Statistical Mechanics of Non-Abelian Chern-Simons Particles

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Abstract

We discuss the statistical mechanics of a two-dimensional gas of non-Abelian Chern-Simons particles which obey the non-Abelian braid statistics. The second virial coefficient is evaluated in the framework of the non-Abelian Chern-Simons quantum mechanics.

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One of the novel features of anyons [1] is that their thermodynamic character as well as their spin and statistics interpolates between that of bosons to that of fermions as the statistical parameter varies. This feature has been clearly exhibited in the works of Arovas, Schrieffer, Wilczek and Zee [2] where they studied the two-dimensional gas of free anyons in the low-density regime by taking the virial expansion. In particular, the second virial coefficient was evaluated in their study and was found to have periodic, non-analytic behavior as a function of the statistical parameter. Comtet, Georgelin and Ouvry [3] confirmed this result by making use of a harmonic potential as means of a regulator and Blum, Hagen and Ramaswamy [4] discussed further the effects of spin on the second virial coefficient.

In this letter, we will discuss the statistical mechanics of a two dimensional gas of $SU(2)$ non-Abelian Chern-Simons (NACS) particles in the framework of the non-Abelian Chern-Simons quantum mechanics which has been developed recently in refs. [5–7]. The NACS particles are point-like sources which interact with each others through a topological non-Abelian Aharonov-Bohm interaction. Carrying both non-Abelian charges and non-Abelian magnetic fluxes, they acquire fractional spins and obey braid statistics. Since the braid (non-Abelian) statistics is much more generalized than the fractional statistics of anyons, it is interesting to explore their statistical mechanical properties and to compare them with those of anyons. The NACS particle and the non-Abelian Aharonov-Bohm effect have been discussed in connection with various physical phenomena such as the scattering [8] of non-Abelian vortices which form in some spontaneously broken gauge theories, the fractional quantum Hall effect [9], and the (2+1) dimensional gravity [10]. Thus, exploring the statistical mechanics of the NACS particles is important in many respects.

The dynamics of the N-body system of free NACS particles are governed by the Hamiltonian [5,6,11]

$$H_N = -\sum_{\alpha=1}^{N} \frac{1}{\mu_\alpha} \left(\nabla_{\bar{z}_\alpha} \nabla_{z_\alpha} + \nabla_{z_\alpha} \nabla_{\bar{z}_\alpha} \right)$$

$$\nabla_{z_\alpha} = \frac{\partial}{\partial z_\alpha} + \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \hat{Q}_\alpha^a \hat{Q}_\beta^a \frac{1}{z_\alpha - z_\beta} \tag{1}$$

2

$$\nabla_{\bar{z}_\alpha} = \frac{\partial}{\partial \bar{z}_\alpha}$$

where the particles are labeled by $\alpha = 1, \ldots, N$ and their spatial coordinates are denoted by $(q_\alpha^1, q_\alpha^2) = (z_\alpha + \bar{z}_\alpha, -i(z_\alpha - \bar{z}_\alpha))/2$. Here κ is a parameter of the theory such that $4\pi\kappa =$ integer and $\hat{Q}^a$'s are the isovector operators which can be represented by some generators T_l^a in a representation of isospin l. One can construct a classical model [5–7,12] where the NACS particles are described as isospin particles [13,14] of which isospin charges are minimally coupled with the non-Abelian Chern-Simons gauge fields [15]. Solving the Gauss' constraint explicitly and integrating out the gauge fields by choosing the holomorphic gauge [5,6] in the framework of the coherent state quantization [16,17], one obtains the Hamiltonian H_N, Eq.(1).

