

# Generalized Harmonic Oscillator and the Schrödinger Equation with Position-Dependent Mass

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## Abstract

We study the generalized harmonic oscillator which has both the position-dependent mass and the potential depending on the form of mass function in a more general framework. The explicit expressions of the eigenvalue and eigenfunction for such system are given, they have the same forms as those for the usual harmonic oscillator with constant mass. The coherent state and the its properties for the system with PDM are also discussed. We give the corresponding effective potentials for several mass functions, the systems with such potentials are isospectral to the usual harmonic oscillator.

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## I. INTRODUCTION

Recently, the study of quantum systems with position-dependent mass (PDM) has attracted a lot of interests [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. The models have played important roles in description of the electronic properties of semiconductors [1], quantum dots [4], liquid crystals[5] and so on. In the theoretical works on the system with PDM, the main concern was the exact solutions of the corresponding Schrödinger equation or its relativistic generalizations such as Klein-Gordon equation and Dirac equation. Using coordinate transformation method[33, 34, 35], supersymmetric quantum mechanics [36, 37] and other methods, many solvable potentials of the wave equations for different mass functions have been obtained. These potentials can be classified according to the forms of eigenfunctions of the wave equations, which show some similarities between systems with constant mass and with PDM[32, 34].

It is well known that the harmonic oscillator with constant mass is a very important system in quantum mechanics[38]. Apart from its wide applications in problems of condensed matter, atomic, nuclear and elementary particle physics, the harmonic oscillator also has connections with many interesting methods analytically solving Schrödinger equation such as factorization method, supersymmetric quantum mechanics, algebraic method. So, it is natural to generalize the usual harmonic oscillator and related studies to the case of position-dependent mass. There exist some works in this aspect by using point canonical transformation and Lie algebraic approach[25, 26, 27]. However, these investigations only involve the specific potentials and mass functions. In this paper, we consider the so-called generalized harmonic oscillator(GHO) based on the following two requirements: the Hamiltonian for the system with PDM can be factorized in terms of the rising operator and lowering operator which are generalization of the creation and destruction operators for the usual harmonic oscillator, respectively; the rising operator and lowering operator satisfy the same form of commutation relation as that for the creation and destruction operators. With these requirements, we put the discussion of GHO in a more general framework. We will see that many systems with PDM rather than with specific mass functions have the same energy spectrum and same form of eigenfunctions as those of the usual harmonic oscillator.

The paper is organized as follows. In section II, the condition for the Hamiltonian with

PDM to be that for GHO is derived, the eigenproblem of GHO is solved using the operator method. Some studies such as the coherent states related to the harmonic oscillator are generalized to the system with PDM and their properties are discussed. In section III, for some mass functions, the corresponding potentials are given. In the last section, we will make some remarks.

## II. GENERALIZED HARMONIC OSCILLATOR AND ITS SOLUTION

When the mass of a particle depends on its position, the mass and momentum operators no longer commute[2], so there are several ways to define the kinetic energy operator. We start with the kinetic energy introduced by von Roos[2], the Hamiltonian with position-dependent mass  $M(\vec{\mathbf{r}})$  and potential energy  $V(\vec{\mathbf{r}})$  reads

$$\begin{aligned}\hat{H} &= \frac{1}{4} \left[ M(\vec{\mathbf{r}})^\alpha \hat{\mathbf{p}} M(\vec{\mathbf{r}})^\beta \hat{\mathbf{p}} M(\vec{\mathbf{r}})^\gamma + M(\vec{\mathbf{r}})^\gamma \hat{\mathbf{p}} M(\vec{\mathbf{r}})^\beta \hat{\mathbf{p}} M(\vec{\mathbf{r}})^\alpha \right] + V(\vec{\mathbf{r}}) \\ &= -\frac{\hbar^2}{4m_0} \left[ m(\vec{\mathbf{r}})^\alpha \vec{\nabla} m(\vec{\mathbf{r}})^\beta \vec{\nabla} m(\vec{\mathbf{r}})^\gamma + m(\vec{\mathbf{r}})^\gamma \vec{\nabla} m(\vec{\mathbf{r}})^\beta \vec{\nabla} m(\vec{\mathbf{r}})^\alpha \right] + V(\vec{\mathbf{r}}),\end{aligned}\tag{1}$$

