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Integrable boundary conditions and modified Lax equations

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Abstract

We consider integrable boundary conditions for both discrete and continuum classical integrable models. Local integrals of motion generated by the corresponding "transfer" matrices give rise to time evolution equations for the initial Lax operator. We systematically identify the modified Lax pairs for both discrete and continuum boundary integrable models, depending on the classical r-matrix and the boundary matrix.

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1 Introduction

Lax representation of classical dynamical evolution equations [1] is one key ingredient in the modern theory of classical integrable systems [2]–[7] together with the associated notion of classical r-matrix [8, 9]. It takes the generic form of an isospectral evolution equation: $\frac{dL}{dt} = [L,\ A]$, where L encapsulates the dynamical variables and A defines the time evolution. We shall generically consider the situation where L and A depend on a complex (spectral) parameter. The spectrum of the Lax matrix or its extension (transfer matrices), or equivalently the invariant coefficients of the characteristic determinant, thus provide automatically candidates to realize the hierarchy of Poisson-commuting Hamiltonians required by Liouville's theorem [10, 11]. Existence of the classical r-matrix then guarantees Poisson-commutativity of these natural dynamical quantities taken as generators of the algebra of classical conserved charges.

The question of finding an appropriate time evolution matrix A for a given Lax matrix L therefore entails the possibility of systematically constructing classically integrable models once a Lax matrix is defined with suitable properties. Depending on which properties are emphasized, this problem may be approached in several ways. One approach uses postulated Poisson algebra properties of L, specifically the r-matrix structure, as a starting point, and establishes a systematic construction of the time evolution operator A associated to the Hamiltonian evolution obtained from any function in the enveloping algebra of the Poisson-commuting traces of such an L matrix. Such a formulation was proposed long time ago [8, 9] for the bulk case, when the Poisson structure for L is a simple linear or quadratic r-matrix structure.

We shall consider here more general situations when the dynamical system also depends on supplementary parameters encapsulated into a matrix K. We shall restrict ourselves to the situation where these parameters are non-dynamical, i.e. the Poisson brackets with themselves and with the initial "bulk" parameters is zero. For situations where the extra parameters are dynamical see [12, 13]. Note that interpretation of the physical meaning of these c-number parameters will come a posteriori when computing the associated Hamiltonians. In particular any physical interpretation of the K matrix as a description of the "boundary properties" (external fields, ...) may not be appropriate in all cases as shall appear in our discussion of examples. We shall however keep this designation as a book-keeping device throughout this paper.

Our central purpose will be twofold. We shall first of all define generic sets of sufficient algebraic conditions on K also formulated in terms of the "bulk" r-matrix. A "boundary–modified" generating matrix of candidate conserved quantities, hereafter denoted $\mathcal{T}$, will be accordingly constructed as a suitable combination of L and K matrices. The idea for such

a construction naturally arises when considering the semi-classical limit of the well–known Cherednik-Sklyanin reflection algebras preserving bulk quantum integrability [14]. We shall then redefine the time evolution operator A associated to a given Hamiltonian constructed from $\mathcal{T}$ (modified monodromy matrix) from the new basic elements, i.e. the matrix $\mathcal{T}$, the bulk r-matrix or (r, s) pair [9, 15], and the reflection matrix K. Such a construction was exemplified in [16]; what we propose here is however a generic, systematic procedure to obtain modified "boundary" or "folded" classical integrable systems from initial pure "bulk" systems. Note also that the example worked out in [16] is precisely a case which the K-matrix does describe boundary effects.

It must be emphasized here that an alternative, analytic approach to this question, at least in the bulk case, was extensively described in [7] (chapter 3). In this approach one uses instead the analytic properties (location of poles and algebraic structure of residues) of the Lax matrix L as a meromorphic function of the spectral parameter. They provide for a unique consistent form of an associated A matrix. The subsequent Lax equation is then developed into separate equations corresponding to the poles of L and A. The Poisson structure and r matrix structure are only then defined as providing a consistent Hamiltonian interpretation of this Lax equation. Similar reformulation of our results must exist but we shall not consider them here.

We shall expand here the r-matrix approach in three situations. We start with the simpler case where the initial bulk structure is a linear Poisson structure for a Lax matrix parametrized by a single r-matrix. We then develop the cases of both discrete and continuous parametrized Lax matrices relevant to the description of systems on a lattice or on a continuous line. In these last two situations the relevant r-matrix structure is a quadratic Sklyanin-type bracket [17]. We shall in all three cases derive (or actually rederive, in some cases) the form of the generating functional for Poisson-commuting Hamiltonians, and establish the explicit general formula yielding the A operator. Explicit examples shall be developed in all cases, albeit restricting ourselves to the simplest situations of non–dynamical non–constant double–pole rational r-matrices. More complicated situations (trigonometric and/or dynamical r-matrices, relevant for e.g. sine-Gordon models or affine Toda field theories) will be left for further studies.

2 Linear Poisson structure

We consider here the original situation [1] where the full dynamical system under study is represented by a single Lax matrix, living in a representation of a finite-dimensional Lie algebra or a loop algebra. In this last case the Lax matrix also depends on a complex parameter

λ known as "spectral parameter". In all cases the requirement of Poisson-commutativity for candidate Hamiltonians TrL^n is equivalent [18] to the existence of a classical r-matrix [8, 9] realizing a linear Poisson structure for L. Indeed, consider the Lax pair $(L,\ A)$ satisfying

$$\frac{\partial L}{\partial t} = \Big[A,\ L\Big] \tag{2.1}$$

the associated spectral problem

$$L(\lambda)\ \psi = u\ \psi, \qquad \det(L(\lambda) - u) = 0 \tag{2.2}$$

provides the integrals of motion, obtained through the expansion in powers of the spectral parameter λ of $trL(\lambda)$. Alternatively, in particular when no spectral parameter exists, one should consider the traces of powers of the Lax matrix $trL^n(\lambda)$ as natural Hamiltonians.

Assuming that the L matrix satisfies the fundamental relation

$$\Big\{L_a(\lambda),\ \ L_b(\mu)\Big\} = \Big[r_{ab}(\lambda-\mu),\ L_a(\lambda) + L_b(\mu)\Big] \tag{2.3}$$

it is shown using (2.3) that for any integer n, m:

$$\Big\{trL^n(\lambda),\ trL^m(\mu)\Big\} = 0. \tag{2.4}$$

The reciprocal property was shown in [18]. Let us recall [8, 9, 18] how one may identify the A-operator associated to the various charges in involution. Bearing in mind the fundamental relation (2.3) it is shown that

$$\Big\{tr_a\ L_a^n(\lambda),\ L_b(\mu)\Big\} = n\ tr_a\ \Big(L_a^{n-1}(\lambda)\ r_{ab}(\lambda-\mu)\Big)\ L_b(\mu) - n\ L_b(\mu)\ tr_a\ \Big(L_a^{n-1}(\lambda) r_{ab}(\lambda-\mu)\Big) \tag{2.5}$$

From (2.5) one extracts A:

$$A_n(\lambda,\ \mu) = n\ tr_a\ \Big(L_a^{n-1}(\lambda) r_{ab}(\lambda-\mu)\Big). \tag{2.6}$$

In the case of the simplest rational non-dynamical r-matrices [19]

$$r(\lambda) = \frac{\mathbb{P}}{\lambda} \quad \text{where} \quad \mathbb{P} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji} \tag{2.7}$$

$\mathbb{P}$ is the permutation operator, and $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, we end up with a simple form for A_n:

$$A_n(\lambda,\ \mu) = \frac{n}{\lambda-\mu} L^{n-1}(\lambda) \tag{2.8}$$

and as usual to obtain the Lax pair associated to each local integral of motion one has to expand $A_n = \sum_i \frac{A_n^{(i)}}{\lambda^i}$. Note that generically, using the dual formulation of the classical r-matrix [9] one also has:

$$A(\lambda,\mu) = Tr(r(\lambda,\mu) dH) \tag{2.9}$$

where H is the Hamiltonian expressed as any function in the enveloping algebra generated by TrL^n.

Ultimately we would like to consider an extended classical algebra in analogy to the quantum boundary algebras arising in integrable systems with non-trivial boundary conditions that preserve integrability. Subsequently we shall deal with two types of algebras which may be associated with the two types of known quantum boundary conditions. These boundary conditions are known as soliton preserving (SP), traditionally studied in the framework of integrable quantum spin chains (see e.g. [14], [20]–[23]), and soliton non-preserving (SNP) originally introduced in the context of classical integrable field theories [16], and further investigated in [24, 25]. SNP boundary conditions have been also introduced and studied for integrable quantum lattice systems [26]–[30]. From the algebraic perspective the two types of boundary conditions are associated with two distinct algebras, i.e. the reflection algebra [14] and the twisted Yangian respectively [31, 32] (see also [25, 29, 30, 33, 34]). The classical versions of both algebras will be defined subsequently in the text (see section 3.2). It will be convenient for our purposes here to introduce some useful notation:

$$\begin{aligned}
&\hat{r}_{ab}(\lambda) = r_{ba}(\lambda) \quad \text{for SP}, \qquad \hat{r}_{ab}(\lambda) = r^{t_a t_b}_{ba}(\lambda) \quad \text{for SNP} \\
&r^{*}_{ab}(\lambda) = r_{ab}(\lambda) \quad \text{for SP}, \qquad r^{*}_{ab}(\lambda) = r^{t_b}_{ba}(-\lambda) \quad \text{for SNP} \\
&\hat{r}^{*}_{ab}(\lambda) = r_{ba}(\lambda) \quad \text{for SP}, \qquad \hat{r}^{*}_{ab}(\lambda) = r^{t_a}_{ab}(-\lambda) \quad \text{for SNP}
\end{aligned} \tag{2.10}$$

together with:

$$\hat{L}(\lambda) = -L(-\lambda) \quad \text{for SP}, \qquad \hat{L}(\lambda) = L^{t}(-\lambda) \quad \text{for SNP}. \tag{2.11}$$

