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Research Article

Γ-Semihypergroups and Regular Relations

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In this paper we discuss the structure of Γ-semihypergroups. We prove some basic results and present several examples of Γ-semihypergroups. Also, we obtain some properties of regular and strongly regular relations on a Γ-semihypergroup and construct a Γ-semigroup from a Γ-semihypergroup by using the notion of fundamental relation.

1. Introduction

The algebraic hyperstructure notion was introduced in 1934 by the French mathematician Marty [1], at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, and noncommutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

Around the 1940s, the general aspects of the theory, the connections with groups, and various applications in geometry were studied. The theory knew an important progress starting with the 1970s, when its research area enlarged. A recent book on hyperstructures [2] points out to their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs, and hypergraphs. Many authors studied different aspects of semihypergroups, for instance, Bonansinga and Corsini [3], Corsini [4], Davvaz [5], Davvaz and Poursalavati [6], Fasino and Freni [7], Guţan [8], Hasankhani [9], Leoreanu [10], and Onipchuk [11].

In 1986, Sen and Saha [12] defined the notion of a Γ-semigroup as a generalization of a semigroup. One can see that Γ-semigroups are generalizations of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [13, 14], Hila [15], Saha [16], Sen et al. [12, 17–20], Seth [21], and Sardar et al. [22]. In [23–25], Anvariyeh et al. introduced the notion of a Γ-semihypergroup as a generalization of a semihypergroup. Many classical notions of semigroups and semihypergroups have been extended to Γ-semihypergroups and a lot of results on Γ-semihypergroups are obtained.

2. Basic Definitions

In this section, we recall certain definitions and results needed for our purpose.

Let A and B be two nonempty sets, M the set of all mapping from A to B, and Γ a set of some mappings from B to A. The usual composition of two elements of M cannot be defined. But if we take f, g from M and α from Γ, then the usual mapping composition $f\alpha g$ can be defined. Also, we see that $f\alpha g \in M$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for $f, g, h \in M$ and $\alpha, \beta \in \Gamma$. Sen and Saha [12] defined the notion of a Γ-semigroup as a generalization of a semigroup as follows.

Definition 1. Let $M = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. Then M is called a Γ-*semigroup* if there exists a mapping $M \times \Gamma \times M \rightarrow M$ written as $(a, \gamma, b) \mapsto a\gamma b$ satisfying the following identity:

$$(a\alpha b)\,\beta c = a\alpha\,(b\beta c) \quad \forall a, b, c \in M,\ \forall \alpha, \beta \in \Gamma. \tag{1}$$

Let K be a nonempty subset of M. Then K is called a *sub-Γ-semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Example 2. If M is the set of $m \times n$ matrices and Γ is a set of some $n \times m$ matrices over the field of real numbers, then we can define $A_{m,n}\alpha_{n,m}B_{m,n}$ such that

$$\left(A_{m,n}\alpha_{n,m}B_{m,n}\right)\beta_{n,m}C_{m,n} = A_{m,n}\alpha_{n,m}\left(B_{m,n}\beta_{n,m}C_{m,n}\right), \quad (2)$$

where $A_{m,n}, B_{m,n}, C_{m,n} \in M$ and $\alpha_{n,m}, \beta_{n,m} \in \Gamma$. An algebraic system that satisfying the associativity property of the above type is a Γ-semigroup.

Example 3. Let $M = [0,1]$ and $\Gamma = \{1/n \mid n \text{ is a positive integer}\}$. Then M is a gamma-semigroup under the usual multiplication. Next, let $K = [0,1]$. We have K as a nonempty subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub-Γ-semigroup of M.

More examples of Γ-semigroup can be found in [12, 16].

Let H be a nonempty set and let $\wp^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \to \wp^*(H)$ and the couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H, then we denote $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

Definition 4. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H one has $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v. \quad (3)$$

A semihypergroup $(H, \circ)$ is called a *hypergroup* if for all $x \in H$, one has

$$x \circ H = H \circ x = H. \quad (4)$$

3. γ-Semihypergroups

In this section, we introduce the notion of a Γ-semihypergroup which is a generalization of the notion of Γ-semigroup and semihypergroup.