where  $m_0$  is a constant mass and  $m(\vec{\mathbf{r}})$  is a dimensionless position-dependent mass  $M(\vec{\mathbf{r}}) = m_0 m(\vec{\mathbf{r}})$ ,  $\alpha, \beta, \gamma$  are parameters and satisfy the condition of  $\alpha + \beta + \gamma = -1$ . Using natural units ( $m_0 = \hbar = 1$ ) and only considering one-dimensional system, we can rewrite Hamiltonian (1) as

$$\hat{H} = -\frac{1}{2} \frac{d}{dx} \frac{1}{m} \frac{d}{dx} + V_{eff}(x) = -\frac{1}{2m} \frac{d^2}{dx^2} + \frac{m'}{2m^2} \frac{d}{dx} + V_{eff}(x),\tag{2}$$

where the effective potential is

$$V_{eff}(x) = V(x) + \frac{1}{4}(\beta + 1) \frac{m''(x)}{m^2} - \frac{1}{2}[\alpha(\alpha + \beta + 1) + \beta + 1] \frac{m'(x)^2}{m(x)^3},\tag{3}$$

and  $m'(x) = \frac{dm}{dx}$ ,  $m''(x) = \frac{d^2m}{dx^2}$ .

There are many debates for the choice of the parameters  $\alpha, \beta, \gamma$ . Morrow and Brownstein[3] have shown that  $\alpha = \gamma$  based on the comparison between the experimental results and the analytical solutions of some models. In the following, we use this kind of condition for the parameters as a basic fact. We will introduce the lowering and rising operators  $\hat{A}$  and  $\hat{A}^+$  so that the Hamiltonian (2) can be factorized in terms of these operators and its solution be obtained by using the operator method. For this purpose, we define the

lowering and rising operators  $\hat{A}$  and  $\hat{A}^+$  as follows[17], respectively

$$\hat{A} = \frac{1}{\sqrt{2}} \left[ m(x)^{\frac{\beta}{2}} \frac{d}{dx} m(x)^{-\frac{1}{2}(\beta+1)} + \mu(x) \right], \quad (4)$$

$$\hat{A}^+ = \frac{1}{\sqrt{2}} \left[ -m(x)^{-\frac{1}{2}(\beta+1)} \frac{d}{dx} m(x)^{\frac{\beta}{2}} + \mu(x) \right], \quad (5)$$

where the function  $\mu(x)$  will be determined by the requirement that

$$[\hat{A}, \hat{A}^+] = 1. \quad (6)$$

The constrain (6) makes our discussion be restricted to GHO, so  $\mu(x)$  in Eqs.(4) and (5) is not the so-called superpotential, which is different from that in [17, 25, 26].

Using Eqs.(4) and (5) and performing some calculations, we have

$$[\hat{A}, \hat{A}^+] = \frac{\mu'(x)}{\sqrt{m(x)}} - \frac{1}{4m(x)}(2\beta + 1) \left[ \frac{m''(x)}{m(x)} - \frac{3}{2} \left( \frac{m'(x)}{m(x)} \right)^2 \right]. \quad (7)$$

It follows from Eqs.(6) and (7) that  $\mu(x)$  and  $\beta$  are given by the following relations, respectively

$$\beta = -\frac{1}{2}, \quad \mu(x) = \int^x \sqrt{m(x)} dx. \quad (8)$$

Now, Eqs.(4) and (5) can be rewritten as follows,

$$\begin{aligned} \hat{A} &= \frac{1}{\sqrt{2}} \left[ m(x)^{-\frac{1}{4}} \frac{d}{dx} m(x)^{-\frac{1}{4}} + \mu(x) \right], \\ \hat{A}^+ &= \frac{1}{\sqrt{2}} \left[ -m(x)^{-\frac{1}{4}} \frac{d}{dx} m(x)^{-\frac{1}{4}} + \mu(x) \right]. \end{aligned} \quad (9)$$

It is easy to see that the operators  $\hat{A}$  and  $\hat{A}^+$  will become the destruction and creation operators for the harmonic oscillator respectively when the mass  $m(x)$  is a constant.

If we assume that the potential function  $V(x)$  in Eq.(2) depends on mass  $m(x)$  and is given by

$$V(x) = \frac{1}{2}[\mu(x)]^2, \quad (10)$$

then the Hamiltonian (2) can be rewritten as

$$H = \hat{A}^+ \hat{A} + \frac{1}{2} = \hat{N} + \frac{1}{2}, \quad (11)$$

where  $\hat{N} = \hat{A}^+ \hat{A}$  may be called number-like operator because it reduces to the number operator for the harmonic oscillator when the mass  $m(x)$  is independent of  $x$ . It is also

evident that with  $m(x) = 1$  the potential (10) and the Hamiltonian (11) become the potential and Hamiltonian for the harmonic oscillator, respectively. In this sense, we call the above system with PDM a generalized harmonic oscillator.

