

CAUCHY'S EQUATIONS AND ULAM'S PROBLEM

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ABSTRACT. Our aim is to study the Ulam's problem for Cauchy's functional equations. First, we present some new results about the superstability and stability of Cauchy exponential functional equation and its Pexiderized for class functions on commutative semigroup to unitary complex Banach algebra. In connection with the problem of Th. M. Rassias and our results, we generalize the theorem of Baker and theorem of L. Székelyhidi. Then the superstability of Cauchy additive functional equation can be prove for complex valued functions on commutative semigroup under some suitable conditions. This result is applied to the study of a superstability result for the logarithmic functional equation, and to give a partial affirmative answer to problem 18, in the thirty-first ISFE. The hyperstability and asymptotic behaviors of Cauchy additive functional equation and its Pexiderized can be study for functions on commutative semigroup to a complex normed linear space under some suitable conditions. As some consequences of our results, we give some generalizations of Skof's theorem, S.-M. Joung's theorem, and another affirmative answer to problem 18, in the thirty-first ISFE. Also we study the stability of Cauchy linear equation in general form and in connection with the problem of G. L. Forti, in the 13th ICFEI (2009), we consider some systems of homogeneous linear equations and our aim is to establish some common Hyers-Ulam-Rassias stability for these systems of functional equations and presenting some applications of these results.

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1. Introduction

1.1. Cauchy's Functional Equations and Ulam's Problem: Stability of Functional Equations. In general, a functional equation is any equation that specifies a function or some functions in implicit form, where an implicit equation is a relation of the form $F(x_1, \dots, x_n) = 0$, where F is a function of several variables (often a polynomial). However, in this note, we deal with Cauchy's functional equations in one and two variables that specifies at most three functions. According to the our study and for the readers convenience and explicit later use, we give the following definitions for functional equations and some types of stability. In our study, we will provide some reasons for these definitions. Note that in this note, all theorems, where has previously been proven by some authors, we will present all this theorems based of our notions and definitions.

Definition 1.1. *Let S and B be nonempty sets. Let $g_i : S^2 \rightarrow S$ for $i \in \{1, \dots, 4\}$ and $G_j : B^2 \rightarrow B$ for $j \in \{1, 2\}$ be functions. Then the following equation*

$$(1.1) \quad G_1[f(g_1(x, y)), f(g_2(x, y))] = G_2[f(g_3(x, y)), f(g_4(x, y))]$$

is called functional equation that specifies unknown functions $f : S \rightarrow B$. We use the symbol $\mathfrak{S}(S, B)$ for equation (1.1) as a functional equation that specifies unknown functions $f : S \rightarrow B$ and denote the set all solutions of this functional equation with $Z_{\mathfrak{S}(S, B)}$. Similarly, we can give to definition of functional equation in single variable.

Definition 1.2. *Let S and B be nonempty sets such that (B, d) be a metric space. Assuming that $\varphi : S^2 \rightarrow [0, \infty)$ and $\phi : S \rightarrow [0, \infty)$ are functions and $\mathfrak{S}(S, B)$ is a functional equation (of the form equation (1.1)). If for every functions $f : S \rightarrow B$ satisfying the inequality*

$$(1.2) \quad d(G_1[f(g_1(x, y)), f(g_2(x, y))], G_2[f(g_1(x, y)), f(g_2(x, y))]) \leq \varphi(x, y)$$

for all $x, y \in S$, there exists $T \in Z_{\mathfrak{S}(S, B)}$ such that

$$(1.3) \quad d(f(x), T(x)) \leq \phi(x)$$

for all $x \in S$, then we say that the functional equation $\mathfrak{S}(S, B)$ is Hyers-Ulam-Rassias stable on (S, B) with control functions (φ, ϕ) and we denoted it by "HUR-stable" on (S, B) with controls (φ, ϕ) . Also we call the function T as "HUR-stable function". If $\varphi(x, y)$ in (1.2) and $\phi(x)$ in (1.3) are replaced by real's $\delta > 0$ and $\varepsilon > 0$ respectively, then we say

that corresponding phenomenon of the functional equation $\mathfrak{S}(S, B)$ is the Hyers-Ulam stable on (S, B) and we denoted it by "HU-stable" on (S, B) with controls (δ, ε) , and we call the function T as "HU-stable function". Similarly, we can define of above concepts for functional equations in single variable.

Definition 1.3. Let S and B be nonempty sets such that (B, d) be a metric space. Assuming that $\varphi : S^2 \rightarrow [0, \infty)$ and $\phi : S \rightarrow [0, \infty)$ are functions and $\mathfrak{S}(S, B)$ is a functional equation. If for every functions $f : S \rightarrow B$ satisfying the inequality

$$(1.4) \quad d(G_1[f(g_1(x, y)), f(g_2(x, y))], G_2[f(g_3(x, y)), f(g_4(x, y))]) \leq \varphi(x, y)$$

for all $x, y \in S$, then either $f \in Z_{\mathfrak{S}(S, B)}$ or

$$(1.5) \quad d(f(x)) \leq \phi(x)$$

for all $x \in S$, then we say that the functional equation $\mathfrak{S}(S, B)$ is "superstable" on (S, B) with control functions (φ, ϕ) .

Definition 1.4. Let S and B be nonempty sets such that (B, d) be a metric space. Assuming that $\varphi : S^2 \rightarrow [0, \infty)$ is a function and $\mathfrak{S}(S, B)$ is a functional equation. If for every functions $f : S \rightarrow B$ satisfying the inequality

$$(1.6) \quad d(G_1[f(g_1(x, y)), f(g_2(x, y))], G_2[f(g_3(x, y)), f(g_4(x, y))]) \leq \varphi(x, y)$$

for all $x, y \in S$, then $f \in Z_{\mathfrak{S}(S, B)}$, then we say that the functional equation $\mathfrak{S}(S, B)$ is "hyperstable" on (S, B) with control (φ) .

In early 19th century, Cauchy has determined the general continuous solution of each of the functional equations

$$(1.7) \quad f(x + y) = f(x) + f(y),$$

$$(1.8) \quad f(x + y) = f(x) \cdot f(y),$$

$$(1.9) \quad f(x \cdot y) = f(x) + f(y),$$

$$(1.10) \quad f(x \cdot y) = f(x) \cdot f(y),$$

for real-valued functions on some subsets of \mathbb{R} (is the real numbers field) and showing that they are, respectivel, ax , a^x , $a \log x$, x^a , where in each case a is an arbitrary constant. These functional equations, its often called the Cauchy additive, exponential, logarithmic, and multiplicative functional equations respectively, in honor of A. L. Cauchy. For some

first generalization of these functional equations see [19] and for recent achievements see [42]. If no further conditions are imposed on f , then (assuming the axiom of choice) there are infinitely many other functions that satisfy these functional equations. This was proved in 1905 by Georg Hamel using Hamel bases. Such functions are sometimes called Hamel functions. For more information see ([45], Chap. 5). The fifth problem on Hilbert's list is a generalisation of equation (1.7). Functions where there exists a real number c such that $f(cx) \neq cf(x)$ are known as Cauchy-Hamel functions and are used in Dehn-Hadwiger invariants which are used in the extension of Hilbert's third problem from 3-D to higher dimensions [15], also see [58]. The properties of Cauchy's functional equations are frequently applied to the development of theories of other functional equations. Moreover, the properties of Cauchy's functional equations are powerful tools in almost every field of natural and social sciences.

The starting point of the stability theory of functional equations was the problem formulated by S. M. Ulam in 1940 (see [62]). More precisely, S. M. Ulam gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

Let (S, \cdot) be a group and (B, \cdot, d) be a metric group, let $\mathfrak{S}(S, B)$ be the functional equation $f(xy) = f(x)f(y)$. Does for every $\varepsilon > 0$, there exists a $\delta > 0$ such that the functional equation $\mathfrak{S}(S, B)$ is HU-stable on (S, B) with controls (δ, ε) .

For the first in 1941, D. H. Hyers [29] gave an affirmative partial answer to this problem for the case where S and B are assumed to be Banach spaces. The result of Hyers is stated as follows:

Theorem 1.5. *Suppose that S and B are two real Banach spaces and $\mathfrak{S}(S, B)$ be the functional equation $f(x + y) = f(x) + f(y)$. Then for every $\varepsilon > 0$, the functional equation $\mathfrak{S}(S, B)$ is HU-stable on (S, B) with controls $(\varepsilon, \varepsilon)$ and also proved that HU-stable function is unique.*

This is the reason for which today this type of stability is called Hyers-Ulam stability of functional equations. And also this is a reason for definition of (1.2). Here, note that Hyers considered only bounded control functions for Cauchy additive functional equation. T. Aoki [8] introduced unbounded one and generalized a result of Theorem (1.5). Th. M. Rassias, who independently introduced the unbounded control functions was the first to prove the stability of the linear mapping between

Banach spaces. Taking this fact into account, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam-Rassias stability on two Banach spaces, and this is a reason for definition of (1.2). This terminology is also applied to the case of other functional equations. For more detailed definitions of such terminology one can refer to [24] and [30]. Th. M. Rassias [52] generalized Hyers's Theorem as follows:

Theorem 1.6. *Suppose that S and B are two real Banach spaces and $\mathfrak{S}(S, B)$ be the functional equation $f(x + y) = f(x) + f(y)$. Then for every $\epsilon > 0$ and $0 \leq p < 1$, the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\epsilon(\|x\|^p + \|y\|^p), \frac{2\epsilon}{2-2^p}\|x\|^p)$ and also proved that HUR-stable function is unique.*

Th. M. Rassias [56] noticed that the proof of this theorem also works for $p < 0$. In fourth section, we show that for this case the Cauchy additive functional equation is hyperstable and also give some generalizations. Following Th. M. Rassias and P. Šemrl [57] generalized the result of (1.6) and obtained stability result for the case $p \geq 0$ and $p \neq 1$. For the case $p = 1$, Z. Gajda in his paper [26] showed that the theorem of Th. Rassias (1.6) is false for some special control function and give the following counterexample.

Theorem 1.7. *Let $S = B$ be real field \mathbb{R} and $\mathfrak{S}(\mathbb{R}, \mathbb{R})$ be the functional equation $f(x + y) = f(x) + f(y)$. Then for every $\theta > 0$ there is no constant $\delta \in [0, \infty)$ such that the functional equation $\mathfrak{S}(\mathbb{R}, \mathbb{R})$ is HUR-stable on (\mathbb{R}, \mathbb{R}) with controls $(\theta(|x| + |y|), \delta|x|)$.*

M. S. Moslehian and Th. M. Rassias [48] generalized the Theorem (1.5) and Theorem (1.6) in non-Archimedean spaces. Also, H. G. Dales and M. S. Moslehian in [49] introduced multi-normed spaces and study some properties of multi-bounded mappings on such spaces. Then they proved some generalized Hyers-Ulam-Rassias stability theorems associated to the Cauchy additive functional equation for mappings from linear spaces into multi-normed spaces. The Hyers-Ulam stability of mappings is in development and several authors have remarked interesting applications of this theory to various mathematical problems. For the first, L. Cădariu and V. Radu proved the Hyers-Ulam-Rassias stability of the additive Cauchy equation by using the fixed point method (see [18] and [51]). This method appears to be powerful and successful. In fact the Hyers-Ulam stability has been mainly used to study problems concerning approximate isometries or quasi-isometries, the stability of Lorentz

and conformal mappings, the stability of stationary points, the stability of convex mappings, or of homogeneous mappings, etc. For more information about theory of stability of functional equations one can refer to [38], [50], and [53]. For important special functional equations, we can refer to [9, 10, 31, 32, 33, 34, 35, 54], and [61]. The stability problem of functional equations has been extended in various directions and studied by several mathematicians. So, we are necessary to introduce exact definitions of some stability which is applicable to all functional equations in this note at the first. So, we present these notions and definitions based on paper [53] and book [38].

1.2. Superstability of Functional Equations: Cauchy Exponential Functional Equations. In 1979, another type of stability was observed by J. Baker, J. Lawrence and F. Zorzitto [12]. Indeed, they proved that if a real-valued function f on a real vector space V satisfies the functional inequality

$$(1.11) \quad |f(x+y) - f(x)f(y)| \leq \epsilon$$

for some $\delta > 0$ and for all $x, y \in V$, then f is either bounded or exponential. In fact, they prove that Cauchy exponential functional equation is superstable on (V, \mathbb{R}) . This result was the first result concerning the superstability phenomenon of functional equations and the definition of (1.3) is based on this result. Later, J. Baker [11] (see also [20] and [27]) generalized this famous result as follows:

Theorem 1.8. *Let (S, \cdot) be semigroup, B be the field of complex numbers, and $\mathfrak{S}(S, \mathbb{C})$ be the functional equation $f(x \cdot y) = f(x)f(y)$. Then for every $\epsilon > 0$, the functional equation $\mathfrak{S}(S, \mathbb{C})$ is superstable on (S, \mathbb{C}) with controls $(\epsilon, (1 + \frac{\sqrt{1+4\epsilon}}{2}))$*

In the proof of the preceding theorem, the multiplicative property of the norm was crucial. Indeed, the proof above works also for functions $f : S \rightarrow A$, where A is a normed algebra in which the norm is multiplicative, i.e., $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. Examples of such real normed algebras are the quaternions and the Cayley numbers. In the same paper Baker gives the following example to show that this result fails if the algebra does not have the multiplicative norm property. Let $\epsilon > 0$, choose $\delta > 0$ so that $|\delta - \delta^2| = \epsilon$ and let $f : \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C}$ be defined as

$$f(\lambda) = (e^\lambda, \delta), \quad \lambda \in \mathbb{C}.$$

Then, with the nonmultiplicative norm given by $\|(\lambda, \mu)\| = \max\{|\lambda|, |\mu|\}$, we have $\|f(\lambda + \mu) - f(\lambda)f(\mu)\| = \epsilon$ for all complex λ and μ , f is unbounded, but it is not true that $f(\lambda + \mu) = f(\lambda)f(\mu)$ for all complex λ and μ .

