Coverage Probability of Random Intervals *

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July 2007

Abstract

In this paper, we develop a general theory on the coverage probability of random intervals defined in terms of discrete random variables with continuous parameter spaces. The theory shows that the minimum coverage probabilities of random intervals with respect to corresponding parameters are achieved at discrete finite sets and that the coverage probabilities are continuous and unimodal when parameters are varying in between interval endpoints. The theory applies to common important discrete random variables including binomial variable, Poisson variable, negative binomial variable and hypergeometrical random variable. The theory can be used to make relevant statistical inference more rigorous and less conservative.

1 Binomial Random Intervals

Let X be a Bernoulli random variable defined in a probability space $(\Omega, \mathscr{F}, \Pr)$ such that $\Pr\{X = 1\} = p$ and $\Pr\{X = 0\} = 1 - p$ where $p \in (0,1)$. Let $X_1, \cdots, X_n$ be n identical and independent samples of X. In many applications, it is important to construct a confidence interval (L, U) such that $\Pr\{L < p < U \mid p\} \approx 1 - \delta$ with $\delta \in (0,1)$. Here $L = L(n, \delta, K)$ and $U = U(n, \delta, K)$ are multivariate functions of n, δ and random variable $K = \sum_{i=1}^n X_i$. To simply notations, we drop the arguments and write $L = L(K)$ and $U = U(K)$. Also, we use notation $\Pr\{L(K) < p < U(K) \mid p\}$ to represent the probability when the binomial parameter assumes value p. Such notation is used in a similar way throughout this paper. We would thus advise the reader to distinguish this notation from conventional notation of conditional probability.

Clearly, the construction of confidence interval is independent of the binomial parameter p. But, for fixed n and δ, the quantity $\Pr\{L(K) < p < U(K) \mid p\}$ is a function of p and is conventionally referred to as the coverage probability. In many situations, it is desirable to know what is the worst-case coverage probability for p belonging to interval $[a, b] \subset (0,1)$. For this purpose, we have

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Theorem 1 *Suppose that both $L(k)$ and $U(k)$ are monotone functions of $k \in \{0, 1, \cdots, n\}$. Then, the minimum of $\Pr\{L(K) < p < U(K) \mid p\}$ with respect to $p \in [a, b]$ is attained at the discrete set $\{a, b\} \cup \{L(k) \in (a, b) : 0 \leq k \leq n\} \cup \{U(k) \in (a, b) : 0 \leq k \leq n\}$.*

We would like to emphasis that the only assumption in Theorem 1 is that both $L(k)$ and $U(k)$ are either non-decreasing or non-increasing with respect to k. The interval $(L(K), U(K))$ can be general *random interval* without being restricted to the context of confidence intervals. Theorem 1 can be generalized as Theorem 7 in Section 4. The application of the theorem is discussed in the full version of our paper [4]. Specially, Theorem 1 can be applied to the sample size problems studied in [1].

For closed confidence interval $[L, U]$, it is interesting to compute the infimum of $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ with respect to $p \in [a, b] \subset (0, 1)$. For this purpose, we have

Theorem 2 *Suppose that both $L(k)$ and $U(k)$ are monotone functions of $k \in \{0, 1, \cdots, n\}$. Then, the infimum of $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ with respect to $p \in [a, b]$ equals the minimum of the set $\{C(a),\ C(b)\} \cup \{C_U(p) : p \in \mathscr{S}_U\} \cup \{C_L(p) : p \in \mathscr{S}_L\}$, where*

$$\mathscr{S}_U = \{U(k) \in (a, b) : 0 \leq k \leq n\}, \quad \mathscr{S}_L = \{L(k) \in (a, b) : 0 \leq k \leq n\}, \quad C(p) = \Pr\{L(K) \leq p \leq U(K) \mid p\},$$

$$C_U(p) = \Pr\{L(K) \leq p < U(K) \mid p\} \text{ and } C_L(p) = \Pr\{L(K) < p \leq U(K) \mid p\}.$$

It should be noted that the only assumption in the above theorem is that both $L(k)$ and $U(k)$ are either non-decreasing or non-increasing with respect to k. The interval $[L(K), U(K)]$ can be general *random interval* without being restricted to the context of confidence intervals. Theorem 2 can be considered as a specialized result of Theorem 7 in Section 4.

2 Poisson Random Intervals

Let X be a Poisson random variable defined in a probability space $(\Omega, \mathscr{F}, \Pr)$ such that

$$\Pr\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, \qquad k = 0, 1, 2, \cdots$$

where $\lambda > 0$ is called the Poisson parameter. Let $X_1, \cdots, X_n$ be n identical and independent samples of X. It is a frequent problem to construct a confidence interval (L, U) such that $\Pr\{L < \lambda < U \mid \lambda\} \approx 1 - \delta$ with $\delta \in (0, 1)$. Here $L = L(n, \delta, K)$ and $U = U(n, \delta, K)$ are multivariate functions of n, δ and random variable $K = \sum_{i=1}^n X_i$. For simplicity of notations, we drop the arguments and write $L = L(K)$ and $U = U(K)$. For fixed n and δ, the coverage probability $\Pr\{L(K) < \lambda < U(K) \mid \lambda\}$ is a function of λ. The worst-case coverage probability with respect to λ belonging to interval $[a, b] \subset (0, \infty)$ can be obtained by the following theorem.

Theorem 3 *Suppose that both $L(k)$ and $U(k)$ are monotone functions of non-negative integer k. Then, the minimum of $\Pr\{L(K) < \lambda < U(K) \mid \lambda\}$ with respect to $\lambda \in [a, b]$ is attained at the discrete set $\{a, b\} \cup \{L(k) \in (a, b) : k \geq 0\} \cup \{U(k) \in (a, b) : k \geq 0\}$.*

It should be emphasized that the interval $(L(K), U(K))$ can be general *random interval* without being restricted to the context of confidence intervals. The only assumption in the above theorem is that both $L(k)$ and $U(k)$ are either non-decreasing or non-increasing with respect to k. Theorem 3 can be generalized as Theorem 7 in Section 4. The application of the theorem is discussed in the full version of our paper [4] for the sample size problems studied in [2].

For the exact computation of the infimum of coverage probability $\Pr\{L(K) \leq \lambda \leq U(K) \mid \lambda\}$ for the closed confidence interval $[L, U]$, we have

Theorem 4 *Suppose that both $L(k)$ and $U(k)$ are monotone functions of non-negative integer k. Then, the infimum of $\Pr\{L(K) \leq \lambda \leq U(K) \mid \lambda\}$ with respect to $\lambda \in [a, b]$ equals the minimum of the set $\{C(a),\ C(b)\} \cup \{C_U(\lambda) : \lambda \in \mathscr{S}_U\} \cup \{C_L(\lambda) : \lambda \in \mathscr{S}_L\}$ where*

$$\mathscr{S}_U = \{U(k) \in (a, b) : k \geq 0\}, \qquad \mathscr{S}_L = \{L(k) \in (a, b) : k \geq 0\}, \qquad C(\lambda) = \Pr\{L(K) \leq \lambda \leq U(K) \mid \lambda\},$$

$$C_U(\lambda) = \Pr\{L(K) \leq \lambda < U(K) \mid \lambda\}, \qquad C_L(\lambda) = \Pr\{L(K) < \lambda \leq U(K) \mid \lambda\}.$$

In Theorem 4, the interval $[L(K), U(K)]$ can be general *random interval* without being restricted to the context of confidence intervals. This theorem is a special case of Theorem 7 in Section 4.

3 Negative-Binomial Random Intervals

Let K be a negative binomial random variable such that

$$\Pr\{K = k\} = \binom{k + r - 1}{k} p^r (1 - p)^k, \qquad k = 0, 1, \cdots \tag{1}$$

with parameter $p \in (0, 1)$ and $r > 0$. In the special case that $r = 1$, a negative binomial random variable becomes a geometrical random variable. For the coverage probability of open random interval $(L(K), U(K))$ for a negative binomial random variable K, we have

Theorem 5 *Suppose that both $L(k)$ and $U(k)$ are monotone functions of non-negative integer k. Then, the minimum of $\Pr\{L(K) < p < U(K) \mid p\}$ with respect to $p \in [a, b] \subset (0, 1)$ is attained at the discrete set $\{a, b\} \cup \{L(k) \in (a, b) : k \geq 0\} \cup \{U(k) \in (a, b) : k \geq 0\}$.*

This theorem can be readily obtained by applying Theorem 7 of Section 4. For the coverage probability of closed random interval $[L(K), U(K)]$ for a negative binomial random variable K, we have

Theorem 6 *Suppose that both $L(k)$ and $U(k)$ are monotone functions of non-negative integer k. Then, the infimum of $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ with respect to $p \in [a, b] \subset (0, 1)$ equals the minimum of the set $\{C(a),\ C(b)\} \cup \{C_U(p) : p \in \mathscr{S}_U\} \cup \{C_L(p) : p \in \mathscr{S}_L\}$, where*

$$\mathscr{S}_U = \{U(k) \in (a, b) : k \geq 0\}, \quad \mathscr{S}_L = \{L(k) \in (a, b) : k \geq 0\}, \quad C(p) = \Pr\{L(K) \leq p \leq U(K) \mid p\},$$

$$C_U(p) = \Pr\{L(K) \leq p < U(K) \mid p\} \text{ and } C_L(p) = \Pr\{L(K) < p \leq U(K) \mid p\}.$$

This theorem can be easily deduced from Theorem 7 of next section.

4 Fundamental Theorem of Random Intervals

In previous sections, we discuss coverage probability of random intervals for specific random variables. Actually, the results can be generalized to a large class of discrete random variables. In this direction, we have recently established in [4] the following fundamental theorem of random intervals.