In addition the "boundary conditions", to be interpreted on specific examples, are parametrized by a single non-dynamical matrix $\mathrm{k}(\lambda)$. We propose here the following set of algebraic relations:

$$\begin{aligned}
&\Big\{\mathcal{T}_1(\lambda_1),\ \mathcal{T}_2(\lambda_2)\Big\} = \mathrm{r}^{-}_{12}(\lambda_1,\lambda_2)\ \mathcal{T}_1(\lambda_1) - \mathcal{T}_1(\lambda_1)\ \tilde{\mathrm{r}}^{-}_{12}(\lambda_1,\lambda_2) \\
&+\mathrm{r}^{+}_{12}(\lambda_1,\lambda_2)\ \mathcal{T}_2(\lambda_2) - \mathcal{T}_2(\lambda_2)\ \tilde{\mathrm{r}}^{+}_{12}(\lambda_1,\lambda_2)
\end{aligned} \tag{2.12}$$

where we define:

$$\begin{aligned}
\mathrm{r}^{-}_{12}(\lambda_1,\lambda_2) &= r_{12}(\lambda_1-\lambda_2)\ \mathrm{k}_2(\lambda_2) - \mathrm{k}_2(\lambda_2)\ r^{*}_{12}(\lambda_1+\lambda_2) \\
\tilde{\mathrm{r}}^{-}_{12}(\lambda_1,\lambda_2) &= \mathrm{k}_2(\lambda_2)\ \hat{r}_{12}(\lambda_1-\lambda_2) - \hat{r}^{*}_{12}(\lambda_1+\lambda_2)\ \mathrm{k}_2(\lambda_2) \\
\mathrm{r}^{+}_{12}(\lambda_1,\lambda_2) &= r_{12}(\lambda_1-\lambda_2)\ \mathrm{k}_1(\lambda_1) + \mathrm{k}_1(\lambda_1)\ \hat{r}^{*}_{12}(\lambda_1+\lambda_2) \\
\tilde{\mathrm{r}}^{+}_{12}(\lambda_1,\lambda_2) &= \mathrm{k}_1(\lambda_1)\ \hat{r}_{12}(\lambda_1-\lambda_2) + r^{*}_{12}(\lambda_1+\lambda_2)\ \mathrm{k}_1(\lambda_1),
\end{aligned} \tag{2.13}$$

for a k matrix satisfying:

$$\Big\{\mathrm{k}_1(\lambda),\ \mathrm{k}_2(\mu)\Big\} = 0, \qquad \Big\{\mathrm{k}_1(\lambda),\ L_2(\mu)\Big\} = 0, \tag{2.14}$$

$$r_{12}(\lambda_1-\lambda_2)\ \mathrm{k}_1(\lambda)\ \mathrm{k}_2(\lambda_2)+\mathrm{k}_1(\lambda_1)\ \hat{r}^*_{12}(\lambda_1+\lambda_2)\mathrm{k}_2(\lambda_2)$$
$$=\mathrm{k}_1(\lambda_1)\ \mathrm{k}_2(\lambda_2)\ \hat{r}_{12}(\lambda_1-\lambda_2)+\mathrm{k}_2(\lambda_2)\ r^*_{12}(\lambda_1+\lambda_2)\ \mathrm{k}_1(\lambda_1). \tag{2.15}$$

We then show:

Theorem 2.1.: The quantity

$$\mathcal{T}(\lambda)=L(\lambda)\ \mathrm{k}(\lambda)+\mathrm{k}(\lambda)\ \hat{L}(\lambda) \tag{2.16}$$

is a representation of the algebra defined by (2.12), with k obeying (2.14), (2.15).

Proof: We shall need in addition to (2.3) the following exchange relations:

$$\Big\{\hat{L}_a(\lambda),\ L_b(\mu)\Big\}=\hat{L}_a(\lambda)\hat{r}^*_{ab}(\lambda+\mu)+\hat{r}^*_{ab}(\lambda+\mu)L_b(\mu)-\hat{r}^*_{ab}\lambda+\mu)\hat{L}_a(\lambda)-L_b(\mu)\hat{r}^*_{ab}(\lambda+\mu)$$
$$\Big\{L_a(\lambda),\ \hat{L}_b(\mu)\Big\}=L_a(\lambda)r^*_{ab}(\lambda+\mu)+r^*_{ab}(\lambda+\mu)\hat{L}_b(\mu)-r^*_{ab}(\lambda+\mu)L_a(\lambda)-\hat{L}_b(\mu)r^*_{ab}(\lambda+\mu)$$
$$\Big\{\hat{L}_a(\lambda),\ \hat{L}_b(\mu)\Big\}=\Big[\hat{r}_{ab}(\lambda-\mu),\ \hat{L}_a(\lambda)+\hat{L}_b(\mu)\Big]. \tag{2.17}$$

By explicit use of the algebraic relations (2.15) and (2.17) we obtain:

$$\Big\{L_a(\lambda)\mathrm{k}_a(\lambda)+\hat{L}_a(\lambda)\mathrm{k}_a(\lambda),\ L_b(\mu)\mathrm{k}_b(\mu)+\mathrm{k}_b(\mu)\hat{L}_b(\mu)\Big\}=\dots$$
$$=\mathrm{r}^-_{ab}(\lambda,\mu)\Big(L_a(\lambda)\mathrm{k}_a(\lambda)+\mathrm{k}_a(\lambda)\hat{L}_a(\lambda)\Big)-\Big(L_a(\lambda)\mathrm{k}_a(\lambda)+\mathrm{k}_a(\lambda)\hat{L}_a(\lambda)\Big)\tilde{\mathrm{r}}^-_{ab}(\lambda,\mu)$$
$$+\mathrm{r}^+_{ab}(\lambda,\mu)\Big(L_b(\mu)\mathrm{k}_b(\mu)+\mathrm{k}_b(\mu)\hat{L}_b(\mu)\Big)-\Big(L_b(\mu)\mathrm{k}_b(\mu)+\mathrm{k}_b(\mu)\hat{L}_b(\mu)\Big)\tilde{\mathrm{r}}^+_{ab}(\lambda,\mu)$$
$$+\Big(L_a(\lambda)+L_b(\mu)\Big)\Big(-r_{ab}(\lambda-\mu)\mathrm{k}_a(\lambda)\mathrm{k}_b(\mu)-\mathrm{k}_a(\lambda)\hat{r}^*_{ab}(\lambda+\mu)\mathrm{k}_b(\mu)$$
$$+\mathrm{k}_b(\mu)r^*_{12}(\lambda+\mu)\mathrm{k}_a(\lambda)+\mathrm{k}_a(\lambda)\mathrm{k}_b(\mu)\hat{r}_{12}(\lambda-\mu)\Big)$$
$$+\Big(-r_{ab}(\lambda-\mu)\mathrm{k}_a(\lambda)\mathrm{k}_b(\mu)-\mathrm{k}_a(\lambda)\hat{r}^*_{ab}(\lambda+\mu)\mathrm{k}_b(\mu)$$
$$+\mathrm{k}_b(\mu)r^*_{12}(\lambda+\mu)\mathrm{k}_a(\lambda)+\mathrm{k}_a(\lambda)\mathrm{k}_b(\mu)\hat{r}_{12}(\lambda-\mu)\Big)\Big(\hat{L}_a(\lambda)+\hat{L}_b(\mu)\Big). \tag{2.18}$$

Bearing however in mind that the k-matrix obeys (2.15) we conclude that the last four lines of the equation above disappear, which shows that (2.16) satisfies (2.12), and this concludes our proof. ■

Define $\mathbb{T}(\lambda)=\mathrm{k}^{-1}(\lambda)\ \mathcal{T}(\lambda)$, we now prove:

Theorem 2.2.:

$$\Big\{tr_a\mathbb{T}^N_a(\lambda),\ tr_b\mathbb{T}^M_b(\mu)\Big\}=0. \tag{2.19}$$

Proof:

$$\Big\{tr_a\mathbb{T}_a^N(\lambda),\ tr_b\mathbb{T}_b^M(\mu)\Big\} = \sum_{n,m} tr_{ab}\ \mathbb{T}_a^{N-n}(\lambda)\mathbb{T}_b^{M-m}(\mu)\Big\{\mathbb{T}_a(\lambda),\ \mathbb{T}_b(\mu)\Big\}\mathbb{T}_a^{n-1}(\lambda)\mathbb{T}_b^{m-1}(\mu) \quad (2.20)$$

employing (2.12) the preceding expression becomes

$$\begin{aligned}
&\ldots \propto \\
&tr_{ab}\mathbb{T}_a^{N-1}(\lambda)\mathbb{T}_b^{M-1}(\mu)\mathrm{k}_a^{-1}\mathrm{k}_b^{-1}\Big(\mathrm{r}_{ab}^-(\lambda,\mu)\mathcal{T}_a(\lambda) - \mathcal{T}_a(\lambda)\tilde{\mathrm{r}}_{ab}^-(\lambda,\mu) + \mathrm{r}_{ab}^+(\lambda,\mu)\mathcal{T}_b(\mu) - \mathcal{T}_b(\mu)\tilde{\mathrm{r}}_{ab}^+(\lambda,\mu)\Big) \\
&= \ldots = 0. \qquad (2.21)
\end{aligned}$$

Note that in order to show that the latter expression is zero we moved appropriately the factors in the products within the trace and we used (2.15). ■

We finally identify the modified Lax formulation associated to the generalized algebra (2.14)–(2.12) as:

Theorem 2.3.: Defining Hamiltonians as: $tr_a\mathbb{T}^n(\lambda) = \sum_i \frac{\mathcal{H}_n^{(i)}}{\lambda^i}$ the classical equations of motion for $\mathbb{T}$:

$$\dot{\mathbb{T}}(\mu) = \Big\{\mathcal{H}_n^{(i)},\ \mathbb{T}(\mu)\Big\} \quad (2.22)$$

take a zero curvature form

$$\dot{\mathbb{T}}(\mu) = \Big[\mathbb{A}(\lambda,\mu),\ \mathbb{T}(\mu)\Big], \quad (2.23)$$

where $\mathbb{A}_n$ is identified as:

$$\mathbb{A}_n(\lambda,\mu) = n\ tr_a\Big(\mathbb{T}_a^{n-1}(\lambda)\mathrm{k}_a^{-1}(\lambda)\tilde{\mathrm{r}}_{ab}^+(\lambda,\mu)\Big). \quad (2.24)$$

Proof:

$$\begin{aligned}
&\Big\{tr_a\mathbb{T}_a^n(\lambda),\ \mathbb{T}_b(\mu)\Big\} = \ldots = \\
&n\ tr_a\Big(\mathbb{T}_a^{n-1}(\lambda)\mathrm{k}_a^{-1}(\lambda)\mathrm{k}_b^{-1}(\mu)(\mathrm{r}_{ab}^-(\lambda,\mu)\mathcal{T}_a(\lambda) - \mathcal{T}_a(\lambda)\tilde{\mathrm{r}}_{ab}^-(\lambda,\mu) + \mathrm{r}_{ab}^+(\lambda,\mu)\mathcal{T}_b(\mu) - \mathcal{T}_b(\mu)\tilde{\mathrm{r}}_{ab}^+(\lambda,\mu))\Big) \\
&= n\ tr_a\Big(\mathbb{T}_a^{n-1}\mathrm{k}_a^{-1}(\lambda)\mathrm{k}_b(\mu)(\mathrm{r}_{ab}^-(\lambda,\mu)\mathcal{T}_a(\lambda) - \mathcal{T}_a(\lambda)\tilde{\mathrm{r}}_{ab}^-(\lambda,\mu))\Big) \\
&+n\ tr_a\Big(\mathbb{T}_a^{n-1}\mathrm{k}_a^{-1}(\lambda)\mathrm{k}_b(\mu)(\mathrm{r}_{ab}^+(\lambda,\mu)\mathcal{T}_b(\mu) - \mathcal{T}_b(\mu)\tilde{\mathrm{r}}_{ab}^+(\lambda,\mu))\Big). \qquad (2.25)
\end{aligned}$$

Taking into account (2.15) we see that the first term of RHS of the equality above disappears and the last term may be appropriately rewritten such as:

$$\Big\{tr_a\mathbb{T}_a^n(\lambda),\ \mathbb{T}_b(\mu)\Big\} = n\ tr_a\Big(\mathbb{T}^{n-1}(\lambda)\mathrm{k}_a^{-1}(\lambda)\tilde{\mathrm{r}}_{ab}^+(\lambda,\mu)\Big)\ \mathbb{T}_b(\mu) - n\ \mathbb{T}_b(\mu)\ tr_a\Big(\mathbb{T}^{n-1}(\lambda)\mathrm{k}_a^{-1}(\lambda)\tilde{\mathrm{r}}_{ab}^+(\lambda,\mu)\Big). \quad (2.26)$$

From the latter formula (2.24) is deduced. ■

Finally, let $\mathcal{T}(\lambda),\ \mathcal{T}'(\lambda)$ be two representations of (2.12), and let also

$$\Big\{\mathcal{T}_a(\lambda),\ \mathcal{T}'_b(\mu)\Big\} = 0. \tag{2.27}$$

It is then straightforward to show, based solely on the fact that $\mathcal{T},\ \mathcal{T}'$ satisfy (2.12) and (2.27) , that the sum $\mathcal{T}(\lambda) + \mathcal{T}'(\lambda)$ is also a representation of (2.12).