The result of Baker, Lawrence and Zorzitto [12] was generalized by L. Székelyhidi [60] in another way and he obtained the following result.

Theorem 1.9. *Let (S, \cdot) be an Abelian group with identity 1, B be the field of complex numbers, and let $\mathfrak{S}(S, \mathbb{C})$ be the functional equation $f(1)f(x \cdot y) = g(x)f(y)$, where $g : S \rightarrow \mathbb{C}$ is a function. Assume that $M_1, M_2 : S \rightarrow [0, \infty)$ are two functions. Then there exists $\delta > 0$ such that the functional equation $\mathfrak{S}(S, \mathbb{C})$ is superstable on (S, \mathbb{C}) with controls $(\min\{M_1(x), M_2(y)\}, \delta)$ such that $f(x) = g(x)f(1)$ for all $x \in S$.*

During the thirty-first International Symposium on Functional Equations, Th. M. Rassias [55] introduced the term *mixed stability* of the function $f : E \rightarrow \mathbb{R}$ (or \mathbb{C}), where E is a Banach space, with respect to two operations addition and multiplication among any two elements of the set $\{x, y, f(x), f(y)\}$. Especially, he raised an open problem concerning the behavior of solutions of the inequality

$$\|f(x \cdot y) - f(x)f(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

In connection with this open problem, P. Găvruta [28] gave an answer to the problem suggested by Rassias concerning the mixed stability.

Theorem 1.10. *Let S and B be a real normed space and a normed algebra with multiplicative norm, respectively. Let $\mathfrak{S}(S, B)$ be the functional equation $f(x + y) = f(x)f(y)$. Then for every $\theta > 0$ and $p > 0$, the functional equation $\mathfrak{S}(S, B)$ is superstable on (S, B) with controls $(\theta(\|x\|^p + \|y\|^p), g(x))$, where $g(x) = \frac{1}{2}(2^p + \sqrt{4^p + 8\theta})\|x\|^p$ with $\|x\| \geq 1$.*

In the second section, first we give another proof of Theorem (1.8), where its important idea for other results. We study the superstability of Cauchy exponential functional equation. As a consequence of our results and in connection with problem of T. H. Rassias, we extend the results of Baker and Székelyhidi in unitary complex Banach algebra. Also we present this result for the Pexiderized Cauchy exponential equation. More precisely, we proved the superstability and stability of Cauchy exponential functional equation and its Pexiderized when the controls functions is not bounded. Furthermore, we consider the superstability and stability for the functional equation of the form $f(x + y) = g(x)f(y)$, in which f is a function from a commutative semigroup to an complex

Banach space and g is function from a commutative to complex field and next we consider the superstability and stability for the equations of the forms $f(x + y) = g(x)f(y)$ and $f(x + y) = g(x)h(x)$ when f, g and h are three functions from a commutative semigroup to an unitary complex Banach algebra. Also this Results is applied to the study of homogeneous functional equation and its Pexiderized.

Also during the thirty-first International Symposium on Functional Equations (ISFE), the following question arises. Let (S, \cdot) be an arbitrary semigroup or group and let a mapping $f : S \rightarrow \mathbb{R}$ (the set of reals) be such that the set $\{f(x \cdot y) - f(x) - f(y) \mid x, y \in S\}$ is bounded. Is it true that there is a mapping $T : S \rightarrow R$ that satisfies

$$(1.12) \quad T(x \cdot y) - T(x) - T(y) = 0$$

for all $x, y \in S$ and that the set $\{T(x) - f(x) \mid x \in S\}$ is bounded? G. L. Forti in [25] gave a negative answer to this problem. In third section the superstability of the Cauchy equation (in the sense of additive) can be proved for complex valued functions on commutative semigroup under some suitable conditions and the result is applied to the study of a superstability result for the logarithmic functional equation. Furthermore, these results is partial affirmative answers to problem 18, in the thirty-first ISFE. Also in fourth section, we give another partial affirmative answers to this problem under some suitable conditions.

1.3. Asymptotic Behavior of Functional Equations. Several authors have used asymptotic conditions in stating approximations to Cauchy's functional equation. P. D. T. A. Elliott [23] showed that if the real function f belongs to the class $L^p(0, z)$ for every $z \geq 0$, where $p \geq 1$, and satisfies the asymptotic condition

$$\lim_{z \rightarrow \infty} \frac{\int_0^z \int_0^z |f(x + y) - f(x) - f(y)|^p dx dy}{z} = 0,$$

then there is a constant c such that $f(x) = cx$ almost everywhere on \mathbb{R}^+ . One of the theorems of J. R. Alexander, C. E. Blair and L. A. Rubel [2] states that if $f \in L^1(0, b)$ for all $b > 0$, and if for almost all $x > 0$

$$\lim_{u \rightarrow \infty} \frac{\int_0^y [f(x + y) - f(x) - f(y)] dy}{u} = 0,$$

then for some real number c , $f(x) = cx$ for almost all $x \geq 0$.

F. Skof [59] proved the following theorem and applied the result to the study of an asymptotic behavior of additive functions.

Theorem 1.11. *Let E_1 and E_2 be a normed space and a Banach space, respectively. Given $a > 0$, suppose a function $f : E_1 \rightarrow E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in E_1$ with $\|x\| + \|y\| > a$. Then there exists a unique additive function $A : E_1 \rightarrow E_2$ such that

$$\|f(x) - A(x)\| \leq 9\delta$$

for all $x \in E_1$.

Using this theorem, F. Skof [59] has studied an interesting asymptotic behavior of additive functions as we see in the following theorem.

Theorem 1.12. *Let E_1 and E_2 be a normed space and a Banach space, respectively. For a function $f : E_1 \rightarrow E_2$ the following two conditions are equivalent:*

- (1) $\|f(x+y) - f(x) - f(y)\| \rightarrow 0$ as $\|x\| + \|y\| \rightarrow \infty$;
- (2) $f(x+y) - f(x) - f(y) = 0$

for all $x, y \in E_1$.

S.-M. Joung [36], proved that the Hyers-Ulam stability for Jensen's equation on a restricted domain and the result applied to the study of an interesting asymptotic behavior of the additive mappings. More precisely, he proved that a mapping $f : E_1 \rightarrow E_2$ satisfying $f(0) = 0$ is additive if and only if

- (1) $\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \rightarrow 0$ as $\|x\| + \|y\| \rightarrow \infty$.

In fourth section, we study the hyperstability of Cauchy additive functional equation and its Pexiderized for class functions on commutative semigroups to complex normed spaces. As a consequence of our results, we present some new results about the asymptotic behavior of Cauchy additive functional equation and its Pexiderized. Also, we give a simple proofs of Skof's theorem (1.12) and S.-M. Joung's theorem and show that these results is true when E_2 be a complex normed linear space. Furthermore, we present some generalization of Skof's and S.-M. Joung's theorems and give another affirmative answer to problem 18, in the thirty-first ISFE.

1.4. Stability and Common Stability of Functional Equations.

Of the most importance is the linear functional equation or Cauchy linear equation in general form

$$(1.13) \quad f(\rho(x)) = p(x)f(x) + q(x)$$

where ρ , p and q are given functions on an interval I and f is unknown. When $q(x) \equiv 0$ this equation, i.e.,

$$(1.14) \quad f(\rho(x)) = p(x)f(x)$$

is called homogeneous linear equation. We refer the reader to [46] and [1] for numerous results and references concerning this equation and its stability in the sense of Ulam.

In 1991 Baker [13] discussed Hyers-Ulam stability for linear equations (1.13). More concretely, the Hyers-Ulam stability and the generalized Hyers-Ulam-Rassias stability for equation

$$(1.15) \quad f(x + p) = kf(x)$$

were discussed by Lee and Jun [47]. Also the gamma functional equation is a special form of homogeneous linear equation (1.14) were discussed by S. M. Jung [39, 40, 41] proved the modified Hyers-Ulam stability of the gamma functional equation. Thereafter, the stability problem of gamma functional equations has been extended and studied by several mathematicians [7, 14, 37, 43, 44].

Assume that S is a nonempty set, $F = \mathbb{Q}, \mathbb{R}$ or \mathbb{C} , B is a Banach space over F , $\psi : S \rightarrow \mathbb{R}^+$, $f, g : S \rightarrow B$, $p : S \rightarrow K \setminus \{0\}$, $q : S \rightarrow B$ are functions, and $\sigma : S \rightarrow S$ is a arbitrary map.

In the fifth section of this note, we present some results about Hyers-Ulam-Rassias stability via a fixed point approach for the linear functional equation in general form (1.13) and its Pexiderized

$$(1.16) \quad f(\rho(x)) = p(x)g(x) + q(x)$$

under some suitable conditions. Note that the main results of this section can be applied to the well known stability results for the gamma, beta, Abel, Schröder, iterative and G-function type's equations, and also to certain other forms.

During the 13st International Conference on Functional Equations and Inequalities (ICFEI) 2009 , G. L. Forti posed following problem (see [[63], pp. 144]).

Consider functional equations of the form

$$(1.17) \quad \sum_{i=1}^n a_i f\left(\sum_{k=1}^{n_i} b_{ik} x_k\right) = 0 \quad \sum_{i=1}^n a_i \neq 0$$

and

$$(1.18) \quad \sum_{i=1}^m \alpha_i f\left(\sum_{k=1}^{m_i} \beta_{ik} x_k\right) = 0 \quad \sum_{i=1}^m \beta_i \neq 0$$

where all parameters are real's number and $f : \mathbb{R} \rightarrow \mathbb{R}$. Assume that the two functional equations are equivalent, i.e., they have the same set of solutions. Can we say something about the common stability? More precisely, if (1.17) is stable, what can we say about the stability of (1.18). Under which additional conditions the stability of (1.17) implies that of (1.18)?

In connection the above problem, we consider the term of common stability for systems of functional equations. We give the definition of this type stability in fifth section. In connection with the problem of G. L. Forti, we consider some systems of homogeneous linear equations and our aim is to establish some common Hyers-Ulam-Rassias stability for these systems of functional equations. As a consequence of these results, we give some superstability results for the exponential functional equation.

For the readers convenience and explicit later use, we will recall a fundamental results in fixed point theory.

Definition 1.13. *The pair (X, d) is called a generalized complete metric space if X is a nonempty set and $d : X^2 \rightarrow [0, \infty]$ satisfies the following conditions:*

- (1) $d(x, y) \geq 0$ and the equality holds if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$;
- (4) every d -Cauchy sequence in X is d -convergent.

for all $x, y \in X$.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Theorem 1.14. [22] *Let (X, d) be a generalized complete metric space and $J : X \rightarrow X$ be strictly contractive mapping with the Lipschitz constant L . Then for each given element $x \in X$, either*

$$d(J^n(x), J^{n+1}(x)) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n(x), J^{n+1}(x)) < \infty$, for all $n \geq n_0$;

- (2) the sequence $\{J^n(x)\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}(x), y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-L}d(J(y), y)$.

Theorem 1.15. (*Banachs contraction principle*) Let (X, d) be a complete metric space and let $J : X \rightarrow X$ be strictly contractive mapping. Then

- (1) the mapping J has a unique fixed point $x^* = J(x^*)$;
- (2) the fixed point x^* is globally attractive, i.e.,

$$\lim_{n \rightarrow \infty} J^n(x) = x^*$$

for any starting point $x \in X$;

- (3) one has the following estimation inequalities:

$$\begin{aligned} d(J^n(x), x^*) &\leq L^n d(x, x^*), \\ d(J^n(x), x^*) &\leq \frac{1}{1-L} d(J^n(x), J^{n+1}(x)), \\ d(x, x^*) &\leq \frac{1}{1-L} d(J(x), x) \end{aligned}$$

for all nonnegative integers n and all $x \in X$.

2. Superstability of Cauchy Exponential Functional Equation

Some of the results of this section was proved by Mohsen Alimohammady and Ali Sadeghi, which was published see [3]. In the first, we give another proof of Baker's Theorem (1.8). In general, this method appears to be powerful and successful for our aims of this section.

Proof. Let $f : S \rightarrow \mathbb{C}$ be a unbounded function. Assume that $a \in S$ such that $|f(a)| > 1$. Let $\mathfrak{S}'(S, B)$ be the functional equation $f(a \cdot y) = f(a)f(y)$.

