ON THE EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A FOURTH-ORDER SEMILINEAR ELLIPTIC PROBLEM

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By means of Minimax theory, we study the existence of one nontrivial solution and multiple nontrivial solutions for a fourth-order semilinear elliptic problem with Navier boundary conditions.

1. Introduction

Let us consider the problem

$$\begin{aligned} &\triangle^2 u + c\triangle u = f(x,u), \quad x \in \Omega, \\ &u = 0, \quad \triangle u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{P}$$

where, $\triangle^2$ is the biharmonic operator and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. This fourth-order semilinear elliptic problem can be considered as an analogue of a class of second order problems which have been studied by many authors. In [3], there was a survey of results obtained in this direction.

Known results about (P) were concerned with the case $c < \lambda_1$ the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$. In [8], the author proved the existence of a negative solution of (P) by a degree theory with $f(x,u) = b[(u+1)^+ - 1]$. [4] showed that there existed multiple solutions for $f(x,u) = bg(x,u)$ by using variational approach. Our recent work obtained a positive solution and a negative solution of (P) by Mountain Pass Theorem, and one more nontrivial solution by Morse theory. It is natural to ask what additional phenomena if c goes beyond λ_1. In [5], the author considered the problem (P) with $f(x,u) = bg(x,u)$, and got two solutions by using a "variation of linking" theorem under certain conditions. In the present work, we study the problem (P) with $c \geq \lambda_1$ by using variational approach.

In Section 2, we prove the existence of one nontrivial solution by Linking Theorem including the Saddle Point Theorem, whether c is one of the eigenvalues λ_k of $(-\triangle, H_0^1(\Omega))$ or not. In Section 3, we obtain two nontrivial solutions by using a "variation of linking" theorem. Section 4 is devoted to prove the multiplicity of nontrivial solutions, by using the pseudo-index theory. Of course, our results are still valid for second-order semilinear elliptic problem under weaker conditions.

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2. The existence of one nontrivial solution

Let $H := H^2 \bigcap H_0^1(\Omega)$. Denote $0 < \lambda_1 < \lambda_2 \leq \cdots$ to be the eigenvalues of $-\triangle$ in $H_0^1(\Omega)$, and each eigenvalue is repeated according to its multiplicity. Let e_k be the eigenfunction corresponding to λ_k orthogonal in $L^2(\Omega)$, we can choose $e_1 > 0$ in Ω. Set $\Lambda_k = \lambda_k(\lambda_k - c)$. If $c \geq \lambda_1$, define

$$\begin{aligned} (u,v)_H &= \int_\Omega \triangle u \triangle v + \nabla u \nabla v, \\ J_c(u) &= \frac{1}{2}\int_\Omega (|\triangle u|^2 - c|\nabla u|^2) - \int_\Omega F(x,u), \end{aligned} \tag{2.1}$$

where, $F(x,t) = \int_0^t f(x,s)ds$. If $c < \lambda_1$, then $(\int_\Omega (|\triangle u|^2 - c|\nabla u|^2)dx)^{1/2}$ can be taken as a norm on H, one can use the Mountain Pass theorem to establish the existence of a weak solution of (P) (and even a positive solution). However, if $c \geq \lambda_1$, our previous mechanism fails, we will apply the Linking Theorem to obtain the weak solution of (P).

Linking Theorem [9, Theorem 2.12]. *Let $X = Y \bigoplus Z$ be a Banach space with* $\dim Y < \infty$. *Let $\rho > r > 0$ and $z \in Z$ such that $\|z\| = r$. Define*

$$\begin{aligned} M &= \{u = y + \lambda z : \|u\| \leq \rho,\ \lambda > 0,\ y \in Y\}, \\ M_0 &= \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0 \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\}, \\ N &= \{u \in Z : \|u\| = r\}. \end{aligned} \tag{2.2}$$

Let $J \in C^1(X,\mathbb{R})$ be such that

$$b := \inf_N J > a := \max_{M_0} J. \tag{2.3}$$

If J satisfies the (PS) condition,then J has a critical point whose critical value not smaller than b.