2.1 Examples

We shall present here a simple example, starting from the classical rational Gaudin model [35]. Details on the so called "dual" description of the Toda chain [36, 37] and the DST model [38, 39], both associated to the $A^{(1)}_{\mathcal{N}-1}$ r-matrix [40], will be presented in a forthcoming publication. Before we proceed with the particular example let us rewrite $\mathbb{A}_n$. We consider a situation where the Poisson brackets are parametrized by the simplest rational non-dynamical r-matrix [19]. After substituting the rational r-matrix in (2.24) we get for both types of boundary conditions:

$$\begin{aligned}
\mathbb{A}_n(\lambda,\mu) &= n\ \Big(\frac{\mathbb{T}^{n-1}(\lambda)}{\lambda-\mu} + \frac{\mathrm{k}(\lambda)\mathbb{T}^{n-1}(\lambda)\mathrm{k}^{-1}(\lambda)}{\lambda+\mu}\Big) \quad \text{for SP,} \\
\mathbb{A}_n(\lambda,\mu) &= n\ \Big(\frac{\mathbb{T}^{n-1}(\lambda)}{\lambda-\mu} - \frac{(\mathrm{k}(\lambda)\mathbb{T}^{n-1}(\lambda)\mathrm{k}^{-1}(\lambda))^t}{\lambda+\mu}\Big) \quad \text{for SNP.}
\end{aligned} \tag{2.28}$$

The L-matrix associated to the classical $gl_{\mathcal{N}}$ Gaudin model, and satisfying the algebraic relation (2.3) is

$$L(\lambda) = \sum_{\alpha,\beta=1}^{\mathcal{N}} \sum_{i=1}^{N} \frac{\mathcal{P}^{(i)}_{\alpha\beta}}{\lambda - z_i} E_{\alpha\beta} \tag{2.29}$$

where $\mathcal{P}_{\alpha\beta} \in gl_{\mathcal{N}}$. Recall that in the "bulk" case the integrals of motion are obtained through $tr\ L^n(\lambda)$. The first non-trivial example reads

$$t^{(2)}(\lambda) = tr\ L^2(\lambda) = \sum_{\alpha,\beta=1}^{\mathcal{N}} \sum_{i,j=1}^{N} \frac{\mathcal{P}^{(i)}_{\alpha\beta}\mathcal{P}^{(j)}_{\beta\alpha}}{(\lambda - z_i)(\lambda - z_j)} \tag{2.30}$$

and by taking the residue of the latter expression for $\lambda \to z_i$ we obtain the Gaudin Hamiltonian:

$$H^{(2)} = \sum_{i\neq j=1}^{N} \sum_{\alpha,\beta=1}^{\mathcal{N}} \frac{\mathcal{P}^{(i)}_{\alpha\beta}\mathcal{P}^{(j)}_{\beta\alpha}}{z_i - z_j}. \tag{2.31}$$

Let us now come to the generic algebra (2.12), (2.14), (2.15) considering both SP and SNP cases. Based on the definitions (2.11) we have:

$$\hat{L}(\lambda) = \sum_{\alpha,\beta=1}^{\mathcal{N}} \sum_{i=1}^{N} \frac{\mathcal{Q}_{\alpha\beta}^{(i)}}{\lambda + z_i} E_\alpha \quad \text{where}$$
$$\mathcal{Q}_{\alpha\beta} = \mathcal{P}_{\alpha\beta} \quad \text{for SP}, \qquad \mathcal{Q}_{\alpha\beta} = -\mathcal{P}_{\beta\alpha} \quad \text{for SNP}. \tag{2.32}$$

Let us take as a representation $\mathcal{T}(\lambda) = L(\lambda)\mathrm{k}(\lambda) + \mathrm{k}(\lambda)\hat{L}(\lambda) + K(\lambda)$ where k is a solution of (2.14), (2.15), and K is a c-number representation of (2.12) with zero Poisson bracket. To obtain the relevant Hamiltonian we now formulate $tr_a \mathbb{T}^2(\lambda)$ (where $\mathbb{T} = k^{-1}\mathcal{T}$ and we also define $\tilde{K} = \mathrm{k}^{-1}K$), and the Hamiltonian arises as the residue of the latter expression at $\lambda = z_i$:

$$\mathcal{H}^{(2)} = \sum_{i\neq j=1}^{N} \sum_{\alpha,\beta=1}^{\mathcal{N}} \frac{\mathcal{P}_{\alpha\beta}^{(i)}\mathcal{P}_{\beta\alpha}^{(j)}}{z_i - z_j} + \sum_{i,j=1}^{N} \sum_{\alpha,\beta,\gamma,\delta,\epsilon=1}^{\mathcal{N}} \frac{\mathrm{k}_{\alpha\gamma}\mathrm{k}_{\delta\epsilon}\mathcal{P}_{\gamma\delta}^{(i)}\mathcal{Q}_{\epsilon\alpha}^{(j)}}{z_i + z_j} + \sum_{i=1}^{N} \sum_{\alpha,\beta,\gamma,\delta,\epsilon=1}^{\mathcal{N}} \tilde{K}_{\epsilon\alpha}\mathrm{k}_{\alpha\gamma}^{-1}\mathrm{k}_{\delta\epsilon}\mathcal{P}_{\gamma\delta}^{(i)} \tag{2.33}$$

A special case of the generic algebra (2.12) is discussed in [41, 42]. Note also that expression (2.33) may be also obtained as the semiclassical limit of the quantum $gl_{\mathcal{N}}$ inhomogeneous open spin chain for special boundary conditions, (see e.g. [43]), z_j being the inhomogeneities attached to each site j. However, here the Hamiltonians are obtained directly from the representations of our new classical algebra (2.12), (2.14), (2.15). It appears in this example that the parameters $\mathrm{k}(\lambda)$ play the role of "coupling constants" consistent with the integrable "folding" of a $2N$ site Gaudin model, and not the role of boundary parameters.

3 Quadratic Poisson structures: the discrete case

Quadratic Poisson structures first appeared as the well-known Sklyanin bracket [44]. A more general form, characterized by a pair of respectively skew symmetric and symmetric matrices $(r,\ s)$ appeared in [15] in the formulation of consistent Poisson structures for non-ultralocal classical integrable field theories. Finally it was shown [45] that this was the natural quadratic form a la Sklyanin for a non-skew-symmetric r-matrix, reading:

$$\Big\{L_1,\ L_2\Big\} = \Big[r - r^\pi,\ L_1 L_2\Big] + L_1(r + r^\pi)L_2 - L_2(r + r^\pi)L_1. \tag{3.1}$$

A typical situation when one considers naturally a quadratic Poisson structure for the Lax matrix occurs when considering discrete (on a lattice) or continuous (on a line) integrable systems where the Lax matrix depends on either a discrete or a continuous variable; the Lax pair is thus associated to a point on the space-like lattice or continuous line [6, 17]. Let us first examine the discrete case where one considers a finite set of Lax matrices L_n labelled by $n \in \mathbb{N}$.

3.1 Periodic boundary conditions

Introduce the Lax pair $(L,\ A)$ for discrete integrable models [4] (see also [46] for statistical systems), and the associated auxiliary problem (see e.g. [6])

$$\begin{aligned} \psi_{n+1} &= L_n\ \psi_n \\ \dot{\psi}_n &= A_n\ \psi_n. \end{aligned} \tag{3.2}$$

From the latter equations one may immediately obtain the discrete zero curvature condition:

$$\dot{L}_n = A_{n+1}\ L_n - L_n\ A_n. \tag{3.3}$$

The monodromy matrix arises from the first equation (3.2) (see e.g. [6])

$$T_a(\lambda) = L_{aN}(\lambda) \dots L_{a1}(\lambda) \tag{3.4}$$

where index a denotes the auxiliary space, and the indices $1, \dots, N$ denote the sites of the one dimensional classical discrete model.

Consider now a skew symmetric classical r-matrix which is a solution of the classical Yang-Baxter equation [8, 9]

$$\Big[r_{12}(\lambda_1 - \lambda_2),\ r_{13}(\lambda_1) + r_{23}(\lambda_2)\Big] + \Big[r_{13}(\lambda_1),\ r_{23}(\lambda_2)\Big] = 0, \tag{3.5}$$

and let L satisfy the associated Sklyanin bracket

$$\Big\{L_a(\lambda),\ L_b(\mu)\Big\} = \Big[r_{ab}(\lambda - \mu),\ L_a(\lambda) L_b(\mu)\Big]. \tag{3.6}$$

It is then immediate that that (3.4) also satisfies (3.6). Use of the latter equation shows that the quantities $trT(\lambda)^n$ provide charges in involution, that is

$$\Big\{tr\ T^n(\lambda),\ tr\ T^m(\mu)\Big\} = 0 \tag{3.7}$$

which again is trivial by virtue of (3.6). In the simple sl_2 case the only non trivial quantity is $trT(\lambda) = t(\lambda)$, that is the usual "bulk" transfer matrix.

Let us now briefly review how the classical A-operator and the corresponding classical equations of motion are obtained from the expansion of the monodromy matrix in the simple case of periodic boundary conditions. Let us first introduce some useful notation

$$T_a(n, m; \lambda) = L_{an}(\lambda) L_{an-1}(\lambda) \dots L_{am}(\lambda), \qquad n > m. \tag{3.8}$$

To formulate $\{t(\lambda),\ L(\mu)\}$ or even better $\{\ln\ t(\lambda),\ L(\mu)\}$, given that usually $\ln\ trT(\lambda)$ gives rise to local integrals of motion, we first derive the quantity $\{T_a(\lambda),\ L_{bn}(\mu)\}$.

$$\Big\{T_a(\lambda),\ L_{bn}(\mu)\Big\} = T_a(N,n+1;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n,1;\lambda)\ L_{bn}(\mu) - L_{bn}(\mu)\ T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda). \quad (3.9)$$

Taking the trace over the auxiliary space a, and then the logarithm we conclude

$$\Big\{\ln\ t(\lambda),\ L(\mu)\Big\} = t^{-1}(\lambda)\ \Big(tr_a\{T_a(N,n+1;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n,1;\lambda)\}\ L(\mu) - L(\mu)\ tr_a\{T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\}\Big) \quad (3.10)$$

the auxiliary index b is suppressed from the latter expression for simplicity. Then plugging the latter forms into the classical equations of motions for all integrals of motion

$$\dot{L}_n(\mu) = \Big\{t(\lambda),\ L_n(\mu)\Big\} \quad (3.11)$$

we conclude that the zero curvature condition (3.3) is realized by:

$$A_n(\lambda,\mu) = t^{-1}(\lambda)\ tr_a\{T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\}. \quad (3.12)$$

Let us focus on the classical rational r-matrix (2.7). In this case A_n takes the simple form:

$$A_n(\lambda,\mu) = \frac{t^{-1}(\lambda)}{\lambda-\mu}\ T(n-1,1;\lambda)\ T(N,n;\lambda). \quad (3.13)$$

In the open case, as we shall see in the subsequent section, it is sufficient to consider the expansion of $t(\lambda)$ in order to obtain the local integrals of motion.