Definition 5. Let S and Γ be two non-empty sets. S is called a Γ-semihypergroup if every $\gamma \in \Gamma$ be a hyperoperation on S, that is, $x\gamma y \subseteq S$ for every $x, y \in S$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$ one has $x\alpha(y\beta z) = (x\alpha y)\beta z$.

If every $\gamma \in \Gamma$ is an operation, then S is a Γ-semigroup.

If (S, γ) is a hypergroup for every $\gamma \in \Gamma$, then S is called a Γ-*hypergroup*.

Let A and B be two non-empty subsets of S and $\gamma \in \Gamma$, we define the following:

$$A\gamma B = \cup \{a\gamma b \mid a \in A, b \in B\}. \quad (5)$$

Also

$$A\Gamma B = \cup \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\} = \bigcup_{\gamma\in\Gamma} A\gamma B. \quad (6)$$

A nonempty subset A of S is called a *sub-*Γ-*semihypergroup* of S if $A\Gamma A \subseteq A$.

A is called a *right (left) hyperideal* of S if $A\Gamma S \subseteq A$ ($S\Gamma A \subseteq A$), and is called a *hyperideal* of S if it is both a right and a left hyperideal. Clearly, every right (left) hyperideal is a sub-Γ-semihypergroup.

A Γ-semihypergroup S is called *commutative* if for all $x, y \in S$ and $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$.

Example 6. Let $(S, \circ)$ be a semihypergroup and Γ be a nonempty set. We define $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then, S is a Γ-semihypergroup.

Example 7. Let $(S, \circ)$ be a semihypergroup and Γ be a nonempty subset of S. We define $x\gamma y = x \circ \gamma \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ-semihypergroup.

Example 8. Let $S = [0,1]$ and $\Gamma = \mathbb{N}$, for every $x, y \in S$ and $\gamma \in \Gamma$. We define

$$\gamma : S \times S \longrightarrow \mathscr{P}^*(S)$$
$$x\gamma y = \left[0, \frac{xy}{\gamma}\right]. \quad (7)$$

Then, γ is a hyperoperation. For every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ we have

$$(x\alpha y)\beta z = \left[0, \frac{xyz}{\alpha\beta}\right] = x\alpha(y\beta z). \quad (8)$$

This means that S is a Γ-semihypergroup.

Also, if $\gamma \in \Gamma$, then (S, γ) is not a hypergroup, because for every $0 \le t < 1$ we have $t\gamma S = [0, t/\gamma] \ne S$. So S is not a Γ-hypergroup.

Example 9. Let $S = \{0,1\}$ and $\Gamma = \{\alpha, \beta\}$ be defined as follows:

α	0	1
0	0	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

β	0	1
0	$\{0,1\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

(9)

It is not difficult to see that S is a Γ-hypergroup.

Example 10. Let $S = \{0,1\}$ and $\Gamma = \{\beta, \gamma\}$ be defined as follows:

β	0	1
0	$\{0,1\}$	$\{0,1\}$
1	$\{0,1\}$	$\{0,1\}$

γ	0	1
0	0	1
1	1	0

(10)

It is not difficult to see that S is a Γ-hypergroup. Also, (S, β) is a hypergroup and (S, γ) is a group.

Example 11. Let $(G, \circ)$ be a group and $H_1, \ldots, H_k$ be normal subgroups of G such that $H_1 \subseteq H_2 \subseteq \cdots H_k$. Set $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ where for every $1 \leq i \leq k$, the hyperoperation γ_i be as follows:

$$x\gamma_i y = H_i \circ x \circ y, \quad \forall x, y \in G. \tag{11}$$

For every $1 \leq i, j \leq k$ we have

$$\begin{aligned} x\gamma_i \left(y\gamma_j z\right) &= x\gamma_i \left(H_j \circ y \circ z\right) \\ &= H_i \circ x \circ \left(H_j \circ y \circ z\right) \\ &= H_i \circ H_j \circ x \circ y \circ z \\ &= H_{\max\{i,j\}} \circ x \circ y \circ z. \end{aligned} \tag{12}$$

Also, $x\gamma_i(y\gamma_j z) = H_{\max\{i,j\}} \circ x \circ y \circ z$ and so G is a Γ-semihypergroup. It is easy to see that G is a Γ-hypergroup.