With Eq.(6), we can easily get the commutators between operators  $\hat{A}$ ,  $\hat{A}^+$  and  $\hat{N}$

$$[\hat{N}, \hat{A}] = -\hat{A}, \quad [\hat{N}, \hat{A}^+] = \hat{A}^+. \quad (12)$$

The following procedure solving the eigenproblem of Eq.(11) is similar to that of the harmonic oscillator[39]. If we assume that  $\psi_n(x)$  is an eigenstate of  $\hat{N}$  with eigenvalue  $n$ , that is

$$\hat{N}\psi_n(x) = n\psi_n(x), \quad (13)$$

then using Eq.(12), we have the relation

$$\hat{N}\hat{A}^+\psi_n(x) = (n+1)\hat{A}^+\psi_n(x). \quad (14)$$

Eq.(14) indicates that  $\hat{A}^+\psi_n(x)$  is also an eigenstate of  $\hat{N}$  with eigenvalue  $(n+1)$ , so  $\hat{A}^+$  can be regarded as a raising operator. By a similar reasoning,  $\hat{A}$  is a lowering operator which means that  $\hat{A}\psi_n(x)$  is an eigenstate of  $\hat{N}$  with eigenvalue  $(n-1)$ . If we assume the eigenstate of  $\hat{N}$  has a lower bound, that is there exists a state  $\psi_0(x)$  satisfying the relation

$$\hat{A}\psi_0(x) = \frac{1}{\sqrt{2}} \left[ m(x)^{-\frac{1}{4}} \frac{d}{dx} (m(x)^{-\frac{1}{4}} \psi_0(x)) + \mu(x) \psi_0(x) \right] = 0, \quad (15)$$

then all eigenstates of  $\hat{N}$  can be obtained by successive application of the operator  $\hat{A}^+$  on the state  $\psi_0(x)$ . The existence of the lower bound is related to the requirement that  $|\hat{A}\psi_n(x)|^2$  is real and positive. Solving Eq.(15), we get the analytical expression of  $\psi_0(x)$

$$\psi_0(x) = [m(x)]^{\frac{1}{4}} e^{-\frac{1}{2}[\mu(x)]^2}. \quad (16)$$

Now, acting  $\psi_0(x)$  on the left with the operator  $\hat{A}^+$ , we have

$$\psi_1(x) = \hat{A}^+\psi_0(x) = \sqrt{2}\mu(x)\psi_0(x) = \frac{\sqrt{2}}{2} [m(x)]^{\frac{1}{4}} e^{-\frac{1}{2}[\mu(x)]^2} H_1(\mu(x)). \quad (17)$$

Similarly, we get

$$\psi_2(x) = (\hat{A}^+)^2\psi_0(x) = \{2[\mu(x)]^2 - 1\}\psi_0(x) = \frac{1}{2} [m(x)]^{\frac{1}{4}} e^{-\frac{1}{2}[\mu(x)]^2} H_2(\mu(x)). \quad (18)$$

By induction, one can show that all eigenstates of  $\hat{H}$  are given by

$$\psi_n(x) = \mathcal{N}_n [m(x)]^{\frac{1}{4}} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots), \quad (19)$$

where  $\mathcal{N}_n$  is the normalization coefficient,  $\mu(x)$  is defined by Eq.(8), and  $H_n(\mu(x))$  is the Hermite polynomial. The normalization of  $\psi_n$  means that

$$\begin{aligned} \int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = 1 &= \mathcal{N}_n^2 \int_{-\infty}^{\infty} [m(x)]^{\frac{1}{2}} e^{-[\mu(x)]^2} H_n^2(\mu(x)) dx \\ &= \mathcal{N}_n^2 \int_{\mu_{min}}^{\mu_{max}} e^{-[\mu(x)]^2} H_n^2(\mu(x)) d\mu(x), \end{aligned} \quad (20)$$

where  $\mu_{min} = \mu(-\infty)$  and  $\mu_{max} = \mu(\infty)$ . If  $\mu_{min} = 0$  or  $-\infty$  and  $\mu_{max} = \infty$ , we can get the explicit expression of  $\mathcal{N}_n$  as follows

$$\mathcal{N}_n = \begin{cases} \frac{1}{\sqrt{2^n n! \sqrt{\pi}}}, & (\mu_{min} = -\infty) \\ \frac{1}{\sqrt{2^{n-1} n! \sqrt{\pi}}}. & (\mu_{min} = 0) \end{cases} \quad (21)$$

The requirement for the parameters  $\mu_{min}, \mu_{max}$  will input constraint on the mass functions. However, if we require that  $\psi_n(x)$  is orthogonal for different  $n$ , then there is only one choice of  $\mu_{min} = -\infty$ .