We may remove the interaction terms in H_N by a similarity transformation

$$H_N \longrightarrow U H_N U^{-1} = H_N^{\text{free}} = -\sum_\alpha^N \frac{2}{\mu_\alpha} \partial_{\bar{z}_\alpha} \partial_{z_\alpha}$$

$$\Psi_H \longrightarrow U \Psi_H = \Psi_A \tag{2}$$

where $U(z_1, \ldots, z_N)$ satisfies the Knizhnik-Zamolodchikov (KZ) equation [18]

$$\left(\frac{\partial}{\partial z_\alpha} - \frac{1}{2\pi\kappa} \sum_{\beta \neq \alpha} \hat{Q}_\alpha^a \hat{Q}_\beta^a \frac{1}{z_\alpha - z_\beta} \right) U(z_1, \ldots, z_N) = 0 \tag{3}$$

and $\Psi_H(z_1, \ldots, z_N)$ denotes the wave function of the N-body system of the NACS particles in the holomorphic gauge. Comparing Eq.(3) with the KZ equation which is satisfied by the Green's functions in the conformal field theory, we find that $(4\pi\kappa - 2)$ corresponds to the level of the underlying $SU(2)$ current algebra. Note that Ψ_A obeys the braid statistics due to the transformation function $U(z_1, \ldots, z_N)$ [5,6] while Ψ_H satisfies ordinary statistics. In analogy with the Abelian Chern-Simons particle theory Ψ_A may be called the NACS particle wave function in the anyon gauge. The transformation function also defines the inner product in the holomorphic gauge

$$< \Psi_1 | \Psi_2 > = \int d^{2N} \zeta \Psi_1(\zeta)^\dagger U^\dagger(\zeta) U(\zeta) \Psi_2(\zeta) \tag{4}$$

3

where $\zeta = (z_1, \ldots, z_N)$. This inner product renders the Hamiltonian in the holomorphic gauge H_N Eq.(1) Hermitian, which does not look manifestly Hermitian.

Defining a matrix (operator) valued 1-from

$$\omega = \sum_{\alpha < \beta} \frac{1}{2\pi\kappa} \Omega_{\alpha\beta} \, d \ln(\zeta_\alpha - \zeta_\beta) \tag{5}$$

where $\Omega_{\alpha\beta} = \hat{Q}_\alpha^a \hat{Q}_\beta^a$, we find that the transformation function U corresponds to the monodromy of ω [19]

$$U(z_1, \ldots, z_N) = I + \int_\Gamma \omega + \int_\Gamma \omega\omega + \ldots \tag{6}$$

where Γ is a contour in the N-dimensional complex space with one end point-fixed and the other being $(z_1, \ldots, z_N)$. The integrability condition

$$\left[\nabla_{z_\alpha}, \nabla_{z_\beta} \right] = 0 \tag{7}$$

for the transformation function U to exist leads to

$$d\omega + \omega \wedge \omega = 0. \tag{8}$$

This condition is fulfilled, since $\Omega_{\alpha\beta}$ satisfy the infinitesimal pure braid relations [19] which are relevant to the classical Yang-Baxter equation.

Having defined the non-Abelian Chern-Simons quantum mechanics, we turn to the statistical mechanics of the NACS particles. The grand partition function Ξ is defined as usual in terms of the N-body Hamiltonian H_N and the fugacity ν by

$$\Xi = \sum_{N=0}^\infty \nu^N \, \text{Tr} \, e^{-\beta H_N} \tag{9}$$

where $\beta = 1/kT$. In the low-density regime, a cluster expansion can be applied to Ξ

$$\Xi = \exp\left(V \sum_{n=1}^\infty b_n \nu^n \right) \tag{10}$$

where V is the volume (the area A for the two-dimensional gas) and b_n is the n-th cluster integral. Comparing the two expressions for Ξ, Eqs.(9) and (10), we have

$$b_1 = \frac{1}{V} Z_1, \quad b_2 = \frac{1}{V} \left(Z_2 - Z_1^2/2 \right) \tag{11}$$

and

$$Z_N = \frac{1}{N!} \int d^{2N} \zeta \sum_{\{m_i\}} \sum_P < (m_1, z_1) \dots (m_N, z_N) | e^{-\beta H_N} | P(m_1, z_1) \dots P(m_N, z_N) > \tag{12}$$

where m_i, $i = 1, \dots, N$ denotes the isospin quantum number of the i-th particle. We note that P denotes the operation of permutation and the nontrivial inner product Eq.(4) is to be appropriately taken into account in Eq.(12).