3.2 Open boundary conditions

We now generalize the procedure described in the preceding section to the case of generic integrable "boundary conditions". We propose a construction of two types of monodromy and transfer matrices, and associated Lax-type evolution equations, of the form (3.6)–(3.12), albeit incorporating a supplementary set of non-dynamical parameters encapsulated into a "reflection" matrix $K(\lambda)$. In some examples they may indeed be interpreted as boundary effects consistent with integrability of an open chain-like system. We should stress that this is the first time to our knowledge that such an investigation is systematically undertaken. There are related studies regarding particular examples of open spin chains such as XXZ, XYZ and 1D Hubbard models [47, 48]. However, the derivation of the corresponding Lax pair is restricted to the Hamiltonian only and not to all associated integrals of motion of the open chain. In this study we present a generic description independent of the choice of

model, and we derive the Lax pair for each one of the entailed boundary integrals of motion. Particular examples are also presented.

The two types of monodromy matrices will respectively represent the classical version of the reflection algebra $\mathbb{R}$, and the twisted Yangian $\mathbb{T}$ written in the compact form: (see e.g. [14, 15, 49]):

$$\Big\{\mathcal{T}_1(\lambda_1),\ \mathcal{T}_2(\lambda_2)\Big\} = r_{12}(\lambda_1-\lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2) - \mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)\hat{r}_{12}(\lambda_1-\lambda_2)$$
$$+\mathcal{T}_1(\lambda_1)\hat{r}^*_{12}(\lambda_1+\lambda_2)\mathcal{T}_2(\lambda_2) - \mathcal{T}_2(\lambda_2)r^*_{12}(\lambda_1+\lambda_2)\mathcal{T}_1(\lambda_1) \tag{3.14}$$

where $\hat{r},\ r^*,\ \hat{r}^*$ are defined in (2.10). The latter equation may be thought of as the semi-classical limit of the reflection equation [14]. In most well known physical cases, such as the $A^{(1)}_{\mathcal{N}-1}$ r-matrices $r^{t_1t_2}_{12} = r_{21}$ implying that in the SNP case $r^*_{ab} = \hat{r}^*_{ab}$. In the case of the Yangian r-matrix $r_{12} = r_{21}$, hence all the expressions above may be written in a more symmetric form. These two distinct algebras are respectively associated with the SP boundary conditions ($\mathbb{R}$) and the SNP boundary conditions ($\mathbb{T}$).

In order to construct representations of (3.14) yielding the generating function of the integrals of motion one now introduces c-number representations ($K^{\pm}$) of the algebra $\mathbb{R}$ ($\mathbb{T}$) satisfying (3.14) for SP and SNP respectively, and also the non-dynamical condition:

$$\Big\{K^{\pm}_1(\lambda_1),\ K^{\pm}_2(\lambda_2)\Big\} = 0. \tag{3.15}$$

Taking now as $T(\lambda)$ any bulk monodromy matrix (3.4) built from local L matrices obeying (3.6) and defining in addition

$$\hat{T}(\lambda) = T^{-1}(-\lambda) \quad \text{for SP}, \qquad \hat{T}(\lambda) = T^t(-\lambda) \quad \text{for SNP}. \tag{3.16}$$

one gets:

Theorem 3.1.: Representations of the corresponding algebras $\mathbb{R},\ \mathbb{T}$, are given by the following expression see e.g. [14, 13]:

$$\mathcal{T}(\lambda) = T(\lambda)\ K^-(\lambda)\ \hat{T}(\lambda). \tag{3.17}$$

For a detailed proof see e.g. [13]. ■

Define now as generating function of the involutive quantities

$$t(\lambda) = tr\{K^+(\lambda)\ \mathcal{T}(\lambda)\}. \tag{3.18}$$

Theorem 3.2.: Due to (3.14) it is shown that [14, 13]

$$\Big\{t(\lambda_1),\ t(\lambda_2)\Big\} = 0, \quad \lambda_1,\ \lambda_2 \in \mathbb{C}. \tag{3.19}$$

■

The expansion of $t(\lambda)$ naturally gives rise to the integrals of motions i.e.

$$t(\lambda) = \sum_i \frac{\mathcal{H}^{(i)}}{\lambda^i}. \tag{3.20}$$

Usually one considers the quantity ln $t(\lambda)$ to get *local* integrals of motion, however for the examples we are going to examine here the expansion of $t(\lambda)$ is enough to provide the associated local quantities as will be transparent in the subsequent section. Finally one has:

Theorem 3.3.: Time evolution of the local Lax matrix L_n under generating Hamiltonian action of $t(\lambda)$ is given by:

$$\dot{L}_n(\mu) = \mathbb{A}_{n+1}(\lambda,\mu)\ L_n(\mu) - \mathbb{A}_n(\lambda,\mu)\ L_n(\mu), \tag{3.21}$$

where $\mathbb{A}_n$ is the modified (boundary) quantity,

$$\begin{aligned}\mathbb{A}_n(\lambda,\mu) &= tr_a\Big(K_a^+(\lambda)\ T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\ K_a^-(\lambda)\ \hat{T}_a(\lambda) \\ &+ K_a^+(\lambda)\ T_a(\lambda)\ K_a^-(\lambda)\ \hat{T}_a(1,n-1;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n,N;\lambda)\Big).\end{aligned} \tag{3.22}$$

Proof: We need in addition to (3.6) one more fundamental relation i.e.

$$\Big\{\hat{L}_a(\lambda), L_b(\mu)\Big\} = \hat{L}_a(\lambda)\hat{r}^*_{ab}(\lambda)L_b(\mu) - L_b(\mu)\hat{r}^*_{ab}(\lambda+\mu)\hat{L}_a(\lambda). \tag{3.23}$$

where the notation $\hat{L}$ is self-explanatory from (3.16). Taking into account the latter expressions we derive for $\hat{T}$

$$\begin{aligned}\Big\{\hat{T}_a(\lambda),\ L_{bn}(\mu)\Big\} &= \hat{T}_a(1,n;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n+1,N;\lambda)\ L_{bn}(\mu) \\ &- L_{bn}(\mu)\ \hat{T}_a(1,n-1;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n,N;\lambda).\end{aligned} \tag{3.24}$$

The next step is to formulate $\Big\{t(\lambda),\ L_{bn}(\mu)\Big\}$; indeed recalling (3.9), (3.24), (3.17) and (3.18) we conclude:

$$\begin{aligned}\Big\{t(\lambda),\ L_{bn}(\mu)\Big\} &= tr_a\Big(K_a^+(\lambda)\ T_a(N,n+1;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n,1;\lambda)\ K_a^-(\lambda)\ \hat{T}(\lambda) \\ &+ K_a^+(\lambda)\ T_a(\lambda)\ K_a^-(\lambda)\hat{T}_a(1,n;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n+1,N;\lambda)\Big)L_{bn}(\mu) \\ &- L_{bn}(\mu)\ tr_a\Big(K_a^+(\lambda)\ T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\ K_a^-(\lambda)\hat{T}(\lambda) \\ &+ K_a^+(\lambda)\ T_a(\lambda)\ K_a^-(\lambda)\ \hat{T}_a(1,n-1;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n,N;\lambda)\Big).\end{aligned} \tag{3.25}$$

Expression (3.22) is readily extracted from (3.25). ■

Special care should be taken at the boundary points $n = 1$ and $n = N + 1$. Indeed going back to formula (3.22) restricting ourselves to $n = 1$ and $n = N + 1$ and taking into account that $T(N, N+1, \lambda) = T(0, 1, \lambda) = \hat{T}(1, 0, \lambda) = \hat{T}(N+1, N, \lambda) = \mathbb{I}$ we obtain the explicit form for $\mathbb{A}_1,\ \mathbb{A}_{n+1}$. We should stress that the derivation of the boundary Lax pair is universal, namely the expressions (3.22) are generic and independent of the choice of $L,\ r$. Note that expansion of the latter expressions of $\mathbb{A}_n(\lambda, \mu) = \sum \frac{\mathbb{A}_n^{(i)}}{\lambda^i}$ provides all the $\mathbb{A}_n^{(i)}$ associated to the corresponding integrals of motion $\mathcal{H}^{(i)}$, which follow from the expansion of $t(\lambda)$. This will become quite transparent in the examples presented in the subsequent section.

Remark: A different construction of (3.14) was already given in a very general setting in [49]. It is related to the formulation of non-ultralocal integrable field theories on a lattice and extends the analysis of [15]. The essential difference with our construction is that the k matrix (denoted γ) in [49], is sandwiched between the "local" monodromy matrices $T_{n,n-1}$ so as to obtain an overall Poisson bracket of the form (3.14) for the "dressed" monodromy matrix $...T_{n+1,n}\gamma T_{n,n-1}\gamma....$ The matrix γ allows to take into account the effects of the non-ultralocal part $\delta'(x-y)$ of the Poisson bracket structure. In this respect γ also corresponds to "boundary" effects, although multiple and internal.

3.3 Examples

We shall now examine a simple example, i.e. the open generalized DST model, which may be seen as a lattice version of the generalized (vector) NLS model, (see also [39, 50, 51, 52, 13] for further details). The open Toda chain will be also discussed as a limit of the DST model. We shall explicitly evaluate the "boundary" Lax pairs for the first integrals of motion. If we focus on the special case (2.7), which will be our main interest here, the latter expression reduces to the following expressions for SP and SNP boundary conditions. In particular, $\mathbb{A}_n$ for SP boundary conditions reads:

$$\begin{aligned}\mathbb{A}_n(\lambda,\mu) &= \frac{1}{\lambda-\mu}\, T(n-1,1;\lambda)\; K^-(\lambda)\; \hat{T}(\lambda)\; K^+(\lambda)\; T(N,n;\lambda) \\ &+ \frac{1}{\lambda+\mu}\, \hat{T}(n,N;\lambda)\; K_a^+(\lambda)\; T(\lambda)\; K^-(\lambda)\; \hat{T}(1,n-1;\lambda) \quad \text{for SP} \end{aligned} \tag{3.26}$$

and for SNP boundary conditions:

$$\begin{aligned}\mathbb{A}_n(\lambda,\mu) &= \frac{1}{\lambda-\mu}\, T(n-1,1;\lambda)\; K^-(\lambda)\; \hat{T}(\lambda)\; K^+(\lambda)\; T(N,n;\lambda) \\ &- \frac{1}{\lambda+\mu}\, \hat{T}^t(1,n-1;\lambda)\; K_a^{-t}(\lambda)\; T^t(\lambda)\; K^{+t}(\lambda)\; \hat{T}^t(n,N;\lambda) \quad \text{for SNP,} \end{aligned} \tag{3.27}$$

where we recall that for the special points $n = 1,\ N+1$ we take into account that: $T(N, N+1, \lambda) = T(0, 1, \lambda) = \hat{T}(1, 0, \lambda) = \hat{T}(N+1, N, \lambda) = \mathbb{I}$.

