Spacetime Dependent Lagrangians and Electrogravity Duality

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Abstract

We apply the spacetime dependent lagrangian formalism [1] to the action in general relativity. We obtain a Barriola-Vilenkin type monopole solution by exploiting the electrogravity duality of the vacuum Einstein equations and using a modified definition of empty space. An *upper bound* is obtained on the monopole mass M, $M \leq e^{(1-\alpha)/\alpha}/(1-\alpha)^2 G$ where $\alpha = 2k$ is the global monopole charge.

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1.Introduction

It has been shown that electromagnetic duality as well as weak-strong duality in equations of motion can be obtained by introducing explicit spacetime dependence of the lagrangian [1]. Duality symmetry, which hinges on nonperturbative aspects of a theory, is implemented through certain general set of characteristics of a spacetime dependent function -representing the spacetime dependence of the lagrangian- at large values of the spacetime coordinates i.e. on the boundary. The solutions for the fields obtained under these conditions are topological in nature. Electrogravity duality is a topological defect [2] generating process. Under electrogravity duality transformations on the Einstein equations, vacuum solutions get mapped onto solutions with a global monopole.An example of a global monopole solution is the Barriola-Vilenkin (B-V)monopole [3].In this work we show that, using the formalism of [1], one can get a B-V type metric in asymptotic regions defined by $r \approx e^{1/\alpha}$ where the global monopole charge α is very close to zero. We also additionally obtain an *upper* bound for the monopole mass: $M \leq e^{(1-\alpha)/\alpha}/(1-\alpha)^2 G$ where G is Newton's gravitational constant. Interestingly one also finds an analogue of the holographic principle operating in this example.

Vacuum Einstein equations are invaiant under $G_{ik} \leftrightarrow R_{ik}$ where $G_{ik} = R_{ik} - (1/2)g_{ik}R$ is the Einstein tensor, R_{ik} is Ricci tensor,R is the Ricci scalar and G the Einstein scalar. This signifies the existence of electrogravity duality symmetry. Like the electromagnetic field, the gravitational field can be resolved into "electric" (due to charge) and "magnetic" (motion of charge) parts. For gravity the analogues are mass-energy and its motion respectively. Gravity entails two kinds of charges– non-gravitational matter-energy

("active" part) and gravitational field energy ("passive" part).Electrogravity duality means that vacuum equations are invariant under the interchange of the active and passive "electric" parts. Mathematically this entails interchange of Ricci and Einstein curvatures. Under such electrogravity duality transformations, vacuum solutions get mapped onto topological solutions.

In this context,there is a definition of vacuum (empty space) [4] which is less restrictive than $R_{ik} = 0$ (i.e. vanishing of all Ricci components). Vacuum, instead, is characterised by energy density (relative to a static observer) $\rho = 0$, timelike convergence density (relative to a static observer) $\rho_t = 0$, null convergence density (relative to radial null geodesic) $\rho_n = 0$ and the absence of any energy flux $P_c = 0$. For a spherically symmetric metric $\rho_n = P_c = 0$ imply $R_{01} = 0$, $R_0^0 = R_1^1$ which in turn implies that $g_{00} = f(r) = -g^{11}$; while $\rho = 0$ means $R_2^2 = 0$ which integrates to give the Schwarzschild solution. So empty space is characterised by

$$R_2^2 = 0 = R_1^0 \; ; R_0^0 = R_1^1 \tag{1a}$$

It has been shown [4] that if we replace ρ by ρ_t and use $\rho_t = \rho_n = P_c = 0$, then this implies that

$$G_2^2 = 0 = G_1^0 \; ; G_0^0 = G_1^1 \tag{1b}$$

Then replacing the Ricci tensor by the Einstein tensor leads to the solution of the metric tensor as that of the Barriola-Vilenkin monopole. G is the electrogravity dual of R . So the global monopole metric is electrogravity dual of the Schwarzschild metric.

In the presence of non-gravitational sources we take a modified action shown in equation (4) and field equations obtained from equation (2). These are motivated by Ref.[1a].

