Cohomological constraint to deformations of compact Kähler manifolds

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Dedicated to the memory of Fabio Bardelli

Abstract

We prove that for every compact Kähler manifold X the cup product

$$H^*(X,T_X)\otimes H^*(X,\Omega^*_X)\to H^*(X,\Omega^{*-1}_X)$$

can be lifted to an L_∞-morphism from the Kodaira-Spencer differential graded Lie algebra to the suspension of the space of linear endomorphisms of the singular cohomology of X. As a consequence we get an algebraic proof of the principle “obstructions to deformations of compact Kähler manifolds annihilate ambient cohomology”.

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Introduction

In this paper we give an algebraic proof of the principle “obstructions to deformations of compact Kähler manifolds annihilate ambient cohomology” recently proved, in a different way, by Herb Clemens [4] and Ziv Ran [18].

Let X be a fixed compact Kähler manifold of dimension n and consider the graded vector space $M_X=\operatorname{Hom}^*_{\mathbb{C}}(H^*(X,\mathbb{C}),H^*(X,\mathbb{C}))$ of linear endomorphisms of the singular cohomology of X. The Hodge decomposition gives natural isomorphisms

$$M_X=\bigoplus_i M^i_X,\qquad M^i_X=\bigoplus_{r+s=p+q+i}\operatorname{Hom}_{\mathbb{C}}(H^p(\Omega^q_X),H^r(\Omega^s_X))$$

and the composition of the cup product and the contraction operator $T_X\otimes\Omega^p_X\xrightarrow{\vdash}\Omega^{p-1}_X$ gives natural linear maps

$$\theta_p\colon H^p(X,T_X)\to\bigoplus_{r,s}\operatorname{Hom}^*_{\mathbb{C}}(H^r(\Omega^s_X),H^{r+p}(\Omega^{s-1}_X))\subset M[-1]^p_X=M^{p-1}_X.$$

The Dolbeaut's complex of the holomorphic tangent bundle T_X

$$KS_X=\bigoplus_p KS^p_X,\qquad KS^p_X=\Gamma(X,\mathcal{A}^{0,p}(T_X))$$

has a natural structure of differential graded Lie algebra (DGLA), [3], [8], [11, 3.4.1], called the Kodaira-Spencer algebra of X. By Dolbeaut's theorem $H^*(KS_X)=H^*(X,T_X)$ and then the maps θ_i give a morphism of graded vector spaces $\theta\colon H^*(KS_X)\to M[-1]_X$. This morphism is generally nontrivial: consider for instance a Calabi-Yau manifold where the map θ_p induces an isomorphism $H^p(X,T_X)=\operatorname{Hom}_{\mathbb{C}}(H^0(\Omega^n_X),H^p(\Omega^{n-1}_X))$.

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Theorem A. *In the above notation, consider $M[-1]_X$ as a differential graded Lie algebra with trivial differential and trivial bracket.*
Every choice of a Kähler metric on X induces a canonical lifting of θ to an L_∞-morphism from KS_X to $M[-1]_X$.

The above theorem, together some standard and purely formal results in Schlessinger's theory, gives immediate applications to the study of deformations of X. In fact the deformations of X are governed by the Kodaira-Spencer differential graded Lie algebra KS_X and every L_∞-morphism between DGLAs induces a natural transformation between the associated deformation functors. The triviality of the DGLA structure on $M[-1]_X$ allows to prove easily the following:

Corollary B. *Let $f\colon \mathcal{Y} \to \mathcal{B}$ be the semiuniversal deformation of a compact Kähler manifold Y and let $X \xrightarrow{\pi} Y$ be a finite unramified covering. For every $p \geq 0$ denote by α_p the composite linear map*

$$\alpha_p\colon H^p(Y, T_Y) \xrightarrow{\pi^*} H^p(X, T_X) \xrightarrow{\theta_p} \bigoplus_{r,s} \mathrm{Hom}_{\mathbb{C}}(H^r(\Omega^s_X), H^{r+p}(\Omega^{s-1}_X)).$$

Then:

1. *If α_1 is injective then $f\colon \mathcal{Y} \to \mathcal{B}$ is universal.*
2. *There exists a morphisms of complex analytic singularities $q\colon (H^1(Y, T_Y), 0) \to (\ker \alpha_2, 0)$ such that $\mathcal{B}$ is isomorphic to $q^{-1}(0)$. In particular if α_2 is injective then $\mathcal{B}$ is smooth.*

As an example, if Y is a projective manifold with torsion canonical bundle and $\pi\colon X \to Y$ is the canonical covering, then all the maps α_p are injective.