Assume $\lambda_n < c < \lambda_{n+1}$, $n \geq 1$. let

$$Y := \operatorname{span}\{e_1,\dots,e_n\}, \qquad Z := \{u \in H : (u,v) = 0,\ \forall v \in Y\}. \tag{2.4}$$

Since $c < \lambda_{n+1}$, $\|w\|^2 = \int_\Omega (|\triangle w|^2 - c|\nabla w|^2)$ and $\|w\|_H^2$ are norms equivalent in the space Z, denote $\|w\|^2 = \|w\|_H^2$ for convenience. Then $H = Y \bigoplus Z$.

The conditions imposed on $f(x,t)$ are as follows:

(f_1) *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and for some $1 < p < 2^* = (2N/N-4)$, $c_0 > 0$,*

$$|f(x,u)| \leq c_0(1 + |u|^{p-1}). \tag{2.5}$$

(f_2) *There exists $\alpha > 2$, for $|u| \gg 1$,*

$$0 < \alpha F(x,u) \leq u f(x,u). \tag{2.6}$$

(f_3) *$f(x,u) = o(|u|)$, $|u| \to 0$ uniformly on Ω.*

(f_4) $(\Lambda_n/2)u^2 \le F(x,u) = \int_0^u f(x,t)dt$.

LEMMA 2.1. *Under* (f_1)-(f_2), *any sequence* $(u_n) \subset H$ *such that*

$$d := \sup J_c(u_n) < \infty, \qquad J_c'(u_n) \longrightarrow 0, \tag{2.7}$$

contains a convergent subsequence.

Proof. First of all, we observe that

$$\nabla J_c(u) = u + i^*\big((1+c)\triangle u - f(x,u)\big), \tag{2.8}$$

where, $i^* : L^2(\Omega) \to H$ is a compact operator (i^* is the adjoint of the immersion $i : H \hookrightarrow L^2(\Omega)$).

It is enough to prove that $(\|u_n\|)_{n\in N}$ is bounded, because of (2.8) and (f_1). We consider the case $N \ge 5$. Form (f_2), we obtain the existence $c_1 > 0$ such that

$$c_1\big(|u|^\alpha - 1\big) \le F(x,u). \tag{2.9}$$

Let $\beta \in (\alpha^{-1}, 2^{-1})$, for n large enough and $c_2, c_3 > 0$, we have

$$\begin{aligned} d + 1 + ||u_n||_H &\ge J_c(u_n) - \beta\langle J_c'(u_n,u_n)\rangle \\ &= \int_\Omega \Big[\Big(\frac{1}{2}-\beta\Big)\Big(|\triangle u_n|^2 - c|\nabla u_n|^2\Big) + \beta f(x,u_n)u_n - F(x,u_n)\Big]dx \\ &\ge \Big(\frac{1}{2}-\beta\Big)\Big(||z_n||_H^2 + \Lambda_1|y_n|_2^2\Big) + (\alpha\beta - 1)\int_\Omega F(x,u_n)dx - c_2 \\ &\ge \Big(\frac{1}{2}-\beta\Big)\Big(||z_n||_H^2 + \Lambda_1|y_n|_2^2\Big) + c_1(\alpha\beta-1)|u_n|_\alpha^\alpha - c_3, \end{aligned} \tag{2.10}$$

where, $u_n = y_n + z_n$, $y_n \in Y$, $z_n \in Z$. It is easy to verify that (u_n) is bounded in H using the fact that $\dim Y$ is finite.

A standard argument shows that $\{u_n\}$ has a convergent subsequence in H. Therefore, J satisfies the (PS) condition. □

THEOREM 2.2. *Assume* (f_1)–(f_4), *then problem (P) has at least one nontrivial solution.*

Proof. (1) We consider the case $N \ge 5$. We will verify the assumptions of the Linking Theorem. The (PS) condition follows form the preceding Lemma 2.1.