Theorem 7 *Let K be an integer-valued random variable parameterized by $\theta \in \Theta$. Let $L(K)$ and $U(K)$ be functions of random variable K. Let $[a,b]$ be an interval contained in Θ. Let $\mathscr{S}_L$ denote the intersection of the interval (a,b) and the support of $L(K)$. Let $\mathscr{S}_U$ denote the intersection of the interval (a,b) and the support of $U(K)$. Suppose that, for any $\vartheta \in \Theta$, $\Pr\{L(K) \leq \vartheta \leq U(K) \mid \theta\}$ is a continuous and unimodal function of $\theta \in \Theta$. Then, the minimum of $\Pr\{L(K) < \theta < U(K) \mid \theta\}$ with respect to $\theta \in [a,b]$ is attained at the set $\mathscr{S}_L \cup \mathscr{S}_U \cup \{a,b\}$ and the infimum of $\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\}$ with respect to $\theta \in [a,b]$ is equal to the minimum of the set $\{C_L(\theta) : \theta \in \mathscr{S}_L\} \cup \{C_U(\theta) : \theta \in \mathscr{S}_U\} \cup \{C(a),\ C_U(a),\ C(b),\ C_L(b)\}$, where $C_L(\theta) = \Pr\{L(K) < \theta \leq U(K) \mid \theta\}$, $C_U(\theta) = \Pr\{L(K) \leq \theta < U(K) \mid \theta\}$ and $C(\theta) = \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\}$. Moreover, for both open random interval $((L(K), U(K))$ and closed random interval $[L(K), U(K)]$, the coverage probability is continuous and unimodal for $\theta \in (\theta', \theta'')$, where θ' and θ'' are arbitrary consecutive distinct elements of $\mathscr{S}_L \cup \mathscr{S}_U \cup \{a,b\}$.*

Theorem 7 is proved in Appendices A. The concepts of support and unimodal functions have been used in Theorem 7. The support of a random variable is actually the set of all possible values assumed by that random variable. A function is said to be a unimodal function of $\theta \in \Theta$ if there exists θ^* such that the function is non-decreasing for $\theta \in \Theta$ no greater than θ^* and non-increasing for $\theta \in \Theta$ no less than θ^*. It should be noted that a monotone function can be considered as a special case of unimodal function by specifying θ^* as the infimum or supremum of Θ. Based on such notion of unimodal function, the coverage theory stated in Theorem 7 applies to one-sided random intervals such as $(-\infty, U(K)]$, $[L(K), \infty)$, $(-\infty, U(K))$, $(L(K), \infty)$.

Under the assumption that $\{L(K) \leq \vartheta \leq U(K)\}$ is an event that K is contained in an interval, it can be readily shown that the assumption of Theorem 7 is satisfied for common discrete random variables such as binomial random variable, Poisson random variable, geometrical random variable, negative binomial random variable, etc.

Let $C_L(\theta)$ and $C_U(\theta)$ be defined as in Theorem 7. By the same argument as that for proving Theorem 7, we can show that the infimum of $\Pr\{L(K) < \theta \leq U(K) \mid \theta\}$ with respect to $\theta \in [a,b]$ is equal to the minimum of the set $\{C_L(\theta) : \theta \in \mathscr{S}_L\} \cup \{C_U(\theta) : \theta \in \mathscr{S}_U\} \cup \{C(a),\ C_U(a),\ C(b),\ C_L(b)\}$, where $C(\theta) = \Pr\{L(K) < \theta \leq U(K) \mid \theta\}$. Similarly, the infimum of $\Pr\{L(K) \leq \theta < U(K) \mid \theta\}$ with respect to $\theta \in [a,b]$ is equal to the minimum of the set $\{C_L(\theta) : \theta \in \mathscr{S}_L\} \cup \{C_U(\theta) : \theta \in \mathscr{S}_U\} \cup \{C(a),\ C_U(a),\ C(b),\ C_L(b)\}$, where $C(\theta) = \Pr\{L(K) \leq \theta < U(K) \mid \theta\}$.

5 Infimum Coverage Probability over Parameter Space

In previous sections, we have considered the infimum of coverage probability over a closed interval $[a, b]$ contained in the parameter space Θ. In many cases, the parameter space Θ is an open set and consequently, the infimum of coverage probability over Θ needs to be treated differently.

As an application of Theorem 7, we have obtained the following results for binomial random intervals.

Theorem 8 *Let $K = \sum_{i=1}^n X_i$, where $X_1, \cdots, X_n$ are i.i.d. samples of Bernoulli random variable X such that $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0, 1)$. Let $L(k)$ and $U(k)$ be functions of nonnegative integer k such that $0 = L(0) < U(0) < 1,\ 0 < L(n) < U(n) = 1$ and that, for any $\theta \in (0, 1)$, there exist two numbers u and v such that $\{L(K) \leq \theta \leq U(K)\} = \{u \leq K \leq v\}$. Let $\mathscr{S}_L = \{L(k) \in (0, 1) : k = 1, \cdots, n\}$, $\mathscr{S}_U = \{U(k) \in (0, 1) : k = 0, 1, \cdots, n-1\}$ and $\mathscr{S} = \mathscr{S}_L \cup \mathscr{S}_U$. Then, $\inf_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\}$ is equal to $\min_{p\in\mathscr{S}} \Pr\{L(K) < p < U(K) \mid p\}$. Moreover, $\inf_{p\in(0,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ is equal to the minimum of $\min_{p\in\mathscr{S}_L} \Pr\{L(K) < p \leq U(K) \mid p\}$ and $\min_{p\in\mathscr{S}_U} \Pr\{L(K) \leq p < U(K) \mid p\}$. Furthermore, $\inf_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\} = \inf_{p\in(0,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ under additional assumption that $\mathscr{S}_{\mathscr{L}} \cap \mathscr{S}_L = \emptyset$.*

See Appendix B for a proof.

Theorem 8 reveals a counterintuitive fact. That is, the infimum of the coverage probability of an open random interval is not necessarily equals to the infimum of the coverage probability of the corresponding closed random interval. This discovery can be confirmed by investigating random intervals with

$$L(K) = \max\left\{\frac{K}{n} - \frac{1}{n}, 0\right\}, \qquad U(K) = \min\left\{\frac{K}{n} + \frac{1}{n}, 1\right\},$$

where K is defined in Theorem 8. For $n = 3$, we can show by direct computation that

$$\Pr\{L(K) < p < U(K) \mid p\} = \begin{cases} (1-p)^3 + 3p(1-p)^2 & \text{for } 0 < p < \frac{1}{3}, \\ \frac{4}{9} & \text{for } p = \frac{1}{3}, \\ 3p(1-p) & \text{for } \frac{1}{3} < p < \frac{2}{3}, \\ \frac{4}{9} & \text{for } p = \frac{2}{3}, \\ 3p^2(1-p) + p^3 & \text{for } \frac{2}{3} < p < 1 \end{cases}$$

$$\Pr\{L(K) \leq p \leq U(K) \mid p\} = \begin{cases} (1-p)^3 + 3p(1-p)^2 & \text{for } 0 < p < \frac{1}{3}, \\ \frac{26}{27} & \text{for } p = \frac{1}{3}, \\ 3p(1-p) & \text{for } \frac{1}{3} < p < \frac{2}{3}, \\ \frac{26}{27} & \text{for } p = \frac{2}{3}, \\ 3p^2(1-p) + p^3 & \text{for } \frac{2}{3} < p < 1 \end{cases}$$

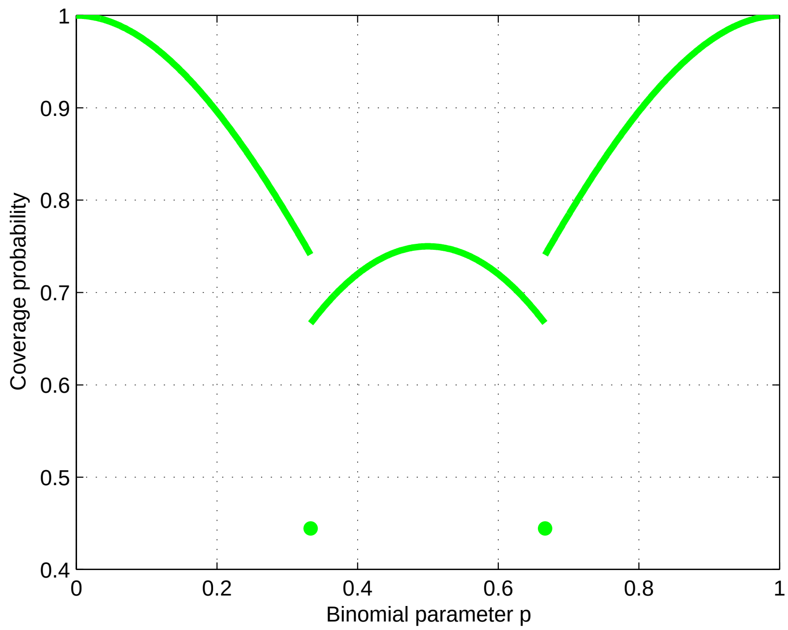

Figure 1: Coverage probability of open random interval

and that

$$\inf_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\} = \min_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\} = \frac{4}{9},$$
$$\inf_{p\in(\frac{1}{3},\frac{2}{3})} \Pr\{L(K) < p < U(K) \mid p\} = \frac{2}{3} > \min_{p\in\{\frac{1}{3},\frac{2}{3}\}} \Pr\left\{L(K) < p < U(K) \mid p\right\} = \frac{4}{9},$$
$$\inf_{p\in(\frac{1}{3},\frac{2}{3})} \Pr\{L(K) \leq p \leq U(K) \mid p\} = \frac{2}{3} < \min_{p\in\{\frac{1}{3},\frac{2}{3}\}} \Pr\left\{L(K) \leq p \leq U(K) \mid p\right\} = \frac{26}{27},$$
$$\inf_{p\in(0,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\} = \frac{2}{3} > \inf_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\} = \frac{4}{9}. \qquad (2)$$

In particular, (2) shows that the infimum of coverage probabilities for the open and closed random intervals are not equal. This is quite surprising. The coverage probabilities $\Pr\{L(K) < p < U(K) \mid p\}$ and $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ are shown by Figure 1 and Figure 2 respectively.