Probably the main interesting aspect of Theorem A is that it gives a concrete construction of a morphism whose existence is predicted by the general philosophy of extended deformation theory.
Roughly speaking, to every deformation problem over a field of characteristic 0, it is associated a differential graded Lie algebra L, unique up to quasiisomorphism, and a formal pointed quasismooth dg-manifold $\mathcal{M}$ quasiisomorphic to L as L_∞-algebra. The differential graded Lie algebra L governs the deformation problem via the solutions Maurer-Cartan modulo gauge action and the truncation in degree 0 of $\mathcal{M}$ is the classical moduli space (cf. [15], Section 2 of [2] and references therein).
Moreover, according to this general philosophy, every natural morphism between moduli spaces (e.g. the period map from deformations of a compact Kähler manifold to deformations of its Hodge decomposition) should extend to a morphism of their extended moduli spaces and therefore induces an L_∞-morphism between the associated differential graded Lie algebras.

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Notation

For every holomorphic vector bundle E on a complex manifold we denote by $\mathcal{A}^{p,q}(E)$ the sheaf of differential (p,q)-forms with coefficients in E.
For every vector space V and every linear functional $\alpha\colon V \to \mathbb{C}$ we denote by $\alpha\vdash\colon \bigwedge^k V \to \bigwedge^{k-1} V$ the contraction operator

$$\alpha \vdash (v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha(v_i) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k.$$

We point out for later use that $\alpha\vdash$ is a derivation of degree -1 of the graded algebra $(\bigwedge^* V,\wedge)$.
We denote by Σ_m the symmetric group of permutations of the set $\{1,2,\dots,m\}$ and, for every $0\le p\le m$ by $S(p,m-p)\subset\Sigma_m$ the set of unshuffles of type $(p,m-p)$. By definition $\sigma\in S(p,m-p)$ if and only if $\sigma_1<\sigma_2<\dots<\sigma_p$ and $\sigma_{p+1}<\sigma_{p+2}<\dots<\sigma_m$.

1 L_∞-morphisms

Let $V=\oplus V^i$ be a $\mathbb{Z}$-graded vector space, for every integer n we denote by $V[n]=\oplus V[n]^i$ the graded vector space where $V[n]^i=V^{n+i}$. The space $V[-1]$ is also called the suspension of V and $V[1]$ the unsuspension.
The graded m-th symmetric power of V is denoted by $\bigodot^m V$. If $\sigma\in\Sigma_m$ and $a_1,\dots,a_m\in V$ are homogeneous elements, the Koszul sign $\epsilon(V,\sigma;a_1,\dots,a_m)=\pm1$ is defined by the rule

$$a_{\sigma_1}\odot\dots\odot a_{\sigma_m}=\epsilon(V,\sigma;a_1,\dots,a_m)a_1\odot\dots\odot a_m\in\bigodot^m V.$$

For simplicity of notation we write $\epsilon(V,\sigma)$ when the elements $a_1,\dots,a_m$ are clear from the context. If $a\in V$ is homogeneous we denote by $\deg(a,V)$ its degree; we also write $\deg(a,V)=\overline{a}$ when there is no ambiguity about V. Note that $\deg(a,V[n])=\deg(a,V)-n$.
We denote by $C(V)$ the reduced graded symmetric coalgebra generated by $V[1]$; more precisely it is the graded vector space

$$C(V)=\overline{S}(V[1])=\bigoplus_{m=1}^{\infty}\bigodot^m(V[1])$$

endowed with the coproduct $\Delta\colon C(V)\to C(V)\otimes C(V)$, $\Delta(a)=0$ for every $a\in V[1]$ and

$$\Delta(a_1\odot\dots\odot a_m)=\sum_{r=1}^{m-1}\sum_{\sigma\in S(r,m-r)}\epsilon(V[1],\sigma)(a_{\sigma_1}\odot\dots\odot a_{\sigma_r})\otimes(a_{\sigma_{r+1}}\odot\dots\odot a_{\sigma_m})$$

for every $a_1,\dots,a_m\in V[1]$, $m\ge 2$.
Assume now that V has a structure of differential graded Lie algebra with differential d and bracket $[\ ,\ ]$, then the linear map

$$Q\colon\bigodot^2(V[1])\to V[1],\qquad Q(a\odot b)=(-1)^{\deg(a,V[1])}[a,b]$$

has degree 1 and the map $\delta\colon C(V)\to C(V)$ defined by

$$\begin{aligned}\delta(a_1\odot\dots\odot a_m)=&\sum_{\sigma\in S(1,m-1)}\epsilon(V[1],\sigma;a_1,\dots,a_m)da_{\sigma_1}\odot a_{\sigma_2}\odot\dots\odot a_{\sigma_m}+\\&+\sum_{\sigma\in S(2,m-2)}\epsilon(V[1],\sigma;a_1,\dots,a_m)Q(a_{\sigma_1}\odot a_{\sigma_2})\odot a_{\sigma_3}\odot\dots\odot a_{\sigma_m}\end{aligned}\qquad(1)$$