Since the NACS particles are described in the regular gauges, for instance, the Coulomb, axial and holomorphic gauges, as bosons interacting with each other through the topological terms induced by the Chern-Simons gauge fields, the expression of the N-particle partition function Eq.(12) holds generally in cases of the regular gauges. However, in the anyon gauge where the non-Abelian braid statistics of the NACS becomes manifest, this expression for Z_N is not valid and some modification is necessary. We may rewrite Eq.(12) as

$$Z_N = \sum_{\{m_i\}} \int d^{2N} \zeta \, \Psi^*_{\{m_i\}}(\zeta) U^\dagger(\zeta) U(\zeta) e^{-\beta H_N} \mathbf{P} \Psi_{\{m_i\}}(\zeta), \tag{13}$$

$$\mathbf{P} = \frac{1}{N!} \sum_{i=0}^{N-1} S_i,$$

in terms of the exchange operators; $S_0 = I$ and S_i, $i = 1, \dots, N-1$

$$S_i \Psi_{m_1, \dots, m_i, m_{i+1}, \dots, m_N}(z_1, \dots, z_i, z_{i+1}, \dots, z_N)$$

$$= \Psi_{m_1, \dots, m_{i+1}, m_i, \dots, m_N}(z_1, \dots, z_{i+1}, z_i, \dots, z_N) \tag{14}$$

where $\Psi_{\{m_i\}}(\zeta)$ denotes $\Psi_{m_1, \dots, m_N}(z_1, \dots, z_N) = < (m_1, z_1) \dots (m_N, z_N) | \Psi >$. In the anyon gauge we may obtain a similar expression for Z_N. But the exchange operator is no longer represented by the simple permutation P and a nontrivial exchange factor, which satisfies the Yang-Baxter equation, must be introduced [6]

$$S_i^A \Psi_A(z_1, \dots, z_i, z_{i+1}, \dots, z_N) = \mathcal{R}(z_i, z_{i+1}) \Psi_A(z_1, \dots, z_{i+1}, z_i, \dots, z_N) \tag{15}$$

where Ψ_A denotes the wave function in the anyon gauge. Thus, defining $\mathbf{P}^A = 1/N! \sum_{i=0}^{N-1} S_i^A$, $S_0^A = I$ we have

5

$$Z_N = \sum_{\{m_i\}} \int d^{2N}\zeta\, \Psi^{A*}_{\{m_i\}}(\zeta) e^{-\beta H^A_N} \mathbf{P}^A \Psi^A_{\{m_i\}}(\zeta) \tag{16}$$

$$= \int d^{2N}\zeta\, \mathrm{tr} < z_1, \ldots, z_N | e^{-\beta H^A_N} \mathbf{P}^A | z_1, \ldots, z_N >$$

where H^A_N is the N-body Hamiltonian in the anyon gauge.

The virial expansion, i.e., the expansion of the equation of state in powers of the density ρ is given as

$$P = \rho kT \left(1 + B_2(T)\rho + B_3(T)\rho^2 + \ldots \right) \tag{17}$$

where $B_n(T)$ is the n-th virial coefficient. The second virial coefficient $B_2(T)$ which is the main subject of this letter is written as

$$B_2(T) = -\frac{b_2}{b_1^2} = A \left(\frac{1}{2} - \frac{Z_2}{Z_1^2} \right). \tag{18}$$

Thus evaluation of $B_2(T)$ only involves the one-particle and two-particle particle partition functions. If we assume that all the NACS particles belong to the same isospin multiplet $\{m = -l, \ldots, l; |l, m >\}$ and have the same mass 2μ,

$$Z_1 = \mathrm{Tr}\, e^{-\beta H_1} = (2l+1)A\lambda_T^{-2} \tag{19}$$

where $\lambda_T = \sqrt{2\pi\hbar^2/\mu kT}$ is the thermal wavelength. Since the two-body Hamiltonian can be expressed as