Step 1. We prove that the functional equation $\mathfrak{S}'(S, B)$ is HU-stable on (S, B) with controls $(\varepsilon, \frac{\varepsilon}{|f(a)|-1})$ and HU-stable function is unique.

we have

$$(2.1) \quad |f(a \cdot y) - f(a)f(y)| \leq \varepsilon$$

for all $y \in S$. Let us consider the set $A := \{u : S \rightarrow \mathbb{C}\}$ and introduce a metric on A as follows:

$$d(u, h) = \sup_{y \in S} \frac{|u(y) - h(y)|}{\varepsilon}.$$

It is easy to show that (A, d) is a complete metric space. Now we define the function $J : A \rightarrow A$ with

$$J(h(y)) = \frac{1}{f(a)}h(a \cdot y)$$

for all $h \in A$ and $y \in S$. So

$$\begin{aligned} d(J(u), J(h)) &= \sup_{y \in S} \frac{|u(a \cdot y) - h(a \cdot y)|}{|f(a)|\varepsilon} \\ &\leq L \sup_{y \in S} \frac{|u(a \cdot y) - h(a \cdot y)|}{\varepsilon} \leq Ld(u, h) \end{aligned}$$

for all $u, h \in A$, that is J is a strictly contractive selfmapping of A , with the Lipschitz constant $L = \frac{1}{|f(a)|}$. From (2.1), we get

$$\left| \frac{f(a \cdot y)}{f(a)} - f(y) \right| \leq L\varepsilon,$$

for all $y \in S$, which says that $d(J(f), f) \leq L < \infty$. So by Theorem 1.15, there exists a mapping $T : S \rightarrow \mathbb{C}$ such that

(1) T is a fixed point of J , i.e.,

$$(2.2) \quad T(a \cdot y) = f(a)T(y)$$

for all $y \in S$. The mapping T is a unique fixed point of J . So, $T \in \mathfrak{S}'(S, B)$ and is unique HU-stable function.

(2) $d(J^n(f), T) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$T(y) = \lim_{n \rightarrow \infty} \frac{f(a^n \cdot y)}{f(a)^n}$$

for all $y \in S$.

(3) $d(f, T) \leq \frac{1}{1-L}d(J(f), f)$, which implies that $d(f, T) \leq \frac{L}{1-L}$ or

$$(2.3) \quad |f(y) - T(y)| \leq \frac{\varepsilon}{|f(a)| - 1}.$$

for all $y \in S$. The proof of step 1 is complete.

Step 2. We prove that the HU-stable function T have the property $T(x \cdot y) = T(x)f(y)$ and then $f(x \cdot y) = f(x)f(y)$ for all $x, y \in S$.

Let $x, y \in S$ be two arbitrary fixed elements, we have

$$|f(a^n \cdot x \cdot y) - f(a^n \cdot x)f(y)| \leq \varepsilon$$

and dividing by $|f(a)|^n$,

$$\left| \frac{f(a^n \cdot x \cdot y)}{f(a)^n} - \frac{f(a^n \cdot x)}{f(a)^n} f(y) \right| \leq \frac{\varepsilon}{|f(a)|^n}$$

and letting n to infinity, we get

$$(2.4) \quad T(x \cdot y) = T(x)f(y)$$

for all $x, y \in S$. Let $x, y, z \in S$ be arbitrary elements, then

$$T(x \cdot y \cdot z) = T(x)f(y \cdot z)$$

and

$$T(x \cdot y \cdot z) = T(x \cdot y)f(z) = T(x)f(y)f(z)$$

or

$$T(x)(f(y \cdot z) - f(y)f(z)) = 0$$

for all $x, y, z \in S$. Since f is an unbounded function, then from (2.3) implies that the function T is nonzero. Therefore, we have $f(x \cdot y) = f(x)f(y)$ for all $x, y \in S$ (i.e., $f \in \mathfrak{S}(S, B)$). The proof is complete.

In the following, first, we study the stability and superstability of the a Pexider type of Cauchy exponential functional equation

$$f(x \cdot y) = g(x)f(y),$$

for class functions f on commutative semigroup to complex Banach space and given complex-valued function g . For the readers convenience and explicit later use in this section, we present the some notions.

Definition 2.1. Let (S, \cdot) be a semigroup and let $g : S \rightarrow \mathbb{C}$ and $\psi : S^2 \rightarrow [0, \infty)$ be functions, then we denote the set $N_{g, \psi}$ as the all $a \in S$, which $|g(a)| > 1$ and

$$(2.5) \quad \psi(x, y \cdot a) \leq \psi(x, y)$$

for all $x, y \in S$.

Definition 2.2. Let (S, \cdot) be a semigroup, let B be a complex Banach algebra with unit 1_B , and let $f : S \rightarrow B$ be a function, then denote the set M_f as follows:

$$M_f = \{a \in S : f(a) \in \mathbb{C} \times \{1_B\}\}.$$

Also, we introduce the function $\hat{f} : S \rightarrow \mathbb{C}$, where $\hat{f}(a) \times 1_B = f(a)$ if $a \in M_f$ and $\hat{f}(a) = 1$ for another elemnts.

Theorem 2.3. *Let (S, \cdot) be commutative semigroup and B be a complex Banach space, and let $\mathfrak{S}(S, B)$ be the functional equation $f(x \cdot y) = g(x)f(y)$, where $g : S \rightarrow \mathbb{C}$ is a given function. If $N_{g,\psi} \neq \emptyset$, then the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls*

$$(\psi(x, y), \inf_{a \in N_{g,\psi}} \frac{\psi(a, y)}{|g(a)| - 1})$$

such that the HUR-stable function such T is unique and

$$(2.6) \quad (g(x \cdot y) - g(x)g(y))T(z) = 0$$

for all $x, y, z \in S$.

Proof. Let $a \in N_{g,\psi}$ be fixed and let $\mathfrak{S}^a(S, B)$ be the functional equation $f(a \cdot y) = g(a)f(y)$.

Step (1). We prove that the functional equation $\mathfrak{S}^a(S, B)$ is HUR-stable on (S, B) with controls $(\psi(a, y), \frac{\psi(a, y)}{|g(a)| - 1})$ and HUR-stable function such T_a is unique.

We have

$$(2.7) \quad \|f(a \cdot y) - g(a)f(y)\| \leq \psi(a, y)$$

for all $y \in S$. Let us consider the set $A := \{g : S \rightarrow B\}$ and introduce the generalized metric on A :

$$d(g, h) = \sup_{\{y \in S ; \psi(a, y) \neq 0\}} \frac{\|g(y) - h(y)\|}{\psi(a, y)}.$$

It is easy to show that (A, d) is complete metric space. Now we define the function $J_a : A \rightarrow A$ as follows:

$$J_a(h(y)) = \frac{1}{g(a)}h(y \cdot a)$$

for all $h \in A$ and $y \in S$. So

$$\begin{aligned} d(J_a(u), J_a(h)) &= \sup_{\{y \in S ; \psi(a, y) \neq 0\}} \frac{\|u(y \cdot a) - h(y \cdot a)\|}{|g(a)|\psi(a, y)} \\ &\leq \sup_{\{y \in S ; \psi(a, y) \neq 0\}} \frac{\|u(y \cdot a) - h(y \cdot a)\|}{|g(a)|\psi(a, y \cdot a)} \leq \frac{1}{|g(a)|}d(u, h) \end{aligned}$$

for all $u, h \in A$, that is J is a strictly contractive selfmapping of A , with the Lipschitz constant $\frac{1}{|g(a)|}$. From (2.7), we get

$$\left\| \frac{f(y \cdot a)}{g(a)} - f(y) \right\| \leq \frac{\psi(a, y)}{|g(a)|}$$

for all $y \in S$, which says that $d(J(f), f) < \frac{1}{|g(a)|} < \infty$. By Theorem (1.14), there exists a mapping $T_a : S \rightarrow B$ such that

(1) T_a is a fixed point of J , i.e.,

$$(2.8) \quad T_a(y \cdot a) = g(a)T_a(y)$$

for all $y \in S$. The mapping T_a is a unique fixed point of J in the set $\tilde{A} = \{h \in A : d(f, h) < \infty\}$. Hence, $T_a \in \mathfrak{S}^a(S, B)$ and is unique HUR-stable function.

(2) $d(J^n(f), T_a) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$T_a(y) = \lim_{n \rightarrow \infty} \frac{f(y \cdot n^a)}{g(a)^n}$$

for all $x \in S$.

(3) $d(f, T_a) \leq \frac{1}{1 - \frac{1}{|g(a)|}} d(J(f), f)$, which implies,

$$\|f(y) - T_a(y)\| \leq \frac{\psi(a, y)}{|g(a)| - 1}$$

for all $y \in S$ and the proof of step (1) is complete.

Step (2). We prove that the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x, y), \inf_{a \in N_{g, \psi}} \frac{\psi(a, y)}{|g(a)| - 1})$ such that the HUR-stable function such T is unique and $(g(x \cdot y) - g(x)g(y))T(z) = 0$.

From (2.7), its easy to show that following inequality

$$(2.9) \quad \|f(y \cdot a^n) - g(a)^n f(y)\| \leq \sum_{i=0}^{n-1} \psi(a, y \cdot a^i) |g(a)|^{n-1-i}$$

for all $y, a \in S$ and $n \in \mathbb{N}$. Now since

$$\psi(x, y \cdot a) \leq \psi(x, y)$$

for all $y \in S$ and all $a \in N_{g, \psi}$, so

$$(2.10) \quad \psi(a, y \cdot a^m) \leq \psi(a, y)$$

for all $x \in S$ and $m \in \mathbb{N}$, thus from (2.9), we obtain

$$(2.11) \quad \|f(y \cdot a^n) - g(a)^n f(y)\| \leq \psi(a, y) \frac{|g(a)|^n - 1}{|g(a)| - 1}$$

for all $y \in S$. Our aim is to prove that $T_a = T_b$ for each $a, b \in N_{g,\psi}$. We have from inequality (2.11)

$$(2.12) \quad \|f(y \cdot a^n) - g(a)^n f(y)\| \leq \psi(a, y \cdot a^n) \frac{|g(a)|^n - 1}{|g(a)| - 1}$$

$$(2.13) \quad \|f(y \cdot b^n) - g(b)^n f(y)\| \leq \psi(b, y \cdot b^n) \frac{|g(b)|^n - 1}{|g(b)| - 1}$$

for all $y \in S$. On the replacing y by $y \cdot b^n$ in (2.12) and y by $y \cdot a^n$ in (2.13) and so from (2.10), we get

$$\|f(y \cdot (a \cdot b)^n) - g(a)^n f(y \cdot b^n)\| \leq \psi(a, y) \frac{|g(a)|^n - 1}{|g(a)| - 1}$$

$$\|f(y \cdot (a \cdot b)^n) - g(b)^n f(y \cdot a^n)\| \leq \psi(b, y) \frac{|g(b)|^n - 1}{|g(b)| - 1}.$$

Thus,

$$\|g(a)^n f(y \cdot b^n) - g(b)^n f(y \cdot a^n)\| \leq \psi(a, y) \frac{|g(a)|^n}{|g(a)| - 1} + \psi(b, y) \frac{|g(b)|^n}{|g(b)| - 1}$$

and dividing by $|g(a)^n g(b)^n|$

$$\left\| \frac{f(y \cdot a^n)}{g(a)^n} - \frac{f(y \cdot b^n)}{g(b)^n} \right\| \leq \frac{\psi(a, y)}{|g(b)|^n (|g(a)| - 1)} + \frac{\psi(b, y)}{|g(a)|^n (|g(b)| - 1)}$$

and since $|g(a)| > 1$ for any $a \in N_{g,\psi}$, letting n to infinity, we obtain $T_a(y) = T_b(y)$ for all $y \in S$. So, there a unique function T such that $T = T_a$ for every $a \in N_{g,\psi}$ and

$$\|f(y) - T(y)\| \leq \frac{\psi(a, y)}{|g(a)| - 1}$$

for all $y \in S$ and $a \in N_g$. Since $a \in N_{g,\psi}$ is an arbitrary element, so

$$\|f(y) - T(y)\| \leq \inf_{a \in N_g} \frac{\psi(a, y)}{|g(a)| - 1}$$

for all $y \in S$.

Let $x, y \in S$ and $a \in N_{g,\psi}$ be three arbitrary fixed elements, we have

$$\|f(x \cdot y \cdot a^n) - g(x) f(y \cdot a^n)\| \leq \psi(x, y \cdot a^n),$$

from (2.10) and dividing last inequality with $|g(a)|^n$, we get

$$\left\| \frac{f(x \cdot y \cdot a^n)}{g(a)^n} - g(y) \frac{f(x \cdot a^n)}{g(a)^n} \right\| \leq \frac{\psi(x, y)}{|g(a)|^n}$$

and letting n to infinity, we get $T(x \cdot y) = g(x)T(y)$.

Let $x, y, z \in S$ be arbitrary elements, then

$$T(x \cdot y \cdot z) = g(x \cdot y)T(z)$$

and

$$T(x \cdot y \cdot z) = g(x)T(y \cdot z) = g(x)g(y)T(z)$$

or

$$(g(x \cdot y) - g(x)g(y))T(z) = 0$$

for all $x, y, z \in S$. The proof is complete.

In connection with the problem of Th. M. Rassias and Theorem (2.3), in the following, we prove some extensions of Baker's theorem (1.8) and also we prove a generalized version of L. Székelyhidi's theorem (1.9). Note that in the Definition (2.1), if the function ψ is constant such ϵ , then either f is bounded or unbounded function if and only if either $f(N_{f,\epsilon})$ is bounded or unbounded set.

Corollary 2.4. *Let (S, \cdot) be commutative semigroup and let $\mathfrak{S}(S, \mathbb{C})$ be the functional equation $f(x \cdot y) = f(x)f(y)$. If $f : S \rightarrow \mathbb{C}$ is a function such that $f(N_{f,\psi})$ is an unbounded set, then $f \in Z_{\mathfrak{S}(S,\mathbb{C})}$.*

Proof. In Theorem (2.3), if we put $B := \mathbb{C}$ and $g := f$, then we have

$$|f(y) - T(y)| \leq \inf_{a \in N_{f,\psi}} \frac{\psi(a, y)}{|f(a)| - 1}$$

and

$$T(x \cdot y) = f(x)T(y)$$

for all $x, y \in S$. Now if $f(N_{f,\psi})$ is an unbounded set, then $f = T$ and so $f \in Z_{\mathfrak{S}(S,B)}$.