Theorem 12. *Let S be a Γ-semihypergroup and exists $\alpha \in \Gamma$ such that (S, α) be a hypergroup. Then, for every $\gamma \in \Gamma$, (S, γ) is a hypergroup.*

Proof. For every $x, z \in S$ and $\gamma \in \Gamma$ there exists $y \in S$ such that

$$z \in (x\gamma x)\alpha y = x\gamma(x\alpha y). \tag{13}$$

Now, there exists $u \in x\alpha y$ such that $z \in x\gamma u$ and so (S, γ) is a hypergroup. □

Remark 13. Let S be a Γ-group and for some $\gamma \in \Gamma$, element e be an identity of semigroup (S, γ). Then, in general case e is not an identity element of (S, β), where $\beta \in \Gamma$. For example, let $S = \{0, 1\}$ and $\Gamma = \{\beta, \gamma\}$ be as follows:

β	0	1
0	1	0
1	0	1

γ	0	1
0	0	1
1	1	0

(14)

0 is an identity element of (S, γ) and 1 is an identity element of (S, γ).

Lemma 14. *Let S be a Γ-semihypergroup and $\alpha, \beta \in \Gamma$. If e be a scalar identity of (S, α) and (S, β), then $\alpha = \beta$.*

Proof. We prove that for every $x, y \in \Gamma$, $x\alpha y = x\beta y$. We have the following:

$$x\alpha y = x\alpha(e\beta y) = (x\alpha e)\beta y = x\beta y. \tag{15}$$

□

Theorem 15. *Let S be a Γ-semihypergroup and e be a scalar identity of S, that is, for every $(x, \gamma) \in S \times \Gamma$, one has $x\gamma e = x = e\gamma x$. Then* card$(\Gamma) = 1$ *and S is a semihypergroup.*

Proof. It is obtained from Lemma 14. □

Example 16. Theorem 15 is not true for identity elements of Γ-semihypergroups. In Example 9, we see that 0 and 1 are identity elements and $\alpha \neq \beta$.

Theorem 17. *Let S be a Γ-semihypergroup and for every $\gamma \in \Gamma$, and semihypergroup (S, γ) has a scalar identity $e_\gamma \in S$. Then, one has* card$(\Gamma) \leq$ card(S).

Proof. By Lemma 14, we have

$$\operatorname{card}(\Gamma) = \operatorname{card}\left(\left\{e_\gamma \mid \gamma \in \Gamma\right\}\right) \leq \operatorname{card}(S). \tag{16}$$

□

In general, the above theorem is not true for a Γ-semihypergroup with identity elements.

Example 18. Let $S = \{0, 1\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ be as follows:

β	0	1
0	1	0
1	0	1

β	0	1
0	$\{0, 1\}$	$\{0, 1\}$
1	$\{0, 1\}$	$\{0, 1\}$

γ	0	1
0	0	1
1	1	0

(17)

In Γ-hypergroup S we have 0 as a scalar identity of (S, α), an identity of (S, β), and 1 is an identity of (S, β) and an scalar identity of (S, γ), but

$$\operatorname{card}(\Gamma) = 3 > \operatorname{card}(S) = 2. \tag{18}$$

Definition 19. Let S be a Γ-semihypergroup and S' be a Γ'-semihypergroup. A map $f : S \to S'$ is called a *homomorphism* if there exists a bijective map $h : \Gamma \to \Gamma'$ such that for every $\gamma \in \Gamma$ and $x, y \in S$:

$$f(x\gamma y) = f(x)h(\gamma)f(y). \tag{19}$$

If we set $h(\gamma) = \gamma'$, then $f(x\gamma y) = f(x)\gamma' f(y)$.

Lemma 20. *Let S be a Γ-semihypergroup, S' be a Γ'-semihypergroup, and $f : S \to S'$ be a homomorphism. Then one has the following.*

(1) *If T is a sub-Γ-semihypergroup of S, then $f(T)$ is a sub-Γ'-semihypergroup of S'.*

(2) *If T' is a sub-Γ'-semihypergroup of S' and $f^{-1}(T') \neq \emptyset$, then $f^{-1}(T')$ is a sub-Γ-semihypergroup of S.*

Proof. It is straightforward. □

4. Regular and Strongly Regular Relations

If S is a Γ-semihypergroup and $R \subseteq S \times S$ is an equivalence relation on S, then for all pairs (A, B) of nonempty subsets of S, we set $A\overline{R}B$ if and only if for every $a \in A$ there exists $b \in B$ such that aRb and for every $b \in B$ there exists $a \in A$ such that aRb. The equivalence relation R is said to be *regular to the right* if xRy implies $x\gamma a\overline{R}y\gamma a$ for all $(x, y, a) \in S^3$ and $\gamma \in \Gamma$. Analogously, we can define *regular to the left*. Moreover, R is called *regular* if it is regular to the right and to the left.