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2.Global monopole and an upper bound for the global monopole mass

We first recall our formalism [1]. Let the lagrangian L' be a function of fields η_ρ, their derivatives $\eta_{\rho,\nu}$ *and the spacetime coordinates* x_ν, i.e. $L' = L'(\eta_\rho, \eta_{\rho,\nu}, x_\nu)$.Variational principle yields $\int dV \left(\partial_\eta L' - \partial_\mu \partial_{\partial_\mu \eta} L' \right) = 0$. Assuming a separation of variables : $L'(\eta_\sigma, \eta_{\sigma,\nu}, ..x_\nu) = \Lambda(x_\nu) L(\eta_\sigma, \eta_{\sigma,\nu})$, $(\Lambda(x_\nu)$ is the x_ν dependent part and is a finite non-vanishing function) gives

$$\int dV \left(\partial_\eta (\Lambda L) - \partial_\mu \partial_{\partial_\mu \eta} (\Lambda L) \right) = 0 \tag{2}$$

The usual action in gravity is (in units where velocity of light $c = 1$)

$$S = -(1/16\pi\mathtt{G}) \int d^4x R\sqrt{-g} + (1/2) \int d^4x g^{ik} T_{ik} \sqrt{-g} \tag{3}$$

g is the determinant of the metric tensor, $\mathtt{G}$ is Newton's gravitational constant and T_{ik} the energy-momentum tensor. Consider the modified action

$$S_\Lambda = -(1/16\pi\mathtt{G}) \int d^4x \Lambda(x) R\sqrt{-g} \tag{4}$$

In our formalism this will lead to equations of motion which when solved in the light of Dadhich's approach yields a solution for the metric for a global monopole as well as Λ. The spacetime dependence (after the overall dependence on R) is expressed by Λ (assumed to be a function of r only; $r = \infty$ is a boundary of the theory). Λ is *not* dynamical and is a finite, non-vanishing function at all x_ν. It is like some external field and equations of motion for Λ meaningless at the length scales (classical gravity) under consideration. Λ is finite at infinity. The finite behaviour of Λ on the boundary *encodes the electrogravity duality of the theory within the boundary* providing an analogue of 't Hooft's holographic principle [5].

Using the action as defined in (4) the equations of motion that follow using (2) are

$$\Lambda[R_{ik} - (1/2)g_{ik}R] - (\delta/\delta g^{ik})[(\partial_m\Lambda)(g^{pm}\delta\Gamma^l_{pl} - g^{pq}\delta\Gamma^m_{pq})] = 0$$

where Γ^i_{jk} are the affine connections. This can be recast into the form

$$\Lambda(R_{ik} - (1/2)g_{ik}R) - M_{ik} = 0$$

i.e.

$$\Lambda G_{ik} = M_{ik} \tag{5a}$$

with

$$M_{ik} = (1/2)(g^{mn}\partial_n g_{ik}\partial_m\Lambda) - \partial_n(g^{mn}g_{ik}\partial_m\Lambda)$$

$$-(g^{mn}\partial_m g_{kn}\partial_i\Lambda) + (1/2)(g^{mn}\partial_k g_{mn}\partial_i\Lambda) + \partial^n(g_{mi}g_{nk}\partial^m\Lambda) \tag{5b}$$

M_{ik} is like an effective stress tensor.

We shall take the spherically symmetric metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{6}$$

(i.e. $g_{00} = B(r)$; $g_{11} = -A(r)$). Then

$$G^2_2 = M^2_2/\Lambda = (\frac{1}{Ar})\frac{\partial_1\Lambda}{\Lambda} - (\frac{1}{A^2})(\partial_1 A)(\frac{\partial_1\Lambda}{\Lambda}) + (\frac{1}{A})\frac{\partial_1^2\Lambda}{\Lambda} \tag{7a}$$

$$G^0_0 = M^0_0/\Lambda = [(\frac{1}{A^2})\partial_1 A - (\frac{1}{2AB})(\partial_1 B)]\frac{\partial_1\Lambda}{\Lambda} - (\frac{1}{A})\frac{\partial_1^2\Lambda}{\Lambda} \tag{7b}$$

$$G^1_1 = M^1_1/\Lambda = [-(\frac{1}{A^2})\partial_1 A - (\frac{1}{2AB})(\partial_1 B)]\frac{\partial_1\Lambda}{\Lambda} - (\frac{2}{Ar})\frac{\partial_1\Lambda}{\Lambda} \tag{7c}$$

Λ is determined such that (1b) is obeyed which translated in our formalism means (M_{ik} as defined in equation (5b)) $M^2_2 = 0$; $M^0_0 = M^1_1 = -\alpha\Lambda/r^2$ i.e.