The following result establishes the *nonexistence* of local minima for the coverage probability of binomial random intervals under mild conditions.

Theorem 9 *Let* $K = \sum_{i=1}^{n} X_i$*, where* $X_1, \cdots, X_n$ *are i.i.d. samples of Bernoulli random variable* X *such that* $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0,1)$*. Let* $L(k)$ *and* $U(k)$ *be nondecreasing functions of nonnegative integer* k *such that* $L(0) = 0$*,* $U(n) = 1$ *and* $L(k) \leq U(k)$ *for* $k = 0, 1, \cdots, n$*. Then, there exists no local minima for* $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ *with respect to* $p \in (0,1)$*.*

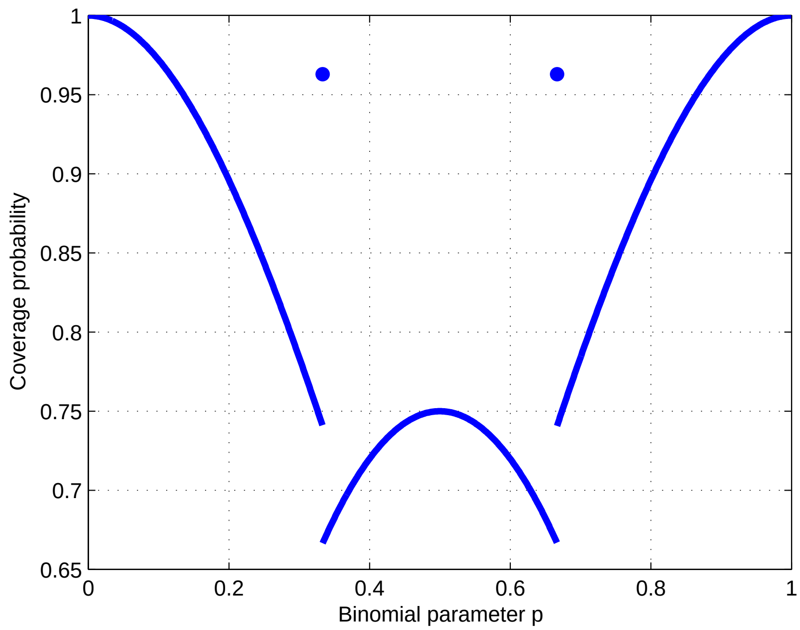

Figure 2: Coverage probability of closed random interval

The proof of Theorem 9 is available in Appendix C.

By similar argument as that for proving Theorems 7 and 8, we have established Theorems 10–13 in the sequel.

For one-sided binomial random intervals, we have the following results.

Theorem 10 *Let $K = \sum_{i=1}^n X_i$, where $X_1, \cdots, X_n$ are i.i.d. samples of Bernoulli random variable X such that $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0, 1)$. Let $L(k)$ and $U(k)$ be nondecreasing functions of nonnegative integer k such that $0 = L(0) < L(n) < 1$ and $0 < U(0) < U(n) = 1$. Let $\mathscr{S}_L = \{L(k) \in (0, 1) : k = 1, \cdots, n\}$ and $\mathscr{S}_U = \{U(k) \in (0, 1) : k = 0, 1, \cdots, n - 1\}$. Then,*

$$\inf_{p \in (0,1)} \Pr\{L(K) < p \mid p\} = \inf_{p \in (0,1)} \Pr\{L(K) \leq p \mid p\} = \min_{p \in \mathscr{S}_L} \Pr\{L(K) < p \mid p\},$$
$$\inf_{p \in (0,1)} \Pr\{p < U(K) \mid p\} = \inf_{p \in (0,1)} \Pr\{p \leq U(K) \mid p\} = \min_{p \in \mathscr{S}_U} \Pr\{p < U(K) \mid p\}.$$

For Poisson random intervals, we have the following results.

Theorem 11 *Let K be a Poisson random variable of mean $\lambda > 0$. Let $L(k)$ and $U(k)$ be functions of nonnegative integer k. Let $\mathscr{S}_L = \{L(k) \in (0, \infty) : k \geq 1\}$, $\mathscr{S}_U = \{U(k) \in (0, \infty) : k \geq 0\}$ and $\mathscr{S} = \mathscr{S}_L \cup \mathscr{S}_U$. Suppose that $0 = L(0) < U(0)$, $\mathscr{S}_L \neq \emptyset$, $\sup \mathscr{S}_U = \infty$ and that, for any $\theta \in (0, \infty)$, there exist two numbers u and v such that $\{L(K) \leq \theta \leq U(K)\} = \{u \leq K \leq v\}$. Then, $\inf_{\lambda \in (0,\infty)} \Pr\{L(K) < \lambda < U(K) \mid \lambda\}$ is equal to $\inf_{\lambda \in \mathscr{S}} \Pr\{L(K) < \lambda < U(K) \mid \lambda\}$. Moreover, $\inf_{\lambda \in (0,\infty)} \Pr\{L(K) \leq \lambda \leq U(K) \mid \lambda\}$ is equal to the minimum of $\inf_{\lambda \in \mathscr{S}_L} \Pr\{L(K) < \lambda \leq U(K) \mid \lambda\}$ and $\inf_{\lambda \in \mathscr{S}_U} \Pr\{L(K) \leq \lambda < U(K) \mid \lambda\}$.*

For one-sided Poisson random intervals, we have the following results.

Theorem 12 *Let K be a Poisson random variable of mean $\lambda > 0$. Let $L(k)$ and $U(k)$ be nondecreasing functions of nonnegative integer k. Let $\mathscr{S}_L = \{L(k) \in (0,\infty) : k \geq 1\}$ and $\mathscr{S}_U = \{U(k) \in (0,\infty) : k \geq 0\}$. Suppose that $0 = L(0) < U(0)$, $\mathscr{S}_L \neq \emptyset$ and $\sup \mathscr{S}_U = \infty$. Then,*

$$\inf_{\lambda \in (0,\infty)} \Pr\{L(K) < \lambda \mid \lambda\} = \inf_{\lambda \in (0,\infty)} \Pr\{L(K) \leq \lambda \mid \lambda\} = \inf_{\lambda \in \mathscr{S}_L} \Pr\{L(K) < \lambda \mid \lambda\},$$
$$\inf_{\lambda \in (0,\infty)} \Pr\{\lambda < U(K) \mid \lambda\} = \inf_{\lambda \in (0,\infty)} \Pr\{\lambda \leq U(K) \mid \lambda\} = \inf_{\lambda \in \mathscr{S}_U} \Pr\{\lambda < U(K) \mid \lambda\}.$$

For negative binomial random intervals, we have the following results.

Theorem 13 *Let K be a negative binomial random variable defined by (1). Let $L(k)$ and $U(k)$ be non-increasing functions of nonnegative integer k. Let $\mathscr{S}_L = \{L(k) \in (0,1) : k \geq 1\}$, $\mathscr{S}_U = \{U(k) \in (0,1) : k \geq 0\}$ and $\mathscr{S} = \mathscr{S}_L \cup \mathscr{S}_U$. Suppose that $0 < L(0) < U(0) = 1$ and $\lim_{k\to\infty} L(k) = 0 < \lim_{k\to\infty} U(k) < 1$. Then, $\inf_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\}$ is equal to $\inf_{p\in\mathscr{S}} \Pr\{L(K) < p < U(K) \mid p\}$. Moreover, $\inf_{p\in(0,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ is equal to the minimum of $\inf_{p\in\mathscr{S}_L} \Pr\{L(K) < p \leq U(K) \mid p\}$ and $\inf_{p\in\mathscr{S}_U} \Pr\{L(K) \leq p < U(K) \mid p\}$. Furthermore,*

$$\inf_{p \in (0,1)} \Pr\{L(K) < p \mid p\} = \inf_{p \in (0,1)} \Pr\{L(K) \leq p \mid p\} = \inf_{p \in \mathscr{S}_L} \Pr\{L(K) < p \mid p\},$$
$$\inf_{p \in (0,1)} \Pr\{p < U(K) \mid p\} = \inf_{p \in (0,1)} \Pr\{p \leq U(K) \mid p\} = \inf_{p \in \mathscr{S}_U} \Pr\{p < U(K) \mid p\}.$$

6 Hypergeometrical Random Intervals

So far what we have addressed are random intervals of variables with continuous parameter spaces. In this section, we shall consider random intervals when the parameter space is discrete. We focus on the important hypergeometrical random variable.

Consider a finite population of N units, among which M units have a certain attribute. Let K be the number of units found to have the attribute in a sample of n units obtained by sampling without replacement. The number K is known to be a random variable of hypergeometrical distribution.