Also, we set $A\overline{\overline{R}}B$ if and only if $a\rho b$ for all $a \in A$ and $b \in B$. The equivalence relation R is said to be *strongly regular to the right* if xRy implies that $x\gamma a\overline{\overline{R}}y\gamma a$ for all $(x, y, a) \in S^3$ and $\gamma \in \Gamma$. Analogously, we can define *strongly regular to the left*. Moreover, R is called *strongly regular* if it is strongly regular to the right and to the left.

Let S be a Γ-semihypergroup and R be an equivalence relation on S. Let $R(a)$ be the equivalence class of a with respect to R and let $S/R = \{R(a) \mid a \in S\}$. For every $\gamma \in \Gamma$ we define γ/R on S/R as follows:

$$R(a)\frac{\gamma}{R}R(b) = \{R(x) \mid x \in a\gamma b\}. \tag{20}$$

Now, we set $\Gamma/R = \{\gamma/R \mid \gamma \in \Gamma\}$.

Theorem 21. *Let S be a Γ-semihypergroup and R be an equivalence relation on S. The following conditions are equivalent:*

(1) *R is a regular relation,*

(2) *S/R is a Γ/R-semihypergroup.*

Proof. ($1 \Rightarrow 2$): We will prove that for every $\gamma \in \Gamma$ the hyperoperation γ/R is welldefined. Let $\gamma \in \Gamma$ and $x, y \in S$. If $x' \in R(x)$ and $y' \in R(y)$, we prove that $R(x')(\gamma/R)R(y') = R(x)(\gamma/R)R(y)$. We have $x'Rx, y'Ry$, and R is regular so $x\gamma y\overline{R}x\gamma y'$ and $x\gamma y'\overline{R}x'\gamma y'$. Therefore, for every $w \in x\gamma y$, there exists $u \in x\gamma y'$ such that wRu and for every $v \in x\gamma y'$, there exists $z \in x'\gamma y'$ such that zRv. Thus, for every $w \in x\gamma y$ there exists $v \in x'\gamma y'$ such that wRz and $R(w) = R(z)$. Therefore, for every $R(w) \in R(x')(\gamma/R)R(y')$ there exists $z \in x\gamma y$ such that $R(w) = R(z) \in R(x)(\gamma/R)R(y)$, that is,

$$R(x')\frac{\gamma}{R}R(y') \subseteq R(x)\frac{\gamma}{R}R(y). \tag{21}$$

Also, by a similar way, we have

$$R(x)\frac{\gamma}{R}R(y) \subseteq R(x')\frac{\gamma}{R}R(y'). \tag{22}$$

Now, we see that for every $\alpha/R, \beta/R \in \Gamma/R$ and $R(x)$, $R(y)$, $R(z) \in S/R$,

$$R(x)\frac{\alpha}{R}\left(R(y)\frac{\beta}{R}R(z)\right) = \left(R(x)\frac{\alpha}{R}R(y)\right)\frac{\beta}{R}R(z). \tag{23}$$

Let $R(w) \in R(x)(\alpha/R)(R(y)(\beta/R)R(z))$. Then there exists $R(v) \in R(y)(\beta/R)R(z)$ such that $R(w) \in R(x)(\alpha/R)R(v)$. Hence, we can suppose that $v \in y\beta z$ and $w \in x\alpha(y\beta z) = (x\alpha y)\beta z$. From this it follows that $w \in x\alpha v$, as a consequence $R(w) \in (R(x)(\alpha/R)R(y))(\beta/R)R(z)$. In a similar way, we obtain

$$\left(R(x)\frac{\alpha}{R}R(y)\right)\frac{\beta}{R}R(z) \subseteq R(x)\frac{\alpha}{R}\left(R(y)\frac{\beta}{R}R(z)\right). \tag{24}$$