The Lax operator of the $gl(\mathcal{N})$ DST model has the following form:

$$L(\lambda) = (\lambda - \sum_{j=1}^{\mathcal{N}-1} x^{(j)} X^{(j)}) E_{11} + b \sum_{j=2}^{\mathcal{N}} E_{jj} + b \sum_{j=2}^{\mathcal{N}} x^{(j-1)} E_{1j} - \sum_{j=2}^{\mathcal{N}} X^{(j-1)} E_{j1} \tag{3.28}$$

with $x_n^{(j)},\ X_n^{(j)}$ being canonical variables

$$\begin{aligned} &\left\{x_n^{(i)},\ x_m^{(j)}\right\} = \left\{X_n^{(i)},\ X_m^{(j)}\right\}, \qquad \left\{x_n^{(i)},\ X_m^{(j)}\right\} = \delta_{nm}\delta_{ij} \\ &i,\ j \in \{1,\ldots,\mathcal{N}\}, \quad n,\ m \in \{1,\ldots,N\}. \end{aligned} \tag{3.29}$$

In [13] the first non-trivial integral of motion for the SNP case, choosing the simplest consistent value $K^{\pm} = \mathbb{I}$ was explicitly computed:

$$\begin{aligned} &\mathcal{H} = -\frac{1}{2}\sum_{n=1}^{N} \mathbb{N}_n^2 - b \sum_{n=1}^{N}\sum_{j=1}^{\mathcal{N}-1} X_n^{(j)} x_{n+1}^{(j)} - \frac{1}{2}\sum_{j=1}^{\mathcal{N}-1} (X_N^{(j)} X_N^{(j)} + b^2 x_1^{(j)}) \\ &\text{where} \quad \mathbb{N}_n = \sum_{j=1}^{\mathcal{N}-1} x_n^{(j)} X_n^{(j)}. \end{aligned} \tag{3.30}$$

Our aim is now to determine the modified Lax pair induced by the non-trivial integrable boundary conditions. We shall focus here on the case of SNP boundary conditions, basically because in the particular example we consider here such boundary conditions are technically easier to study. Moreover, the SNP boundary conditions have not been so much analyzed in the context of lattice integrable models, which provides an extra motivation to investigate them. Taking into account (3.27) we explicitly derive the modified Lax pair for the generalized DST model with SNP boundary conditions. Indeed, after expanding (3.27) in powers of λ^{-1} we obtain the quantity associated to the Hamiltonian (3.30)[3]:

$$\begin{aligned} &\mathbb{A}_n^{(2)} = \lambda E_{11} - \sum_{j\neq 1} X_{n-1}^{(j-1)} E_{j1} + b \sum_{j\neq 1} x_n^{(j-1)} E_{1j}, \quad n \in \{2,\ldots N\} \\ &\mathbb{A}_1^{(2)} = \lambda E_{11} - b \sum_{j\neq 1} x_1^{(j-1)} E_{j1} + b \sum_{j\neq 1} x_1^{(j-1)} E_{1j}, \\ &\mathbb{A}_{N+1}^{(2)} = \lambda E_{11} - \sum_{j\neq 1} X_N^{(j-1)} E_{j1} + \sum_{j\neq 1} X_N^{(j-1)} E_{1j}. \end{aligned} \tag{3.31}$$

Let us now consider the simplest possible case, i.e. the sl_2 DST model. It is worth stressing that in this case the SP and SNP boundary coincide given that

$$L^{-1}(-\lambda) = V\ L^t(-\lambda)\ V, \qquad V = \text{antid}(1,\ldots,1). \tag{3.32}$$

[3]To obtain both $\mathcal{H}^{(2)}$ and $\mathbb{A}_n^{(2)}$ we divided the original expressions by a factor two.

In this particular case the Hamiltonian (3.30) reduces into:

$$\mathcal{H}^{(2)} = -\frac{1}{2}\sum_{n=1}^{N} x_n^2 X_n^2 - b\sum_{n=1}^{N-1} x_{n+1}X_n - \frac{b^2}{2}x_1^2 - \frac{1}{2}X_N^2. \tag{3.33}$$

The equations of motion associated to the latter Hamiltonian may be readily extracted by virtue of[4]

$$\dot{L} = \Big\{\mathcal{H}^{(2)},\ L\Big\}. \tag{3.35}$$

It is deduced from (3.31) for the sl_2 case:

$$\begin{aligned}
&\mathbb{A}_n^{(2)}(\lambda) = \begin{pmatrix} \lambda & bx_n \\ -X_{n-1} & 0 \end{pmatrix}, \quad n \in \{2,\dots,N\} \\
&\mathbb{A}_1^{(2)}(\lambda) = \begin{pmatrix} \lambda & bx_1 \\ -bx_1 & 0 \end{pmatrix}, \quad \mathbb{A}_{N+1}^{(2)}(\lambda) = \begin{pmatrix} \lambda & X_N \\ -X_N & 0 \end{pmatrix}.
\end{aligned} \tag{3.36}$$

Alternatively the equations of motion may be derived from the zero curvature condition

$$\frac{\partial L_n}{\partial t} = \mathbb{A}_{n+1}^{(i)}\ L_n - L_n\mathbb{A}_n^{(i)} \tag{3.37}$$

which the modified Lax pair satisfies. It is clear that to each one of the higher local charges a different quantify $\mathbb{A}_n^{(i)}$ is associated. Both equations (3.35), (3.37) lead naturally to the same equations of motion, which for this particular example read as:

$$\begin{aligned}
&\dot{x}_n = x_n^2 X_n + bx_{n+1}, \quad \dot{X}_n = -x_n X_n^2 - bX_{n-1}, \quad n \in \{2,\dots N-1\} \\
&\dot{x}_1 = x_1^2 X_1 + bx_2, \quad \dot{X}_1 = -x_1 X_1^2 - bx_1 \\
&\dot{x}_N = x_N^2 X_N + X_N, \quad \dot{X}_N = -x_N X_N^2 - bX_{N-1}.
\end{aligned} \tag{3.38}$$

The Toda model may be seen as an appropriate limit of the DST model (see also [53]). Indeed consider the following limiting process as $b \to 0$:

$$X_n \to e^{-q_n}, \quad x_n \to e^{q_n}(b^{-1} + p_n) \tag{3.39}$$

It is clear that the harmonic oscillator algebra defined by $(x_n,\ X_n,\ x_nX_n)$ reduces to the Euclidian Lie algebra $(e^{\pm q_n},\ p_n)$, and consequently the Lax operator reduces to:

$$L_n(\lambda) = \begin{pmatrix} \lambda - p_n & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix} \tag{3.40}$$

[4]The associated Poisson brackets for both DST and Toda models are defined as:

$$\begin{aligned}
&\Big\{A,\ B\Big\} = \sum_n \Big(\frac{\partial A}{\partial x_n}\ \frac{\partial B}{\partial X_n} - \frac{\partial A}{\partial X_n}\ \frac{\partial B}{\partial x_n}\Big), \quad \text{DST model} \\
&\Big\{A,\ B\Big\} = \sum_n \Big(\frac{\partial A}{\partial q_n}\ \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\ \frac{\partial B}{\partial q_n}\Big), \quad \text{Toda chain.}
\end{aligned} \tag{3.34}$$

where $q_n,\ p_n$ are canonical variables. In this case the corresponding Hamiltonian may be readily extracted and takes the form

$$\mathcal{H}^{(2)} = -\frac{1}{2}\sum_{n=1}^{N} p_n^2 - \sum_{n=1}^{N-1} e^{q_{n+1}-q_n} - \frac{1}{2}e^{2q_1} - \frac{1}{2}e^{-2q_N} \tag{3.41}$$

and the corresponding $\mathbb{A}_n^{(2)}$ are expressed as

$$\begin{aligned}
&\mathbb{A}_n^{(2)}(\lambda) = \begin{pmatrix} \lambda & e^{q_n} \\ -e^{q_{n-1}} & 0 \end{pmatrix}, \qquad n \in \{2,\dots,N\} \\
&\mathbb{A}_1^{(2)}(\lambda) = \begin{pmatrix} \lambda & e^{q_1} \\ -e^{q_1} & 0 \end{pmatrix}, \qquad \mathbb{A}_{N+1}^{(2)}(\lambda) = \begin{pmatrix} \lambda & e^{-q_N} \\ -e^{-q_N} & 0 \end{pmatrix}.
\end{aligned} \tag{3.42}$$

In this case as well both formulas (3.37) and (3.35) lead to the following set of equations of motions:

$$\begin{aligned}
&p_n = \dot{q}_n, \qquad \ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}, \qquad n \in \{2,\dots,N-1\} \\
&p_1 = \dot{q}_1, \qquad \ddot{q}_1 = e^{q_2-q_1} - e^{2q_1} \\
&p_N = \dot{q}_N, \qquad \ddot{q}_N = e^{-2q_N} - q^{q_N-q_{N-1}}.
\end{aligned} \tag{3.43}$$

4 The continuous case

4.1 Periodic boundary conditions

Let us now recall the basic notions regarding the Lax pair and the zero curvature condition for a continuous integrable model following essentially [6]. Define Ψ as being a solution of the following set of equations (see e.g. [6])

$$\frac{\partial\Psi}{\partial x} = \mathbb{U}(x,t,\lambda)\Psi \tag{4.1}$$

$$\frac{\partial\Psi}{\partial t} = \mathbb{V}(x,t,\lambda)\Psi \tag{4.2}$$

$\mathbb{U},\ \mathbb{V}$ being in general $n\times n$ matrices with entries defined as functions of complex valued dynamical fields, their derivatives, and the spectral parameter λ. Compatibility conditions of the two differential equation (4.1), (4.2) lead to the zero curvature condition [3]–[5]

$$\dot{\mathbb{U}} - \mathbb{V}' + \Big[\mathbb{U},\ \mathbb{V}\Big] = 0, \tag{4.3}$$

giving rise to the corresponding classical equations of motion of the system under consideration. The monodromy matrix may be written from (4.1) as:

$$T(x,y,\lambda) = \mathcal{P}exp\Big\{\int_y^x \mathbb{U}(x',t,\lambda)dx'\Big\}, \tag{4.4}$$

where apparently $T(x,x,\lambda)=1$. The fact that the monodromy matrix satisfies equation (4.1) is extensively used to get the relevant integrals of motion and the associated Lax pairs.