Using Eqs. (11) and (13), we have

$$\hat{H}\psi_n(x) = \left(n + \frac{1}{2}\right) \psi_n(x), \quad (n = 0, 1, 2, \dots), \quad (22)$$

which means that  $\psi_n(x)$  is the eigenstate of  $\hat{H}$  with eigenvalue  $E_n = n + \frac{1}{2}$ . In a sense, GHO is in the same class as the harmonic oscillator considering that they have the same energy spectrum but with different potentials.

Now, we can calculate the matrix elements  $\langle n|\hat{A}^+|n'\rangle$ ,  $\langle n|\hat{A}|n'\rangle$  which are the same as those for the harmonic oscillator

$$\begin{aligned} \langle n|\hat{A}^+|n'\rangle &= \int_{-\infty}^{\infty} \psi_n \hat{A}^+ \psi_{n'} dx = \sqrt{n'+1} \int_{-\infty}^{\infty} \psi_n \psi_{n'+1} dx = \sqrt{n'+1} \delta_{n,n'+1}, \\ \langle n|\hat{A}|n'\rangle &= \int_{-\infty}^{\infty} \psi_n \hat{A} \psi_{n'} dx = \sqrt{n'} \int_{-\infty}^{\infty} \psi_n \psi_{n'-1} dx = \sqrt{n'} \delta_{n,n'-1}. \end{aligned} \quad (23)$$

Because the operators  $\hat{A}$  and  $\hat{A}^+$  have the same kind of properties of the creation and destruction operators, many studies on the latter may be generalized to the former. For example, the canonical coherent state is defined to be the eigenstate of the destruction operator[40]. If we generalize this definition of the coherent state to the operator  $\hat{A}$ , that is,

$$\hat{A}|z\rangle = z|z\rangle, \quad (24)$$

where  $z$  is a complex number, and similarly we expand  $|z\rangle$  in term of the state  $|n\rangle$ ,  $|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$ , then we have

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \quad (25)$$

where the factor  $e^{-\frac{|z|^2}{2}}$  comes from the convention of  $\langle z|z\rangle = 1$ . The completeness relations, non-orthogonality for different eigenstates  $z$  and  $z'$ , and other properties for the coherent state of system with constant mass are also hold for the system with PDM. However, due to the fact that we cannot express  $x$  in terms of the combinations of the operators  $\hat{A}$  and  $\hat{A}^+$  as in the case of harmonic oscillator, it is difficult to calculate the matrix elements such as  $\langle z|x|z\rangle$ ,  $\langle z|x^2|z\rangle$ . From Eqs.(9), we have

$$\begin{aligned} \mu(x) &= \frac{1}{\sqrt{2}}(\hat{A} + \hat{A}^+), \\ \hat{\pi}(x) &= m^{-\frac{1}{4}} \hat{p} m^{-\frac{1}{4}} = -im^{-\frac{1}{4}} \frac{d}{dx} m^{-\frac{1}{4}} = -\frac{i}{\sqrt{2}}(\hat{A} - \hat{A}^+), \end{aligned} \quad (26)$$

where  $\hat{\pi}$  is the so-called deformed momentum[21]. With Eq.(24), it is easy to get the following expressions of the matrix elements

$$\begin{aligned} \langle z|\mu(x)|z\rangle &= \frac{1}{\sqrt{2}}(z + z^*), & \langle z|\mu(x)^2|z\rangle &= \frac{1}{2}[(z + z^*)^2 + 1], \\ \langle z|\hat{\pi}(x)|z\rangle &= -\frac{i}{\sqrt{2}}(z - z^*), & \langle z|\hat{\pi}(x)^2|z\rangle &= -\frac{1}{2}[(z - z^*)^2 - 1], \end{aligned} \quad (27)$$

and

$$\begin{aligned} \Delta\mu(x) &= \sqrt{\langle z|\mu(x)^2|z\rangle - (\langle z|\mu(x)|z\rangle)^2} = \frac{1}{\sqrt{2}}, \\ \Delta\hat{\pi}(x) &= \sqrt{\langle z|\hat{\pi}(x)^2|z\rangle - (\langle z|\hat{\pi}(x)|z\rangle)^2} = \frac{1}{\sqrt{2}}. \end{aligned} \quad (28)$$

Noticing that  $\mu(x)$  and  $\hat{\pi}(x)$  are functions of  $x$  except that  $\hat{\pi}(x)$  is related to the momentum  $\hat{p}$ , Eqs.(28) show that there exist more quantities with minimum uncertainty in the coherent states for systems with PDM than for those with constant mass.