($2 \Rightarrow 1$): Let xRx' then $R(x) = R(x')$ and so for every $y \in S$, we have $R(x)(\gamma/R)R(y) = R(x')(\gamma/R)R(y)$. Now, let $u \in x\gamma y$. Then $R(u) \in R(x)(\gamma/R)R(y) = R(x')(\gamma/R)R(y)$ and so there exists $v \in x'\gamma y$ such that $R(v) = R(u)$. In the same way, for every $z \in x'\gamma y$ there exists $w \in x\gamma y$ such that $R(w) = R(z)$. Therefore, $x\gamma y\overline{R}x'\gamma y$ and so R is regular. □

Theorem 22. *Let S be a Γ-semihypergroup and R be a regular equivalence relation on S. Then, the canonical projection $\pi : S \to S/R$ is an epimorphism and if S is a Γ-hypergroup, then S/R is a Γ/R-hypergroup.*

Proof. First, we show that π is a homomorphism. Let $R(z) \in \pi(x\gamma y)$, where $\gamma \in \Gamma$. Thus, there exists $z' \in x\gamma y$ such that $R(z) = \pi(z')$, and so $R(z) = R(z') \in \pi(x)(\gamma/R)\pi(y)$, hence $\pi(x\gamma y) \subseteq \pi(x)\,\gamma/R\,\pi(y)$.

Now, let $R(z) \in \pi(x)(\gamma/R)R(y)$ thus there exists $z' \in R(z)$ such that $z' \in x\gamma y$, so $R(z) = \pi(z') \in \pi(x\gamma y)$. Therefore, $\pi(x\gamma y) = \pi(x)(\gamma/R)\pi(y)$. Finally, if S is a Γ-hypergroup, then for all $x \in S$ and $\gamma \in \Gamma$, we have

$$\begin{aligned} \pi(S)\frac{\gamma}{R}\pi(x) &= \pi(S\gamma x) = \pi(S) \\ &= \frac{S}{R} = \pi(S) = \pi(x)\frac{\gamma}{R}\pi(S) = \pi(x\gamma S). \end{aligned} \tag{25}$$

This implies that S/R is a Γ/R-hypergroup. □

Theorem 23. *Let S be a Γ-hypergroup, S' be a Γ'-hypergroup and $f : S \to S'$ be a homomorphism. Then the equivalence relation R associated with f, that is,*

$$xRy \Longleftrightarrow f(x) = f(y), \tag{26}$$

is regular and the function $\psi : f(S) \to S/R$, $\psi(f(x)) = R(x)$ is an isomorphism.

Proof. Let $x, y \in S$ and xRy. Then, for every $a \in S$ and $\gamma \in \Gamma$ we have

$$f(x\gamma a) = f(x)\gamma' f(a) = f(y)\gamma' f(a) = f(y\gamma a). \tag{27}$$

So for every $u \in x\gamma a$, there exists $v \in y\gamma a$ such that $f(u) = f(v)$, this means that uRv. Also, for every $w \in y\gamma a$, there exists $z \in x\gamma a$ such that $f(z) = f(w)$, this means that zRw. Therefore, R is regular to the right. In a similar way, we obtain

R to be regular to the left and so R is regular. Finally, for every $x, y \in S$ and for every $\gamma \in \Gamma$ we have:

$$\begin{aligned}\psi(f(x)\gamma' f(y)) &= \psi(f(x\gamma y)) \\ &= \{R(z) \mid z \in x\gamma y\} \\ &= R(x)\frac{\gamma}{R}R(y) \\ &= \psi\left(f(x)\frac{\gamma}{R}f(y)\right),\end{aligned} \tag{28}$$

$$\begin{aligned}\psi^{-1}\left(R(x)\frac{\gamma}{R}R(y)\right) &= \psi^{-1}(\{R(z) \mid z \in x\gamma y\}) \\ &= \bigcup_{z\in x\gamma y} f(z) \\ &= f(x\gamma y) = f(x)\gamma' f(y) \\ &= \psi^{-1}(R(x))\gamma'\psi^{-1}(R(y)).\end{aligned} \tag{29}$$

Therefore, ψ is an isomorphism. □

Theorem 24. *Let S be a Γ-semihypergroup and R be an equivalence relation on S. The following conditions are equivalent:*

(1) *R is a strongly regular relation,*

(2) *if x_1Ry_1 and x_2Ry_2 then $x_1\gamma x_2\overline{\overline{R}}y_1\gamma y_2$ for every $\gamma \in \Gamma$,*

(3) *S/R is a Γ/R-semigroup.*

Proof. (1 ⇒ 2) and (2 ⇒ 1) are clear.