$$H_2 = H_{\mathrm{cm}} + H_{\mathrm{rel}}$$

$$= -\frac{1}{2\mu}\partial_Z\partial_{\bar{Z}} - \frac{1}{\mu}(\nabla_z\nabla_{\bar{z}} + \nabla_{\bar{z}}\nabla_z), \tag{20}$$

$$\nabla_z = \partial_z + \frac{\Omega}{z}, \quad \nabla_{\bar{z}} = \partial_{\bar{z}}$$

where $Z = (z_1 + z_2)/2$, $z = z_1 - z_2$ are the center of mass and the relative coordinates respectively, the two-particle partition function are factorized into the contribution of the center of mass coordinates and that of the relative coordinates as usual

$$Z_2 = \mathrm{Tr}\, e^{-\beta H_2} = 2A\lambda_T^{-2}Z_2', \tag{21a}$$

6

$$Z_2' = \mathrm{Tr}_{\mathrm{rel}} \, e^{-\beta H_{\mathrm{rel}}}$$

$$= \frac{1}{2} \int d^2 z \sum_{m_1, m_2} \Big[< m_1, m_2; z | e^{-\beta H_{\mathrm{rel}}} | m_1, m_2; z >$$

$$+ < m_1, m_2; z | e^{-\beta H_{\mathrm{rel}}} | m_2, m_1; -z > \Big]. \tag{21b}$$

In Eq.(20) Ω is a block-diagonal matrix given by

$$\Omega = \hat{Q}_1^a \hat{Q}_2^a / (2\pi\kappa) = \frac{1}{4\pi\kappa} \left((\hat{Q}_1 + \hat{Q}_2)^2 - (\hat{Q}_1)^2 - (\hat{Q}_2)^2 \right)$$

$$= \sum_{j=0}^{2l} \frac{1}{4\pi\kappa} \left(j(j+1) - 2l(l+1) \right) \otimes I_j \tag{22}$$

$$= \sum_{j=0}^{2l} \omega_j \otimes I_j.$$

As observed in the discussion on the scattering of the NACS particles [6], it is convenient to take a similarity transformation given by

$$H_{\mathrm{rel}} \longrightarrow H_{\mathrm{rel}}' = G^{-1} H_{\mathrm{rel}} G,$$

$$\Psi(z, \bar{z}) \longrightarrow \Psi'(z, \bar{z}) = G^{-1} \Psi(z, \bar{z}) \tag{23}$$

where $G(z, \bar{z}) = \exp\left(-\frac{\Omega}{2} \ln(z\bar{z}) \right)$. This similarity transformation renders the inner product trivial and the Hamiltonian H_{rel}' becomes manifestly Hermitian

$$H_{\mathrm{rel}}' = -\frac{1}{\mu} (\nabla_z' \nabla_{\bar{z}}' + \nabla_{\bar{z}}' \nabla_z'), \tag{24}$$

$$\nabla_z' = \partial_z + \frac{\Omega}{2} \frac{1}{z}, \quad \nabla_{\bar{z}}' = \partial_{\bar{z}} - \frac{\Omega}{2} \frac{1}{\bar{z}}.$$

The partition function is invariant under this similarity transformation, i.e., $Z_2' = \mathrm{Tr}_{\mathrm{rel}} \, e^{-\beta H_{\mathrm{rel}}'}$. Rewriting the Hamiltonian H_{rel}' in the polar coordinates and projecting it onto the subspace of total isospin j, we see that it corresponds to the Hamiltonian in the Coulomb gauge for anyons of which statistical parameter is given by $\alpha_s = \omega_j$

$$H_j' = -\frac{1}{2\mu} \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} + i\omega_j \right)^2 \right]. \tag{25}$$

Then it follows from symmetry of the Clebsch-Gordan coefficients in the case of $SU(2)$, the two-particle partition function may be written as

$$Z_2' = \frac{1}{2} \int d^2z \sum_{j=0}^{2l} (2j+1) \left[< z|e^{-\beta H_j'}|z > +(-1)^j < z|e^{-\beta H_j'}| - z > \right]. \qquad (26)$$