(2) By (f_1) (f_2), we have

$$\forall \varepsilon > 0,\ \exists c_\varepsilon > 0 \text{ such that } |F(x,u)| \le \varepsilon|u|^2 + c_\varepsilon|u|^p. \tag{2.11}$$

On Z, we obtain

$$J_c(u) \ge \frac{1}{2}\|u\|_H^2 - \int_\Omega \big(\varepsilon|u|^2 + c_\varepsilon|u|^p\big) = \frac{1}{2}\|u\|_H^2 - \varepsilon|u|_2^2 - c_\varepsilon|u|_p^p. \tag{2.12}$$

By Sobolev imbedding theorem, there exists $r > 0$, such that

$$b := \inf_{\|u\|_H = r} J_c(u) > 0, \quad u \in Z. \tag{2.13}$$

(3) By (f_4), on Y we have

$$J_c(u) \leq \int_\Omega \left[\frac{1}{2}\Lambda_n u^2 - F(x,u)\right] \leq 0. \tag{2.14}$$

Define $z := re_{n+1}/\|e_{n+1}\|_H$. It follows (2.9), for $u = y + \lambda z$ with $\lambda > 0$, we deduce

$$\begin{aligned} J_c(u) &= \frac{1}{2}\int_\Omega \left[|\triangle(y+\lambda z)|^2 - c|\nabla(y+\lambda z)|^2\right] - \int_\Omega F(x,u) \\ &= \frac{1}{2}\int_\Omega (|\triangle y|^2 - c|\nabla y|^2) + \frac{\lambda}{2}\int_\Omega (|\triangle z|^2 - c|\nabla z|) - \int_\Omega F(x,u) \\ &\leq \frac{1}{2}\Lambda_n \int_\Omega y^2 + \frac{\lambda}{2}r - c_1|u|_\alpha^\alpha + c_1|\Omega|. \end{aligned} \tag{2.15}$$

Since on the finite dimensional space $Y \bigoplus \mathbb{R}z$ all norms are equivalent, then we get

$$J_c(u) \longrightarrow -\infty, \quad \|u\|_H \longrightarrow \infty, \quad u \in Y \bigoplus \mathbb{R}z. \tag{2.16}$$

Thus, there exists $\rho > r$ such that

$$\max_{M_0} J_c = 0, \tag{2.17}$$

where M_0 is as above. By the Linking Theorem, there exists a critical point u of J satisfying $J_c(u) \geq b > 0$. Since $J_c(0) = 0$, then u is a nontrivial solution of (P). □

Remark 2.3. If $c < \lambda_1$, it suffices to use the Mountain Pass Theorem [6, Theorem 2.2] instead of Linking Theorem.

Theorem 2.4. *Under* (f_1)*–*(f_3) *with* $c < \lambda_1$, *(P) has a nontrivial solution.*

Proof. Please see [6, Theorem 2.15] for its proof in detail, where $E = H^2 \bigcap H_0^1(\Omega)$, $(u,v)_E = \int_\Omega (\triangle u \triangle v - c\nabla u \nabla v)$ and $I_c(u) = (1/2)\int_\Omega (|\triangle u|^2 - c|\nabla u|^2) - \int_\Omega F(x,u)$. □

Similarly, we can obtain the following corollary.

Corollary 2.5. *Under* (f_1)*–*(f_3) *and*

$(f_4)'$ $f(x,t)t \geq 0$ *for all* $t \in \mathbb{R}$,

problem (P) has a positive solution and a negative solution.

Proof. By the truncation technique and the Mountain Pass theorem, the problem

$$\begin{aligned} &\triangle^2 u + c\triangle u = \bar{f}(x,u), \quad x \in \Omega, \\ &u = 0, \quad \triangle u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{$\bar{P}$}$$

where

$$\bar{f}(x,u) = \begin{cases} f(x,u), & u \geq 0, \\ 0, & u < 0, \end{cases} \tag{2.18}$$

has a solution $u \not\equiv 0$ satisfying

$$\int_\Omega \triangle u \triangle v - c\nabla u \nabla v = \int_\Omega \bar{f}(x,u)v, \quad \forall v \in H. \tag{2.19}$$

Let $A = \{x \in \Omega \mid u(x) < 0\}$, then by the definition of $\bar{f}$,

$$\begin{aligned} &\triangle^2 u + c\triangle u = 0, \quad x \in A, \\ &u = 0, \quad \triangle u = 0, \quad x \in \partial A. \end{aligned} \tag{2.20}$$

By the maximum principle, we have $u \equiv 0$ in A, therefore $A = \emptyset$. Thus $u \geq 0$ a.e. on Ω.