(1 ⇒ 3): Let $x, y \in S$ and $\gamma \in \Gamma$. Since R is strongly regular, then for every $u, v \in x\gamma y$ we have $R(u) = R(v)$ and so $R(x)(\gamma/R)R(y)$ is singleton. Now, by Theorem 21, the proof is completed.

(3 ⇒ 1): Let xRy, $a \in S$ and $\gamma \in \Gamma$, Since $R(x)(\gamma/R)R(a) = \{R(z) \mid z \in x\gamma a\}$, $R(y)(\gamma/R)R(a) = \{R(t) \mid t \in y\gamma a\}$ are singleton, then for every $z \in x\gamma a$ and $t \in y\gamma a$ we have zRt and so R is strongly regular to the right and the same way implies that R is strongly regular to the left. Therefore, R is strongly regular. □

Theorem 25. *Let S be a Γ-hypergroup. Let $\alpha \in \Gamma$ such that a relation R be a strongly regular on the semihypergroup (S, α) and exists $u \in S$ such that for every $(x, \gamma) \in S \times \Gamma$, $x\gamma u\overline{\overline{R}}x\gamma u$ and $u\gamma x\overline{\overline{R}}u\gamma x$. Then, for every $\gamma \in \Gamma$, the relation R is a strongly regular on a semihypergroup (S, γ), that means R is a strongly regular relation on a Γ-semihypergroup S.*

Proof. Let $\gamma \in \Gamma$ and $x, y, a \in S$ such that xRy. Since S is a Γ-hypergroup, then there exists $v \in S$ such that $a \in u\alpha v$. By hypothesis $x\gamma u\overline{\overline{R}}y\gamma u$, thus, we have

$$(x\gamma u)\alpha v\overline{\overline{R}}(y\gamma u)\alpha v \Longrightarrow x\gamma(u\alpha v)\overline{\overline{R}}\,y\gamma(u\alpha v). \tag{30}$$

Since $x\gamma a \subseteq x\gamma(u\alpha v)$ and $y\gamma a \subseteq y\gamma(u\alpha v)$, then $x\gamma a\overline{\overline{R}}\,x\gamma a$. In the same way, we obtain $a\gamma x\ \overline{\overline{R}}\ a\gamma y$. Therefore, R is a strongly regular relation on a Γ-semihypergroup S. □

Theorem 26. *Let S be a Γ-semihypergroup. Then,*

(1) *if H is a Γ'-semigroup and $\phi : S \to H$ is a homomorphism, then the equivalence relation R associated to ϕ is strongly regular;*

(2) *if S is a Γ-hypergroup and R a strongly regular relation, then S/R is a Γ/R-group.*

Proof. (1): We have $xRy \Leftrightarrow \phi(x) = \phi(y)$. Since H is a Γ'-semigroup, we obtain

$$\phi(x\gamma a) \in \phi(x)\gamma'\phi(a), \tag{31}$$

for every $\gamma \in \Gamma$ and $a \in S$, and xRy implies that

$$|\phi(x\gamma a)| = 1 = |\phi(y\gamma a)|. \tag{32}$$

Therefore, $x\gamma a\overline{\overline{R}}y\gamma a$.

(2): It is obtained from Theorems 22 and 24. □

Definition 27. Let S be a Γ-semihypergroup and θ be a relation on S. We say that θ is a *fundamental relation* on S if θ is the smallest strongly regular equivalence relation on S.

In fact, the fundamental relation θ is the smallest equivalence relation on a Γ-semihypergroup S such that the quotient S/θ is a Γ/θ-semigroup.

Definition 28. Let S be a Γ-semihypergroup and ρ be a relation on S as follows:

$x\rho y$ if and only if there exist $n \in \mathbb{N}$, $x_1, \dots x_n \in S$ and $\gamma_1, \dots, \gamma_{n-1} \in \Gamma$ such that

$$\{x, y\} \subseteq x_1\gamma_1x_2\gamma_2\cdots\gamma_{n-2}x_{n-1}\gamma_{n-1}x_n, \tag{33}$$

and let ρ^* be the *transitive closure* of the relation ρ.