is a codifferential of degree 1 on the coalgebra $C(V)$. The differential graded coalgebra $(C(V),\delta)$ is called the L_∞-algebra associated to the DGLA $(V,d,[\ ,\ ])$.
By definition, an L_∞-morphism between two DGLA V,V' is a morphism of differential graded coalgebras $\Theta\colon(C(V),\delta)\to(C(V'),\delta')$.
It is easy to check that if $f\colon V\to V'$ is a morphism of differential graded Lie algebras then the linear map

$$(C(V),\delta)\to(C(V'),\delta'),\qquad a_1\odot\dots\odot a_m\to f(a_1)\odot\dots\odot f(a_m)$$

is an L_∞-morphism. We refer to [11], [12], [13], [20] for the general theory of L_∞-morphisms. In this paper we are interested only in the particular and simple case when V' has trivial differential and trivial bracket: under these assumption $\delta' = 0$ and there exists a bijection between the set of L_∞-morphism $\Theta\colon (C(V),\delta) \to (C(V'),0)$ and the set of morphisms of graded vector spaces $F\colon C(V) \to V'[1]$ such that $F \circ \delta = 0$. The bijection is given by the formulas

$$F = p_1 \circ \Theta, \qquad p_1\colon C(V') \to \bigodot^1 V'[1] = V'[1] \quad \text{the projection}$$

$$\Theta = \sum_{m=1}^{\infty} \frac{1}{m!} F^{\odot m} \circ \Delta^{m-1}_{C(V)}\colon C(V) \to C(V')$$

where $F^{\odot m}$ is the composition of $F^{\otimes m}\colon \bigotimes^m C(V) \to \bigotimes^m (V'[1])$ with the projection onto the symmetric product $\bigotimes^m (V'[1]) \to \bigodot^m (V'[1])$.
Let $F_1\colon V[1] \to V'[1]$ the composition of F with the inclusion $V[1] \hookrightarrow C(V)$. Just to explain the statement of Theorem A we observe that the condition $F \circ \delta = 0$ implies $F_1 \circ d = 0$ and then F_1 induce a map in cohomology $\theta\colon H^*(V) \to H^*(V') = V'$.

2 Proof of Theorem A

Let X be a complex manifold of dimension n; consider the graded vector space $L = \oplus L^p$, where $L^p = \Gamma(X, \mathcal{A}^{0,p+1}(T_X))$, $-1 \le p \le n-1$, and two linear maps of degree $+1$, $d\colon L \to L$, $Q\colon \odot^2 L \to L$ defined in the following way: if $z_1, \dots, z_n$ are local holomorphic coordinates, then

$$d\left(\phi \frac{\partial}{\partial z_i}\right) = (\overline{\partial}\phi)\frac{\partial}{\partial z_i}, \qquad \phi \in \mathcal{A}^{0,*}.$$

If I, J are ordered subsets of $\{1, \dots, n\}$, $a = f d\overline{z}_I \frac{\partial}{\partial z_i}$, $b = g d\overline{z}_J \frac{\partial}{\partial z_j}$, $f, g \in \mathcal{A}^{0,0}$ then

$$Q(a \odot b) = (-1)^{\overline{a}} d\overline{z}_I \wedge d\overline{z}_J \left(f \frac{\partial g}{\partial z_i}\frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_i}\right), \qquad \overline{a} = \deg(a, L).$$

The equation (1), with L in place of $V[1]$, gives a codifferential δ of degree 1 on $\overline{S}(L)$ and the differential graded coalgebra $(\overline{S}(L), \delta)$ is exactly the L_∞-algebra associated to the Kodaira-Spencer DGLA KS_X.

If $\mathrm{Der}^p(\mathcal{A}^{*,*}, \mathcal{A}^{*,*})$ denotes the vector space of $\mathbb{C}$-derivations of degree p of the sheaf of graded algebras $(\mathcal{A}^{*,*}, \wedge)$, where the degree of a (p,q)-form is $p+q$ (note that $\partial, \overline{\partial} \in \mathrm{Der}^1(\mathcal{A}^{*,*}, \mathcal{A}^{*,*})$), then we can define a morphism of graded vector spaces

$$L \xrightarrow{\;\widehat{}\;} \mathrm{Der}^*(\mathcal{A}^{*,*}, \mathcal{A}^{*,*}) = \bigoplus_p \mathrm{Der}^p(\mathcal{A}^{*,*}, \mathcal{A}^{*,*}), \qquad a \to \widehat{a}$$

given in local coordinates by

$$\widehat{\phi \frac{\partial}{\partial z_i}}(\eta) = \phi \wedge \left(\frac{\partial}{\partial z_i} \vdash \eta\right).$$

If $\overline{a} = p$ then $\widehat{a}$ is a bihomogeneous derivation of bidegree $(-1, p+1)$: in particular $\widehat{a}(\mathcal{A}^{0,*}) = 0$.