From $(\mathrm{f}_4)'$, further using the strong maximum principle [2], we deduce $u > 0$, that is, u is the positive solution of (P). □

While for $\lambda_1 < c \in (\lambda_n, \lambda_{n+1})$, by the Linking Theorem, we have the following theorem.

Theorem 2.6. *Under* (f_1)–$(\mathrm{f}_4)'$, *problem (P) has at least a nontrivial solution.*

Proof. Condition $(\mathrm{f}_4)'$ is stronger than (f_4), which is also applied to show that, for $u \in Y$

$$J_c(u) = \frac{1}{2}\int_\Omega \left[|\triangle u|^2 - c|\nabla u|^2\right] - \int_\Omega F(x,u) \leq \frac{1}{2}\Lambda_n \int_\Omega u^2 - \int_\Omega F(x,u) \leq 0. \tag{2.21}$$

As the similar proof of Theorem 2.2, we obtain the result. □

Remark 2.7. In Corollary 2.5, we obtain a positive solution of (P) by using truncation technique, if $c < \lambda_1$. However, if $c \geq \lambda_1$, we cannot expect a positive solution of (P). Indeed, if v_1 is the eigenfunction corresponding to λ_1, we can assume $v_1 > 0$ in Ω. Therefore, if u is a solution of (P), we get

$$\int_\Omega f(x,u)v_1 = \int_\Omega (\triangle^2 u + c\triangle u)v_1 = \int_\Omega (\triangle^2 v_1 + c\triangle v_1)u = \Lambda_1 \int_\Omega v_1 u. \tag{2.22}$$

If u is positive in Ω, the left-hand side of (2.22) is nonnegative by $(\mathrm{f}_4)'$, while the right-hand side is nonpositive, since $c \geq \lambda_1$. Thus, there can only be a positive solution $u(x)$ if $c = \lambda_1$, and $p(x,u(x)) \equiv 0$.

If $c = \lambda_k < \lambda_{k+1}$, we can apply the Saddle Point Theorem to obtain a nontrivial solution of (P).

Saddle Point Theorem [6, Theorem 4.6]. *Let $E = V \bigoplus X$, where E is a real Banach space and $V \not\equiv \{0\}$ is finite dimensional. Suppose $J \in C^1(E,\mathbb{R})$ satisfies (PS) condition, and*

- (I_1) *there is a constant α and a bounded neighborhood D of 0 in V, such that $J|_{\partial D} \le \alpha$,*
- (I_2) *there is a constant $\beta > \alpha$ such that $J|_X \ge \beta$.*

Then J possessed a critical point whose critical value $c \ge \beta$.

Theorem 2.8. *Under the following conditions*

- (i) *$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and for some $C > 0$, $|f(x,t)| \le C$,*
- (ii) *$F(x,\xi) = \int_0^\xi f(x,t)dt \to \infty$ as $|\xi| \to \infty$ uniformly for $x \in \Omega$,*

problem (P) possesses a nontrivial solution.

Proof. Since (i), J_c is of C^1. Let $V := \operatorname{span}\{e_1,\dots,e_k\}$, and $X := \overline{\operatorname{span}\{e_j \mid j \ge k+1\}}$, so $X = V^\perp$. Therefore $H = V \bigoplus X$. We will show that J_c satisfies (i) (ii) and (PS) condition. Then our result follows from the Saddle Point Theorem.