Remark 29. Let $(H, \circ)$ be a semihypergroup. First time, the relation $\beta_\circ$ on H introduced by Koskas [26] is as follows:

$$x\beta_\circ y \Longleftrightarrow \exists x_1, \dots, x_n \in H, \quad \text{such that } \{x, y\} \subseteq x_1 \circ x_2 \circ \cdots \circ x_n. \tag{34}$$

Therefore, the relation ρ is a generalization of the relation β.

Lemma 30. *Let S be a Γ-semihypergroup. Then, for every $\gamma \in \Gamma$ one has $\beta_\gamma \subseteq \rho$.*

Proof. It is straightforward. □

Lemma 31. *The relation ρ is reflexive and symmetric on S and ρ^* is an equivalence relation, moreover one has*

$$\left(\bigcup_{\gamma\in\Gamma}\beta_\gamma\right)^* \subseteq \rho^*. \tag{35}$$

Proof. Clearly, ρ is reflexive and symmetric, so ρ^* is an equivalence relation. Since for every $\gamma \in \Gamma$, $\beta_\gamma \subseteq \rho$, then $\bigcup_{\gamma\in\Gamma}\beta_\gamma \subseteq \rho$ and so $(\bigcup_{\gamma\in\Gamma}\beta_\gamma)^* \subseteq \rho^*$. □

Example 32. Let G be the Γ-hypergroup in Example 11. For every $1 \le i \le n$, we set $\beta_{\gamma_i} = \beta_i$. Then, we have the following:

$$\beta_1 \subseteq \beta_2 \subseteq \cdots \subseteq \beta_k \subseteq \rho. \quad (36)$$

Also, it is not difficult to see that $\beta_k = \rho$. Every (G, γ_i) is a hypergroup and so $\beta = \beta^*$ (see [27]). Therefore, we obtain $\rho = \beta_k = \beta_k^*$ and ρ is an equivalence relation on S.

Theorem 33. *The relation ρ^* is a strongly regular equivalence relation on the Γ-semihypergroup S.*

Proof. By Lemma 31, ρ^* is an equivalence relation. In order to prove that it is strongly regular, we show first that

$$x\rho y \Longrightarrow x\gamma a \overline{\overline{\rho}} y\gamma a,\ a\gamma x \overline{\overline{\rho}} a\gamma y, \quad (37)$$

for every $a \in S$, and $\gamma \in \Gamma$. If $x\rho y$, then there exist $n \in \mathbb{N}$, $x_1, \dots x_n \in S$ and $\gamma_1, \dots, \gamma_{n-1} \in \Gamma$ such that

$$\{x, y\} \subseteq x_1\gamma_1 x_2\gamma_2 \cdots \gamma_{n-2} x_{n-1} \gamma_{n-1} x_n. \quad (38)$$

For every $a \in S$ and $\gamma \in \Gamma$ and every $u \in x\gamma a$ and $v \in y\gamma a$, we obtain:

$$\begin{aligned} u &\in x\gamma a \subseteq x_1\gamma_1 x_2\gamma_2 \cdots \gamma_{n-2} x_{n-1}\gamma_{n-1} x_n \gamma a, \\ v &\in y\gamma a \subseteq x_1\gamma_1 x_2\gamma_2 \cdots \gamma_{n-2} x_{n-1}\gamma_{n-1} x_n \gamma a. \end{aligned} \quad (39)$$

Thus, $u\rho v$ and so $x\gamma a\ \overline{\overline{\rho}}\ y\gamma a$. In the same way, we can show that $x\rho y \ \Rightarrow a\gamma x\ \overline{\overline{\rho}}\ a\gamma y$.