$$M^2_2 = (\frac{1}{Ar})\partial_1\Lambda - (\frac{1}{A^2})(\partial_1 A)(\partial_1\Lambda) + (\frac{1}{A})\partial_1^2\Lambda = 0 \tag{8a}$$

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$$M_0^0 = [(\frac{1}{A^2})\partial_1 A - (\frac{1}{2AB})(\partial_1 B)]\partial_1 \Lambda - (\frac{1}{A})\partial_1^2 \Lambda = -(\frac{\alpha \Lambda}{r^2}) \qquad (8b)$$

$$M_1^1 = [-(\frac{1}{A^2})\partial_1 A - (\frac{1}{2AB})(\partial_1 B)]\partial_1 \Lambda - (\frac{2}{Ar})\partial_1 \Lambda = -(\frac{\alpha \Lambda}{r^2}) \qquad (8c)$$

We shall now obtain solutions of equations (8) under certain physically plausible approximations. Specifically we show below (Case 2) that there exists a region of large r given by $r \sim e^{1/\alpha}$, $\alpha \to 0$, where one has the Barriola-Vilenkin metric for $\Lambda \neq constant$ (Case 2).

Case 1: $r = \infty$; $\alpha = 0$; $\Lambda = constant$

Then $M_{ik} = 0$ for all i, k and (5a) reduces to the usual equation in general relativity. The solution is the usual Schwarzschild solution and global monopole solution is not possible.On the boundary $(r = \infty; \Lambda = constant)$ the action is the usual one

$$S = -(1/16\pi \texttt{G}) \int d^4 x R \sqrt{-g}$$

This is in terms of the Ricci tensor whose solution gives the Schwarzschild solution.

Case 2: $r \neq \infty; \alpha \neq 0; \Lambda \neq constant$

Then $M_{ik} \neq 0$.Postulate $\Lambda = \Lambda(r)$ for $\alpha \neq 0$. We now illustrate that there exists a form for $\Lambda(r)$ which can lead to the B-V monopole under certain assumptions. Consider

$$\Lambda(r) = \frac{1}{(1-\alpha)} ln[r(1-\alpha) - 2\texttt{GM}] \qquad (9)$$

In (9), α *is never unity* so that Λ is always well defined. Also , α *is never zero*, because then we would have $\Lambda = constant$ (Case 1). Further, we shall take α to be small. So $0 < \alpha < 1$.

The equation $(8a)$ implies $\partial_1 \Lambda(r) = P_1 A(r)/r$. Choose the constant of integration $P_1 = 1$. For Λ as in (9) then implies

$$A(r) = \frac{1}{(1 - \alpha - \frac{2\text{GM}}{r})}$$

Adding $(8b)$ and $(8c)$ and simplifying gives

$$lnB = ln[r(1 - \alpha) - 2\text{GM}] - 2lnr$$

$$+ (\frac{2\alpha}{(1 - \alpha)}) \int dr(\frac{1}{r}) ln[r(1 - \alpha) - 2\text{GM}]$$

For large r,

$$ln[r(1 - \alpha) - 2\text{GM}] \approx ln(1 - \alpha) - lnr - \frac{2\text{GM}}{r(1 - \alpha)}$$

Carrying out the integrations and choosing the integration constants to be zero give:

$$lnB(r) = ln(1 - \alpha) - \frac{2\text{GM}}{r(1 - \alpha)} - lnr + \frac{4\alpha\text{GM}}{(1 - \alpha)^2 r}$$

$$+ \frac{2\alpha ln(1 - \alpha)}{(1 - \alpha)} lnr + \frac{\alpha}{(1 - \alpha)}(lnr)^2$$

Now lnr is never zero. The contribution from the terms proportional to lnr will be negligible in the region where

$$lnr = \frac{(1 - \alpha)}{\alpha} - 2ln(1 - \alpha) \tag{10a}$$

i.e.

$$r = \frac{e^{(1-\alpha)/\alpha}}{(1 - \alpha)^2} \tag{10b}$$

Then $r \approx e^{\frac{1}{\alpha}}$ for small α so that α^2 and higher orders are negligible. *This is the region of large r.* In this region it is easy to see that

$$lnB(r) = ln(1 - \alpha) - \frac{(1 - 2\alpha)}{(1 - \alpha)}[2\text{GM}/r(1 - \alpha)]$$