By (i), let $M := \sup_{x\in\overline{\Omega},\xi\in\mathbb{R}} |f(x,\xi)|$, then

$$\left| \int_\Omega F(x,u)dx \right| \le M \int_\Omega |u|dx \le M_1 \|u\|_H, \tag{2.23}$$

for all $u \in H$ via the Hölder and Poincaré inequality. On X, the norms $\|u\|^2 = \int_\Omega (|\triangle u|^2 - c|\nabla u|^2)$ and $\|u\|_H^2$ are equivalent, we have

$$J_c(u) = \frac{1}{2}\int_\Omega [|\triangle u|^2 - c|\nabla u|^2] - \int_\Omega F(x,u) \ge c_1 \|u\|_H^2 - M_1 \|u\|_H, \quad c_1 > 0, \tag{2.24}$$

which shows J_c is bounded from below on X, that is, (I_2) holds.

Next, if $u \in V$, then $u = u^0 + u^-$, where $u^0 \in E^0 := \operatorname{span}\{e_j \mid \lambda_j = c\}$, and $u^- \in E^- := \operatorname{span}\{e_j \mid \lambda_j < c\}$. Then

$$J_c(u) = \frac{1}{2}\int_\Omega \left(|\triangle u^-|^2 - c|\nabla u^-|^2 \right) - \int_\Omega F(x,u^0) - \int_\Omega (F(x,u^0 + u^-) - F(x,u^0)). \tag{2.25}$$

Estimating the last term as in (2.23), since all norms are equivalent on the finite dimensional subspace E^-, we have

$$J_c(u) \le -M_2 ||u^-||_H^2 - \int_\Omega F(x,u^0) + M_1 ||u^-||_H. \tag{2.26}$$

Now, (2.26) and (ii) show $J_c(u) \to -\infty$ as $u \to \infty$ in V. Hence J_c satisfies (I_1).

Lastly to verify (PS) condition, it suffices to show that $|J_c(u_m)| \le K$ and $J_c'(u_m) \to 0$ implies (u_m) is bounded, since (i) and (2.8). Writing $u_m = u_m^0 + u_m^- + u_m^+$, where $u_m^0 \in E^0$, $u_m^- \in E^-$, $u_m^+ \in X$. For large m,

$$|J_c'(u_m)u_m^{\pm}| = \left| \int_\Omega [\triangle u_m \triangle u_m^{\pm} - c\nabla u_m \nabla u_m^{\pm} - f(x,u_m)u_m^{\pm}]dx \right| \le ||u_m^{\pm}||_H. \tag{2.27}$$

Consequently, since $X = V^\perp$, by (2.27) and an estimate like (2.23), we get

$$||u_m^+||_H \ge ||u_m^+||_H^2 - M_1||u_m^+||_H. \tag{2.28}$$

which shows that $\{\|u_m^+\|_H\}$ is bounded. Similarly $\{\|u_m^-\|_H\}$ is bounded. Finally we claim that $\{\|u_m^0\|_H\}$ is bounded. Then (u_m) is bounded in H and we are through. Indeed,

$$\begin{aligned} K \ge |J_c(u_m)| = \Bigg| \int_\Omega \bigg\{ \frac{1}{2}\Big[|\triangle u_m^+|^2 + |\triangle u_m^-|^2 - c|\nabla u_m^+|^2 - c|\nabla u_m^-|^2 \Big] \\ - (F(x,u_m) - F(x,u_m^0)) \bigg\} dx - \int_\Omega F(x,u_m^0)dx \Bigg|. \end{aligned} \tag{2.29}$$

By what has already been shown, the first term on the right is bounded independently of m. Therefore

$$K \ge \left| \int_\Omega F(x,u_m^0) \right| - K_1, \tag{2.30}$$

so $(\int_\Omega F(x,u_m^0)dx)$ is bounded, which implies (u_m^0) is bounded as the proof of Lemma 4.21 [6]. □

Remark 2.9. If (ii) is replaced by $F(x,\xi) \to -\infty$ as $|\xi| \to \infty$, the above arguments can easily be modified to handle this case.

3. The existence of two nontrivial solution

By using the following "a variation of linking" theorem, we can obtain at least two solutions of (P).