Now, if $x\rho^* y$, then there exist $m \in \mathbb{N}$ and $x = w_0, w_1, \dots, w_{m-1}, w_m = y \in S$ such that $x = w_0\rho w_1\rho \cdots \rho w_{m-1}\rho w_m = y$, whence, we obtain

$$x\gamma a = w_0\gamma a \overline{\overline{\rho}} w_1 \gamma a \overline{\overline{\rho}} \cdots \overline{\overline{\rho}} w_{m-1}\gamma a \overline{\overline{\rho}} w_m \gamma a = y\gamma a. \quad (40)$$

Finally, for all $v \in x\gamma a = w_0\gamma a$ and for all $w \in w_m\gamma a = y\gamma a$, taking $z_1 \in w_1\gamma a$, $z_2 \in w_2\gamma a, \dots, z_{m-1} \in w_{m-1}\gamma a$, we have $v\rho z_1\rho z_2\rho \cdots \rho z_{m-1}\rho w$ and so $v\rho^* w$. Therefore,

$$x\rho^* y \Longrightarrow x\gamma a \overline{\overline{\rho^*}} y\gamma a. \quad (41)$$

Similarly, we can prove that $x\rho^* y \Rightarrow a\gamma x \overline{\overline{\rho^*}} a\gamma y$. Hence, ρ^* is strongly regular. □

Corollary 34. *Let S be a Γ-semihypergroup. Then, the quotient S/ρ^* is a Γ/ρ^*-semigroup.*

Proof. By Theorems 24 and 33, the proof is completed. □

Theorem 35. *Let S be a Γ-semihypergroup. Then the equivalence relation ρ^* is the smallest strongly regular equivalence relation on S and so $\rho^* = \theta$, where θ is the fundamental relation on S.*

Proof. Let R be a strongly regular relation. Now, let $x\rho y$ so there exist $n \in \mathbb{N}$, $x_1, \dots x_n \in S$ and $\gamma_1, \dots, \gamma_{n-1} \in \Gamma$ such that

$$\{x, y\} \subseteq x_1\gamma_1 x_2\gamma_2 \cdots \gamma_{n-2} x_{n-1}\gamma_{n-1} x_n. \quad (42)$$

Since R is a strongly regular relation, then S/R is a Γ/R-semigroup and so

$$\begin{aligned} R(x) &= R(x_1)\ddot{\gamma}_1 R(x_2)\ddot{\gamma}_2 \cdots \ddot{\gamma}_{n-2} R(x_{n-1})\ddot{\gamma}_{n-1} R(x_n), \\ R(y) &= R(x_1)\ddot{\gamma}_1 R(x_2)\ddot{\gamma}_2 \cdots \ddot{\gamma}_{n-2} R(x_{n-1})\ddot{\gamma}_{n-1} R(x_n), \end{aligned} \quad (43)$$

where $\ddot{\gamma}_i = \gamma_i/R$. Thus, $R(x) = R(y)$ and so we have $x\rho y \Rightarrow xRy$. Finally, if $x\rho^* y$, then there exist $m \in \mathbb{N}$ and $x = w_0, w_1, \dots, w_{m-1}, w_m = y \in S$ such that $x = w_0\rho w_1\rho \cdots \rho w_{m-1}\rho w_m = y$. Therefore, $x = w_0 R w_1 R \cdots R w_{m-1} R w_m = y$, and transitivity of R implies that xRy. Thus, $\rho^* \subseteq R$. □

Theorem 36. *If S is a Γ-hypergroup, then the quotient S/ρ^* is a Γ/ρ^*-group.*

Proof. Let $\rho^*(x)$, $\rho^*(y) \in S/\rho^*$ and $\gamma/\rho^* \in \Gamma/\rho^*$. Then, there exists $z \in S$ such that $x \in y\gamma z$ so $\rho^*(x) = \rho^*(y)\gamma/\rho^*\rho^*(z)$ and so S/ρ^* is a Γ/ρ^*-group. □

Theorem 37. *If S is a Γ-semihypergroup with an identity element $e \in S$, then* $\mathrm{card}(\Gamma/\rho^*) = 1$ *and so S/ρ^* is a semigroup.*

Proof. Since $\rho^*(e)$ is an identity element of Γ/ρ^*-semigroup S/ρ^*, then by Theorem 15, the proof is completed. □

Theorem 38. *If S is a Γ-hypergroup with a left (right) identity element $e \in S$, then* $\mathrm{card}(\Gamma/\rho^*) = 1$ *and so S/ρ^* is a semigroup.*

Proof. It is easy to see that for every $\gamma \in \Gamma$, $\rho^*(e)$ there is a left identity element of group $(S/\rho^*, \gamma/\rho^*)$ and so $\rho^*(e)$ is an identity. By Theorem 15, the proof is completed. □

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