Corollary 2.5. *Let (S, \cdot) be commutative semigroup and let $\mathfrak{S}(S, \mathbb{C})$ be the functional equation $f(x \cdot y) = f(x)f(y)$. If $f : S \rightarrow \mathbb{C}$ is a function and there exist $a \in N_{f,\psi}$ such that $\psi(a, y) = 0$ for any $y \in S$ or there is $x_0 \in S$ such that the following limit exists and*

$$\lim_{n \rightarrow \infty} \frac{f(x_0 \cdot a^n)}{(f(a))^n} \neq 0,$$

then $f \in Z_{\mathfrak{S}(S,\mathbb{C})}$.

Proof. In Theorem (2.3), if we put $B := \mathbb{C}$ and $g(x) := f(x)$, then we have

$$|f(y) - T(y)| \leq \inf_{a \in N_{f,\psi}} \frac{\psi(a, y)}{|f(a)| - 1}$$

and $(f(x \cdot y) - f(x)f(y))T(z) = 0$ for all $x, y, z \in S$. If $\psi(a, y) = 0$ for any $y \in S$, then $f = T$ and the proof is done. For another case, according to the proof of Theorem (2.3) and assumption, we have $T(x_0) \neq 0$ and so that $f(x \cdot y) = f(x)f(y)$ for all $x, y \in S$, where that is $f \in Z_{\mathfrak{S}(S, \mathbb{C})}$.

Corollary 2.6. *Let (S, \cdot) be commutative semigroup and B be a complex Banach space, and let $\mathfrak{S}(S, B)$ be the functional equation $f(x \cdot y) = g(x)f(y)$, where $g : S \rightarrow \mathbb{C}$ is a given function. If $g(N_{g, \psi})$ is an unbounded set, then the functional equation $\mathfrak{S}(S, \mathbb{C})$ is hyperstable on (S, B) with control $(\psi(x, y))$ and $f(x) = g(x)f(1)$ for all $x \in S$.*

Proof. With Theorem (2.3), if the set $g(N_{g, \psi})$ is an unbounded set, then $f = T$, which implies $f(x) = g(x)f(1)$ for all $x \in S$ and since

$$(g(x + y) - g(x)g(y))T(z) = 0$$

for any $x, y, z \in S$ and $f = T$ is a nonzero function for any nonzero function f , so $f \in Z_{\mathfrak{S}(S, \mathbb{C})}$.

In [11], Baker presented an example to show that

$$\|f(x + y) - f(x)f(y)\| \leq \varepsilon \quad \text{for } x, y \in S$$

implies that f is either bounded or exponential fails if the algebra does not have the multiplicative norm property. Here, we extend this result another way and conditions in unitary complex Banach algebra.

Theorem 2.7. *Let S be commutative semigroup, let B be a complex Banach algebra with unit 1_B , and let $\mathfrak{S}(S, B)$ be the functional equation $f(x \cdot y) = g(x)f(y)$, where $g : S \rightarrow B$ is a given function. If $N_{(\hat{g}, \psi)} \neq \emptyset$, then the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x, y), \inf_{a \in N_{\hat{g}, \psi}} \frac{\psi(a, y)}{|\hat{g}(a)| - 1})$ such that the HUR-stable function such T is unique and*

$$(2.14) \quad (g(x \cdot y) - g(x)g(y))T(z) = 0$$

for all $x, y, z \in S$.

Proof. Let $a \in N_{\hat{g}, \psi}$ be an arbitrary fixed. We have

$$\begin{aligned} \|f(a \cdot y) - g(a)f(y)\| &= \|f(a \cdot y) - \hat{g}(a)(1_B f(y))\| \\ &= \|f(a \cdot y) - \hat{g}(a)f(y)\| \leq \psi(a, y) \end{aligned}$$

thus, $\|f(y \cdot a) - \hat{g}(a)f(y)\| \leq \psi(a, y)$ for all $y \in S$. So from Theorem (2.3), there is a unique function $T : S \rightarrow B$ such that

$$T(x \cdot y) = \hat{g}(x)T(y)$$

$$[\widehat{g}(x \cdot y) - \widehat{g}(x)\widehat{g}(y)]T(z) = 0$$

and satisfying

$$(2.15) \quad \|f(y) - T(y)\| \leq \inf_{a \in N_{\widehat{g}}} \left[\frac{\psi(a, y)}{|\widehat{g}(a)| - 1} \right]$$

for all $x, y, z \in S$. We have

$$(2.16) \quad \|f(x \cdot y \cdot a^n) - g(x)f(y \cdot a^n)\| \leq \psi(x, y \cdot a^n)$$

then on the dividing by $|\widehat{g}(a)|^n$ we see that

$$(2.17) \quad \left\| \frac{f(x \cdot y \cdot a^n)}{\widehat{g}(a)^n} - g(x) \frac{f(y \cdot a^n)}{\widehat{g}(a)^n} \right\| \leq \frac{\psi(x, y \cdot a^n)}{|\widehat{g}(a)|^n} \leq \frac{\psi(x, y)}{|\widehat{g}(a)|^n}.$$

Hence, $T(x \cdot y) = g(x)T(y)$ for all $x, y \in S$. Now let $x, y, z \in S$ be arbitrary elements, then

$$T(x \cdot y \cdot z) = g(x \cdot y)T(z)$$

and

$$T(x \cdot y \cdot z) = g(x)T(y \cdot z) = g(x)g(y)T(z)$$

so,

$$(g(x \cdot y) - g(x)g(y))T(z) = 0$$

for all $x, y, z \in S$. The proof is complete.

In the following, we generalize the well-known Baker's superstability and stability result for exponential mappings with values in the field of complex numbers to the case of an arbitrary unitary complex Banach algebra.

Corollary 2.8. *Let (S, \cdot) be commutative semigroup, B be a complex Banach algebra with unit 1_B , and let $\mathfrak{S}(S, B)$ be the functional equation $f(x \cdot y) = f(x)f(y)$. If $f : S \rightarrow B$ a function such that $f(N_{\widehat{f}, \psi})$ is an unbounded set, then $f \in Z_{\mathfrak{S}(S, B)}$.*

Proof. In Theorem (2.7), if we put $g(x) := f(x)$, then we will had

$$\|f(x) - T(x)\| \leq \inf_{a \in N_{\widehat{f}, \psi}} \left[\frac{\psi(a, x)}{|f(a)| - 1} \right] \text{ and } (f(x + y) - f(x)f(y))T(z) = 0$$

for all $x, y, z \in S$. Now since $\widehat{f}(N_{\widehat{f}, \psi})$ is unbounded set, then we have $f = T$, which says that $f \in Z_{\mathfrak{S}_f}$ and the proof is complete.

Corollary 2.9. *Let (S, \cdot) be commutative semigroup with identity 1, B be a complex Banach algebra with unit 1_B , and let $\mathfrak{S}(S, B)$ be the functional equation $f(1)f(x \cdot y) = g(x)f(y)$, where $g : S \rightarrow \mathbb{C}$ is a given function. If $\widehat{g}(N_{\widehat{g}, \psi})$ is unbounded set, then the functional equation $\mathfrak{S}(S, B)$ is hyperstable on (S, B) with control $(\psi(x, y))$ and $f(x) = g(x)f(1)$ for all $x \in S$.*

Proof. With Theorem (2.7).

In the following Theorem, we consider the superstability of the a Pexiderized of exponential equation

$$f(x + y) = g(x)h(y),$$

in which f, g and h are three functions from a commutative semigroup to an unitary an complex Banach algebra.

Theorem 2.10. *Let (S, \cdot) be commutative semigroup and B be a complex Banach algebra with unit 1_B . Let $f, g, h : S \rightarrow B$ be three functions and $g(x_0) = 1_B$ for a fixed $x_0 \in S$ and also*

$$(2.18) \quad \|f(x \cdot y) - g(x)h(y)\| \leq \psi(x, y)$$

for all $x, y \in S$. If $N_{\widehat{g}, \psi} \neq \emptyset$, then there exists a exactly one function $T : S \rightarrow B$ such that

$$\begin{aligned} T(x \cdot y) &= g(x)T(y), \\ (g(x \cdot y) - g(x)g(y))T(z) &= 0 \end{aligned}$$

and satisfies

$$\begin{aligned} \|f(y) - T(y)\| &\leq \inf_{a \in N_{\widehat{g}, \psi}} \frac{\widetilde{\psi}(a, y)}{|\widehat{g}(a)| - 1}, \\ \|h(y) - T(y)\| &\leq \inf_{a \in N_{\widehat{g}, \psi}} \frac{\widehat{\psi}(a, y)}{|\widehat{g}(a)| - 1} \end{aligned}$$

for all $x, y, z \in S$, in which $\widetilde{\psi}(x, y) = \psi(x, y) + \|g(x)\|\psi(x_0, y)$ and $\widehat{\psi}(x, y) = \psi(x, y) + \psi(x_0, x \cdot y)$ for $x, y \in S$.

Proof. Applying (2.20) we get for all $x, y \in S$

$$\begin{aligned} \|f(x \cdot y) - g(x)f(y)\| &\leq \|f(x \cdot y) - g(x)h(y)\| + \|g(x)f(y) - g(x)h(y)\| \\ &\leq \psi(x, y) + \|g(x)\|\psi(x_0, y) \end{aligned}$$

and

$$\begin{aligned} \|h(x \cdot y) - g(x)h(y)\| &\leq \|h(x \cdot y) - f(x \cdot y)\| + \|f(x \cdot y) - g(x)h(y)\| \\ &\leq \psi(x, y) + |g(x_0)|\psi(x_0, x \cdot y) \end{aligned}$$

We set $\tilde{\psi}(x, y) = \psi(x, y) + \|g(x)\|\psi(x_0, y)$ and $\hat{\psi}(x, y) = \psi(x, y) + \psi(x_0, x \cdot y)$ for $x, y \in S$ and these are obvious that

$$\tilde{\psi}(x, y \cdot a) \leq \tilde{\psi}(x, y)$$

and

$$\hat{\psi}(x, y \cdot a) \leq \hat{\psi}(x, y)$$

for $x, y \in S$ and $a \in N_{\hat{g}, \psi}$. Therefore by Theorem (2.7), then there exists a exactly one function $H : S \rightarrow B$ such that

$$\begin{aligned} H(x + y) &= g(x)H(y) \\ (g(x + y) - g(x)g(y))H(z) &= 0 \end{aligned}$$

and satisfies

$$\|f(y) - H(y)\| \leq \inf_{a \in N_{\hat{g}, \psi}} \frac{\tilde{\psi}(a, y)}{|\hat{g}(a) - 1|}$$

for all $x, y, z \in S$, where $H(x) = \lim_{n \rightarrow \infty} \frac{f(x \cdot a^n)}{\hat{g}(a)^n}$ for all $x \in S$ and any fixed $a \in N_{\hat{g}, \psi}$. And also then there exists a exactly one function $F : S \rightarrow B$ such that

$$\begin{aligned} F(x + y) &= g(x)F(y) \\ (g(x + y) - g(x)g(y))F(z) &= 0 \end{aligned}$$

and satisfies

$$\|h(y) - F(y)\| \leq \inf_{a \in N_{\hat{g}, \psi}} \frac{\hat{\psi}(a, y)}{|\hat{g}(a) - 1|}$$

for all $x, y, z \in S$, where $F(x) = \lim_{n \rightarrow \infty} \frac{h(x \cdot a^n)}{\hat{g}(a)^n}$ for all $x \in S$ and any fixed $a \in N_{\hat{g}, \psi}$. Furthermore, we have

$$\begin{aligned} \left\| \frac{f(x \cdot a^n)}{\hat{g}(a)^n} - \frac{h(x \cdot a^n)}{\hat{g}(a)^n} \right\| &= |\hat{g}(a)|^{-n} \|f(x \cdot a^n) - h(x \cdot a^n)\| \\ &\leq \frac{|g(x_0)|\psi(x_0, x \cdot a^n)}{\hat{g}(a)^n} \leq \frac{|g(x_0)|\psi(x_0, x)}{\hat{g}(a)^n} \end{aligned}$$

for all $x \in S$ and any fixed $a \in N_{\hat{g}, \psi}$. Hence, $H = F$ and so there exists an exactly one function $T : S \rightarrow X$ such that

$$\begin{aligned} T(x \cdot y) &= g(x)T(y), \\ (g(x \cdot y) - g(x)g(y))T(z) &= 0 \end{aligned}$$

and satisfies

$$\|f(y) - T(y)\| \leq \inf_{a \in N_{\hat{g}, \psi}} \frac{\tilde{\psi}(a, y)}{|\hat{g}(a) - 1|},$$

$$\|h(y) - T(y)\| \leq \inf_{a \in N_{\hat{g}, \psi}} \frac{\hat{\psi}(a, y)}{|\hat{g}(a)| - 1}$$

for all $x, y, z \in S$. The proof is complete.