Lemma 2.1. *If* $[\ ,\ ]$ *denotes the standard bracket on* $\mathrm{Der}^*(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$*, then for every pair of homogeneous* $a,b\in L$ *we have:*

1. $\widehat{da}=[\overline{\partial},\widehat{a}]=\overline{\partial}\widehat{a}-(-1)^{\overline{a}}\widehat{a}\overline{\partial}.$
2. $Q(\widehat{a\odot b})=-[[\partial,\widehat{a}],\widehat{b}]=(-1)^{\overline{a}}\widehat{a}\partial\widehat{b}+(-1)^{\overline{a}\,\overline{b}+\overline{b}}\widehat{b}\partial\widehat{a}\pm\partial\widehat{a}\widehat{b}\pm\widehat{b}\widehat{a}\partial.$

Proof By linearity we may assume $a=fd\overline{z}_I\dfrac{\partial}{\partial z_i}$, $b=gd\overline{z}_J\dfrac{\partial}{\partial z_j}$, $f,g\in\mathcal{A}^{0,0}$. Moreover all the four expressions are derivations vanishing on the subalgebra $\mathcal{A}^{0,*}$ and therefore it is sufficient to check the above equalities when computed on the dz_i's; since $\overline{\partial}dz_i=\partial dz_i=\widehat{a}\widehat{b}dz_i=0$, the computation becomes straightforward and it is left to the reader. □

Remark. The apparent asymmetry in the right hand side of Item 2 of the above lemma is easily understood: in fact $[\widehat{a},\widehat{b}]=0$ and then by Jacobi identity

$$0=[\partial,[\widehat{a},\widehat{b}]]=[[\partial,\widehat{a}],\widehat{b}]-(-1)^{\overline{a}\,\overline{b}}[[\partial,\widehat{b}],\widehat{a}].$$

Assume now that X is compact Kähler, fix a Kähler metric on X and denote by: $A^{p,q}=\Gamma(X,\mathcal{A}^{p,q})$ the vector space of global (p,q)-forms, $\overline{\partial}^*\colon A^{p,q}\to A^{p,q-1}$ the adjoint operator of $\overline{\partial}$, $\Delta_{\overline{\partial}}=\overline{\partial}\,\overline{\partial}^*+\overline{\partial}^*\overline{\partial}$ the $\overline{\partial}$-Laplacian, $G_{\overline{\partial}}$ the associated Green operator, $\mathcal{H}\subset A^{*,*}$ the graded vector space of harmonic forms, $i\colon\mathcal{H}\to A^{*,*}$ the inclusion and $h=Id-\Delta_{\overline{\partial}}G_{\overline{\partial}}=Id-G_{\overline{\partial}}\Delta_{\overline{\partial}}\colon A^{*,*}\to\mathcal{H}$ the harmonic projector.

We identify the graded vector space M_X with the space of endomorphisms of harmonic forms $\mathrm{Hom}^*_{\mathbb{C}}(\mathcal{H},\mathcal{H})$. We also denote by $N=\mathrm{Hom}^*_{\mathbb{C}}(A^{*,*},A^{*,*})$ the graded associative algebra of linear endomorphisms of the space of global differential forms on X.
For notational simplicity we identify $\mathrm{Der}^*(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$ with its image into N.

Setting $\tau=G_{\overline{\partial}}\overline{\partial}^*\partial\in N^0$ we have by Kähler identities (cf. [10], [21]):

$$h\partial=\partial h=\tau h=h\tau=\partial\tau=\tau\partial=0$$

$$[\partial,\overline{\partial}^*]=[\partial,G_{\overline{\partial}}]=[\overline{\partial},G_{\overline{\partial}}]=0,\qquad[\overline{\partial},\tau]=\overline{\partial}G_{\overline{\partial}}\overline{\partial}^*\partial-G_{\overline{\partial}}\overline{\partial}^*\partial\overline{\partial}=G_{\overline{\partial}}\Delta_{\overline{\partial}}\partial=\partial.$$