Hamiltonian formulation of the equations of motion is available again under the r-matrix approach. In this picture the underlying classical algebra is manifestly analogous to the quantum case. Let us first recall this method for a general classical integrable system on the full line. The existence of the Poisson structure for $\mathbb{U}$ realized by the classical r-matrix, satisfying the classical Yang-Baxter equation (3.5), guarantees the integrability of the classical system. Indeed assuming that the operator $\mathbb{U}$ satisfies the following ultralocal form of Poisson brackets

$$\Big\{\mathbb{U}_a(x,\lambda),\ \mathbb{U}_b(y,\mu)\Big\}=\Big[r_{ab}(\lambda-\mu),\ \mathbb{U}_a(x,\lambda)+\mathbb{U}_b(y,\mu)\Big]\ \delta(x-y), \tag{4.5}$$

$T(x,y,\lambda)$ satisfies:

$$\Big\{T_a(x,y,t,\lambda_1),\ T_b(x,y,t,\lambda_2)\Big\}=\Big[r_{ab}(\lambda_1-\lambda_2),\ T_a(x,y,t,\lambda_1)T_b(x,y,t,\lambda_2)\Big]. \tag{4.6}$$

Making use of the latter equation one may readily show for a system on the full line:

$$\Big\{\ln tr\{T(x,y,\lambda_1)\},\ \ln tr\{T(x,y,\lambda_2)\}\Big\}=0 \tag{4.7}$$

i.e. the system is integrable, and the charges in involution –local integrals of motion– are obtained by expansion of the generating function $\ln tr\{T(x,y,\lambda)\}$, based essentially on the fact that T satisfies (4.1).

Let us now recall how one constructs the $\mathbb{V}$-operator associated to given local integrals of motion. One easily proves the following identity using (4.5)

$$\Big\{T_a(L,-L,\lambda),\ \mathbb{U}_b(x,\mu)\Big\}=\frac{\partial M(x,\lambda,\mu)}{\partial x}+\Big[M(x,L,-L,\lambda,\mu),\ \mathbb{U}_b(x,\mu)\Big] \tag{4.8}$$

where we define

$$M(x,\lambda,\mu)=T_a(L,x,\lambda)\ r_{ab}(\lambda-\mu)\ T_a(x,L,\lambda). \tag{4.9}$$

For more details on the proof of the formula above we refer the interested reader to [6]; (4.8) may be seen as the continuum version of relation (4.6). Recall that $t(\lambda)=trT(\lambda)$ then it naturally follows from (4.8), and (4.3), that

$$\Big\{\ln\ t(\lambda),\ \mathbb{U}(x,\lambda)\Big\}=\frac{\partial\mathbb{V}(x,\lambda,\mu)}{\partial x}+\Big[\mathbb{V}(x,\lambda,\mu),\ \mathbb{U}(x,\lambda)\Big] \tag{4.10}$$

with

$$\mathbb{V}(x,\lambda,\mu)=t^{-1}(\lambda)\ tr_a\Big(T_a(L,x,\lambda)\ r_{ab}(\lambda,\mu)\ T_a(x,-L,\lambda)\Big) \tag{4.11}$$

and in the rational case (4.11) reduces to

$$\mathbb{V}(x,\lambda,\mu)=\frac{t^{-1}(\lambda)}{\lambda-\mu}\ T(x,-L,\lambda)\ T(L,x,\lambda). \tag{4.12}$$

4.2 General integrable boundary conditions

Our aim here is to consider integrable models on the interval with consistent "boundary conditions", and derive rigorously the Lax pairs associated to the entailed boundary local integrals of motion as a continuous extension of theorems 3.1.–3.3. For this purpose we follow the line of action described in [6], using now Sklyanin's formulation for the system on the interval or on the half line. We briefly describe this process below for any classical integrable system on the interval. In this case one constructs a modified 'monodromy' matrix $\mathcal{T}$, based on Sklyanin's formulation and satisfying again the Poisson bracket algebras $\mathbb{R}$ or $\mathbb{T}$. To construct the generating function of the integrals of motion one also needs c-number representations of the algebra $\mathbb{R}$ or $\mathbb{T}$ satisfying (3.14) for SP and SNP respectively, such that:

$$\Big\{K_1^{\pm}(\lambda_1),\ K_2^{\pm}(\lambda_2)\Big\} = 0. \tag{4.13}$$

Theorem 4.1.: The modified 'monodromy' matrices, realizing the corresponding algebras $\mathbb{R}$, $\mathbb{T}$, are given by the following expressions [14] ($\hat{T}$ being defined in (3.16)):

$$\mathcal{T}(x,y,t,\lambda) = T(x,y,t,\lambda)\ K^-(\lambda)\ \hat{T}(x,y,t,\lambda). \tag{4.14}$$

■

The generating function of the involutive quantities is defined as

$$t(x,y,t,\lambda) = tr\{K^+(\lambda)\ \mathcal{T}(x,y,t,\lambda)\}. \tag{4.15}$$

Indeed one shows:
Theorem 4.2.:

$$\Big\{t(x,y,t,\lambda_1),\ t(x,y,t,\lambda_2)\Big\} = 0, \quad \lambda_1,\ \lambda_2 \in \mathbb{C}. \tag{4.16}$$

■

In the case of open boundary conditions, exactly as in the discrete integrable models, and taking into account (3.24) we prove

$$\Big\{\mathcal{T}_a(0,-L,\lambda),\ \mathbb{U}_b(x,\mu)\Big\} = \mathbb{M}'_a(x,\lambda,\mu) + \Big[\mathbb{M}_a(x,\lambda,\mu),\ \mathbb{U}_b(x,\ \mu)\Big] \tag{4.17}$$

where we define

$$\begin{aligned}\mathbb{M}(x,\lambda,\mu) &= T(0,x,\lambda)r_{ab}(\lambda-\mu)T(x,-L,\lambda)K^-(\lambda)\hat{T}(0,-L,\lambda)\\ &+ T(0,-L,\lambda)K^-(\lambda)\hat{T}(x,-L,\lambda)\hat{r}^*_{ab}(\lambda+\mu)\hat{T}(0,x,\lambda).\end{aligned} \tag{4.18}$$

Finally bearing in mind the definition of $t(\lambda)$ and (4.17) we conclude with:

Theorem 4.3.:

$$\Big\{ \ln\ t(\lambda),\ \mathbb{U}(x,\mu)\Big\} = \frac{\partial \mathbb{V}(x,\lambda,\mu)}{\partial x} + \Big[\mathbb{V}(x,\lambda,\mu),\ \mathbb{U}(x,\mu)\Big] \tag{4.19}$$

where

$$\mathbb{V}(x,\lambda,\mu) = t^{-1}(\lambda)\ tr_a\Big(K^+(\lambda)\ \mathbb{M}_a(x,\lambda,\mu)\Big). \tag{4.20}$$

■

As in the discrete case particular attention should be paid to the boundary points $x = 0,\ -L$. Indeed, for these two points one has to simply take into account that $T(x,x,\lambda) = \hat{T}(x,x,\lambda) = \mathbb{I}$. Moreover, the expressions derived in (4.18), (4.20) are universal, that is independent of the choice of model. As was remarked upon when discussing the discrete case, a quadratic algebra of the form (3.14) was initially obtained in the continuous case [15] when extending the derivation of 4.1 to situations where the Poisson brackets (4.5) are non-ultralocal, exhibiting $\delta'(x-y)$ terms. Connection to "boundary" effects was discussed previously (see Section 3.2).

4.3 Example

We shall now examine a particular example associated to the rational r-matrix (2.7), that is the $gl_{\mathcal{N}}$ NLS model. Although in [13] an extensive analysis for both types of boundary conditions is presented, here we shall focus on the simplest diagonal ($K^{\pm} = \mathbb{I}$) boundary conditions. The Lax pair is given by the following expressions [6, 54]:

$$\mathbb{U} = \mathbb{U}_0 + \lambda \mathbb{U}_1, \quad \mathbb{V} = \mathbb{V}_0 + \lambda \mathbb{V}_1 + \lambda^2 \mathbb{V}_2 \tag{4.21}$$

where

$$\begin{aligned}
&\mathbb{U}_1 = \frac{1}{2i}(\sum_{i=1}^{\mathcal{N}-1} E_{ii} - E_{\mathcal{N}\mathcal{N}}), \qquad \mathbb{U}_0 = \sum_{i=1}^{\mathcal{N}-1}(\bar{\psi}_i E_{i\mathcal{N}} + \psi_i E_{\mathcal{N}i}) \\
&\mathbb{V}_0 = i\sum_{i,\ j=1}^{\mathcal{N}-1}(\bar{\psi}_i\psi_j E_{ij} - |\psi_i|^2 E_{\mathcal{N}\mathcal{N}}) - i\sum_{i=1}^{\mathcal{N}-1}(\bar{\psi}_i' E_{i\mathcal{N}} - \psi_i' E_{\mathcal{N}i}), \\
&\mathbb{V}_1 = -\mathbb{U}_0, \quad \mathbb{V}_2 = -\mathbb{U}_1
\end{aligned} \tag{4.22}$$

and $\psi_i,\ \bar{\psi}_j$ satisfy[5]:

$$\Big\{\psi_i(x),\ \psi_j(y)\Big\} = \Big\{\bar{\psi}_i(x),\ \bar{\psi}_j(y)\Big\} = 0, \qquad \Big\{\psi_i(x),\ \bar{\psi}_j(y)\Big\} = \delta_{ij}\ \delta(x-y). \tag{4.24}$$

Note that we have suppressed the constant κ from (4.22) compared e.g. to [13] by rescaling the fields $(\psi_i,\ \bar{\psi}_i) \to \sqrt{\kappa}(\psi_i,\ \bar{\psi}_i)$.

From the zero curvature condition (4.3) the classical equations of motion for the generalized NLS model with periodic boundary conditions are entailed i.e.

$$i\frac{\partial\psi_i(x,t)}{\partial t} = -\frac{\partial^2\psi_i(x,t)}{\partial^2 x} + 2\kappa\sum_j |\psi_j(x,t)|^2\psi_i(x,t), \qquad i,\ j \in \{1,\ldots,\mathcal{N}-1\}. \tag{4.25}$$

It is clear that for $\mathcal{N}=2$ the equations of motion of the usual NLS model are recovered. The boundary Hamiltonian for the generalized NLS model may be expressed as

$$\begin{aligned}\mathcal{H} &= \int_{-L}^{0} dx\ \sum_{i=1}^{\mathcal{N}-1}\Big(\kappa|\psi_i(x)|^2\sum_{j=1}^{\mathcal{N}-1}|\psi_j(x)|^2 + \psi_i'(x)\bar{\psi}_i'(x)\Big) \\ &- \sum_{i=1}^{\mathcal{N}-1}\Big(\psi_i'(0)\bar{\psi}_i(0) + \psi_i(0)\bar{\psi}_i'(0)\Big) + \sum_{i=1}^{\mathcal{N}-1}\Big(\psi_i'(-L)\bar{\psi}_i(-L) + \psi_i(-L)\bar{\psi}_i'(-L)\Big).\end{aligned} \tag{4.26}$$

One sees here that the K-matrix indeed contributes as a genuine boundary effect. The Hamiltonian, obtained as one of the charges in involution (see e.g. [13] for further details) provides the classical equations of motion by virtue of:

$$\begin{aligned}&\frac{\partial\psi_i(x,t)}{\partial t} = \Big\{\mathcal{H}(0,-L),\ \psi_i(x,t)\Big\}, \quad \frac{\partial\bar{\psi}_i(x,t)}{\partial t} = \Big\{\mathcal{H}(0,-L),\ \bar{\psi}_i(x,t)\Big\}, \\ &-L \le x \le 0.\end{aligned} \tag{4.27}$$

Indeed considering the Hamiltonian $\mathcal{H}$, we end up with the following set of equations with Dirichlet type boundary conditions

$$\begin{aligned}&i\frac{\partial\psi_i(x,t)}{\partial t} = -\frac{\partial^2\psi_i(x,t)}{\partial^2 x} + 2\kappa\sum_{j=1}^{\mathcal{N}-1}|\psi_j(x,t)|^2\psi_i(x,t) \\ &\psi_i(0) = \psi_i(-L) = 0 \qquad i \in \{1,\ldots,\mathcal{N}-1\}.\end{aligned} \tag{4.28}$$

For a detailed and quite exhaustive analysis of the various integrable boundary conditions of the NLS model see [13]. Note also that the $\mathcal{N}=2$ case was investigated classically on the half line in [55], whereas the NLS equation on the interval was studied in [56].