Corollary 2.11. *Let (S, \cdot) be commutative semigroup and B be complex field. Let $f, g, h : S \rightarrow \mathbb{C}$ be three functions and $g(x_0) = 1$ for a fixed $x_0 \in S$ and also*

$$(2.19) \quad \|f(x \cdot y) - g(x)h(y)\| \leq \psi(x, y)$$

for all $x, y \in S$. If $g(N_{g, \psi})$ is an unbounded set, then $f(x \cdot y) = g(x)h(y)$ for all $x, y \in S$, $f = h$, and $f(x) = f(1)g(x)$ for all $x \in S$.

Proof. Applying Theorem (2.10), we get, there exists a exactly one function $T : S \rightarrow \mathbb{C}$ such that

$$\begin{aligned} T(x \cdot y) &= g(x)T(y), \\ (g(x \cdot y) - g(x)g(y))T(z) &= 0 \end{aligned}$$

and satisfies

$$\begin{aligned} \|f(y) - T(y)\| &\leq \inf_{a \in N_{\hat{g}, \psi}} \frac{\tilde{\psi}(a, y)}{|\hat{g}(a)| - 1}, \\ \|h(y) - T(y)\| &\leq \inf_{a \in N_{\hat{g}, \psi}} \frac{\hat{\psi}(a, y)}{|\hat{g}(a)| - 1} \end{aligned}$$

for all $x, y, z \in S$. Since $g = \hat{g}$ and $g(N_{g, \psi})$ is an unbounded set, so $f = h = T$ and so that $f(x) = g(x)f(1)$ for all $x \in S$. The proof is complete.

Therefore, with above Corollary, we give a version of Baker's Theorem (1.8) for Pexiderized of exponential functional equation.

Corollary 2.12. *Let (S, \cdot) be commutative semigroup and B be complex field. Let $f, g, h : S \rightarrow \mathbb{C}$ be three functions and $g(x_0) = 1$ for a fixed $x_0 \in S$ and also*

$$(2.20) \quad \|f(x \cdot y) - g(x)h(y)\| \leq \epsilon$$

for all $x, y \in S$ for some $\epsilon > 0$. If the function g is an unbounded function, then $f(x \cdot y) = g(x)h(y)$ for all $x, y \in S$, $f = h$, and $f(x) = f(1)g(x)$ for all $x \in S$.

Proof. If we set $\psi(x, y) := \epsilon$ for all $x, y \in S$. Its obvious that $N_{g, \epsilon} = \{a \in S : |g(a)| > 1\}$, where implies that $g(N_{g, \epsilon})$ is an unbounded set and so that Corollary (2.11) complete the proof.

Some Remarks About Homogeneous Functional Equation

The functional equation

$$(2.21) \quad f(yx) = y^k f(x)$$

(where k is a fixed real constant) is called the "homogeneous functional equation" of degree k . In the case when $k = 1$ in the equation (2.21), the equation is simply called the "homogeneous functional equation". In general, " ϕ -homogeneous functional equation" is the following equation:

$$(2.22) \quad f(\alpha x) = \phi(\alpha) f(x)$$

for suitable class function and spaces. In [21], S. Czerwik considered the Pexiderized homogeneous functional equation and he obtained the following result:

Theorem 2.13. *Let V be a real linear space and E a real Banach space. Let $f, g : V \rightarrow E$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R} \times E \rightarrow \mathbb{R}^+$ be given functions. Assume that*

$$\|f(\alpha x) - \phi(\alpha)g(x)\| \leq h(\alpha, x)$$

for all $(\alpha, x) \in \mathbb{R} \times V$ and $\phi(1) = 1$. Suppose that there exists $\beta \in \mathbb{R}$ such that $\phi(\beta) \neq 0$ and the series

$$\sum_{n=1}^{\infty} |\phi(\beta)|^{-n} H(\beta, \beta^n x)$$

converges pointwise for all $x \in V$, and

$$\liminf_{n \rightarrow \infty} |\phi(\beta)|^{-n} H(\alpha, \beta^n x) = 0$$

for all $(\alpha, x) \in \mathbb{R} \times V$, where $H(\alpha, x) := h(\alpha, x) + |\phi(\alpha)|h(1, x)$. Then there exists exactly one ϕ -homogeneous function $A : V \rightarrow E$:

$$A(\alpha x) = \phi(\alpha)A(x)$$

for all $(\alpha, x) \in \mathbb{R} \times V$ such that

$$\|f(x) - A(x)\| \leq \sum_{n=1}^{\infty} |\phi(\beta)|^{-n} H(\beta, \beta^{n-1}x),$$

$$\|g(x) - A(x)\| \leq \sum_{n=1}^{\infty} |\phi(\beta)|^{-n} G(\beta, \beta^{n-1}x),$$

for all $x \in V$, where $G(\alpha, x) := h(\alpha, x) + h(1, \alpha x)$.

For more information about this subject one can refer to ([38], Chap. 5). Since in this section, we study the exponential functional equation and its Pexiderized on semigroup domain, so the results (2.3), (2.6), (2.7), (2.9), (2.10), (2.11), and (2.12) is applied for study of superstability and stability of the homogeneous functional equation and its Pexiderized. Hence, we don't present the similarly results for homogeneous functional equation.

3. Superstability of the Cauchy Additive Functional Equation on Semigroups

The results of this section was proved by Mohsen Alimohammady and Ali Sadeghi, which was published see [4]. Throughout of this section, assume that (S, \cdot) is an arbitrary commutative semigroup, \mathbb{C} is the field of all complex numbers, \mathbb{R} is real field, $\widehat{\mathbb{C}} = \{z \in \mathbb{C} \mid -\pi < \arg(z) \leq \pi\}$, $\psi : S \rightarrow \mathbb{R}^+$ and $\phi : S^2 \rightarrow \mathbb{R}^+$ are some functions. Also for the function $f : S \rightarrow \widehat{\mathbb{C}}$, then $f(S)^+$ is a subset of $\widehat{\mathbb{C}}$, where $f(S)^+ = \{p \in S \mid \operatorname{Re}(f(p)) > 0\}$. In this section, we call $f : S \rightarrow \widehat{\mathbb{C}}$ is a Cauchy function, if

$$f(x \cdot y) - f(x) - f(y) = 0$$

for all $x, y \in S$.

Theorem 3.1. *Let $f : S \rightarrow \widehat{\mathbb{C}}$ is a function and*

$$(3.1) \quad |f(x \cdot y) - f(x) - f(y)| \leq \phi(x, y);$$

$$(3.2) \quad |f(x)| \leq \psi(x)$$

for all $x, y \in S$. Assume that there exists $p \in f(S)^+$ such that

$$(3.3) \quad \sum_{m=0}^{\infty} \phi(p, p^{m+1}) < \infty;$$

$$(3.4) \quad \psi(x \cdot p) \leq \psi(x)$$

for all $x \in S$. Then f is a Cauchy function.

Note that the above Theorem is partial affirmative answer to problem 18, in the thirty-first ISFE. Moreover, we present a superstability result for the logarithmic functional equation.

Proof of Theorem. Let $E : \mathbb{C} \rightarrow \mathbb{C}$ be exponential function, where $E(a) = \exp(a)$ for each $a \in \mathbb{C}$. Now from (3.2), we have

$$\begin{aligned} |E(f(x \cdot y)) - E(f(x) + f(y))| &\leq |E(f(x \cdot y))| + |E(f(x) + f(y))| \\ &\leq E(|f(x \cdot y)|) + E(|f(x) + f(y)|) \\ &\leq E(\psi(x \cdot y)) + E(\psi(x) + \psi(y)) \end{aligned}$$

for all $x, y \in S$. So, the function $\widehat{E} : S \rightarrow \mathbb{C}$ with $\widehat{E} = (E \circ f)(x) = \exp(f(x))$ for all $x \in S$, satisfies the following inequality

$$(3.5) \quad |\widehat{E}(x \cdot y) - \widehat{E}(x)\widehat{E}(y)| \leq \varphi(x, y)$$

for all $x, y \in S$, in which $\varphi(x, y) := E(\psi(x \cdot y)) + E(\psi(x) + \psi(y))$ for all $x, y \in S$. From assumption $\psi(x \cdot p) \leq \psi(x)$ for all $x \in S$. So, its easy to show that

$$(3.6) \quad \varphi(x, y \cdot p) \leq \varphi(x, y)$$

for all $x, y \in S$. We are going to show that $\widehat{E}(x \cdot y) - \widehat{E}(x)\widehat{E}(y) = 0$ for all $x, y \in S$. Now let us consider the set $\mathcal{A} := \{h : S \rightarrow \mathbb{C}\}$ and introduce the generalized metric on \mathcal{A} :

$$d(u, h) = \sup_{x \in S} \frac{|u(x) - h(x)|}{\varphi(p, x)}.$$

It is easy to show that (\mathcal{A}, d) is generalized complete metric space. Now we define the function $J : \mathcal{A} \rightarrow \mathcal{A}$ with

$$J(h(x)) = \frac{1}{\widehat{E}(p)} h(p \cdot x)$$

for all $h \in \mathcal{A}$ and $x \in S$. Since $\varphi(x, y \cdot p) \leq \varphi(x, y)$ for all $x \in S$, so

$$\begin{aligned} d(J(u), J(h)) &= \sup_{x \in S} \frac{|u(p \cdot x) - h(p \cdot x)|}{|\widehat{E}(p)|\varphi(p, x)} \\ &\leq \sup_{x \in S} \frac{|u(p \cdot x) - h(p \cdot x)|}{|\widehat{E}(p)|\varphi(p, p \cdot x)} \\ &\leq \frac{1}{|\widehat{E}(p)|} d(u, h) \end{aligned}$$

for all $u, h \in \mathcal{A}$, that is J is a strictly contractive selfmapping of \mathcal{A} , with the Lipschitz constant $L = \frac{1}{|\widehat{E}(p)|}$ (note that $|\widehat{E}(p)| = |\exp(f(p))| =$

$\exp(\operatorname{Re}(f(p))) > 1$). From (3.5), we get

$$\left| \frac{\widehat{E}(p \cdot x)}{\widehat{E}(p)} - \widehat{E}(x) \right| \leq \frac{\varphi(p, x)}{|\widehat{E}(p)|}$$

for all $x \in S$, which says that $d(J(\widehat{E}), \widehat{E}) \leq L < \infty$. By Theorem (1.14), there exists a mapping $T : G \rightarrow \mathbb{C}$ such that

(1) T is a unique fixed point of J , i.e.,

$$(3.7) \quad T(p \cdot x) = \widehat{E}(p)T(x)$$

for all $x \in S$.

(2) $d(J^n(\widehat{E}), T) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$(3.8) \quad T(x) = \lim_{n \rightarrow \infty} \frac{\widehat{E}(p^n \cdot x)}{\widehat{E}^n(p)}$$

for all $x \in S$.

(3) $d(\widehat{E}, T) \leq \frac{1}{1-L}d(J(\widehat{E}), \widehat{E})$, which implies,

$$d(\widehat{E}, T) \leq \frac{1}{|\widehat{E}(p)| - 1}.$$

Let $x, y \in S$ be two arbitrary fixed elements, from (3.5) and (3.6)

$$|\widehat{E}(p^n \cdot (x \cdot y)) - \widehat{E}(x)\widehat{E}(p^n \cdot y)| \leq \varphi(x, p^n \cdot y)$$

and dividing by $|\widehat{E}(p)|^n$,

$$\left| \frac{\widehat{E}(p^n \cdot (x \cdot y))}{\widehat{E}(p)^n} - \widehat{E}(x) \frac{\widehat{E}(p^n \cdot y)}{\widehat{E}(p)^n} \right| \leq \frac{\varphi(x, p^n \cdot y)}{|\widehat{E}(p)|^n} \leq \frac{\varphi(x, y)}{|\widehat{E}(p)|^n}$$

and letting n to infinity, we get $T(x \cdot y) = \widehat{E}(x)T(y)$, which says that T is a \widehat{E} -homogeneous function.

Let $x, y, z \in S$ be arbitrary elements, then

$$T((x \cdot y) \cdot z) = \widehat{E}(x \cdot y)T(z)$$

and

$$T((x \cdot y) \cdot z) = \widehat{E}(x)T(y \cdot z) = \widehat{E}(x)\widehat{E}(y)T(z)$$

or

$$(3.9) \quad (\widehat{E}(x \cdot y) - \widehat{E}(x)\widehat{E}(y))T(z) = 0.$$

Now we show that $T(p) \neq 0$. From (3.1), its easy to show that following inequality

$$|Re(f(p^{n+1}) - (n+1)f(p))| \leq |f(p^{n+1}) - (n+1)f(p)| < \sum_{i=0}^{n-1} \phi(p, p^{i+1})$$

for all $n \in \mathbb{N}$. Also, from relation (3.8), we obtain

$$(3.10) \quad |T(p)| = \lim_{n \rightarrow \infty} \left| \frac{\widehat{E}(p^{n+1})}{\widehat{E}^n} \right|$$

$$(3.11) \quad = \lim_{n \rightarrow \infty} |exp(f(p^{n+1}) - (n+1)f(p))|$$

$$(3.12) \quad = \lim_{n \rightarrow \infty} exp(Re(f(p^n) - (n)f(p))).$$

Since $|T(p)| < \infty$, So $\lim_{n \rightarrow \infty} exp(Re(f(p^n) - (n)f(p)))$ there exist. Now from (3.6) and (3.3), we obtain

$$\lim_{n \rightarrow \infty} |Re(f(p^{n+1}) - (n+1)f(p))| < \infty,$$

where it implies that $|T(p)| \neq 0$. From (3.9), we get

$$\widehat{E}(x \cdot y) = \widehat{E}(x)\widehat{E}(y)$$

or

$$\exp(f(x+y)) = \exp(f(x) + f(y))$$

for all $x, y \in S$. Since exponential function is one-to-one on $\widehat{\mathbb{C}}$, so f is a Cauchy function. The proof is complete.