It is a basic problem to construct a confidence interval (L, U) with $L = L(N, n, \delta, K)$ and $U = U(N, n, \delta, K)$ such that $\Pr\{L < M < U \mid M\} \approx 1 - \delta$. Here, U and L *only assume integer values*. For notational simplicity, we write $L = L(K)$ and $U = U(K)$. In practice, it is useful to know the minimum of coverage probability $\Pr\{L < M < U \mid M\}$ with respect to $M \in [a, b]$, where a and b are integers taken values in between 0 and N. For this purpose, we have

Theorem 14 *Suppose that $L(0) \leq L(1) \leq \cdots \leq L(n)$ and $U(0) \leq U(1) \leq \cdots \leq U(n)$. Then, the minimum of $\Pr\{L(K) < M < U(K) \mid M\}$ with respect to $M \in [a, b]$ is attained at the discrete set I_{UL}, where $I_{UL} = \{a, b\} \cup \{L(k) \in (a, b) : 0 \leq k \leq n\} \cup \{U(k) \in (a, b) : 0 \leq k \leq n\}$. Moreover, $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M in between consecutive distinct elements of I_{UL}.*

For a proof, see Appendix D. In Theorem 14, the interval $(L(K), U(K))$ can be general *random interval* without being restricted to the context of confidence intervals. This theorem can be applied to the sample size problems discussed in [3].

A Proof of Theorem 7

We need some preliminary results.

Lemma 1 *Suppose that* $\{\theta' < L(K) < \theta''\} = \{\theta' < U(K) < \theta''\} = \emptyset$. *Then,*

$$\{L(K) < \theta < U(K)\} = \{L(K) \leq \theta \leq U(K)\} = \{L(K) \leq \theta' < U(K)\} = \{L(K) < \theta'' \leq U(K)\}$$

for any $\theta \in (\theta', \theta'')$.

Proof. By the assumption of the lemma, we have $\{L(K) < \theta''\} = \{L(K) \leq \theta'\} \cup \{\theta' < L(K) < \theta''\} = \{L(K) \leq \theta'\}$ and $\{\theta' < L(K) < \theta\} \subseteq \{\theta' < L(K) \leq \theta\} \subseteq \{\theta' < L(K) < \theta''\} = \emptyset$ for any $\theta \in (\theta', \theta'')$. Consequently,

$$\{L(K) \leq \theta\} = \{L(K) \leq \theta'\} \cup \{\theta' < L(K) \leq \theta\} = \{L(K) \leq \theta'\} = \{L(K) < \theta''\}, \tag{3}$$

$$\{L(K) < \theta\} = \{L(K) \leq \theta'\} \cup \{\theta' < L(K) < \theta\} = \{L(K) \leq \theta'\} \tag{4}$$

for any $\theta \in (\theta', \theta'')$. Combining (3) and (4) yields

$$\{L(K) \leq \theta\} = \{L(K) < \theta\} = \{L(K) \leq \theta'\} = \{L(K) < \theta''\}, \qquad \forall \theta \in (\theta', \theta''). \tag{5}$$

Again by the assumption of the lemma, we have $\{U(K) > \theta'\} = \{U(K) \geq \theta''\} \cup \{\theta' < U(K) < \theta''\} = \{U(K) \geq \theta''\}$ and $\{\theta < U(K) < \theta''\} \subseteq \{\theta \leq U(K) < \theta''\} \subseteq \{\theta' < U(K) < \theta''\} = \emptyset$ for any $\theta \in (\theta', \theta'')$. Consequently,

$$\{U(K) \geq \theta\} = \{U(K) \geq \theta''\} \cup \{\theta \leq U(K) < \theta''\} = \{U(K) \geq \theta''\} = \{U(K) > \theta'\}, \tag{6}$$

$$\{U(K) > \theta\} = \{U(K) \geq \theta''\} \cup \{\theta < U(K) < \theta''\} = \{U(K) \geq \theta''\} \tag{7}$$

for any $\theta \in (\theta', \theta'')$. Combining (6) and (7) yields

$$\{U(K) \geq \theta\} = \{U(K) > \theta\} = \{U(K) > \theta'\} = \{U(K) \geq \theta''\}, \qquad \forall \theta \in (\theta', \theta''). \tag{8}$$

Taking intersection of events and making use of (5) and (8), we have

$$\{L(K) < \theta < U(K)\} = \{L(K) \leq \theta \leq U(K)\} = \{L(K) \leq \theta' < U(K)\} = \{L(K) < \theta'' \leq U(K)\}$$

for any $\theta \in (\theta', \theta'')$. This completes the proof of the lemma. □

Now we are in a position to prove Theorem 7. First, we shall show the first statement regarding the minimum of $\Pr\{L(K) < \theta < U(K) \mid \theta\}$ for $\theta \in [a, b]$. Let $\theta' < \theta''$ be two consecutive distinct elements of $\{a,b\} \cup \mathscr{S}_U \cup \mathscr{S}_L$. Let $\vartheta = \frac{\theta' + \theta''}{2}$. Then, $\{\theta' < L(K) < \theta''\} = \{\theta' < U(K) < \theta''\} = \emptyset$ and by Lemma 1, we have

$$\{L(K) < \theta < U(K)\} = \{L(K) \leq \theta \leq U(K)\} \tag{9}$$
$$= \{L(K) \leq \vartheta \leq U(K)\} = \{L(K) \leq \theta' < U(K)\} = \{L(K) < \theta'' \leq U(K)\} \tag{10}$$

for any $\theta \in (\theta', \theta'')$. By the assumption of the theorem, $\Pr\{L(K) \leq \vartheta \leq U(K) \mid \theta\}$ is a continuous and unimodal function of $\theta \in \Theta$. It follows from (9) and (10) that both $\Pr\{L(K) \leq \theta' < U(K) \mid \theta\}$ and $\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}$ are continuous and unimodal functions of $\theta \in (\theta', \theta'')$. Hence, for $\theta \in (\theta', \theta'')$, letting $0 < \epsilon < \min(\theta - \theta',\ \theta'' - \theta,\ \frac{\theta'' - \theta'}{2})$, we have $\theta' + \epsilon < \theta < \theta'' - \epsilon$ and

$$\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \geq \min\left(\Pr\{L(K) < \theta'' \leq U(K) \mid \theta' + \epsilon\},\ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta'' - \epsilon\}\right). \tag{11}$$

By virtue of the continuity of $\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}$ with respect to $\theta \in (\theta', \theta'')$, we have

$$\lim_{\epsilon \downarrow 0} \Pr\{L(K) < \theta'' \leq U(K) \mid \theta' + \epsilon\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta'\}, \tag{12}$$
$$\lim_{\epsilon \downarrow 0} \Pr\{L(K) < \theta'' \leq U(K) \mid \theta'' - \epsilon\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\} \tag{13}$$

It follows from (11), (12) and (13) that

$$\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \geq \min\left(\Pr\{L(K) < \theta'' \leq U(K) \mid \theta'\},\ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\right) \tag{14}$$

for any $\theta \in (\theta', \theta'')$. Combining (9), (10) and (14) yields

$$\Pr\{L(K) < \theta < U(K) \mid \theta\} = \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \tag{15}$$
$$\geq \min\left(\Pr\{L(K) < \theta'' \leq U(K) \mid \theta'\},\ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\right) \tag{16}$$
$$= \min\left(\Pr\{L(K) \leq \theta' < U(K) \mid \theta'\},\ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\right) \tag{17}$$
$$\geq \min\left(\Pr\{L(K) < \theta' < U(K) \mid \theta'\},\ \Pr\{L(K) < \theta'' < U(K) \mid \theta''\}\right) \tag{18}$$

for any $\theta \in (\theta', \theta'')$. It can be seen from (15), (16), (17) and (18) that the minimum of $\Pr\{L(K) < \theta < U(K) \mid \theta\}$ with respect to $\theta \in [\theta', \theta'']$ is achieved at either θ' or θ''. This implies that the minimum of $\Pr\{L(K) < \theta < U(K) \mid \theta\}$ for $\theta \in [a, b]$ is attained at $\mathscr{S}_L \cup \mathscr{S}_U \cup \{a, b\}$.

Next, we shall show the second statement regarding the infimum of $\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\}$ for $\theta \in [a, b]$. As before, let $\theta' < \theta''$ be two consecutive distinct elements of $\{a,b\} \cup \mathscr{S}_U \cup \mathscr{S}_L$. For simplicity of notations, let

$$\alpha = \inf_{\theta \in [\theta', \theta'']} \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\},$$
$$\beta = \min\left(\Pr\{L(K) \leq \theta' < U(K) \mid \theta'\},\ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\right).$$

Making use of (15), (16), (17) and the observation that

$$\beta \leq \min(\Pr\{L(K) \leq \theta' \leq U(K) \mid \theta'\},\ \Pr\{L(K) \leq \theta'' \leq U(K) \mid \theta''\}),$$

we have $\alpha \geq \beta$. Now we need to show that α is actually equal to β. Suppose, to get a contradiction, that α is greater than β. Then,

$$\Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} > \frac{\alpha+\beta}{2}, \qquad \forall \theta \in [\theta', \theta'']. \tag{19}$$

As a consequence of (9), (10) and (19),

$$\Pr\{L(K) \leq \theta' < U(K) \mid \theta\} = \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} > \frac{\alpha+\beta}{2}, \qquad \forall \theta \in (\theta', \theta''). \tag{20}$$

By virtue of (20) and recalling that both $\Pr\{L(K) \leq \theta' < U(K) \mid \theta\}$ and $\Pr\{L(K) < \theta'' \leq U(K) \mid \theta\}$ are continuous with respect to $\theta \in (\theta', \theta'')$, we have

$$\beta = \min\left(\lim_{\theta \downarrow \theta'} \Pr\{L(K) \leq \theta' < U(K) \mid \theta\}, \ \lim_{\theta \uparrow \theta''} \Pr\{L(K) < \theta'' \leq U(K) \mid \theta\} \right) \geq \frac{\alpha+\beta}{2},$$

leading to $\beta \geq \alpha$, which contradicts to $\alpha > \beta$. Therefore, it must be true that $\alpha = \beta$. That is,

$$\inf_{\theta \in [\theta', \theta'']} \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} = \min\left(\Pr\{L(K) \leq \theta' < U(K) \mid \theta'\}, \ \Pr\{L(K) < \theta'' \leq U(K) \mid \theta''\}\right).$$