We introduce the morphism

$$F_1\colon L\to M_X,\qquad F_1(a)=h\widehat{a}i.$$

We note that F_1 is a morphism of complexes, in fact $F_1(da)=h\widehat{da}i=h(\overline{\partial}\widehat{a}\pm\widehat{a}\overline{\partial})i=0$. Next we define, for every $m\geq 2$, the morphisms of graded vector spaces

$$f_m\colon\bigotimes^m L\to M_X,\qquad F_m\colon\bigodot^m L\to M_X,\qquad F=\sum_{m=1}^{\infty}F_m\colon\overline{S}(L)\to M_X,$$

$$f_m(a_1\otimes a_2\otimes\ldots\otimes a_m)=h\widehat{a_1}\tau\widehat{a_2}\tau\widehat{a_3}\ldots\tau\widehat{a_m}i.$$

$$F_m(a_1\odot a_2\odot\ldots\odot a_m)=\sum_{\sigma\in\Sigma_m}\epsilon(L,\sigma;a_1,\ldots,a_m)f_m(a_{\sigma_1}\otimes\ldots\otimes a_{\sigma_m}).$$

Theorem 2.2. *In the above notation* $F\circ\delta=0$ *and therefore*

$$\Theta=\sum_{m=1}^{\infty}\frac{1}{m!}F^{\odot m}\circ\Delta^{m-1}_{C(KS_X)}\colon(C(KS_X),\delta)\to(C(M[-1]_X),0)$$

is an L_∞*-morphism with linear term* F_1*.*

Proof We need to prove that for every $m \geq 2$ and $a_1, \ldots, a_m \in L$ we have

$$F_m\left(\sum_{\sigma \in S(1,m-1)} \epsilon(L,\sigma) da_{\sigma_1} \odot a_{\sigma_2} \odot \ldots \odot a_{\sigma_m}\right) =$$

$$= -F_{m-1}\left(\sum_{\sigma \in S(2,m-2)} \epsilon(L,\sigma) Q(a_{\sigma_1} \odot a_{\sigma_2}) \odot a_{\sigma_3} \odot \ldots \odot a_{\sigma_m}\right),$$

where $\epsilon(L,\sigma) = \epsilon(L,\sigma; a_1, \ldots, a_m)$.
It is convenient to introduce the auxiliary operators $q\colon \bigotimes^2 L \to N[1]$, $q(a \otimes b) = (-1)^{\overline{a}} \widehat{a} \partial \widehat{b}$ and $g_m\colon \bigotimes^m L \to M[1]_X$,

$$g_m(a_1 \otimes \ldots \otimes a_m) = -\sum_{i=0}^{m-2} (-1)^{\overline{a_1}+\overline{a_2}+\ldots+\overline{a_i}} h\widehat{a_1}\tau \ldots \widehat{a_i}\tau q(a_{i+1} \otimes a_{i+2})\tau \widehat{a_{i+3}} \ldots \tau \widehat{a_m} i.$$

Since for every choice of operators $\alpha = h, \tau$ and $\beta = \tau, i$ and every $a, b \in L$ we have

$$\alpha Q\widehat{(a \odot b)}\beta = \alpha((-1)^{\overline{a}} \widehat{a}\partial\widehat{b} + (-1)^{\overline{a}\,\overline{b}+\overline{b}} \widehat{b}\partial\widehat{a})\beta = \alpha(q(a \otimes b) + (-1)^{\overline{a}\,\overline{b}} q(b \otimes a))\beta,$$

a straightforward computation about symmetrization and unshuffles gives

$$\sum_{\sigma \in \Sigma_m} \epsilon(L,\sigma) g_m(a_{\sigma_1} \otimes \ldots \otimes a_{\sigma_m}) = -F_{m-1}\left(\sum_{\sigma \in S(2,m-2)} \epsilon(L,\sigma) Q(a_{\sigma_1} \odot a_{\sigma_2}) \odot a_{\sigma_3} \odot \ldots \odot a_{\sigma_m}\right).$$