Corollary 3.2. *Let S be a multiplicative semigroup of \mathbb{R} and $L : S \rightarrow \mathbb{R}$ be a function such that*

$$(3.13) \quad |L(xy) - L(x) - L(y)| \leq \phi(x, y);$$

$$(3.14) \quad |L(x)| \leq \psi(x)$$

for all $x, y \in S$. Assume that there exists $p \in S$ such that $L(p) \neq 0$ and

$$(3.15) \quad \sum_{m=0}^{\infty} \phi(p, p^{m+1}) < \infty;$$

$$(3.16) \quad \psi(xp) \leq \psi(x)$$

for all $x \in S$. Then L is a Cauchy function or a logarithmic function i.e.; $L(xy) = L(x) + L(y)$ for all $x, y \in S$.

Proof. If $L(p) < 0$, set $f := -L$ and if $L(p) > 0$, set $f := L$. Now applying Theorem (3.1), we get the result.

4. Asymptotic Behavior of Cauchy Additive Functional Equations

The results of this section was proved by Mohsen Alimohammady and Ali Sadeghi, which was published see [5]. Throughout this section, assume that $(S, +)$ is a commutative semigroup, E_1, E_2 are two complex normed spaces, \mathbb{R} is real field, \mathbb{N} is all positive integers, and $\psi : S^2 \rightarrow [0, \infty)$ is a function.

The following Theorem is a affirmative answer to problem 18, in the thirty-first ISFE.

Theorem 4.1. *Let $\mathfrak{S}(S, E_1)$ be the functional equation $f(x+y) = f(x) + f(y)$. Assume that*

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0;$
- $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$

for any fixed $x_0, y_0, x, y \in S$. Then the functional equation $\mathfrak{S}(S, E_1)$ is hyperstable on (S, E_1) with control (ψ) .

Proof. Let x_0 be any fixed element of S and we have

$$\|f(x + x_0) - f(x_0) - f(x)\| \leq \psi(x, x_0)$$

for all $x \in S$. From last inequality, its easy to show that the following inequality

$$\|f(x + nx_0) - nf(x_0) - f(x)\| \leq \sum_{i=0}^{n-1} \psi(x + ix_0, x_0)$$

for each fixed $x \in S$ and $n \in \mathbb{N}$. Now by assumption $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0$, so

$$f(x_0) = \lim_{n \rightarrow \infty} \frac{f(x + nx_0)}{n}$$

for any fixed $x \in S$. Let x_0, y_0 be any two fixed elements of S , then we have

$$\|f(x + y + n(x_0 + y_0)) - f(x + nx_0) - f(y + ny_0)\| \leq \psi(x + nx_0, y + ny_0)$$

for any fixed $x, y \in S$. Now since $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$, thus

$$f(x_0 + y_0) = f(x_0) + f(y_0),$$

which says that $f \in Z_{\mathfrak{S}(S, B)}$ and the proof is complete.

As a consequence of the following result, its show that the Theorem of Rassias (1.6) for the case $p < 0$ is hypestable.

Corollary 4.2. *Let $\mathfrak{S}(E_1, E_2)$ be the functional equation $f(x + y) = f(x) + f(y)$. Then for every real's $\epsilon > 0$, $p < 0$ and $q \leq 1$, the functional equation $\mathfrak{S}(S, B)$ is hyperstable on (S, B) with control $(\epsilon(\|x\|^p + \|y\|^q))$.*

Proof. Set $\psi(x, y) := (\epsilon(\|x\|^p + \|y\|^q))$ for all $x, y \in S$. Since the sequence $\sum_{i=0}^{n-1} \psi(x + ix_0, x_0)$ for any fixed x, y, x_0, y_0 is increasing sequences and also $p < 0$ and $q \leq 1$, so that obvious that the followings relations:

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0$;
- $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$

for any fixed $x_0, y_0, x, y \in S$. Now Theorem (4.1) implies that the result and the proof is complete.

In the following, by using Theorem (4.1), we give a simple proof of Skof theorem (1.12) and also we show that Skof theorem is true when E_2 be a complex normed space.

Theorem 4.3. *For a function $f : E_1 \rightarrow E_2$ the following two conditions are equivalent:*

- (1) $\|f(x + y) - f(x) - f(y)\| \rightarrow 0$ as $\|x\| + \|y\| \rightarrow \infty$;
- (2) $f(x + y) - f(x) - f(y) = 0$

for all $x, y \in E_1$.

Proof. Set $\psi(x, y) := \|f(x + y) - f(x) - f(y)\|$ for all $x, y \in E_1$. Now let $x_0, y_0 \in E_1$ be two arbitrary fixed elements. Since $\|x + nx_0\| + \|y + ny_0\| \rightarrow \infty$ for each fixed $x, y \in E_1$, so

$$\lim_{n \rightarrow \infty} \psi(x + nx_0, y + ny_0) = 0,$$

for each fixed $x, y \in E_1$, hence its easy to show that the following relations

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0$;
- $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$

for each fixed $x, y \in E_1$. Now with Theorem (4.1) implies that f is an additive mapping (i.e., $f(x + y) = f(x) + f(y)$). The proof is complete.

Let \mathfrak{S} be set all function $\rho : E_1^2 \rightarrow [0, \infty)$ such that

- (1) $\rho(x + nx_0, y + ny_0) \rightarrow \infty$ as $n \rightarrow \infty$

for any fixed $x_0, y_0, x, y \in E_1$. Not that the functions $\rho_1, \rho_2, \rho_3 \in \mathfrak{S}$, in which $\rho_1(x, y) := \|x\| + \|y\|$, $\rho_2(x, y) := \|x + y\|$ and $\rho_3(x, y) :=$

$\max\{\|x\|, \|y\|\}$ for all $x, y \in E_1$. We now apply Theorem (4.1) to a generalization of Skof theorem.

Corollary 4.4. *For a function $f : E_1 \rightarrow E_2$ the following two conditions are equivalent:*

- (1) $\|f(x+y) - f(x) - f(y)\| \rightarrow 0$ as $\rho(x, y) \rightarrow \infty$;
- (2) $f(x+y) - f(x) - f(y) = 0$

for all $x, y \in E_1$, in which $\rho \in \mathfrak{S}$.

Proof. Set $\psi(x, y) := \|f(x+y) - f(x) - f(y)\|$ for all $x, y \in E_1$. Now let $x_0, y_0 \in E_1$ be two arbitrary fixed elements. Since $\rho \in \mathfrak{S}$, so

$$\rho(x + nx_0, y + ny_0) \rightarrow \infty$$

as $n \rightarrow \infty$ for any fixed $x_0, y_0, x, y \in E_1$. Thus

$$\lim_{n \rightarrow \infty} \psi(x + nx_0, y + ny_0) = 0$$

for each fixed $x_0, y_0, x, y \in E_1$. Hence, its easy to show that the following relations:

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0$;
- $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$

for any fixed $x_0, y_0, x, y \in E_1$. Now with Theorem (4.1) implies that f is an additive mapping (i.e., $f(x+y) = f(x) + f(y)$). The proof is complete.

4.1. Asymptotic Behavior of Pexiderized Cauchy Additive Functional Equation.

Theorem 4.5. *Let S be with identity e and $f, g, h : S \rightarrow E_1$ be three functions such that $g(e) = h(e) = 0$ and*

$$(4.1) \quad \|f(x+y) - g(x) - h(y)\| \leq \psi(x, y)$$

for all $x, y \in S$. Assume that

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0$;
- $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$

for any fixed $x_0, y_0, x, y \in S$. Then f, g and h are additive function and $f(x+y) - g(x) - h(y) = 0$ for all $x, y \in S$.

Proof. Set $\tilde{\psi}(x, y) := \psi(x, y) + \psi(x, e) + \psi(e, y)$ and $\hat{\psi}(x, y) := \psi(x+y, e) + \psi(x, e) + \psi(e, y)$ for all $x, y \in S$. From inequality (4.1) and

assumptions, we obtain the following inequalities

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \psi(x, y) + \|f(x) - g(x)\| + \|f(y) - h(y)\| \\ &\leq \psi(x, y) + \psi(x, e) + \psi(e, y) = \tilde{\psi}(x, y) \end{aligned}$$

and

$$\begin{aligned} \|g(x+y) - g(x) - g(y)\| &\leq \psi(x+y, e) + \|f(x+y) - g(x) - g(y)\| \\ &\leq \tilde{\psi}(x, y) + \hat{\psi}(x, y) \end{aligned}$$

and also

$$\begin{aligned} \|h(x+y) - h(x) - h(y)\| &\leq \psi(x+y, e) + \|f(x+y) - h(x) - h(y)\| \\ &\leq \tilde{\psi}(x, y) + \hat{\psi}(x, y) \end{aligned}$$

for all $x, y \in S$. With assumptions its easy to show that

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(x + ix_0, x_0) = 0$;
- $\lim_{n \rightarrow \infty} \frac{1}{n} \phi(x + nx_0, y + ny_0) = 0$

for any fixed $x_0, y_0, x, y \in S$, in which the function ϕ is $\tilde{\psi}$ or $\tilde{\psi} + \hat{\psi}$. Now with Theorem (4.1) f, g and h is additive mapping and also

- $f(x_0) = \lim_{n \rightarrow \infty} \frac{f(x+nx_0)}{n}$
- $g(x_0) = \lim_{n \rightarrow \infty} \frac{g(x+nx_0)}{n}$
- $h(x_0) = \lim_{n \rightarrow \infty} \frac{h(x+nx_0)}{n}$

for each fixed $x_0, x \in S$. Let x_0, y_0 be any two fixed element of S , then from (4.1), we obtain

$$\|f(x+y+n(x_0+y_0)) - g(x+nx_0) - h(y+ny_0)\| \leq \psi(x+nx_0, y+ny_0)$$

for any fixed $x, y \in S$. Now since $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x+nx_0, y+ny_0) = 0$, thus

$$f(x_0+y_0) = g(x_0) + h(y_0),$$

which says that $f(x+y) - g(x) - h(y) = 0$ for all $x, y \in S$. The proof is complete.

In the following, by using Theorem (4.5), we give a generalization of Skof theorem for Pexiderized additive mapping.

Theorem 4.6. *Assume that $f, g, h : E_1 \rightarrow E_2$ are three functions such that $g(0) = h(0) = 0$, then the following two conditions are equivalent:*

- (1) $\|f(x+y) - g(x) - h(y)\| \rightarrow 0$ as $\rho(x, y) \rightarrow \infty$;
- (2) $f(x+y) - g(x) - h(y) = 0$

for all $x, y \in E_1$, in which $\rho \in \mathfrak{S}$.

Proof. Set $\psi(x, y) := \|f(x + y) - g(x) - h(y)\|$ for all $x, y \in E_1$. Now let $x_0, y_0 \in E_1$ be two arbitrary fixed elements. Since $\rho \in \mathfrak{G}$, so

$$\rho(x + nx_0, y + ny_0) \rightarrow \infty$$

as $n \rightarrow \infty$ for any fixed $x_0, y_0, x, y \in E_1$. Thus

$$\lim_{n \rightarrow \infty} \psi(x + nx_0, y + ny_0) = 0$$

for each fixed $x_0, y_0, x, y \in E_1$. Hence, its easy to show that the following relations

- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(x + ix_0, x_0) = 0$;
- $\lim_{n \rightarrow \infty} \frac{1}{n} \psi(x + nx_0, y + ny_0) = 0$

for any fixed $x_0, y_0, x, y \in E_1$. Now with Theorem (4.5) implies that $f(x + y) - g(x) - h(y) = 0$ for all $x, y \in S$. The proof is complete.

In the following, with using Theorem (4.6), we give a simple proof of S.-M. Joung's theorem (see [36]) and also we show that S.-M. Joung's theorem is true when E_2 be a complex normed space.

Theorem 4.7. *Assume that $J : E_1 \rightarrow E_2$ is a function such that $J(0) = 0$, then the following two conditions are equivalent:*

- (1) $\|2J(\frac{x+y}{2}) - J(x) - J(y)\| \rightarrow 0$ as $\|x\| + \|y\| \rightarrow \infty$;
- (2) $2J(\frac{x+y}{2}) - J(x) - J(y) = 0$

for all $x, y \in E_1$.

Proof. Sets $f(x) := 2J(\frac{x}{2})$, $g(x) := J(x)$, and $\rho(x, y) := \|x\| + \|y\|$ for all $x, y \in E_1$. Now apply Theorem (4.6).

Some Remarks

In 2013, J. Brzdęk [16] proved that the Theorem of Rassias (1.6) for the case $p < 0$ is hypostable. Where we prove this result earlier than J. Brzdęk and also we prove its generalization (4.2), which was published in 2012 see [5]. And also in 2013, M. Piszczek [17] consider a general Cauchy additive functional equation as follows

$$(4.2) \quad g(ax + by) = Ag(x) + Bg(y)$$

for class functions $g : X \rightarrow Y$, where X is a normed space over field \mathbb{F} , Y is Banach space over \mathbb{F} , and \mathbb{F} is the fields of real or complex numbers. He prove that the Theorem of Rassias (1.6) for the case $p < 0$ for functional equation (4.2) is hypostable. Where this result is as consequence of

Theorem (4.5), and also we consider more general calss functions, which was published in 2012 see [5].