THEOREM 3.1 ("a variation of linking") [7, Corollary 2.4]. *Let N be a subspace of a Hilbert space H, such that $0 < \dim N < \infty$, and $M = N^\perp$. Assume J is a continuously differentiable functional on H, which satisfies for some $\alpha < \beta$, $0 < \delta < R$ and $w_0 \in M\backslash\{0\}$,*

$$\begin{gathered} J(v) \le \alpha, \quad v \in N, \quad \|v\| \le R, \\ J(sw_0 + v) \le \alpha, \quad s > 0,\ v \in N, \quad ||sw_0 + v|| = R, \\ J(w) \ge \beta, \quad w \in H, \quad \|w\| = \delta. \end{gathered} \tag{3.1}$$

If J satisfies the (PS) condition, then there are at least two solutions of $J'(u) = 0$, one satisfies $J(u) \le \alpha$ and the other $J(u) \ge \beta$.

THEOREM 3.2. *Assume* $c \in (\lambda_{l-1}, \lambda_l)$ *with* $l \geq 2$, *under the conditions* (f_1) (f_2) *and*

(f_5) $F(x,t) = \int_0^t f(x,s)ds$ *satisfies*

$$\frac{1}{2}\Lambda_{l-1}t^2 - w_0(x) \leq F(x,t) \leq \frac{1}{2}\nu_1 t^2 + V(x)^p|t|^p + w_1(x), \tag{3.2}$$

where $\nu_1 < \Lambda_l$, $p > 2$.

$$\begin{gathered} B_j := \int_\Omega w_j(x)dx < \infty, \quad j = 0,1, \\ |Vu|_p^p \leq C\|u\|_H^p, \quad u \in H, \end{gathered} \tag{3.3}$$

(f_6) *the following inequality holds:*

$$B_0 + B_1 < \frac{1}{2}\left(1 - \frac{2}{p}\right)\left(1 - \frac{\nu_1}{\Lambda_l}\right)^{p/(p-2)}\left(\frac{1}{pC}\right)^{2/(p-2)}, \tag{3.4}$$

problem (P) has at least two nontrivial solutions.

Proof. Note that above conditions allow $f(x,0) \not\equiv 0$.

Under (f_1), it is readily checked that the functional J_c is of C^1. Let N be the subspace spanned by the eigenfunctions corresponding to the eigenvalues $\Lambda_1, \ldots, \Lambda_{l-1}$, and let $M = N^\perp \bigcap H$, the orthogonal complement of N in H. On M we have by (f_5),

$$\begin{aligned} J_c(w) = \frac{1}{2}\|w\|_H^2 - \int_\Omega F(x,u) &\geq \frac{1}{2}\|w\|_H^2 - \frac{\nu_1}{2}|w|_2^2 - |Vw|_p^p - B_1 \\ &\geq \frac{1}{2}\left(1 - \frac{\nu_1}{\Lambda_l}\right)\|w\|_H^2 - C\|w\|_H^p - B_1. \end{aligned} \tag{3.5}$$

If we take $\delta^{p-2} = (1 - (\nu_1/\Lambda_l))/pC$, we get

$$\begin{gathered} J_c(w) \geq \frac{1}{2}\left(1 - \frac{2}{p}\right)\left(1 - \frac{\nu_1}{\Lambda_l}\right)\delta^2 - B_1 = \frac{1}{2}\left(1 - \frac{2}{p}\right)\left(1 - \frac{\nu_1}{\Lambda_l}\right)^{p/(p-2)}\left(\frac{1}{pC}\right)^{2/(p-2)} - B_1, \\ \|w\|_H = \delta, \quad w \in M. \end{gathered} \tag{3.6}$$

On the other hand, on N we have by (f_5),

$$J_c(v) \leq \frac{1}{2}\int_\Omega (|\triangle v|^2 - c|\nabla v|^2) - \frac{1}{2}\Lambda_{l-1}\int_\Omega v^2 + B_0 \leq B_0, \quad v \in N. \tag{3.7}$$