On the other hand

$$f_m\left(\sum_{i=0}^{m-1} (-1)^{\overline{a_1}+\ldots+\overline{a_i}} a_1 \otimes \ldots \otimes a_i \otimes da_{i+1} \otimes \ldots \otimes a_m\right) =$$

$$= \sum_{i=0}^{m-1} (-1)^{\overline{a_1}+\ldots+\overline{a_i}} h\widehat{a_1} \ldots \widehat{a_i}\tau(\overline{\partial}\widehat{a_{i+1}} - (-1)^{\overline{a_{i+1}}} \widehat{a_{i+1}}\overline{\partial})\tau \ldots \tau \widehat{a_m} i$$

$$= \sum_{i=0}^{m-2} (-1)^{\overline{a_1}+\ldots+\overline{a_i}} h\widehat{a_1} \ldots \widehat{a_i}\tau(-(-1)^{\overline{a_{i+1}}} \widehat{a_{i+1}}\overline{\partial}\tau\widehat{a_{i+2}} + (-1)^{\overline{a_{i+1}}} \widehat{a_{i+1}}\tau\overline{\partial}\widehat{a_{i+2}})\tau \ldots \tau \widehat{a_m} i$$

$$= g_m(a_1 \otimes \ldots \otimes a_m).$$

Taking the symmetrization of this equality we get

$$\sum_{\sigma \in \Sigma_m} \epsilon(L,\sigma) g_m(a_{\sigma_1} \otimes \ldots \otimes a_{\sigma_m}) = F_m\left(\sum_{\sigma \in S(1,m-1)} \epsilon(L,\sigma) da_{\sigma_1} \odot a_{\sigma_2} \odot \ldots \odot a_{\sigma_m}\right).$$

□

Since it is clear that F_1 is a morphism of complexes inducing the morphism θ in cohomology, Theorem A is proved.

Remark. If X is a Calabi-Yau manifold with holomorphic volume form Ω, then the composition of F with the evaluation at Ω induces an L_∞-morphism $C(KS_X) \to C(\mathcal{H}[n-1])$.
For every $m \geq 2$, $\mathrm{ev}_\Omega \circ F_m\colon \bigodot^m L \to \mathcal{H}[n]$ vanishes on $\bigodot^m \{a \in L \,|\, \partial(a \vdash \Omega) = 0\}$.

The following corollary gives a formality criterion:

Corollary 2.3. *In the notation of introduction, if $\theta\colon H^*(X, T_X) \to M[-1]_X$ is injective, then KS_X is L_∞-quasiisomorphic to an abelian differential graded Lie algebra.*

Proof Let $H \subset M[-1]_X$ be the image of θ and let $p\colon M[-1]_X \to H$ be a linear projection. Since p is a morphism of DGLA, the composition $C(KS_X)\xrightarrow{\Theta}C(M[-1]_X)\xrightarrow{p}C(H)$ is an L_∞ quasiisomorphism.

3 Applications to deformation theory

All the technical tools used in this section are standard and well exposed in the literature. Let **Art** be the category of local Artinian $\mathbb{C}$-algebras (A, m_A) with residue field $A/m_A = \mathbb{C}$. Following [19], by a functor of Artin rings we intend a covariant functor $\mathcal{F}\colon \mathbf{Art} \to \mathbf{Set}$ such that $\mathcal{F}(\mathbb{C}) = \{0\}$ is a set of cardinality 1.
With the term Schlessinger's condition we mean one of the four conditions $(H_1), \dots, (H_4)$ described in Theorem 2.1 of [19].

Lemma 3.1. *Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a natural transformation of functors of Artin rings; if $\mathcal{F}$ satisfies Schlessinger's conditions (H_1) and (H_2), $\mathcal{G}$ is prorepresentable and $\alpha\colon t_\mathcal{F} \to t_\mathcal{G}$ is injective, then also $\mathcal{F}$ is prorepresentable.*

Proof Since $\mathcal{G}$ is prorepresentable its tangent space $t_\mathcal{G}$ is finite dimensional and then the same holds for $t_\mathcal{F}$. Moreover for every small extension $0 \to J \to A \to B \to 0$ there exists a natural transitive free action (cf. [19, 2.15]) of $t_\mathcal{G} \otimes J$ on the nonempty fibres of $\mathcal{G}(A) \to \mathcal{G}(B)$. Therefore also $t_\mathcal{F} \otimes J$ acts without fixed points on $\mathcal{F}(A)$ and then, according to Theorem 2.11 of [19], $\mathcal{F}$ is prorepresentable. □

For every differential graded complex Lie algebra $K = \oplus K^i$, we denote respectively by $\mathrm{MC}_K, \mathrm{Def}_K\colon \mathbf{Art} \to \mathbf{Set}$ the associated Maurer-Cartan and deformation functors (cf. [7], [8], [15]):

$$\mathrm{MC}_K(A) = \left\{ a \in K^1 \otimes m_A \;\middle|\; da + \frac{1}{2}[a,a] = 0 \right\}, \qquad \mathrm{Def}_K(A) = \frac{\mathrm{MC}_K(A)}{\exp(K^0 \otimes m_A)}.$$

The functors MC_K and Def_K are functors of Artin rings satisfying the Schlessinger's conditions (H_1), (H_2) (cf. [19],[5]), the projection $\mathrm{MC}_K \to \mathrm{Def}_K$ is smooth and the tangent space t_{Def_K} of Def_K is naturally isomorphic to $H^1(K)$.