5. Stability and Common Stability for the Systems of Linear Equations

The results of this section was proved by Mohsen Alimohammady and Ali Sadeghi, which was published see [6]. In this section, First we consider the Hyers-Ulam-Rassias stability via a fixed point approach for the linear functional equation (1.13) and then applying these result we will investigate Pexiderized linear functional equation (1.16).

Assume that S is a nonempty set, $F = \mathbb{Q}, \mathbb{R}$ or \mathbb{C} , B is a Banach spaces over F , $\psi : S \rightarrow \mathbb{R}^+$, $f, g : S \rightarrow B$, $p : S \rightarrow K \setminus \{0\}$, $q : S \rightarrow B$ are functions, and $\sigma : S \rightarrow S$ is a arbitrary map.

Theorem 5.1. *Let $\mathfrak{S}(S, B)$ be functional equation $f(\rho(x)) = p(x)f(x) + q(x)$. If there exists a real $0 < L < 1$ such that*

$$(5.1) \quad \psi(\rho(x)) \leq L|p(\rho(x))|\psi(x)$$

for all $x \in S$. Then the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x), \frac{\psi(x)}{(1-L)|p(x)|})$ and HUR-stable function such T is unique.

Proof. Let us consider the set $\mathcal{A} := \{h : S \rightarrow B\}$ and introduce the generalized metric on \mathcal{A} :

$$d(u, h) = \sup_{\{x \in S ; \psi(x) \neq 0\}} \frac{|p(x)| \|g(x) - h(x)\|}{\psi(x)}.$$

It is easy to show that (\mathcal{A}, d) is generalized complete metric space. Now we define the function $J : \mathcal{A} \rightarrow \mathcal{A}$ with

$$J(h(x)) = \frac{1}{p(x)}h(\rho(x)) - \frac{q(x)}{p(x)}$$

for all $h \in \mathcal{A}$ and $x \in S$. Since $\psi(\rho(x)) \leq L|p(\rho(x))|\psi(x)$ for all $x \in X$, so

$$\begin{aligned} d(J(u), J(h)) &= \sup_{\{x \in X ; \psi(x) \neq 0\}} \frac{|p(x)| \|u(\rho(x)) - h(\rho(x))\|}{|p(x)|\psi(x)} \\ &\leq \sup_{\{x \in X ; \psi(\rho(x)) \neq 0\}} L \frac{|p(\rho(x))| \|u(\rho(x)) - h(\rho(x))\|}{\psi(\rho(x))} \leq Ld(u, h) \end{aligned}$$

for all $u, h \in \mathcal{A}$, that is J is a strictly contractive selfmapping of \mathcal{A} , with the Lipschitz constant L (note that $0 < L < 1$). We have

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \leq \psi(x)$$

for all $x \in S$, we get

$$\left\| \frac{f(\rho(x))}{p(x)} - \frac{q(x)}{p(x)} - f(x) \right\| \leq \frac{\psi(x)}{|p(x)|}$$

for all $x \in S$, which says that $d(J(f), f) \leq 1 < \infty$. So, with Theorem (1.14), there exists a mapping $T : X \rightarrow B$ such that

(1) T is a fixed point of J , i.e.,

$$(5.2) \quad T(\rho(x)) = p(x)T(x) + q(x)$$

for all $x \in S$. The mapping T is a unique fixed point of J in the set $\tilde{\mathcal{A}} = \{h \in \mathcal{A} : d(f, h) < \infty\}$. This implies that $T \in Z_{\mathfrak{S}(S, B)}$ and is unique HUR-stable function. Also there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - T(x)\| \leq C \frac{\psi(x)}{|p(x)|}$$

for all $x \in X$.

(2) $d(J^n(f), T) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$T(x) = \lim_{n \rightarrow \infty} \frac{f(\rho^n(x))}{\prod_{i=0}^{n-1} p(\rho^i(x))} - \sum_{k=0}^{n-1} \frac{q(\rho^k(x))}{\prod_{i=0}^k p(\rho^i(x))}$$

for all $x \in X$.

(3) $d(f, T) \leq \frac{1}{1-L} d(J(f), f)$, which implies,

$$d(f, T) \leq \frac{1}{1-L}$$

or

$$\|f(x) - T(x)\| \leq \frac{\psi(x)}{(1-L)|p(x)|}$$

for all $x \in X$.

Therefore, the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x), \frac{\psi(x)}{(1-L)|p(x)|})$ and HUR-stable function T is unique. The proof is complete.

With the Theorem of Z. Gajda (1.7), its easy to show that the following result.

Corollary 5.2. *Let $S = B$ be real field \mathbb{R} and $\mathfrak{S}(S, B)$ be the functional equation $f(2x) = 2f(x)$. Then for every $\theta > 0$ there is no constant $\delta \in [0, \infty)$ such that the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\theta(|x|), \delta|x|)$.*

Its obvious that the above corollary is a counterexample for the special case of functional equation in the Theorem (5.1), when $L = 1$. With the Theorem (5.1), we have the following Corollary.

Corollary 5.3. *Let $\mathfrak{S}(S, B)$ be functional equation $f(\rho(x)) = p(x)f(x) + q(x)$. If $a \leq p(x)$ for all $x \in S$ and some real $a > 1$, then for every real $\delta > 0$, the functional equation $\mathfrak{S}(S, B)$ is HU-stable on (X, B) with controls $(\delta, \frac{a\delta}{a-1})$ and HU-stable function such T is unique.*

Proof. Sets $\psi(x) := \delta$ for all $x \in S$ and $L := \frac{1}{a}$. Now apply Theorem (5.1).

Similarly, we prove that a Hyers-Ulam-Rassias stability for the linear functional equation with another suitable conditions.

Theorem 5.4. *Let $\mathfrak{S}(S, B)$ be functional equation $f(\rho(x)) = p(x)f(x) + q(x)$. Let there exists a positive real $L < 1$ such that*

$$(5.3) \quad |p(x)|\psi(\rho^{-1}(x)) \leq L\psi(x)$$

for all $x \in S$ and also ρ be a permutation of S . Then the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x), \frac{1}{1-L}\psi(\rho^{-1}(x)))$ and HUR-stable function such T is unique.

Proof. Let us consider the set $\mathcal{A} := \{h : S \rightarrow B\}$ and introduce the generalized metric on \mathcal{A} :

$$d(u, h) = \sup_{\{x \in S ; \psi(x) \neq 0\}} \frac{\|g(x) - h(x)\|}{\psi(\rho^{-1}(x))}.$$

It is easy to show that (\mathcal{A}, d) is generalized complete metric space. Now we define the function $J : \mathcal{A} \rightarrow \mathcal{A}$ with

$$J(h(x)) = p(\rho^{-1}(x))h(\rho^{-1}(x)) + q(\rho^{-1}(x))$$

for all $h \in \mathcal{A}$ and $x \in X$. Since $|p(x)|\psi(\rho^{-1}(x)) \leq L\psi(x)$ for all $x \in S$ and ρ is a permutation of S , so

$$\begin{aligned} d(J(u), J(h)) &= \sup_{\{x \in S ; \psi(x) \neq 0\}} \frac{|p(\rho^{-1}(x))| \|u(\rho^{-1}(x)) - h(\rho^{-1}(x))\|}{\psi(\rho^{-1}(x))} \\ &\leq \sup_{\{x \in S ; \psi(\rho^{-1}(x)) \neq 0\}} L \frac{\|u(\rho^{-1}(x)) - h(\rho^{-1}(x))\|}{\psi(\rho^{-2}(x))} \leq Ld(u, h) \end{aligned}$$

for all $u, h \in \mathcal{A}$, that is J is a strictly contractive selfmapping of \mathcal{A} , with the Lipschitz constant L (note that $0 < L < 1$). We have

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \leq \psi(x)$$

for all $x \in S$, we get

$$\|f(x) - p(\rho^{-1}(x))f(\rho^{-1}(x)) + q(\rho^{-1}(x))\| \leq \psi(\rho^{-1}(x))$$

for all $x \in S$, which says that $d(J(f), f) \leq 1 < \infty$. So, by Theorem (1.14), there exists a mapping $T : X \rightarrow B$ such that

(1) T is a fixed point of J , i.e.,

$$(5.4) \quad T(\rho(x)) = p(x)T(x) + q(x)$$

for all $x \in S$. The mapping T is a unique fixed point of J in the set $\tilde{\mathcal{A}} = \{h \in \mathcal{A} : d(f, h) < \infty\}$. This implies that $T \in Z_{\mathfrak{S}(S, B)}$ and is unique HUR-stable function. Also there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - T(x)\| \leq C\psi(\rho^{-1}(x))$$

for all $x \in S$.

(2) $d(J^n(f), T) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$T(x) = \lim_{n \rightarrow \infty} \prod_{i=1}^n p(\rho^{-i}(x)) f(\rho^{-n}(x)) - \sum_{k=1}^n q(\rho^k(x)) \prod_{i=0}^{k-1} p(\rho^{-i}(x))$$

for all $x \in S$.

(3) $d(f, T) \leq \frac{1}{1-L}d(J(f), f)$, which implies,

$$d(f, T) \leq \frac{1}{1-L}.$$

$$\|f(x) - T(x)\| \leq \frac{1}{1-L}\psi(\rho^{-1}(x))$$

for all $x \in S$.

Therefore, the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x), \frac{1}{1-L}\psi(\rho^{-1}(x)))$ and HUR-stable function T is unique. The proof is complete.

Similar to the Corollary (5.2, we get the following result, where its counterexample for the special case of functional equation in the Theorem (5.4), when $L = 1$.

Corollary 5.5. *Let $S = B$ be real field \mathbb{R} and $\mathfrak{S}(S, B)$ be the functional equation $f(\frac{1}{2}x) = \frac{1}{2}f(x)$. Then for every $\theta > 0$ there is no constant $\delta \in [0, \infty)$ such that the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\theta(|x|), \delta|x|)$.*

Corollary 5.6. *Let $\mathfrak{S}(S, B)$ be functional equation $f(\rho(x)) = p(x)f(x) + q(x)$. If $|p(x)| \leq L$ for all $x \in S$, some real $0 < L < 1$, and ρ be a permutation of S . Then for every real $\delta > 0$, the functional equation $\mathfrak{S}(S, B)$ is HU-stable on (S, B) with controls $(\delta, \frac{\delta}{1-L})$ and HU-stable function such T is unique.*

Corollary 5.7. *Let X be a normed linear space over F , let $\mathfrak{S}(S, B)$ be functional equation $f(ax) = kf(x)$ for fixed constants a and k , and let $p \in \mathbb{R}$. If $p \leq 0$, $|a| > 1$ and $|k| > 1$ or $p \leq 0$, $|a| < 1$ and $|k| < 1$ or $p \geq 0$, $|a| > 1$ and $|k| < 1$ or $p \geq 0$, $|a| < 1$ and $|k| > 1$, then the functional equation $\mathfrak{S}(S, B)$ is HUR-stable on (S, B) with controls $(\|x\|^p, \frac{\|x\|^p}{|k|-1})$ and HUR-stable function such T is unique.*

proof. Set $\rho(x) := ax$ and $\psi(x) := \|x\|^p$ for all $x \in S$ and then apply Theorem (5.1) and Theorem (5.4).

Now in the following we consider the Hyers-Ulam-Rassias stability Pexiderized linear functional equation (1.16).

Theorem 5.8. *Let $f, g : S \rightarrow B$ be a function and*

$$(5.5) \quad \|f(\rho(x)) - p(x)g(x) - q(x)\| \leq \psi(x)$$

for all $x \in S$. If there exists a positive real $L < 1$ such that

$$(5.6) \quad \psi(\rho(x)) \leq L|p(\rho(x))|\psi(x);$$

$$(5.7) \quad \|f(\rho(x)) - g(\rho(x))\| \leq L\|f(x) - g(x)\|$$

for all $x \in S$. Then there is an function T such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \leq \frac{\tilde{\psi}(x)}{(1-L)|p(x)|}$$

$$\|g(x) - T(x)\| \leq \frac{L}{1-L} \left[\frac{\tilde{\psi}(x) + \psi(x)}{|p(x)|} \right]$$

for all $x \in S$, in which $\tilde{\psi}(x) = \psi(x) + |p(x)|\|f(x) - g(x)\|$ for all $x \in X$.

Proof. Applying (5.5), we get

$$\begin{aligned} \|f(\rho(x)) - p(x)f(x) - q(x)\| &\leq \psi(x) + |p(x)|\|f(x) - g(x)\| \\ &\leq \tilde{\psi}(x) \end{aligned}$$

for all $x \in S$. From (5.6) and (5.7), its easy to show that the following inequality

$$\tilde{\psi}(\rho(x)) \leq L|p(\rho(x))|\tilde{\psi}(x)$$

for all $x \in S$. So, by Theorem (5.1), there is an unique function $T : X \rightarrow B$ such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \leq \frac{\tilde{\psi}(x)}{(1-L)|p(x)|}$$

for all $x \in S$. So from last inequality, we have

$$\|f(\rho(x)) - T(\rho(x))\| \leq \frac{\tilde{\psi}(\rho(x))}{(1-L)|p(\rho(x))|}$$

for all $x \in S$. We show that T is a linear equation, thus from last inequality and (5.5), we get

$$\|g(x) - T(x)\| \leq \frac{L}{1-L} \left[\frac{\tilde{\psi}(x) + \psi(x)}{|p(x)|} \right]$$

for all $x \in S$. The proof is complete.