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$$\approx ln(1-\alpha) + ln[1 - 2\text{GM}/r(1-\alpha)]$$

$$\approx ln[(1-\alpha)(1 - 2\text{GM}/r(1-\alpha)] \approx ln[1 - \alpha - 2\text{GM}/r]$$

so that $B(r) \approx 1 - \alpha - 2\text{GM}/r$. Here we have assumed that αG is of second order of smallness compared to α. *Therefore for $\alpha \neq 0$ but very small, Λ as in (9), and asymptotic region defined by $r \approx e^{1/\alpha}$ we get the Barriola-Vilenkin metric for a global monopole:*

$$A(r) = \frac{1}{(1 - \alpha - \frac{2\text{GM}}{r})} \tag{11a}$$

$$B(r) \approx 1 - \alpha - 2\text{GM}/r \tag{11b}$$

Given the way we have defined the asymptotic region, the statement "$2\text{GM}/r$ is small for large r" now reads "$2\text{GM}/e^{1/\alpha}$ is small for large r (i.e. $r \sim e^{1/\alpha}$) with $\alpha \to 0$."

Further justification of the solutions (11) is as follows. Subtracting (8c) from (8b) and simplifying gives $\Lambda = -\frac{constant}{3r^3}$ and we now show that in the asymptotic region as defined by us this is readily consistent with the solution (9). Expanding the logarithm in (9) upto the third order in r gives

$$\Lambda = \frac{1}{(1-\alpha)}lnr + \frac{1}{(1-\alpha)}ln(1-\alpha) - \frac{2\text{GM}}{(1-\alpha)\text{r}} - (\frac{1}{2})[\frac{2\text{GM}}{(1-\alpha)\text{r}}]^2$$

$$-(\frac{1}{3})[\frac{2\text{GM}}{(1-\alpha)\text{r}}]^3 - ......$$

In the asymptotic region , $r \sim e^{1/\alpha}$, if we want only the third order term to survive we must have the remaining terms upto order two vanish. Plugging in the values of r and lnr from $(10a, b)$ into the above equation implies that

$$\alpha(1-\alpha)^3(2\text{GM})^2 + 2\alpha(1-\alpha)^2 e^{(1-\alpha)/\alpha}(2\text{GM})$$

$$+2\alpha ln(1-\alpha)e^{2(1-\alpha)/\alpha} - 2(1-\alpha)e^{2(1-\alpha)/\alpha} = 0 \tag{12}$$

This is a quadratic in (2GM).Remembering that both G and M are non negative, and keeping terms upto first order in α a solution is

$$2\text{GM} \approx e^{(1-\alpha)/\alpha}[(3/2)\sqrt{2\alpha} - 1] \tag{13}$$

Certain points are to be noted. G is a universal positive constant and M is also positive. This means that $\alpha \geq 2/9 \approx 0.22222...$Then if we take ,for example, $\alpha \sim 0.22223$,we have $\frac{2\text{GM}}{e^{(1-\alpha)/\alpha}} \sim 0.000017$. The aim of this exercise is to show that as per the definition of our asymptotic region $\frac{2\text{GM}}{e^{(1-\alpha)/\alpha}}$ is indeed small even for a finite α. Therefore in the asymptotic region the solution (9) is consistent with the expression $\Lambda = -\frac{constant}{3r^3}$ where the the value of the constant is $\frac{2\text{GM}}{(1-\alpha)^3}$. We have thus demonstrated unambiguously that the Barriola-Vilenkin type metric can be obtained in our formalism under certain approximations.

Usually GM/r is taken to be a small quantity in general relativity for large values of r. So $\text{GM} < \infty$. This scenario is for the usual Schwarzschild metric.For the electrogravity dual theory (in our spacetime dependent lagrangian formalism) the asymptotic region is defined by $r = \frac{e^{(1-\alpha)/\alpha}}{(1-\alpha)^2}$ with $\alpha \to 0$. Then $\text{GM} \leq \frac{e^{(1-\alpha)/\alpha}}{(1-\alpha)^2}$ means

$$\text{M} \leq \frac{e^{(1-\alpha)/\alpha}}{(1-\alpha)^2\text{G}} \tag{14}$$

. This is an *upper bound* on the monopole mass and can be made very small by choosing α large. This is consistent with that of Harari and Lousto [3b], while avoiding the physically undesirable feature of the monopole mass becoming negative.