It follows that

$$\begin{aligned}
&\inf_{\theta \in [a,b]} \Pr\{L(K) \leq \theta \leq U(K) \mid \theta\} \\
&= \min\{C(a), \ C_U(a), \ C(b), \ C_L(b)\} \cup \{C_U(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\} \cup \{C_L(\theta) : \theta \in \mathscr{S}_L \cup \mathscr{S}_U\}.
\end{aligned} \tag{21}$$

Let

$$\mathscr{S}' = \mathscr{S}_U \cap \mathscr{S}_L, \qquad \mathscr{S}'_U = \mathscr{S}_U \setminus \mathscr{S}', \qquad \mathscr{S}'_L = \mathscr{S}_L \setminus \mathscr{S}'.$$

Then,

$$\begin{aligned}
& \{C_U(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\} \cup \{C_L(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\} \\
= & \left\{C_U(\theta) : \theta \in \mathscr{S}'_U\right\} \cup \left\{C_U(\theta) : \theta \in \mathscr{S}'\right\} \cup \left\{C_L(\theta) : \theta \in \mathscr{S}'_L\right\} \cup \left\{C_L(\theta) : \theta \in \mathscr{S}'\right\} \\
& \cup \left\{C_U(\theta) : \theta \in \mathscr{S}'_L\right\} \cup \left\{C_L(\theta) : \theta \in \mathscr{S}'_U\right\}.
\end{aligned} \tag{22}$$

For $\theta \in \mathscr{S}'_U$, we have $0 \leq \Pr\{L(K) = \theta < U(K) \mid \theta\} \leq \Pr\{L(K) = \theta \mid \theta\} = 0$ and thus

$$\begin{aligned}
C_U(\theta) - C_L(\theta) &= \Pr\{L(K) \leq \theta < U(K) \mid \theta\} - \Pr\{L(K) < \theta \leq U(K) \mid \theta\} \\
&= \Pr\{L(K) = \theta < U(K) \mid \theta\} - \Pr\{L(K) < \theta = U(K) \mid \theta\} \\
&= -\Pr\{L(K) < \theta = U(K) \mid \theta\} \leq 0,
\end{aligned}$$

which implies that

$$\min\left\{C_U(\theta) : \theta \in \mathscr{S}'_U\right\} \leq \min\left\{C_L(\theta) : \theta \in \mathscr{S}'_U\right\}. \tag{23}$$

For $\theta \in \mathscr{S}'_L$, we have $0 \leq \Pr\{L(K) < \theta = U(K) \mid \theta\} \leq \Pr\{U(K) = \theta \mid \theta\} = 0$ and thus

$$\begin{aligned}
C_U(\theta) - C_L(\theta) &= \Pr\{L(K) = \theta < U(K) \mid \theta\} - \Pr\{L(K) < \theta = U(K) \mid \theta\} \\
&= \Pr\{L(K) = \theta < U(K) \mid \theta\} \geq 0,
\end{aligned}$$

which implies that

$$\min\left\{C_U(\theta) : \theta \in \mathscr{S}'_L\right\} \geq \min\left\{C_L(\theta) : \theta \in \mathscr{S}'_L\right\}. \tag{24}$$

Combing (22), (23) and (24) leads to

$$\begin{aligned}
& \min\{C_U(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\} \cup \{C_L(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\} \\
= \; & \min\left\{C_U(\theta) : \theta \in \mathscr{S}'_U\right\} \cup \left\{C_U(\theta) : \theta \in \mathscr{S}'\right\} \cup \left\{C_L(\theta) : \theta \in \mathscr{S}'_L\right\} \cup \left\{C_L(\theta) : \theta \in \mathscr{S}'\right\} \\
= \; & \min\{C_U(\theta) : \theta \in \mathscr{S}_U\} \cup \{C_L(\theta) : \theta \in \mathscr{S}_L\}, \qquad (25)
\end{aligned}$$

which implies that the minimum of the set $\{C(a),\ C_U(a),\ C(b),\ C_L(b)\} \cup \{C_U(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\} \cup \{C_L(\theta) : \theta \in \mathscr{S}_U \cup \mathscr{S}_L\}$ equals the minimum of $\{C(a),\ C_U(a),\ C(b),\ C_L(b)\} \cup \{C_U(\theta) : \theta \in \mathscr{S}_U\} \cup \{C_L(\theta) : \theta \in \mathscr{S}_L\}$. This proves the second statement of Theorem 7.

Clearly, the third statement of Theorem 7 is already justified in the course of proving the first two statements. This concludes the proof of Theorem 7.

B Proof of Theorem 8

We shall first show that $\inf_{p \in (0,1)} \Pr\{L(K) < p < U(K) \mid p\}$ is equal to $\min_{p \in \mathscr{S}} \Pr\{L(K) < p < U(K) \mid p\}$. Clearly, as a consequence of the assumption that $0 = L(0) < U(0) < 1,\ 0 < L(n) < U(n) = 1$, the sets $\mathscr{S}_L, \mathscr{S}_U$ and $\mathscr{S}$ are nonempty. Let a and b be the minimum and maximum of $\mathscr{S}$ respectively. Then, $0 < a \leq b < 1$ and $\inf_{p \in (0,1)} \Pr\{L(K) < p < U(K) \mid p\}$ is equal to the minimum among $\inf_{p \in (0,a)} \Pr\{L(K) < p < U(K) \mid p\}$, $\inf_{p \in (b,1)} \Pr\{L(K) < p < U(K) \mid p\}$ and $\inf_{p \in [a,b]} \Pr\{L(K) < p < U(K) \mid p\}$. By the assumption that, for any $\theta \in (0,1)$, there exist two numbers u and v such that $\{L(K) \leq \theta \leq U(K)\} = \{u \leq K \leq v\}$, it can be easily shown by differentiation that, for any $\vartheta \in (0,1)$, $\Pr\{L(K) \leq \vartheta \leq U(K) \mid p\}$ is a continuous and unimodal function of $p \in (0,1)$. Hence, by virtue of Theorem 7, we have that

$$\inf_{p \in [a,b]} \Pr\{L(K) < p < U(K) \mid p\} = \min_{p \in \mathscr{S}} \Pr\{L(K) < p < U(K) \mid p\}. \tag{26}$$

By Lemma 1 in Appendix A, we have that $\{L(K) < p < U(K)\} = \{L(K) < a \leq U(K)\} = \{L(K) \leq \frac{a}{2} \leq U(K)\}$ for any $p \in (0,a)$. By the assumption that $0 = L(0) < U(0) < 1$, we have $U(0) \geq a > \frac{a}{2}$, which implies that $\{K = 0\} \subseteq \{L(K) \leq \frac{a}{2} \leq U(K)\}$. Invoking the assumption that, for any $\theta \in (0,1)$, there exist two numbers u and v such that $\{L(K) \leq \theta \leq U(K)\} = \{u \leq K \leq v\}$, we can conclude that there exists a nonnegative integer w such that $\{L(K) \leq \frac{a}{2} \leq U(K)\} = \{0 \leq K \leq w\}$. Therefore, $\Pr\{L(K) < p < U(K) \mid p\} = \Pr\{L(K) < a \leq U(K) \mid p\} = \Pr\{0 \leq K \leq w \mid p\}$ for any $p \in (0,a)$. It can be easily shown that $\Pr\{0 \leq K \leq w \mid p\}$ is monotonically decreasing with respect to $p \in (0,a)$. This implies that $\Pr\{L(K) < a \leq U(K) \mid p\}$ is monotonically decreasing with respect to $p \in (0,a)$. Consequently, $\inf_{p \in (0,a)} \Pr\{L(K) < p < U(K) \mid p\} = \inf_{p \in (0,a)} \Pr\{L(K) < a \leq U(K) \mid p\} = \lim_{p \uparrow a} \Pr\{L(K) < a \leq U(K) \mid p\} = \Pr\{L(K) < a \leq U(K) \mid a\}$ and it immediately follows that

$$\inf_{p \in (0,a)} \Pr\{L(K) < p < U(K) \mid p\} \geq \Pr\{L(K) < a < U(K) \mid a\}. \tag{27}$$

By a similar argument, we can show that

$$\inf_{p\in(b,1)} \Pr\{L(K) < p < U(K) \mid p\} \geq \Pr\{L(K) < b < U(K) \mid b\}. \tag{28}$$

Combining (26), (27) and (28) leads to the conclusion that $\inf_{p\in(0,1)} \Pr\{L(K) < p < U(K) \mid p\}$ is equal to $\min_{p\in\mathscr{S}} \Pr\{L(K) < p < U(K) \mid p\}$.

Next, we shall show that $\inf_{p\in(0,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ is equal to the minimum of $\min_{p\in\mathscr{S}_L} \Pr\{L(K) < p \leq U(K) \mid p\}$ and $\min_{p\in\mathscr{S}_U} \Pr\{L(K) \leq p < U(K) \mid p\}$. By a similar argument as above, we can show that

$$\inf_{p\in(0,a)} \Pr\{L(K) \leq p \leq U(K) \mid p\} = \Pr\{L(K) < a \leq U(K) \mid a\} < C\left(\frac{a}{2}\right) \tag{29}$$

and

$$\inf_{p\in(b,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\} = \Pr\{L(K) \leq b < U(K) \mid b\} < C\left(\frac{b+1}{2}\right), \tag{30}$$

where the notion of $C(.)$ is the same as that in Theorem 7.