[5]The Poisson structure for the generalized NLS model is defined as:

$$\Big\{A,\ B\Big\} = i\sum_i\int_{-L}^{L} dx\Big(\frac{\delta A}{\delta\psi_i(x)}\ \frac{\delta B}{\delta\bar{\psi}_i(x)} - \frac{\delta A}{\delta\bar{\psi}_i(x)}\ \frac{\delta B}{\delta\psi_i(x)}\Big) \tag{4.23}$$

As mentioned our ultimate goal here is to derive the boundary Lax pair, in particular the $\mathbb{V}$ operator. In general for any $gl_{\mathcal{N}}$ r-matrix we may express (4.20), taking also into account (4.19), for the two types of boundary conditions already described in the first section, i.e.

$$\begin{aligned}\mathbb{V}(x,\lambda,\mu) &= \frac{t^{-1}(\lambda)}{\lambda-\mu}T(x,-L,\lambda)K^{-}(\lambda)\hat{T}(0,-L,\lambda)K^{+}(\lambda)T(0,x,\lambda)\\ &+ \frac{t^{-1}(\lambda)}{\lambda+\mu}\hat{T}(0,x,\lambda)K^{+}(\lambda)\ T(0,-L,\lambda)K^{-}(\lambda)\hat{T}(x,-L,\lambda) \quad \text{for SP} \quad (4.29)\end{aligned}$$

and for the SNP boundary conditions we obtain:

$$\begin{aligned}\mathbb{V}(x,\lambda,\mu) &= \frac{t^{-1}(\lambda)}{\lambda-\mu}T(x,-L,\lambda)K^{-}(\lambda)\hat{T}(0,-L,\lambda)K^{+}(\lambda)T(0,x,\lambda)\\ &+ \frac{t^{-1}(\lambda)}{\lambda+\mu}\hat{T}^{t}(x,-L,\lambda)K^{-t}(\lambda)\ T^{t}(0,-L,\lambda)K^{+t}(\lambda)\hat{T}^{t}(0,x,\lambda) \quad \text{for SNP} \quad (4.30)\end{aligned}$$

Again for the boundary points $x=0,\ -L$ we should bear in mind that $T(x,x,\lambda)=\hat{T}(x,x,\lambda)=\mathbb{I}$. Ultimately we wish to expand $T(\lambda),\ \hat{T}(\lambda)$ in powers of λ^{-1} in order to determine the Lax pair for each one of the integrals of motion. For a detailed description of the derivation of the boundary integrals of motion for the generalized NLS models see [13]. Hereafter we shall focus on the SP case with the simplest boundary conditions i.e. $K^{\pm}=\mathbb{I}$. Expanding the expression (4.29) in powers of λ^{-1} (we refer the interested reader to Appendix C for technical details) we conclude that $\mathbb{V}^{(3)}(x,\lambda)$ –the bulk part– coincides with $\mathbb{V}$ defined in (4.21), (4.22), and for the boundary points $x_b\in\{0,\ -L\}$ in particular:

$$\mathbb{V}^{(3)}(x_b,\lambda)=-\frac{\lambda^2}{2i}\Big(\sum_{i=1}^{\mathcal{N}-1}E_{ii}-E_{\mathcal{N}\mathcal{N}}\Big)+i\sum_{i,j=1}^{\mathcal{N}-1}\bar{\psi}_i(x_b)\psi_j(x_b)E_{ij}-i\sum_{i,j=1}^{\mathcal{N}-1}\Big(\bar{\psi}_i'(x_b)E_{i\mathcal{N}}-\psi_i'(x_b)E_{\mathcal{N}i}\Big). \tag{4.31}$$

We may alternatively rewrite the latter formula as:

$$\mathbb{V}^{(3)}(x_b,\lambda)=\mathbb{V}(x_b,\lambda)+i\sum_{i=1}^{\mathcal{N}-1}|\psi_i(x_b)|^2E_{\mathcal{N}\mathcal{N}}+\lambda\sum_{i=1}^{\mathcal{N}-1}(\bar{\psi}_i(x_b)E_{i\mathcal{N}}+\psi_i(x_b)E_{\mathcal{N}i}). \tag{4.32}$$

The last two terms additional to $\mathbb{V}$ (4.21), (4.22) are due to the non-trivial boundary conditions; of course more complicated boundary conditions would lead to more intricate modifications of the Lax pair $\mathbb{V}$, however such an exhaustive analysis is beyond the intended scope of the present investigation. It can be shown that the modified Lax pair $(\mathbb{U},\ \mathbb{V}^{(3)})$ gives rise to the classical equations of motion (4.28). It is clear that the 'bulk' quantity $\mathbb{V}$ in the case of SP boundary conditions remains intact. In the SNP case on the other hand we may see that even the bulk part of the Lax pair is drastically modified, due to the fact that the bulk part of the corresponding integrals of motions is also dramatically altered. We shall not further comment on this point, which will be anyway treated in full detail elsewhere.

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A Appendix

We present here technical details on the derivation of the conserved quantities for the generalized NLS model on the full line (see also [13]). Recall that for $\lambda \to \pm i\infty$ one may express T as [6]

$$T(x,y,\lambda) = (\mathbb{I} + W(x,\lambda))\ \exp[Z(x,y,\lambda)]\ (\mathbb{I} + W(y,\lambda))^{-1} \tag{A.1}$$

where W is an off diagonal matrix i.e. $W = \sum_{i\neq j} W_{ij}E_{ij}$, and Z is purely diagonal $Z = \sum_{i=1}^{\mathcal{N}} Z_{ii}E_{ii}$. Also

$$Z_{ii}(\lambda) = \sum_{n=-1}^{\infty} \frac{Z_{ii}^{(n)}}{\lambda^n}, \qquad W_{ij} = \sum_{n=1}^{\infty} \frac{W_{ij}^{(n)}}{\lambda^n}. \tag{A.2}$$

The first step is to insert the ansatz (A.1) in equation (4.1). Then we separate the diagonal and off diagonal part and obtain the following expressions:

$$\begin{aligned} Z' &= \lambda\mathbb{U}_1 + (\mathbb{U}_0 W)^{(D)} \\ W' + WZ' &= \mathbb{U}_0 + (\mathbb{U}_0 W)^{(O)} + \lambda\mathbb{U}_1 W \end{aligned} \tag{A.3}$$

where the superscripts (D), (O) denote the diagonal and off diagonal part of the product $\mathbb{U}_0 W$. Recall that $W = \sum_{i\neq j} W_{ij}E_{ij}$, $Z = \sum_i Z_{ii}E_{ii}$ then it is straightforward to obtain:

$$\begin{aligned} (\mathbb{U}_0 W)^{(D)} &= \sqrt{\kappa}\sum_{i=1}^{\mathcal{N}-1}\Big(\bar{\psi}_i W_{\mathcal{N}i}E_{ii} + \psi_i W_{i\mathcal{N}}E_{\mathcal{N}\mathcal{N}}\Big) \\ (\mathbb{U}_0 W)^{(O)} &= \sqrt{\kappa}\sum_{i\neq j,\ i\neq\mathcal{N},\ j\neq\mathcal{N}}\Big(\bar{\psi}_i W_{\mathcal{N}j}E_{ij} + \psi_i W_{ij}E_{\mathcal{N}j}\Big). \end{aligned} \tag{A.4}$$

Substituting the latter expressions (A.4) in (A.3), we obtain

$$Z(L,-L,\lambda) = -i\lambda L\Big(\sum_{i=1}^{\mathcal{N}-1}E_{ii} - E_{\mathcal{N}\mathcal{N}}\Big) + \sqrt{\kappa}\sum_{i=1}^{\mathcal{N}-1}\int_{-L}^{L} dx\Big(\bar{\psi}_i W_{\mathcal{N}i}E_{ii} + \psi_i W_{i\mathcal{N}}E_{\mathcal{N}\mathcal{N}}\Big). \tag{A.5}$$

The leading contribution in the expansion of $(\ln\ trT)$, (where T is given in (A.1)) for $i\lambda \to \infty$ comes from $Z_{\mathcal{N}\mathcal{N}}$, with a leading term $i\lambda L$ (all other $Z_{ii},\ i \neq \mathcal{N}$ have a $-i\lambda L$ leading term, so when exponentiating such contributions vanish as $i\lambda \to \infty$), indeed

$$Z_{\mathcal{N}\mathcal{N}}(L,-L,\lambda) = i\lambda L + \sqrt{\kappa}\sum_{i=1}^{\mathcal{N}-1}\int_{-L}^{L} dx\ \psi_i(x)W_{i\mathcal{N}}(x). \tag{A.6}$$

Due to (A.6) it is obvious that in this case it is sufficient to derive the coefficients $W_{i\mathcal{N}}$. In any case one can show that the coefficients W_{ij} satisfy the following equations:

$$\sum_{i\neq j} W'_{ij}E_{ij} - i\lambda\sum_{i\neq\mathcal{N}}\Big(W_{\mathcal{N}i}E_{\mathcal{N}i} - W_{i\mathcal{N}}E_{i\mathcal{N}}\Big) + \sqrt{\kappa}\sum_{i\neq\mathcal{N}}\Big(\bar{\psi}_i W^2_{\mathcal{N}i}E_{\mathcal{N}i} + \psi_i W^2_{i\mathcal{N}}E_{i\mathcal{N}}\Big) =$$
$$\sqrt{\kappa}\sum_{i\neq\mathcal{N}}\Big(\bar{\psi}_i E_{i\mathcal{N}} + \psi_i E_{\mathcal{N}i}\Big) + \sqrt{\kappa}\sum_{i\neq j,\ i\neq\mathcal{N},\ j\neq\mathcal{N}}\Big(\bar{\psi}_i W_{\mathcal{N}j}E_{ij} + \psi_i W_{ij}E_{\mathcal{N}j}\Big)$$
$$-\sqrt{\kappa}\sum_{i\neq j,\ i\neq\mathcal{N},\ j\neq\mathcal{N}}\Big(\bar{\psi}_j W_{\mathcal{N}j}W_{ij}E_{ij} + \psi_i W_{i\mathcal{N}}W_{j\mathcal{N}}E_{j\mathcal{N}}\Big). \tag{A.7}$$

Finally setting $W_{ij} = \sum_{n=1}^{\infty}\frac{W^{(n)}_{ij}}{\lambda^n}$ and using (A.7) we find expressions for $W^{(n)}_{i\mathcal{N}}$ i.e.

$$W^{(1)}_{i\mathcal{N}}(x) = -i\sqrt{\kappa}\bar{\psi}_i(x), \qquad W^{(2)}_{i\mathcal{N}}(x) = \sqrt{\kappa}\bar{\psi}'_i(x)$$
$$W^{(3)}_{i\mathcal{N}}(x) = i\sqrt{\kappa}\bar{\psi}''_i(x) - i\kappa^{\frac{3}{2}}\sum_k|\psi_k(x)|^2\bar{\psi}_i(x), \quad \ldots. \tag{A.8}$$