It should be emphasized that above discussion is the generalization of the method used for the usual harmonic oscillator. In this setting, the analytical expressions of the eigenvalues and eigenfunctions for GHO are obtained through the so-called Heisenberg algebra generated by  $\hat{A}^+$ ,  $\hat{A}$ ,  $\hat{N}$  and the unit operator 1 rather than the Lie algebra  $su(1, 1)$  or the nonlinear algebra studied in [25, 26]. It is also noted that the condition (6) was used to determine the superpotential in the framework of supersymmetry quantum mechanics in [25, 26]. However,

the condition (6) has several functions in our discussions: (i) determining the parameters in the Hamiltonian (2); (ii) indicating that the potential in the Hamiltonian (2) is related to the mass function when the Hamiltonian is required to be factorized with the operators obeying the condition (6); (iii) showing that the algebraic methods and some related studies to the usual harmonic oscillator may be applicable to the systems with PDM. As to (iii), the coherent state for the system of PDM discussed above is such an example and it gives a more general result on minimum uncertainty than that for the case of constant mass. In addition, we also study the orthonormality of the wavefunctions and its possible restriction to the form of the mass functions, which is illustrated through examples in the next section. To our knowledge, all those mentioned above are not appeared in the literature for the systems with PDM.

### III. MASS FUNCTIONS AND POTENTIALS

In the last section, we have shown that the generalized harmonic oscillators have the same energy spectrums and the same form eigenstates as those of the harmonic oscillators for a lot of mass functions. In this section, we will give the corresponding potentials for several mass functions that satisfy the requirements for the orthonormality of  $\psi_n(x)$ .

**Example 1.** We first consider the mass function  $m(x)$  to be of the form

$$m(x) = \left( \frac{a + x^2}{1 + x^2} \right)^2, \quad (29)$$

where  $a > 0$ . This effective mass was used by many authors in their studies on solvability of the systems with PDM [9, 14, 15, 17, 25, 26]. It is noted that we have  $m(x) = 1$  for  $a = 1$ , which corresponds to the harmonic oscillator.

Substituting Eq.(29) into Eq.(8), we have

$$\mu(x) = x + (a - 1) \arctan x, \quad (30)$$

and the related potential function given by

$$V_{eff}(x) = \frac{1}{2}[\mu(x)]^2 + \frac{(a - 1) [3x^4 + (4 - 2a)x^2 - a]}{2(a + x^2)^4}, \quad (31)$$

which is the effective-mass analogue of the singular oscillator potential. From Eqs.(19) and (29), we get the eigenstate as follows

$$\psi_n(x) = \mathcal{N}_n \sqrt{\frac{a + x^2}{1 + x^2}} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots). \quad (32)$$



These results are the same as those obtained by the coordinates transformation method, SUSY quantum mechanics method and so on[9, 14, 15, 25].

**Example 2.** The mass for a particle is exponentially increasing or decreasing which has the form[11, 12, 13]

$$m(x) = e^{ax}. \quad (33)$$

This kind of mass function may be used in the study of confine energy states for carriers in semiconductor quantum well structures[12, 13]. For the mass function (33), Eq.(8) gives

$$\mu(x) = \begin{cases} \frac{2}{a}e^{\frac{a}{2}x}, & (a \neq 0), \\ x, & (a = 0), \end{cases} \quad (34)$$

and  $a = 0$  is related to  $m(x) = 1$ , that is to the case of constant mass. From Eq.(10), we have the potential

$$V(x) = \begin{cases} \frac{2}{a^2}e^{ax}, & (a \neq 0), \\ \frac{1}{2}x^2, & (a = 0), \end{cases} \quad (35)$$

which indicates it has a similar behavior as the mass (33) for  $a \neq 0$ [11]. The effective potential corresponding to the mass (33) can be written as

$$V_{eff}(x) = \frac{1}{2}[\mu(x)]^2 - \frac{3a^2}{32}e^{-ax}. \quad (36)$$

The corresponding wavefunctions are

$$\psi_n(x) = \mathcal{N}_n e^{\frac{a}{4}x} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots). \quad (37)$$

It is noted that the orthogonality of the eigenstate requires  $a < 0$  for  $x < 0$  and  $a > 0$  for  $x > 0$ .