Example 3.2. 1. If K has trivial bracket and trivial differential then the gauge action is trivial and therefore, for every $(A, m_A) \in \mathbf{Art}$, $\mathrm{Def}_K(A) = \mathrm{MC}_K(A) = K^1 \otimes m_A$; in particular if K^1 is finite dimensional then Def_K is prorepresented by a smooth germ.

2. If $K = KS_X$ is the Kodaira-Spencer DGLA of a compact complex manifold X then Def_K is isomorphic to the functor Def_X of infinitesimal deformations of X (cf. [8]).

The functor Def_K has a natural obstruction theory with obstruction space $H^2(K)$: this means that for every small extension

$$e: \quad 0 \longrightarrow J \longrightarrow A \xrightarrow{p} B \longrightarrow 0$$

in the category **Art** it is given an "obstruction map" $ob_e\colon \mathrm{Def}_K \to H^2(K) \otimes J$ such that an element $b \in \mathrm{Def}_K(B)$ lifts to $\mathrm{Def}_K(A)$ if and only $ob_e(b) = 0$. Moreover all the obstruction maps behave functorially with respect to morphisms of small extensions (cf. e.g. [1], [5]).
By definition the *primary obstruction map* is the obstruction map $q_2 = ob_\epsilon\colon H^1(K) \to H^2(K)$ relative to the small extension

$$\epsilon: \quad 0 \longrightarrow \mathbb{C} \xrightarrow{t^2} \frac{\mathbb{C}[t]}{(t^3)} \longrightarrow \frac{\mathbb{C}[t]}{(t^2)} \longrightarrow 0.$$

Concretely, if $b \in \mathrm{MC}_K(B)$ and $a \in K^1 \otimes m_A$ is a lifting of b, then by the Jacobi identity $h = da + [a,a]/2 \in K^2 \otimes J$ is a cocycle and its cohomology class $ob_e(b) = [h] \in H^2(K) \otimes J$ does not depend from the choice of a. It is easy to prove that $ob_e(b) = 0$ if and only if b can be lifted to $\mathrm{MC}_K(A)$.
The map ob_e is invariant under the gauge action (this follows from a general result [5, 7.5] but it is also easy to prove directly) and then factors to a map $ob_e\colon \mathrm{Def}_K(B) \to H^2(K) \otimes J$. Since the projection $\mathrm{MC}_K \to \mathrm{Def}_K$ is smooth, we have that the class of b lifts to $\mathrm{Def}_K(A)$ if and only if $ob_e(b) = 0$.
The obstruction space $O_K \subset H^2(K)$ is by definition the vector space generated by the images the maps $(Id \otimes f) \circ ob_e$, where $f \in \mathrm{Hom}_{\mathbb{C}}(J, \mathbb{C})$ and e ranges over all small extension in **Art**.

Remark. If the DGLA K is not formal, it may happen that the primary obstruction map vanishes but $O_K \neq 0$. If $O_K^c \subset O_K$ denotes the subspace generated by the obstructions coming from all the curvilinear small extensions

$$0 \longrightarrow \mathbb{C} \xrightarrow{t^n} \frac{\mathbb{C}[t]}{(t^{n+1})} \longrightarrow \frac{\mathbb{C}[t]}{(t^n)} \longrightarrow 0$$

then, by the (abstract) T^1-lifting theorem [6], Def_K is smooth if and only if $O_K^c = 0$ but in general $O_K^c \neq O_K$ (cf. [5, 5.7]).

Given two differential graded Lie algebras K, M, every L_∞-morphism $\mu\colon C(K) \to C(M)$ induces a natural transformation $\widetilde{\mu}\colon \mathrm{Def}_K \to \mathrm{Def}_M$ (see e.g. [11], [15]). Writing $\mu = \sum_{i \le j} \mu_j^i$, $\mu_j^i\colon \bigodot^j K[1] \to \bigodot^j M[1]$, the morphism μ_1^1 is a morphism of complexes, $H^1(\mu_1^1)\colon H^1(K) \to H^1(M)$ equals the restriction of $\widetilde{\mu}$ on tangent spaces and $H^2(\mu_1^1)\colon H^2(K) \to H^2(M)$ commutes with $\widetilde{\mu}$ and all the obstruction maps.