The expression (14) reminds us of the Bogomolny bound $\text{M}_{\text{monopole}} \geq vg$ for usual monopoles where v is the vacuum expectation value of the Higgs

field and g the monopole charge, except that the Bogomolny bound is a *lower bound* on the monopole mass.Carrying the analogy further, $(1/\mathtt{G})$ is like the vacuum expectation value of some field hitherto undiscovered while $\alpha = 2k$ is defined as the global monopole charge.

The functional forms for Λ are different in the two cases and one cannot go from Case 1 to Case 2 (or *vice-versa*) by varying r. The same Λ cannot encompass both regions by a variation of r.This is expected because Case 1 corresponds to usual space whereas Case 2 is a consequence of electrogravity duality which is a topological defect generating process.

3. The electrogravity dual action

Consider now the respective actions. On the boundary ($r = \infty$; $\alpha = 0$; $\Lambda = constant$) it is the usual one :

$$S = -(1/16\pi\mathtt{G}) \int d^4 x R \sqrt{-g}$$

This is in terms of the Ricci tensor whose solution gives the usual Schwarzschild solution.

Within the boundary, ($r \neq \infty$;$\alpha \neq 0$; $\Lambda \neq constant$) *the action should be in terms of* M i.e.

$$S_\Lambda^{dual} = -(1/16\pi\mathtt{G}) \int d^4 x M \sqrt{-g}$$

$$= -(1/16\pi\mathtt{G}) \int d^4 x \Lambda G \sqrt{-g} = -(1/16\pi\mathtt{G}) \int d^4 x \Lambda (-R) \sqrt{-g}$$

$$= -(1/16\pi\mathtt{G}^{\mathtt{dual}}) \int d^4 x \Lambda R \sqrt{-g} \tag{15}$$

Therefore, the action in the dual theory ($\Lambda \neq constant$) *in our formalism must be defined with Newton's constant* $\mathtt{G}$ *replaced by* $\mathtt{G}^{\mathtt{dual}} = -\mathtt{G}$. *Then* (4)

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and (15) *are equivalent definitions for the action and will lead to same field equations* (5*a*).

Apparently the Ricci tensor for the theory described by the metric (11*a, b*) is $R = -M/\Lambda = 2\alpha/r^2$. However, taking into account the discussion following eq. (15), the action should be defined with $\mathtt{G}' = -\mathtt{G}$. With this definition the Ricci tensor for (11*a, b*) is $R = -2\alpha/r^2$. Alternatively, electrogravity duality interchanges G and R, so the object to look for is G i.e. $G = -R = -2\alpha/r^2$. Thus whichever way one looks, things turn out to be consistent.

The metric also has a spacetime singularity at $r = 0$ and represents a global monopole metric obtained as the dual of the Schwarzschild metric. On the boundary (Case 1),$\alpha = 0$, $\Lambda = constant$; one has the usual Schwarzschild metric. Within the boundary (Case 2) in the asymptotic region defined by $r \approx e^{1/\alpha}$, $\alpha \neq 0, \Lambda \neq constant$; we have the electrogravity dual solution *viz.* the Barriola-Vilenkin metric. This again reminds us of the 't Hooft's holographic principle [5].

4.Conclusion

Thus the spacetime dependent lagrangian formalism can accommodate electrogravity duality in all its aspects. This is illustrated by obtaining the Barriola-Vilenkin monopole metric from the equations of motion under certain approximations. No scalar field is necessary.α may as well originate from other types of non-gravitational matter-energy sources hitherto unknown. A scalar field is merely one such realisation. The solution follows from electrogravity duality which can be precisely formulated in our formalism by stating that *the solutions for* $\Lambda(r) \neq constant$ *are the electrogravity dual of solutions*

for $\Lambda(r) = constant.$ An analogue of the holographic principle is again illustrated. The global monopole mass is predicted to satisfy an *upper bound.* Another point need to be stressed, *viz.* the empty space definition (1). This definition is naturally adaptable to our formalism which has already been shown to be successful in a full quantum theory exhibiting weak-strong duality [1b]. There seems to exist some deeper significance in this definition *vis-a-vis* quantum field theory. This aspect requires further investigation.

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