Let w_0 be an eigenfunction corresponding to the eigenvalue λ_l with unit norm, and let N_1

denote the subspace spanned by N and w_0. On N_1 we have by (f_2),

$$J_c(u) \le \frac{1}{2}\int_\Omega \Big[|\triangle(v+sw_0)|^2 - c|\nabla(v+sw_0)|^2\Big] - \int_\Omega F(x,u) \\ \le \frac{1}{2}\Lambda_{l-1}\int_\Omega v^2 + \frac{s}{2} - c_1|u|_\alpha^\alpha + c_1|\Omega|, \tag{3.8}$$

where $v \in N$, $s > 0$.

In particular, we see that

$$J_c(u) \longrightarrow -\infty \text{ as } \|u\|_H \longrightarrow \infty, \quad u \in N_1, \tag{3.9}$$

since all norms are equivalent on the finite dimensional space N_1.

Take R so large that $R > \delta$ and

$$J_c(u) \le B_0, \quad \|u\| \ge R, \quad u \in N_1. \tag{3.10}$$

If β denote the right-hand of (3.5), we see that $B_0 < \beta$ by (f_6). Under (f_2), Lemma 2.1 has shown that J_c satisfies the (PS) condition, then our results will follow from Theorem 3.1, that is, J_c has at least two nontrivial critical points, since $J_c(0) \ne 0$. □

Remark 3.3. Indeed, if $f(x,0) \ne 0$ in Theorems 2.2, 2.6, and 2.8, we also can obtain at least two nontrivial solutions under certain conditions.

4. The existence of multiple solutions

In this section, we will prove an existence result of multiple solutions by using pseudo-index. We first recall the definition of genus and an abstract theorem of [1].

Let E be a Banach space, $J \in C^1(E,\mathbb{R})$ satisfy $J(-u) = J(u)$ for all $u \in E$. Denote Σ to be all the symmetrical and closed sets in E, and $\mathbb{Z}_+$ the set of nonnegative integer. Define

$$\gamma(A) := \inf\{n \in \mathbb{Z}_+ : \text{there is a continuous and odd map } \varphi : A \longrightarrow \mathbb{R}^n\setminus\{0\}\}. \tag{4.1}$$

If for all $n \in \mathbb{N}$, there is no such φ, set $\gamma(A) = +\infty$, while $A = \emptyset$, set $\gamma(A) = 0$. Genus γ has the following properties:

Proposition 4.1 [9, Theorem 3.2 IV]. *The following conditions hold.*

(1^o) *If $E = X_1 \bigoplus X_2$, $\dim X_1 = k$, $\gamma(A) > k$, then $A\bigcap X_2 \ne \emptyset$.*

(2^o) *If Ω is a symmetrical and bounded neighborhood of 0 in $\mathbb{R}^m$, and there exists a mapping $h \in C(A,\partial\Omega)$ with h an odd homeomorphism, then $\gamma(A) = m$, for $A \in \Sigma$.*

(3^o) *If $\gamma(A) = k$, $0 \notin A$, then there exist at least k distinct pairs of points in A.*

Now, we define the pseudo-index i^* by γ,

$$i^*(A) = \inf_{h\in\Lambda_*(\rho)} \gamma\Big(A\bigcap h(\partial B_1)\Big), \tag{4.2}$$

where $A \in \Sigma^* = \{A \in \Sigma : A$ is compact$\}$ and $\Lambda_*(\rho) = \{h \in C(E,E) : h$ is an odd homeomorphism, for some $\rho > 0$, $h(B_1) \subset J^{-1}(0,\infty)\bigcup B_\rho\}$.