Let Q_U denote the intersection of the interval $(\frac{a}{2}, \frac{b+1}{2})$ and the support of $U(K)$. Let Q_L denote the intersection of the interval $(\frac{a}{2}, \frac{b+1}{2})$ and the support of $L(K)$. In the course of proving Theorem 7, we have established (21). Invoking the assumption that, for any $\theta \in (0,1)$, there exist two numbers u and v such that $\{L(K) \leq \theta \leq U(K)\} = \{u \leq K \leq v\}$, we can conclude from (21) that $\inf_{p\in[\frac{a}{2},\frac{b+1}{2}]} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ is equal to the minimum of $\{C(\frac{a}{2}), C(\frac{b+1}{2})\} \cup \{C_L(p) : p \in Q_L \cup Q_U\} \cup \{C_U(p) : p \in Q_L \cup Q_U\}$, where the meaning of $C(.), C_L(.), C_U(.)$ is the same as that in Theorem 7. Observing that

$$\Pr\{L(K) < a \leq U(K) \mid a\} \geq \min\{C_L(p) : p \in Q_L \cup Q_U\}$$

and

$$\Pr\{L(K) \leq b < U(K) \mid b\} \geq \min\{C_U(p) : p \in Q_L \cup Q_U\},$$

we have that the minimum among $\inf_{p\in(0,a)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$, $\inf_{p\in(b,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ and $\inf_{p\in[\frac{a}{2},\frac{b+1}{2}]} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ is equal to the minimum of $\{C_L(p) : p \in Q_L \cup Q_U\} \cup \{C_U(p) : p \in Q_L \cup Q_U\} = \{C_L(p) : p \in Q_L\} \cup \{C_U(p) : p \in Q_U\} = \{C_L(p) : p \in \mathscr{S}_L\} \cup \{C_U(p) : p \in \mathscr{S}_U\}$, where we have used (25) established in the proof of Theorem 7. It follows that the second statement of Theorem 8 on $\inf_{p\in(0,1)} \Pr\{L(K) \leq p \leq U(K) \mid p\}$ holds true.

Finally, to show the third statement of Theorem 8, it is sufficient to observe that $\Pr\{L(K) < p = U(K) \mid p\} = 0$ for $p \in \mathscr{S}_L$ and that $\Pr\{L(K) = p < U(K) \mid p\} = 0$ for $p \in \mathscr{S}_U$ as a consequence of the assumption that $\mathscr{S}_{\mathscr{L}} \cap \mathscr{S}_L = \emptyset$. The proof of Theorem 8 is thus completed.

C Proof of Theorem 9

For simplicity of notations, let $S_L = \{L(k) \in (0,1) : 0 \leq k \leq n\}$ and $S_U = \{U(k) \in (0,1) : 0 \leq k \leq n\}$. It suffices to consider three exhaustive (but not mutually exclusive) cases as follows.

Case (i): $p \in S_L$;

Case (ii): $p \in S_U$;

Case (iii): $p \notin S_L \cup S_U$.

In Case (i), we can write $\{L(K) \leq p \leq U(K)\} = \{k \leq K \leq l\}$, where $0 \leq k \leq l \leq n$ are integers. Then, $\{L(K) \leq p - \epsilon \leq U(K)\} \subseteq \{k \leq K \leq l-1\}$ for small enough $\epsilon > 0$. Thus,

$$\begin{aligned}
&\Pr\{L(K) \leq p \leq U(K) \mid p\} - \Pr\{L(K) \leq p - \epsilon \leq U(K) \mid p - \epsilon\} \\
&= \Pr\{k \leq K \leq l \mid p\} - \Pr\{k \leq K \leq l \mid p - \epsilon\} + \Pr\{K = l \mid p - \epsilon\} \\
&\to \Pr\{K = l \mid p\} > 0
\end{aligned}$$

as $\epsilon \to 0$. This implies that $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ is greater than $\Pr\{L(K) \leq p - \epsilon \leq U(K) \mid p - \epsilon\}$ for small enough $\epsilon > 0$. Hence, p is not a local minima.

In Case (ii), we can write $\{L(K) \leq p \leq U(K)\} = \{k \leq K \leq l\}$, where $0 \leq k \leq l \leq n$ are integers. Then, $\{L(K) \leq p + \epsilon \leq U(K)\} \subseteq \{k+1 \leq K \leq l\}$ for small enough $\epsilon > 0$. Thus,

$$\begin{aligned}
&\Pr\{L(K) \leq p \leq U(K) \mid p\} - \Pr\{L(K) \leq p + \epsilon \leq U(K) \mid p + \epsilon\} \\
&= \Pr\{k \leq K \leq l \mid p\} - \Pr\{k \leq K \leq l \mid p + \epsilon\} + \Pr\{K = k \mid p + \epsilon\} \\
&\to \Pr\{K = k \mid p\} > 0
\end{aligned}$$

as $\epsilon \to 0$. This implies that $\Pr\{L(K) \leq p \leq U(K) \mid p\}$ is greater than $\Pr\{L(K) \leq p + \epsilon \leq U(K) \mid p + \epsilon\}$ for small enough $\epsilon > 0$. Hence, p is not a local minima.

In Case (iii), since $p \in (0, 1) \subseteq \cup_{k=0}^{n} [L(k), U(k)]$, there must exist an integer $k \in \{0, 1, \cdots, n\}$ such that $p \in [L(k), U(k)]$. Thus, $\{L(K) \leq p \leq U(K)\}$ is not an impossible event. As a result, we can write $\{L(K) \leq p \leq U(K)\} = \{k \leq K \leq l\}$, where $0 \leq k \leq l \leq n$ are integers. Since $p \notin S_L \cup S_U$, we have that $\{L(K) \leq p + \epsilon \leq U(K)\} = \{k \leq K \leq l\}$ and $\{L(K) \leq p - \epsilon \leq U(K)\} = \{k \leq K \leq l\}$ for small enough $\epsilon > 0$. Observing that $\Pr\{k \leq K \leq l \mid \theta\}$ is a continuous and strictly monotone or unimodal function of $\theta \in (0, 1)$, we can conclude that p is not a local minima. The proof of the theorem is thus completed.

D Proof of Theorem 14

For the simplicity of notations, define

$$\binom{m}{z} = \begin{cases} \frac{m!}{z!(m-z)!} & \text{if } 0 \leq z \leq m, \\ 0 & \text{if } z < 0 \text{ or } z > m \end{cases}$$

for non-negative integer m and arbitrary integer z. We now establish some preliminary results.

Lemma 2 *Let* $0 \leq M < N$. *Define* $T(k, M, N, n) = \binom{M}{k}\binom{N-M-1}{n-k-1} \Big/ \binom{N}{n}$. *Then,* $\Pr\{K \leq k \mid M\} - \Pr\{K \leq k \mid M+1\} = T(k, M, N, n)$ *for any integer* k.

Proof. We first show the equation for $0 \le k \le M$. We perform induction on k. For $k = 0$, we have

$$\begin{aligned}
\Pr\{K \le k \mid M\} - \Pr\{K \le k \mid M+1\} &= \Pr\{K = 0 \mid M\} - \Pr\{K = 0 \mid M+1\} \\
&= \frac{\binom{M}{0}\binom{N-M}{n}}{\binom{N}{n}} - \frac{\binom{M+1}{0}\binom{N-M-1}{n}}{\binom{N}{n}} \\
&= \frac{\binom{N-M-1}{n-1}}{\binom{N}{n}} \qquad (31) \\
&= \frac{\binom{M}{0}\binom{N-M-1}{n-0-1}}{\binom{N}{n}} = T(0, M, N, n),
\end{aligned}$$

where (31) follows from the fact that, for non-negative integer m,

$$\binom{m+1}{z+1} = \binom{m}{z} + \binom{m}{z+1} \qquad (32)$$

for any integer z.

Now suppose the lemma is true for $k-1$ with $1 \le k \le M$, i.e.,

$$\Pr\{K \le k-1 \mid M\} - \Pr\{K \le k-1 \mid M+1\} = \frac{\binom{M}{k-1}\binom{N-M-1}{n-k}}{\binom{N}{n}}.$$

Then,

$$\begin{aligned}
\Pr\{K \le k \mid M\} - \Pr\{K \le k \mid M+1\} &= \Pr\{K \le k-1 \mid M\} - \Pr\{K \le k-1 \mid M+1\} \\
&\quad + \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} - \frac{\binom{M+1}{k}\binom{N-M-1}{n-k}}{\binom{N}{n}} \\
&= \frac{\binom{M}{k-1}\binom{N-M-1}{n-k}}{\binom{N}{n}} + \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} - \frac{\binom{M+1}{k}\binom{N-M-1}{n-k}}{\binom{N}{n}} \\
&= \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} - \left[\frac{\binom{M+1}{k}\binom{N-M-1}{n-k}}{\binom{N}{n}} - \frac{\binom{M}{k-1}\binom{N-M-1}{n-k}}{\binom{N}{n}}\right] \\
&= \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} - \frac{\binom{M}{k}\binom{N-M-1}{n-k}}{\binom{N}{n}} \qquad (33) \\
&= \frac{\binom{M}{k}\binom{N-M-1}{n-k-1}}{\binom{N}{n}} \qquad (34)
\end{aligned}$$

where (33) and (34) follows from (32). Therefore, we have shown the lemma for $0 \le k \le M$.

For $k > M$, we have $\Pr\{K \le k \mid M\} = \Pr\{K \le k \mid M+1\} = 1$ and $T(k, M, N, n) = 0$. For $k < 0$, we have $\Pr\{K \le k \mid M\} = \Pr\{K \le k \mid M+1\} = 0$ and $T(k, M, N, n) = 0$. Thus, the lemma is true for any integer k.

□

Lemma 3 *Let $1 \le M \le N$ and $k \le l$. Then,*

$$\Pr\{k \le K \le l \mid M\} - \Pr\{k \le K \le l \mid M-1\} = T(k-1, M-1, N, n) - T(l, M-1, N, n).$$

Proof. To show the lemma, it suffices to consider 6 cases as follows.

