} - \varepsilon)qm}$, so

$$J_3 \leq C_8 \sum_{m=0}^{n-1} e^{(-\chi_\mathrm{u} - \chi_\mathrm{s}(s-1) + s\varepsilon)qm}.$$

We conclude that J_3 is bounded as a function of n provided that $-\chi_\mathrm{u} - (s-1)\chi_\mathrm{s} + s\varepsilon < 0$ and $s < 2$ or equivalently

$$s < 1 - \frac{\chi_\mathrm{u} - \varepsilon}{\chi_\mathrm{s} - \varepsilon} \quad \text{and} \quad s < 2. \tag{14}$$

We have therefore obtained that, if s satisfies (14), then J_1 and $J_2 = J_3 + J_4$ are bounded, and so the integral

$$\iint \min\left\{M,\ \frac{1}{|x-y|^s}\right\} \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

is uniformly bounded and hence converges as $M \to \infty$. This proves that

$$\iint \frac{1}{|x-y|^s} \, \mathrm{d}\mu(x) \mathrm{d}\mu(y) < \infty,$$

provided that (14) holds. Hence

$$\dim_{\mathrm{H}} \Lambda \geq \min\left\{2,\, 1 - \frac{\chi_{\mathrm{u}} - \varepsilon}{\chi_{\mathrm{s}} - \varepsilon}\right\}.$$

Let $\varepsilon \to 0$.

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