Introducing a harmonic potential $\frac{\mu}{2}\epsilon^2 r^2$ to the Hamiltonian for the purpose of regularization, we find that the spectrum for the relative Hamiltonian H_j' is discrete [1] and can be classified into two classes: Type I with energy $E_n^I = \epsilon(2n+1+\omega_j - [\omega_j])$, and degenercy of $n+1$ and type II with energy $E_n^{II} = \epsilon(2n+1-\omega_j + [\omega_j])$, and degenercy of n where n is a non-negative integer and $[\omega_j]$ is the integer such that $0 \le \delta_j = \omega_j - [\omega_j] < 1$. In order to take an appropriate regularization [2,3] we define the second virial coefficient as

$$B_2(\kappa, l, T) - \Delta B_2(l, T) = -\frac{2\lambda_T^2}{(2l+1)^2} \left[Z_2'(\kappa, l, T) - \Delta Z_2'(l, T) \right], \qquad (27)$$

where $\Delta B_2(l, T)$ is given in terms of the well-known virial coefficients of Bose and Fermi systems (B^B, B^F) as

$$\Delta B_2(l, T) = \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\frac{1+(-1)^j}{2} B^B(T) + \frac{1-(-1)^j}{2} B^F(T) \right] \qquad (28)$$

$$= -\frac{1}{4}\lambda_T^2 \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1)(-1)^j.$$

The regularized partition function may be written by

$$Z_2'(\kappa, l, T) - \Delta Z_2'(l, T) = \sum_{j=0}^{2l} (2j+1) \lim_{\epsilon \to 0} \left[\frac{1+(-1)^j}{2} (Z_\epsilon(\delta_j) - Z_\epsilon(0)) \right.$$

$$\left. + \frac{1-(-1)^j}{2} (Z_\epsilon(\delta_j) - Z_\epsilon(1)) \right], \qquad (29)$$

$$Z_\epsilon(\delta_j) = \sum_{n=0}^{\infty} \left[(n+1)e^{-\beta\epsilon(2n+1+\delta_j)} + ne^{-\beta\epsilon(2n+1-\delta_j)} \right]$$

$$= \frac{1}{2} \frac{\cosh(\beta\epsilon(\delta_j - 1))}{\sinh^2 \beta\epsilon}.$$

Then, by some algebra we get

$$B_2(\kappa, l, T) = -\frac{1}{4}\lambda_T^2 + \frac{1}{(2l+1)^2} \sum_{j=0}^{2l} (2j+1) \left[\delta_j - \frac{1}{2}\delta_j^2 \right] \lambda_T^2. \qquad (30)$$

Since the NACS particle is a generalization of the anyon, one may expect that the thermodynamic properties of the NACS particle are similar to those of the anyon. However,

8

this is not the case. Noting that both carry anomalous spins, which are given by $s = l(l+1)/2\pi\kappa$ for the NACS particle and $s = \alpha_s/2$ for the anyon, we may compare the behavior of the virial coefficient of the NACS particle as a function of the spin with that of anyon. We find that the periodicity in s, which is the case for the anyon, does no longer hold and its functional behavior radically differs from that of the anyon even in the large-κ limit as depicted in Fig.1.

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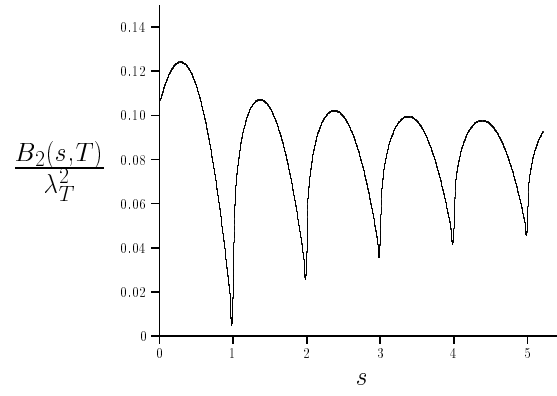
FIGURES

$$\frac{B_2(s,T)}{\lambda_T^2}$$

FIG. 1. The second virial coefficient as a function of the induced spin s in the large-κ limit.