THEOREM 4.2 [1, Theorem 3.6 IV]. *Let E be a Banach space, $J \in C^1(E,\mathbb{R})$ satisfy $J(-u) = J(u)$ for all $u \in E$. Assume*

(I) *there exist ρ, $\alpha_0 > 0$ and a subspace $E_1 \subset E$ with $\dim E_1 = m_1$, such that*

$$J|_{E_1{}^{\perp}\bigcap B_\rho} \geq \alpha_0, \tag{4.3}$$

(II) *there exists a subspace $E_2 \subset E$ with $\dim E_2 = m_2 > m_1$, and $R > 0$ such that $J(u) \leq 0$ for all $u \in E_2 \backslash B_R$,*

and J satisfies (PS) condition, then J has at least $m_2 - m_2$ distinct pairs of critical points with critical value

$$c_n^* = \inf_{i^*(A)\geq n} \sup_{u\in A} J(u). \tag{4.4}$$

THEOREM 4.3. *Assume (f_1), (f_2), (f_3), and*

(f_4) *$f(x,t)$ is odd in t.*

If $\lambda_j < c < \lambda_{j+1}$, then (P) has infinitely many nontrivial solutions.

Proof. Let $E = H := H^2 \bigcap H_0^1(\Omega)$, first we will prove (I) (II) of Theorem 4.2 are satisfied under the conditions of Theorem 4.3.

(I) Let $E_1 := Y_j = \operatorname{span}\{e_1,\dots,e_j\}$, the similar proof of Theorem 2.2 (2), for all $u \in Y_j^{\perp}$,

$$J_c(u) \geq \frac{1}{2}\|u\|_H^2 - \varepsilon|u|_2^2 - c_\varepsilon|u|_P^P, \tag{4.5}$$

thus, there exist ρ, $\alpha_0 > 0$ small enough, such that

$$J_c(u) \geq \alpha_0, \quad \forall u \in Y_j^{\perp}, \quad \|u\|_H = \rho. \tag{4.6}$$

(II) For $m \geq 1$ fixed, since all norms are equivalent on the finite dimensional space Y_{j+m}, by (2.9) there exists a sufficiently big constant $R > \rho$, such that

$$\begin{aligned} J_c(u) &\leq \frac{1}{2}\Lambda_{j+m}\int_\Omega u^2 - c_1|u|_\alpha^\alpha + c_1|\Omega| \\ &= c_2\|u\|_H^2 - c_1\|u\|_H^\alpha + c_1|\Omega| < 0, \quad u \in Y_{j+m}\backslash B_R. \end{aligned} \tag{4.7}$$

Next, by the properties of genus and the definition of c_n^*, we have

$$\alpha_0 \leq c_{j+s}^* < +\infty, \qquad m \geq s \geq 1. \tag{4.8}$$

Indeed, for all $A \in \Sigma^*$ satisfying $i^*(A) \geq j+s$, let $h_0 = \rho \cdot id$, then $h_0 \in \Lambda_*(\rho)$ and

$$\gamma\Big(A\bigcap\partial B_\rho\Big) = \gamma\Big(A\bigcap h_0(\partial B_1)\Big) \geq \inf_{h\in\Lambda_*(\rho)}\gamma\Big(A\bigcap h(\partial B_1)\Big) = i^*(A) > j. \tag{4.9}$$

By (1^o) of Proposition 4.1, $A \bigcap \partial B_\rho \bigcap Y_j^\perp \neq \emptyset$, then (4.6) implies

$$\sup_{u \in \Lambda} J_c(u) \geq \inf_{u \in \partial B_\rho \bigcap Y_j^\perp} J_c(u) \geq \alpha_0. \tag{4.10}$$

Since $A \in \Sigma^*$ is arbitrary, then $c_{j+s}^* \geq \alpha_0$.

As the proof of Theorem 3.6 IV [1], $c_{j+s}^* < +\infty$, since $j+s \leq \dim Y_{j+m}$.

Thus, we have

$$\alpha_0 \leq c_{j+1}^* \leq c_{j+2^*} \leq \cdots \leq c_{j+m}^* < +\infty, \tag{4.11}$$

and the (PS) condition is obtained by Lemma 2.1. Therefore, Theorem 4.2 implies that J_c admits at least m distinct pairs of critical points. Since m is arbitrary and $\lambda_j \to +\infty$ as $j \to \infty$, then (P) has infinitely many nontrivial solutions. □

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