Case (i): $0 < n < k \le l$. In this case, $\Pr\{k \le K \le l \mid M\} = \Pr\{k \le K \le l \mid M-1\} = 0$ and $T(k-1, M-1, N, n) = T(l, M-1, N, n) = 0$.

Case (ii): $k \le l < 0 < n$. In this case, $\Pr\{k \le K \le l \mid M\} = \Pr\{k \le K \le l \mid M-1\} = 0$ and $T(k-1, M-1, N, n) = T(l, M-1, N, n) = 0$.

Case (iii): $k \le 0 < n \le l$. In this case, $\Pr\{k \le K \le l \mid M\} = \Pr\{k \le K \le l \mid M-1\} = 1$ and $T(k-1, M-1, N, n) = T(l, M-1, N, n) = 0$.

Case (iv): $k \le 0 \le l < n$. In this case, $T(k-1, M-1, N, n) = 0$ and, by Lemma 2,

$$\begin{aligned}\Pr\{k \le K \le l \mid M\} - \Pr\{k \le K \le l \mid M-1\} &= \Pr\{K \le l \mid M\} - \Pr\{K \le l \mid M-1\} \\ &= T(k-1, M-1, N, n) - T(l, M-1, N, n).\end{aligned}$$

Case (v): $0 < k \le n \le l$. In this case, $T(l, M-1, N, n) = 0$ and, by Lemma 2,

$$\begin{aligned}\Pr\{k \le K \le l \mid M\} - \Pr\{k \le K \le l \mid M-1\} &= \Pr\{K < k \mid M-1\} - \Pr\{K < k \mid M\} \\ &= T(k-1, M-1, N, n) - T(l, M-1, N, n).\end{aligned}$$

Case (vi): $0 < k \le l < n$. In this case, by Lemma 2,

$$\begin{aligned}&\Pr\{k \le K \le l \mid M\} - \Pr\{k \le K \le l \mid M-1\} \\ =\ & [\Pr\{K \le l \mid M\} - \Pr\{K < k \mid M\}] - [\Pr\{K \le l \mid M-1\} - \Pr\{K < k \mid M-1\}] \\ =\ & [\Pr\{K \le l \mid M\} - \Pr\{K \le l \mid M-1\}] - [\Pr\{K < k \mid M\} - \Pr\{K < k \mid M-1\}] \\ =\ & T(k-1, M-1, N, n) - T(l, M-1, N, n).\end{aligned}$$

□

Lemma 4 *Let $l \ge 0$ and $k < n$. Then, $\left\lfloor \frac{nM}{N+1} \right\rfloor \ge l$ for $M \ge 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$, and $\left\lfloor \frac{nM}{N+1} \right\rfloor \le k-1$ for $M \le 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$.*

Proof. To show the first part of the lemma, observe that $(N+1-n)l \ge 0$, by which we can show $\frac{nNl}{n-1} \ge (N+1)l$. Hence, $n\left(1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor\right) > \frac{nNl}{n-1} \ge (N+1)l$. That is, $\frac{n}{N+1}\left(1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor\right) > l$. It follows that $\left\lfloor \frac{n}{N+1}\left(1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor\right) \right\rfloor \ge l$. Since the floor function is non-decreasing, we have $\left\lfloor \frac{nM}{N+1} \right\rfloor \ge l$ for $M \ge 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$.

To prove the second part of the lemma, note that $(N+1-n)(n-k) > 0$, from which we can deduce $1 + \frac{N(k-1)}{n-1} < \frac{(N+1)k}{n}$. Hence, $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor < \frac{(N+1)k}{n}$, i.e., $\frac{n}{N+1}\left(1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor\right) < k$, leading to $\left\lfloor \frac{n}{N+1}\left(1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor\right) \right\rfloor \le k-1$. Since the floor function is non-decreasing, we have $\left\lfloor \frac{nM}{N+1} \right\rfloor \le k-1$ for $M \le 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$.

□

Lemma 5 *Let $0 \le r \le n$. Then, the following statements hold true.*

(I)

$$T(r-1, M-1, N, n) \le T(r, M-1, N, n) \quad \text{for} \quad 1 \le r \le \left\lfloor \frac{nM}{N+1} \right\rfloor;$$

$$T(r+1, M-1, N, n) \le T(r, M-1, N, n) \quad \text{for} \quad \left\lfloor \frac{nM}{N+1} \right\rfloor \le r \le n-1.$$

(II)

$$T(r, M-2, N, n) \le T(r, M-1, N, n) \quad \text{for} \quad 1 < M \le 1 + \left\lfloor \frac{Nr}{n-1} \right\rfloor;$$

$$T(r, M, N, n) \le T(r, M-1, N, n) \quad \text{for} \quad 1 + \left\lfloor \frac{Nr}{n-1} \right\rfloor \le M < N.$$

Proof. To show statement (I), note that $T(r, M-1, N, n) = 0$ for $\min(M-1, n-1) < r \le n$. Our calculation shows that

$$\frac{T(r-1, M-1, N, n)}{T(r, M-1, N, n)} = \frac{r}{M-r} \frac{N-M+1-(n-r)}{n-r} \le 1 \quad \text{for} \quad 1 \le r \le \frac{nM}{N+1}$$

and

$$\frac{T(r-1, M-1, N, n)}{T(r, M-1, N, n)} > 1 \quad \text{for} \quad \frac{nM}{N+1} < r \le \min(M-1, n-1).$$

To show statement (II), note that $T(r, M-1, N, n) = 0$ for $1 \le M < r+1$, and $T(r, M-1, N, n) \ge T(r, M-2, N, n) = 0$ for $M = r+1$. Direct computation shows that

$$\frac{T(r, M-1, N, n)}{T(r, M-2, N, n)} = \frac{M-1}{M-1-r} \frac{N-M+2-(n-r)}{N-M+1} \ge 1 \quad \text{for} \quad r+1 < M \le 1 + \frac{Nr}{n-1},$$

and

$$\frac{T(r, M-1, N, n)}{T(r, M-2, N, n)} < 1 \quad \text{for} \quad 1 + \frac{Nr}{n-1} < M \le N.$$

□

Lemma 6 *Let $0 \le \mathcal{L} \le \mathcal{U} \le N$. Then, for any integers k and l, $\Pr\{k \le K \le l \mid M\}$ is unimodal with respect to M for $\mathcal{L} \le M \le \mathcal{U}$.*

Proof. Clearly, the lemma is trivially true if $k > l$. Hence, to show the lemma, it suffices to consider 6 cases as follows.

Case (i): $0 < n < k \le l$. In this case, $\Pr\{k \le K \le l \mid M\} = 0$ for any $M \in [\mathcal{L}, \mathcal{U}]$.

Case (ii): $k \le l < 0 < n$. In this case, $\Pr\{k \le K \le l \mid M\} = 0$ for any $M \in [\mathcal{L}, \mathcal{U}]$.

Case (iii): $k \le 0 < n \le l$. In this case, $\Pr\{k \le K \le l \mid M\} = 1$ for any $M \in [\mathcal{L}, \mathcal{U}]$.

Case (iv): $k \le 0 \le l < n$. In this case, $\Pr\{k \le K \le l \mid M\} = \Pr\{K \le l \mid M\}$ is non-increasing with respect to $M \in [\mathcal{L}, \mathcal{U}]$ as can be seen from Lemma 2.

Case (v): $0 < k \le n \le l$. In this case, $\Pr\{k \le K \le l \mid M\} = 1 - \Pr\{K < k \mid M\}$ is non-decreasing with respect to $M \in [\mathcal{L}, \mathcal{U}]$ as can be seen from Lemma 2.

Clearly, the lemma is true for the above five cases.

Case (vi): $0 < k \le l < n$. Define $\Delta(k,l,M,N,n) = \Pr\{k \le K \le l \mid M\} - \Pr\{k \le K \le l \mid M-1\}$. By Lemma 3, $\Delta(k,l,M,N,n) = T(k-1,M-1,N,n) - T(l,M-1,N,n)$.

Invoking Lemma 4, for $M \ge 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$, we have that $\left\lfloor \frac{nM}{N+1} \right\rfloor \ge l$ and thus, by statement (I) of Lemma 5, $T(r,M-1,N,n)$ is non-decreasing with respect to $r \le l$. Consequently, $T(k-1,M-1,N,n) \le T(l,M-1,N,n)$, leading to $\Delta(k,l,M,N,n) \le 0$ for $M \ge 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$.

Similarly, applying Lemma 4, for $M \le 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$, we have that $\left\lfloor \frac{nM}{N+1} \right\rfloor \le k-1$ and thus, by statement (I) of Lemma 5, $T(r,M-1,N,n)$ is non-increasing with respect to $r \ge k-1$. Consequently, $T(k-1,M-1,N,n) \ge T(l,M-1,N,n)$, leading to $\Delta(k,l,M,N,n) \ge 0$ for $M \le 1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor$.

By statement (II) of Lemma 5, for $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor \le M \le 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$, we have that $T(l,M-1,N,n)$ is non-decreasing with respect to M and that $T(k-1,M-1,N,n)$ is non-increasing with respect to M. It follows that $\Delta(k,l,M,N,n)$ is non-increasing with respect to M in this range. Therefore, there exists an integer M^* such that $1 + \left\lfloor \frac{N(k-1)}{n-1} \right\rfloor \le M^* \le 1 + \left\lfloor \frac{Nl}{n-1} \right\rfloor$ and that $\Delta(k,l,M,N,n) \ge 0$ for $0 \le M \le M^*$, and $\Delta(k,l,M,N,n) \le 0$ for $M^* \le M \le N$. This implies that $\Pr\{k \le K \le l \mid M\}$ is non-decreasing for $0 \le M \le M^*$ and non-increasing for $M^* \le M \le N$. This concludes the proof of the lemma.