5.1. Common Stability for the Systems of Homogeneous Linear Equations. Throughout this section, assume that $\{p_i : S \rightarrow K \setminus \{0\}\}_{i \in I}$, $\{\rho_i : S \rightarrow S\}_{i \in I}$ be two family of functions. Here i is a variable ranging over the arbitrary index set I . Also we define the functions $P_{i,n} : S \rightarrow K \setminus \{0\}$ and $\theta_{i,n}(x) : X \rightarrow \mathbb{R}^+$ with

$$P_{i,n}(x) = \prod_{k=0}^{n-1} p_i(\rho_i^k(x))$$

and

$$\theta_{i,n}(x) = \frac{(1 - L_i^n)\psi_i(x)}{(1 - L_i)|p_i(x)|}$$

for a family of positive real's $\{L_i\}_{i \in I}$, all $x \in S$, any index i and positive integer n .

Definition 5.9. Let S and B be nonempty sets. Let $\{\psi_i : X \rightarrow \mathbb{R}^+\}_{i \in I}$ be a family of functions with index set I , $\phi : S \rightarrow [0, \infty)$ be a function, and $\{\mathfrak{S}_i(S, B)\}_{i \in I}$ be a family of functional equations. If for every functions $f : S \rightarrow B$ satisfying the inequality

$$(5.8) \quad d(G_i(f, x), G_i(f, x)) \leq \psi_i(x)$$

for all $x, y \in S$ and any $i \in I$, there exists $T \in \bigcap_{i \in I} Z_{\mathfrak{S}_i(S, B)}$ such that

$$(5.9) \quad d(f(x), T(x)) \leq \phi(x)$$

for all $x \in S$, then we say that the family of functional equations $\{\mathfrak{S}_i(S, B)\}_{i \in I}$ are common Hyers-Ulam-Rassias stable on (S, B) with control functions $(\{\psi_i\}_{i \in I}, \phi)$ and we denoted it by "CHUR-stable" on (S, B) with controls $(\{\psi_i\}_{i \in I}, \phi)$. Also we call the function T as "CHUR-stable function".

In this section, we consider some systems of homogeneous linear equations

$$(5.10) \quad f(\rho_i(x)) = p_i(x)f(x),$$

and our aim is to establish some common Hyers-Ulam-Rassias stability for these systems of functional equations. As a consequence of these results, we give some applications to the study of the superstability result for exponential functional equation to the a family of functional equations. Note that the following Theorem is partial affirmative answer to problem 1, in the 13st ICFEI.

Theorem 5.10. Let $\{\mathfrak{S}_i(S, B)\}_{i \in I}$ be a family of functional equations, in which $\mathfrak{S}_i(S, B)$ be the functional equation $f(\rho_i(x)) = p_i(x)f(x)$ for any $i \in I$. Assume that

- (1) there exists a family of positive real's $\{L_i\}_{i \in I}$ such that $L_i < 1$ and

$$\psi_i(\rho_i(x)) \leq L_i|p_i(\rho_i(x))|\psi_i(x)$$

for all $x \in S$ and $i \in I$;

- (2) $\rho_i \rho_j = \rho_j \rho_i$ for all $i, j \in I$;
- (3) $p_i(\rho_j(x)) = p_i(x)$ for all distinct $i, j \in I$;
- (4) $\lim_{n \rightarrow \infty} \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} = 0$ for all $x \in X$ and every distinct $i, j \in I$.

Then the family of functional equations $\{\mathfrak{S}_i(S, B)\}_{i \in I}$ "CHUR-stable" on (S, B) with controls $(\{\psi_i\}_{i \in I}, \inf_{i \in I} \{\frac{\psi_i(x)}{(1-L_i)|p_i(x)|}\})$ and the CHUR-stable function such T is unique.

Proof. According to the our assumptions, for every $i \in I$, the Theorem (5.1), implies that the functional equation $\mathfrak{S}_i(S, B)$ is HUR-stable on (S, B) with controls $(\psi(x), \frac{\psi(x)}{(1-L)|p(x)|})$ and HUR-stable function such $T_i \in Z_{\mathfrak{S}_i(S, B)}$ is unique. Moreover, The function T_i is given by

$$T_i(x) = \lim_{n \rightarrow \infty} \frac{f(\rho_i^n(x))}{\prod_{k=0}^{n-1} p_i(\rho_i^k(x))} = \lim_{n \rightarrow \infty} J_i^n(f)$$

for all $x \in S$ and any fixed $i \in I$. In the proof of Theorem (5.1), we show that

$$d(J_i(f), f) \leq 1.$$

By induction, its easy to show that

$$d(J_i^n(f), f) \leq \frac{1 - L_i^n}{1 - L_i},$$

which says that

$$\|f(\rho_i^n(x)) - \prod_{k=0}^{n-1} p_i(\rho_i^k(x))f(x)\| \leq (\prod_{k=0}^{n-1} |p_i(\rho_i^k(x))|) \frac{(1 - L_i^n)\psi_i(x)}{(1 - L_i)|p_i(x)|}$$

for all $x \in S$ and $i \in I$. Now we show that $T_i = T_j$ for any $i, j \in I$. Let i and j be two arbitrary fixed indexes of I . So, from last inequality, we obtain

$$(5.11) \quad \|f(\rho_i^n(x)) - P_{i,n}(x)f(x)\| \leq |P_{i,n}(x)|\theta_{i,n}(x);$$

$$(5.12) \quad \|f(\rho_j^n(x)) - P_{j,n}(x)f(x)\| \leq |P_{j,n}(x)|\theta_{j,n}(x)$$

for all $x \in S$. On the replacing x by $\rho_j^n(x)$ in (5.11) and x by $\rho_i^n(x)$ in (5.12)

$$\|f(\rho_i^n(\rho_j^n(x))) - P_{i,n}(\rho_j^n(x))f(\rho_j^n(x))\| \leq |P_{i,n}(\rho_j^n(x))|\theta_{i,n}(\rho_j^n(x));$$

$$\|f(\rho_j^n(\rho_i^n(x))) - P_{j,n}(\rho_i^n(x))f(\rho_i^n(x))\| \leq |P_{j,n}(\rho_i^n(x))|\theta_{j,n}(\rho_i^n(x))$$

for all $x \in X$. From assumptions (2) and (3), its obvious that $f(\rho_i^n(\rho_j^n(x))) = f(\rho_j^n(\rho_i^n(x)))$, $P_{i,n}(\rho_j^n(x)) = P_{i,n}(x)$ and $P_{j,n}(\rho_i^n(x)) = P_{j,n}(x)$ for all $x \in S$. So, from last two inequalities

$$\|P_{i,n}(x)f(\rho_j^n(x)) - P_{j,n}(x)f(\rho_i^n(x))\| \leq |P_{i,n}(x)|\theta_{i,n}(\rho_j^n(x)) + |P_{j,n}(x)|\theta_{j,n}(\rho_i^n(x))$$

or

$$\left\| \frac{f(\rho_j^n(x))}{P_{j,n}(x)} - \frac{f(\rho_i^n(x))}{P_{i,n}(x)} \right\| \leq \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} + \frac{\theta_{j,n}(\rho_i^n(x))}{|P_{i,n}(x)|}$$

for all $x \in X$. From assumption $\lim_{n \rightarrow \infty} \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} = 0$ for all $x \in S$ and every distinct $i, j \in I$, so, its implies that $T_i = T_j$.

Now set $T = T_i$ and since $\|f(x) - T_i(x)\| \leq \frac{\psi_i(x)}{(1-L_i)|p_i(x)|}$ for all $x \in S$ and all $i \in I$, there is a unique function T such that

$$T(\rho_i(x)) = p_i(x)T(x)$$

(i.e., $T \in \bigcap_{i \in I} Z_{\mathfrak{S}_i(S,B)}$) for all $x \in S$ and $i \in I$ and also

$$\|f(x) - T(x)\| \leq \inf_{i \in I} \left\{ \frac{\psi_i(x)}{(1-L_i)|p_i(x)|} \right\}$$

for $x \in S$. The proof is complete.

Corollary 5.11. *Let $\{\mathfrak{S}_i(S, B)\}_{i \in I}$ be a family of functional equations, in which $\mathfrak{S}_i(S, B)$ be the functional equation $f(\rho_i(x)) = c_i f(x)$, where $\{c_i\}_{i \in I}$ is a family of constants. Assume that $\rho_i \rho_j = \rho_j \rho_i$ for all $i, j \in J$ and also there exists a family of positive real's $\{L_i\}_{i \in J}$ such that $0 < L_i < 1$ and*

$$(5.13) \quad \psi_i(\rho_i(x)) \leq L_i |c_i| \psi_i(x)$$

for all $x \in S$ and $i \in J$, in which $J = \{i \in I : |c_i| > 1, L_i |c_i| \in (0, 1]\}$, then the family of functional equations $\{\mathfrak{S}_i(S, B)\}_{i \in J}$ "CHUR-stable" on (S, B) with controls $(\{\psi_i\}_{i \in J}, \inf_{i \in J} \{\frac{\psi_i(x)}{(1-L_i)c_i}\})$ and the CHUR-stable function such T is unique.

Proof. Set $p_i(x) := c_i$ for all $i \in J$, then the conditions (1), (2), and (3) of Theorem (5.10) is holds and now if we show that the condition (4) is hold, then with we obtain the result. We have

$$\theta_{i,n}(\rho_j^n(x)) = \frac{(1-L_i^n)\psi_j(\rho_j^n(x))}{(1-L_i)|c_i|}$$

and

$$P_{j,n}(x) = c_j^n$$

for all $x \in S$ and every $i, j \in J$. Now from (5.13), we get $\psi_i(\rho_j^n(x)) \leq L_i^n |c_i|^n \psi_i(x)$ and so that

$$\lim_{n \rightarrow \infty} \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} \leq \frac{(1-L_i^n)L_i^n |c_i|^n \psi_i(x)}{|c_j|^n}$$

for any $x \in S$ and every distinct $i, j \in J$. Since $0 < |L_i c_i| < 1$ and $|c_i| > 1$ for all $i \in J$, so the above limit approach to zero. The proof is complete.

Now with the above Corolary, in the following, we prove a super-stability result for Cauchy exponential functional equation, where we discussed about it in 2th section

Theorem 5.12. *Let S be commutative semigroup and B be the field of complex numbers \mathbb{C} . Let $\mathfrak{S}(S, B)$ be the functional equation $f(x \cdot y) = g(y)f(x)$, where $g : S \rightarrow \mathbb{C}$ is a function. Let the set J be the elements of $i \in S$, where $|g(a)| > 1$ and and there exists $L_i \in (0, 1)$ with $L_i |g(i)| \in (0, 1]$. Assume that $\phi : S^2 \rightarrow \mathbb{R}^+$ is function, g be is unbounded function*

$$\phi(x, y \cdot i) \leq L_a |g(a)| \phi(x, y)$$

for all $x, y \in S$ and $i \in J$. If $g(J)$ be an unbounded set, then the functional equation $\mathfrak{S}(S, B)$ is hyperstable on (S, B) with control (ϕ) .

Proof. Let g be a unbounded function, then sets $\rho_i(x) := x \cdot i$, $c_i := g(i)$, and $\psi_i := \phi(i, x)$ for all $x \in S$ and any $i \in J$. Since $\rho_i \rho_j = \rho_i \rho_j$ and $\psi_i(\rho_i(x)) \leq L_a |g(a)| \psi_i(x)$ for all $x \in S$ and any $i \in J$, then by Corollary (5.11), the family of functional equations $\{\mathfrak{S}_i(f)\}_{i \in J}$ "CHUR-stable" on (S, B) with controls $(\{\psi_i\}_{i \in J}, \inf_{i \in J} \{\frac{\psi_i(x)}{(1-L_i)c_i}\})$ and the CHUR-stable function such T is unique. Since g is a unbounded function, from last inequality $T = f$ (note that $L_i |g(i)| \in (0, 1]$), which implies that

$$f(\rho_i(x)) = c_i f(x)$$

or

$$(5.14) \quad f(x \cdot i) = g(i) f(x)$$

for all $x \in S$ and $i \in J$. We have

$$(5.15) \quad \|f(x \cdot y) - g(y) f(x)\| \leq \phi(x, y)$$

for all $x, y \in S$. On the replacing y by $y \cdot i^n$ in (5.15)

$$\|f((x \cdot y) \cdot i^n) - g(y \cdot i^n) f(x)\| \leq \phi(x, y \cdot i^n)$$

or

$$\left\| \frac{f((x \cdot y) \cdot i^n)}{g(i)^n} - \frac{g(y) f(x \cdot i^n)}{g(i)^n} \right\| \leq \frac{\phi(x, y \cdot i^n)}{|g(i)|^n}$$

for all $x, y \in S$, any fixed $i \in I$ and positive integer n . From equation (5.14), its easy to show that $f(x \cdot i^n) = g(i)^n f(x)$ and $\phi(x, y \cdot i^n) \leq$

$L_i^n |g(i)|^n \phi(x, y)$ for all $x \in S$, any fixed $i \in J$ and positive integer n . So, we have

$$\|f(x \cdot y) - g(y)f(x)\| \leq L_i^n \phi(x, y)$$

for all $x, y \in S$, any fixed $i \in J$ and positive integer n , which implies that $f(x \cdot y) = g(y)f(x)$ for all $x, y \in S$ (i.e., $f \in Z_{\mathfrak{S}(S, B)}$). The proof is complete.

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