Proposition 3.3. *Let K be a differential graded Lie algebra, $M \oplus M^i$ be a graded vector space considered as a differential graded Lie algebra with trivial bracket and differential and let $\mu = \sum_{i \le j} \mu_j^i\colon C(K) \to C(M)$ be an L_∞-morphism. Then:*

1. *If M^1 is finite dimensional and $H^1(\mu_1^1)$ is injective then* Def_K *is prorepresentable.*
2. *The obstruction space O_K is contained in the kernel of $H^2(\mu_1^1)\colon H^2(K) \to M^2$.*

Proof The first part follows immediately from Lemma 3.1. The second part follows from the fact that all the obstruction maps of the functor Def_M are trivial. □

If X is a compact Kähler manifold we have, in the notation of the Introduction and Section 2, for every $A \in \mathbf{Art}$,

$$\mathrm{Def}_X(A) = \mathrm{Def}_{KS_X}(A) = \frac{\left\{ a \in L^0 \otimes m_A \;\middle|\; da + \frac{1}{2} Q(a \odot a) = 0 \right\}}{\exp(L^{-1} \otimes m_A)},$$

$$\mathrm{Def}_{M[-1]_X}(A) = M_X^0 \otimes m_A$$

and the natural transformation $\widetilde{F}\colon \mathrm{Def}_{KS_X} \to \mathrm{Def}_{M[-1]_X}$ associated to the L_∞-morphism Θ of Theorem 2.2 is induced by

$$\widetilde{\Theta}(a) = \sum_{m=1}^{\infty} \frac{1}{m!} F_m(a^{\odot m}) = F(\exp(a) - 1), \qquad a \in L^0 \otimes m_A.$$

Since $M[-1]_X$ carries the trivial structure of DGLA, Proposition 3.3 gives the following result known as the principle "Obstructions to deformations of compact Kähler manifolds annihilate ambient cohomology" (cf. [4, 10.1], [18, 3.5]).

Corollary 3.4. *Let X be a compact Kähler manifold and denote by O the kernel of*

$$\theta_2\colon H^2(X,T_X)\to\bigoplus_{r,s}\mathrm{Hom}^*_{\mathbb{C}}(H^r(\Omega^s_X),H^{r+2}(\Omega^{s-1}_X)).$$

Then for every small extension $e: 0\longrightarrow J\longrightarrow A\xrightarrow{p}B\longrightarrow 0$ and every $b\in \mathrm{Def}_X(B)$, the obstruction $ob_e(b)$ belongs to $O\otimes J$.

Proof of Corollary B We first recall that, if $\mathcal{Y}\to\mathcal{B}$ is the Kuranishi family of a compact complex manifold Y and $O\subset H^2(Y,T_Y)$ is the subspace generated by all the obstruction to the deformations of Y, then the singularity $\mathcal{B}$ is analytically isomorphic to $q^{-1}(0)$, where $q\colon (H^1(Y,T_Y),0)\to(O,0)$ is the Kuranishi map.
The pull-back of forms and vector fields give a morphism of differential graded Lie algebras $\pi^*\colon KS_Y\to KS_X$. The composition of π^* with Θ gives an L_∞-morphism from KS_Y to $M[-1]_X$. It is now sufficient to apply Proposition 3.3. □

Example 3.5. Let Z be a projective Calabi-Yau manifold of dimension $n\geq 3$ with $H^2(\mathcal{O}_Z)=0$ and let $\pi\colon Y\to Z$ be a smooth Galois double cover. Denoting by $D\subset Z$ the branching divisor, $R\subset Y$ the ramification divisor and $\pi_*\mathcal{O}_Y=\mathcal{O}_Z\oplus\mathcal{O}_Z(-L)$ the eigensheaves decomposition we have (cf. [3], [16]) $\mathcal{O}_Y(R)=K_Y=\pi^*\mathcal{O}_Z(L)$, $\mathcal{O}_Z(D)=\mathcal{O}_Z(2L)$, an exact sequence of sheaves over Y

$$0\longrightarrow T_Y\longrightarrow\pi^*T_Z\longrightarrow\mathcal{O}_R(2R)\longrightarrow 0$$

and, for every i, $H^i(\pi^*T_Z)=H^i(T_Z)\oplus H^i(T_Z(-L))$, $H^i(\mathcal{O}_R(2R))=H^i(\mathcal{O}_D(D))$.
If L is sufficiently ample then $H^1(\mathcal{O}_D(D))=H^2(\mathcal{O}_Z)=0$, $H^2(T_Z(-L))=0$ and then $H^2(T_Y)$ injects into $H^2(T_Z)$. Therefore the cup product with the pull-back of the holomorphic volume form of Z is nondegenerate and then $\theta_2\colon H^2(T_Y)\to M^1$ is injective. Applying Corollary B (with $X=Y$) we get that Y is unobstructed.

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