□

Lemma 7 *Let* $0 \le M < N$. *Then,* $\Pr\{g \le K \le h+1 \mid M+1\} \ge \Pr\{g \le K \le h \mid M\}$ *for any integers* g *and* h.

Proof. Clearly, the lemma is trivially true if $g > h$. Hence, to show the lemma, it suffices to consider the case $g \le h$. Note that, by Lemma 3,

$$\begin{aligned}
& \Pr\{g \le K \le h+1 \mid M+1\} - \Pr\{g \le K \le h \mid M\} \\
= & \binom{M+1}{h+1}\binom{N-M-1}{n-h-1} \Big/ \binom{N}{n} + \Pr\{g \le K \le h \mid M+1\} - \Pr\{g \le K \le h \mid M\} \\
= & \binom{M+1}{h+1}\binom{N-M-1}{n-h-1} \Big/ \binom{N}{n} + T(g-1,M,N,n) - T(h,M,N,n) \\
= & \left[\binom{M+1}{h+1}\binom{N-M-1}{n-h-1} - \binom{M}{h}\binom{N-M-1}{n-h-1}\right] \Big/ \binom{N}{n} + T(g-1,M,N,n) \\
= & \binom{M}{h+1}\binom{N-M-1}{n-h-1} \Big/ \binom{N}{n} + T(g-1,M,N,n) \ge 0,
\end{aligned}$$

where the last equality follows from (32).

□

Lemma 8 *Let* $0 < M \le N$. *Then,* $\Pr\{g-1 \le K \le h \mid M-1\} \ge \Pr\{g \le K \le h \mid M\}$ *for any integers* g *and* h.

Proof. Clearly, the lemma is trivially true if $g > h$. Hence, to show the lemma, it suffices to consider the case $g \leq h$. Note that, by Lemma 3,

$$\begin{aligned}
& \Pr\{g-1 \leq K \leq h \mid M-1\} - \Pr\{g \leq K \leq h \mid M\} \\
= & \binom{M-1}{g-1}\binom{N-M+1}{n-g+1} \Big/ \binom{N}{n} + \Pr\{g \leq K \leq h \mid M-1\} - \Pr\{g \leq K \leq h \mid M\} \\
= & \binom{M-1}{g-1}\binom{N-M+1}{n-g+1} \Big/ \binom{N}{n} + T(h, M-1, N, n) - T(g-1, M-1, N, n) \\
= & \left[\binom{M-1}{g-1}\binom{N-M+1}{n-g+1} - \binom{M-1}{g-1}\binom{N-M}{n-g}\right] \Big/ \binom{N}{n} + T(h, M-1, N, n) \\
= & \binom{M-1}{g-1}\binom{N-M}{n-g+1} \Big/ \binom{N}{n} + T(h, M-1, N, n) \geq 0,
\end{aligned}$$

where the last equality follows from (32).

□

Lemma 9 *Suppose that* $\{M' < L(K) < M''\} = \{M' < U(K) < M''\} = \emptyset$. *Then,* $\Pr\{L(K) < M < U(K) \mid M\}$ *is unimodal with respect to* M *for* $M' \leq M \leq M''$.

Proof. First, we shall show the following facts:

(i) If $\{L(K) = M'\} = \emptyset$, then $\{L(K) < M\} = \{L(K) < M'\} = \{L(K) < M''\}$ for $M' \leq M \leq M''$.

(ii) If $\{L(K) = M'\} \neq \emptyset$, then $\{L(K) < M\} = \{L(K) \leq M'\} = \{L(K) < M''\}$ for $M' < M \leq M''$.

(iii) If $\{U(K) = M''\} = \emptyset$, then $\{U(K) > M\} = \{U(K) > M'\} = \{U(K) > M''\}$ for $M' \leq M \leq M''$.

(iv) If $\{U(K) = M''\} \neq \emptyset$, then $\{U(K) > M\} = \{U(K) > M'\} = \{U(K) \geq M''\}$ for $M' \leq M < M''$.

To show statement (i), making use of $\{L(K) = M'\} = \{M' < L(K) < M''\} = \emptyset$, we have $\{M' \leq L(K) < M\} = \{M' < L(K) < M\} \subseteq \{M' < L(K) < M''\} = \emptyset$ and $\{L(K) < M\} = \{L(K) < M'\} \cup \{M' \leq L(K) < M\} = \{L(K) < M'\}$ for $M' \leq M \leq M''$. On the other hand, $\{L(K) < M\} = \{L(K) < M''\} \setminus \{M \leq L(K) < M''\} = \{L(K) < M''\}$ for $M' \leq M \leq M''$.

To show statement (ii), making use of $\{M' < L(K) < M''\} = \emptyset$, we have $\{M' < L(K) < M\} \subseteq \{M' < L(K) < M''\} = \emptyset$ and $\{L(K) < M\} = \{L(K) \leq M'\} \cup \{M' < L(K) < M\} = \{L(K) \leq M'\}$ for $M' \leq M \leq M''$. On the other hand, $\{L(K) < M\} = \{L(K) < M''\} \setminus \{M \leq L(K) < M''\} = \{L(K) < M''\}$ for $M' < M \leq M''$.

To show statement (iii), using $\{U(K) = M''\} = \{M' < U(K) < M''\} = \emptyset$, we have $\{M' < U(K) \leq M\} \subseteq \{M' < U(K) \leq M''\} = \emptyset$ and $\{U(K) > M\} = \{U(K) > M'\} \setminus \{M' < U(K) \leq$

$M\} = \{U(K) > M'\}$ for $M' \leq M \leq M''$. On the other hand, $\{U(K) > M\} = \{U(K) > M''\} \cup \{M < U(K) \leq M''\} = \{U(K) > M''\}$ for $M' \leq M \leq M''$.

To show statement (iv), note that $\{U(K) > M\} = \{U(K) > M'\}$ for $M' \leq M < M''$. On the other hand, $\{U(K) > M\} = \{U(K) \geq M''\} \cup \{M < U(K) < M''\} = \{U(K) \geq M''\}$ for $M' \leq M < M''$.

Now, to show the lemma, it suffices to consider four cases as follows.

Case (i): $\{L(K) = M'\} = \emptyset$, $\{U(K) = M''\} = \emptyset$.

Case (ii): $\{L(K) = M'\} = \emptyset$, $\{U(K) = M''\} \neq \emptyset$.

Case (iii): $\{L(K) = M'\} \neq \emptyset$, $\{U(K) = M''\} = \emptyset$.

Case (iv): $\{L(K) = M'\} \neq \emptyset$, $\{U(K) = M''\} \neq \emptyset$.

In Case (i), making use of facts (i) and (iii), we have $\{L(K) < M < U(K)\} = \{L(K) < M' < U(K)\} = \{L(K) < M'' < U(K)\}$ for $M' \leq M \leq M''$. Invoking Lemma 6, we have that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' \leq M \leq M''$.

In Case (ii), making use of facts (i) and (iv), we have $\{L(K) < M < U(K)\} = \{L(K) < M' < U(K)\} = \{L(K) < M'' \leq U(K)\}$ for $M' \leq M < M''$. Invoking Lemma 6, we have that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' \leq M < M''$. Since $\{M'' = U(K)\} \neq \emptyset$ and $U(K)$ is monotonically increasing, we have $\{M'' \leq U(K)\} = \{K \geq \underline{k}\}$ and $\{M'' < U(K)\} = \{K \geq \overline{k} + 1\}$, where $\underline{k} = \min\{k : U(k) \geq M''\} \leq \overline{k} = \max\{k : U(k) \leq M''\}$. Therefore, as a result of Lemma 8,

$$\Pr\{L(K) < M'' \leq U(K) \mid M'' - 1\} \geq \Pr\{L(K) < M'' < U(K) \mid M''\}. \tag{35}$$

It follows that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' \leq M \leq M''$.

In Case (iii), making use of facts (ii) and (iii), we have $\{L(K) < M < U(K)\} = \{L(K) \leq M' < U(K)\} = \{L(K) < M'' < U(K)\}$ for $M' < M \leq M''$. Invoking Lemma 6, we have that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' < M \leq M''$. Since $\{M' = L(K)\} \neq \emptyset$ and $L(K)$ is monotonically increasing, we have $\{M' \geq L(K)\} = \{K \leq \overline{k}\}$ and $\{M' > L(K)\} = \{K \leq \underline{k} - 1\}$, where $\underline{k} = \min\{k : L(k) \geq M'\} \leq \overline{k} = \max\{k : L(k) \leq M'\}$. Therefore, as a result of Lemma 8,

$$\Pr\{L(K) < M' < U(K) \mid M'\} \leq \Pr\{L(K) \leq M' < U(K) \mid M' + 1\}. \tag{36}$$

It follows that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' \leq M \leq M''$.

In Case (iv), making use of facts (ii) and (iv), we have $\{L(K) < M < U(K)\} = \{L(K) \leq M' < U(K)\} = \{L(K) < M'' \leq U(K)\}$ for $M' < M < M''$. Invoking Lemma 6, we have that

$\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' < M < M''$. Recalling (35) and (36), we have that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' \leq M \leq M''$.

□

Finally, we are in a position to prove the Theorem 14. Let $M' < M''$ be two consecutive distinct elements of I_{UL}. Then, $\{M' < L(K) < M''\} = \{M' < U(K) < M''\} = \emptyset$. By Lemma 9, we have that $\Pr\{L(K) < M < U(K) \mid M\}$ is unimodal with respect to M for $M' \leq M \leq M''$. Since this argument holds for any consecutive distinct elements of the set I_{UL}, Theorem 14 is established.

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