In the boundary case we shall need in addition to (A.8) the following objects:

$$W^{(1)}_{\mathcal{N}i} = i\sqrt{\kappa}\psi_i, \qquad W^{(2)}_{\mathcal{N}i} = -iW'^{(1)}_{\mathcal{N}i} + \sum_{i\neq j,\ i\neq N,\ j\neq N} W^{(1)}_{\mathcal{N}j}W^{(1)}_{ji}, \qquad W'^{(1)}_{ji} = iW^{(1)}_{j\mathcal{N}}W^{(1)}_{\mathcal{N}i}$$
$$W^{(3)}_{\mathcal{N}i} = -iW'^{(2)}_{\mathcal{N}i} + W^{(1)}_{i\mathcal{N}}W^{(1)}_{\mathcal{N}i}W^{(1)}_{\mathcal{N}i} + \sum_{i\neq j,\ i\neq\mathcal{N},j\neq\mathcal{N}} W^{(1)}_{\mathcal{N}j}W^{(2)}_{ji}$$
$$W'^{(2)}_{ij} = iW^{(1)}_{i\mathcal{N}}W^{(2)}_{\mathcal{N}j} - iW_j\mathcal{N}^{(1)}W^{(1)}_{\mathcal{N}j}W^{(1)}_{ij}. \tag{A.9}$$

Based on the latter formulas, the expression (A.1), and defining $\hat{W}(x,\lambda) = W(x,-\lambda)$ we may rewrite (4.29) as:

$$\mathbb{V}(x,\lambda,\mu) = \frac{1}{\lambda-\mu}(1+W(x))E_{\mathcal{N}\mathcal{N}}(1+W(x))^{-1} + \frac{1}{\lambda+\mu}(1+\hat{W}(x))E_{\mathcal{N}\mathcal{N}}(1+\hat{W}(x))^{-1}$$
$$\mathbb{V}(0,\lambda,\mu) = (X^+_0)^{-1}\Big(\frac{1}{\lambda-\mu}X^+K^+(\lambda) + \frac{1}{\lambda-\mu}K^+(\lambda)X^+\Big)$$
$$\mathbb{V}(0,\lambda,\mu) = (X^-_0)^{-1}\Big(\frac{1}{\lambda-\mu}K^-(\lambda)X^- + \frac{1}{\lambda-\mu}X^-K^-(\lambda)\Big) \tag{A.10}$$

where we define:

$$X^+ = (1+W(0))E_{\mathcal{N}\mathcal{N}}(1+\hat{W}(0))^{-1}, \qquad X^- = (1+\hat{W}(-L))E_{\mathcal{N}\mathcal{N}}(1+W(-L))^{-1}$$
$$X^+_0 = \Big[(1+\hat{W}(0,\lambda))^{-1}K^+(\lambda)(1+W(0,\lambda))\Big]_{\mathcal{N}\mathcal{N}}$$
$$X^-_0 = \Big[(1+W(-L,\lambda))^{-1}K^-(\lambda)(1+\hat{W}(-L,\lambda))\Big]_{\mathcal{N}\mathcal{N}}. \tag{A.11}$$

References

[1] P.D. Lax, Comm. Pure Appl. Math. **21** (1968) 467.

[2] C.S. Gardner, J.M. Greene, M.D. Kruskal and M.D. Miura, Phys. Rev. Lett. **19** (1967) 1095.

[3] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Stud. Appl. Math. **53** (1974) 249.

[4] M.J. Ablowitz and J.F. Ladik, J. Math. Phys. **17** (1976) 1011.

[5] V.E. Zakharov and A.B. Shabat, Anal. Appl. **13** (1979) 13.

[6] L.D. Faddeev, *Integrable models in* $1+1$ *dimensional quentum field theory* in 'Recent Advances in Field Theory and Statistical Mechanics' (J.B. Zuber and R. Stora, Ed.), Les Houches Lectures 1982, North Holland (1984) 561;
L.D. Faddeev and L.A. Takhtakajan, *Hamiltonian Methods in the Theory of Solitons*, (1987) Springer-Verlag.

[7] O. Babelon, D. Bernard and M. Talon, *Introduction to classical Integrable systems*, (2003) Cambridge monographs in Mathematical Physics.

[8] E.G. Sklyanin, Preprint LOMI E-3-97, Leningrad, 1979.

[9] M.A. Semenov-Tian-Shansky, Funct. Anal. Appl. **17** (1983) 259.

[10] J. Liouville, Journal de Mathematiques **XX** (1885) 137.

[11] V.I. Arnold, *Methods of Classical Mechanics*, Graduate Texts in Mathematics 60, (1978) Springer.

[12] P. Baseilhac and G.W. Delius, J. Phys. **A34** (2001) 8259.

[13] A. Doikou, D. Fioravanti and F. Ravanini, Nucl. Phys. **B790** [PM] (2008) 465.

[14] I. Cherednik, Theor. Math. Phys. **61** (1984) 977;
E.K. Sklyanin, J. Phys. **A21** (1988) 2375.

[15] J.-M. Maillet, Phys. Lett. **B162** (1985) 137.

[16] P. Bowcock, E. Corrigan, P.E. Dorey and R.H. Rietdijk, Nucl. Phys. **B445** (1995) 469;
P. Bowcock, E. Corrigan and R.H. Rietdijk, Nucl. Phys. **B465** (1996) 350.

[17] E.K. Sklyanin; Zap. Nauch. Seminarov LOMI **95** (1980), 55;
P.P. Kulish and E.K. Sklyanin, in Tvarminne Lectures, edited by J. Hietarinta and C. Montonen, Springer Lectures in Physics **151** (1982).

[18] O. Babelon and C.M. Viallet, Phys. Lett. **B237** (1990) 411.

[19] C.N. Yang, Phys. Rev. Lett. **19** (1967) 1312.

[20] H.J. de Vega and A. Gonzalez–Ruiz, Nucl. Phys. **B417** (1994) 553;
H.J. de Vega and A. Gonzalez–Ruiz, Phys. Lett. **B332** (1994) 123.

[21] L. Mezincescu and R.I. Nepomechie, Nucl. Phys. **B372** (1992) 597;
S. Artz, L. Mezincescu and R.I. Nepomechie, J. Phys. **A28** (1995) 5131.

[22] A. Doikou and R.I. Nepomechie, Nucl. Phys. **B521** (1998) 547;
A. Doikou and R.I. Nepomechie, Nucl. Phys. **B530** (1998) 641.

[23] W. Galleas and M.J. Martins, Phys. Lett. **A335** (2005) 167;

R. Malara and A. Lima-Santos, J. Stat. Mech. **0609** (2006) P013;
W.-L. Yang and Y.-Z. Zhang, JHEP **0412** (2004) 019.

[24] G.M. Gandenberger, `hep-th/9911178`.

[25] G.W. Delius and N. Mackay, Commun. Math. Phys. **233** (2003) 173.

[26] A. Doikou, J. Phys. **A33** (2000) 8797.

[27] D. Arnaudon, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, Int. J. Mod. Phys. **A21** (2006) 1537.

[28] D. Arnaudon, J. Avan, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, J. Stat. Mech. **0408** (2004) P005;
D. Arnaudon, N. Crampé, A. Doikou, L. Frappat and E. Ragoucy, J. Stat. Mech. **0502** (2005) P007.

[29] A. Doikou, J. Math. Phys. **46** 053504 (2005).

[30] N. Crampé and A. Doikou, J. Math. Phys. **48** 023511 (2007).

[31] G.I. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras* in 'Quantum Groups' (P.P. Kulish, Ed.), Lecture notes in Math. **1510**, Springer (1992) 103;
A.I. Molev, M. Nazarov and G.I. Olshanski, Russ. Math. Surveys **51** (1996) 206.

[32] A.I. Molev, E. Ragoucy and P. Sorba, Rev. Math. Phys. **15** (2003) 789;
A.I. Molev, *Handbook of Algebra*, Vol. 3, (M. Hazewinkel, Ed.), Elsevier, (2003), pp. 907.

[33] G.W. Delius, N.J. MacKay and B.J. Short, Phys. Lett. **B522** (2001) 335; Erratum-ibid. **B524** (2002) 401.

[34] A. Doikou, Nucl. Phys. **B725** (2005) 493.

[35] M. Gaudin, *La Fonction d'Onde de Bethe*, Masson (1983);
M. Gaudin, J. Phys. (Paris) **37** (1976) 1078.

[36] P. van Moerbeke, Invent. Math **37** (1976) 45.

[37] M. Toda, *Theory of non-linear lattices*, (1981) Springer-Verlag.

[38] E.K. Sklyanin, `nlin.SI/0009009`, "Integrable Systems: from classical to quantum"; CRM Proceedings Lecture Notes **28** (2000) 227.

[39] A. Kundu and B. Basu Mallick, Mod. Phys. Lett. **7** (1992) 61.

[40] M. Jimbo, Commun. Math. Phys. **102** (1986) 53

[41] K. Hikami, `solv-int/9506001`.

[42] N. Crampé and C.A.S. Young, J. Phys. **A40** (2007) 5491.

[43] K. Hikami, J. Phys. **A28** (1995) 4997;
W.-L. Yang, Y.-Z. Zhang and M.D. Gould, Nucl. Phys. **B698** (2004) 503;
W.-L. Yang, Y.-Z. Zhang and R. Sasaki, Nucl. Phys. **B729** (2005) 594.

[44] E. Sklyanin, Funct. Anal. Appl. **16** (1982) 263.

[45] L.C. Li and S. Parmentier, Commun. Math. Phys. **125** (1989) 545.

[46] B.M. McCoy and T.T. Wu, Nuovo Cimento **B56** (1966) 311.

[47] H.Q. Zhou, J. Phys. **A29** (1996) L489;
X.W. Guan, D.M. Tong and H.Q. Zhou, J. Phys. Soc. Jap. **65** (1996) 2807;
X.W. Guan, M.S. Wang and S.D. Yang, Nucl. Phys. **B485** (1997) 685.

[48] A. Lima-Santos, `solv-int/9912009`.

[49] L. Freidel and J.M. Maillet, Phys. Lett. **B263** (1991) 403.

[50] V.S. Gerdjikov and M.I. Ivanov, Theor. Math. Phys. **52** (1982) 676

[51] E. Date, M. Jimbo and T. Miwa, J. Phys. Soc. Japan **52** (1983) 761.

[52] I. Merola, O. Ragnisco and G. Zhang Tu, Inverse Problems **10** (1994) 1315;
A. Kundu and O. Ragnisco, J. Phys. **A27** (1994) 6335.

[53] V.B. Kuznetsov, M. Salerno and E.K. Sklyanin, J. Phys. **A33** (2000) 171.

[54] A. Fordy and P.P. Kulish, Commun. Math. Phys. **89** (1983) 427.

[55] V.O. Tarasov Inv. Probl. **7** (1991) 435;
R.F. Bikbaev and V.O. Tarasov, J. Phys. **A24** (1991) 2507.

[56] A.S. Fokas and A.R. Its, `nlin.SI/0412008`.