**Example 3.** We consider another mass which reads as

$$m(x) = 1 + \tanh(ax), \quad (38)$$

where  $a$  is a real parameter. The solution of Schrödinger equation with mass (38) and some smooth potential was studied in Refs.[7] and [8] by solving Schrödinger equation directly. In this case, we get the expression for  $\mu(x)$  in Eq. (8)

$$\mu(x) = \begin{cases} \frac{\sqrt{2}}{a} \ln(e^{ax} + \sqrt{1 + e^{2ax}}), & (a \neq 0), \\ x, & (a = 0). \end{cases} \quad (39)$$

The effective potential corresponding to the mass (38) is

$$V_{eff}(x) = \frac{1}{2}[\mu(x)]^2 - \frac{a^2}{16} \frac{4e^{2ax} + 3}{e^{2ax}(e^{2ax} + 1)}, \quad (40)$$

and the wavefunctions are

$$\psi_n(x) = \mathcal{N}_n [1 + \tanh(ax)]^{\frac{1}{4}} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots). \quad (41)$$

**Example 4.** The mass is a power of  $x$ , that is

$$m(x) = x^a, \quad (42)$$

where  $a$  is a real parameter. For this mass,  $\mu(x)$  in Eq. (8) takes the form

$$\mu(x) = \begin{cases} \frac{2}{a+2} x^{(a+2)/2}, & (a \neq -2), \\ \ln x, & (a = -2). \end{cases} \quad (43)$$

The effective potential and the wavefunctions related to the mass (42) are, respectively

$$V_{eff} = \frac{1}{2}[\mu(x)]^2 - \frac{1}{32} a(3a+4)x^{-(a+2)}, \quad (44)$$

$$\psi_n(x) = \mathcal{N}_n x^{\frac{a}{4}} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots). \quad (45)$$

**Example 5.** The mass is a deformed hyperbolic function of  $x$

$$m(x) = \operatorname{sech}^2(ax), \quad (46)$$

where  $a$  is a positive parameter. Now,  $\mu(x)$  in Eq. (8) is

$$\mu(x) = \frac{2}{a} \arctan e^{ax}. \quad (47)$$

The corresponding effective potential is

$$V_{eff} = \frac{1}{2}[\mu(x)]^2 - \frac{1}{16} [3 \cosh(2ax) + 1]. \quad (48)$$

The wavefunctions are

$$\psi_n(x) = \mathcal{N}_n [\operatorname{sech}(ax)]^{\frac{1}{2}} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots). \quad (49)$$

**Example 6.** The mass function has the form [27]

$$m(x) = \frac{a^2}{(q+x^2)^2}, \quad (50)$$

where  $a$  is a real parameter and  $q > 0$ . When  $q = 0$ , (46) reduces to that of (42). In this case,  $\mu(x)$  in Eq. (8) is

$$\mu(x) = \frac{a}{\sqrt{q}} \arctan\left(\frac{x}{\sqrt{q}}\right). \quad (51)$$

The effective potential is

$$V_{eff} = \frac{a^2}{2q} \left[ \arctan\left(\frac{x}{\sqrt{q}}\right) \right]^2 - \frac{q}{2a^2} - \frac{x^2}{a^2}. \quad (52)$$

The wavefunctions are

$$\psi_n(x) = \mathcal{N}_n \sqrt{\frac{a}{q+x^2}} e^{-\frac{1}{2}[\mu(x)]^2} H_n(\mu(x)), \quad (n = 0, 1, 2, \dots). \quad (53)$$

It is noted that mass (50) has no analogue of the constant mass for whatever  $a$  and  $q$ , which is different from the mass functions in the other five examples. Also, the orthogonality of the wavefunction for GHO requires that  $\mu(x)$  have the property  $\mu_{min} = -\infty$  and  $\mu_{max} = \infty$ , but  $\mu(x)$  in Eqs. (47) and (51) do not so. In this sense, the mass functions (46) and (50) should be excluded from a possible choice of the mass, and the wavefunctions in (49) and (53) are only formal. Note that the wavefunctions in [27] are unnormalized.

#### IV. REMARKS AND DISCUSSIONS

In the above sections, we have discussed a special system with PDM, i.e. the generalized harmonic oscillator that can be solved using operator method. We obtain the analytical expressions of its energy spectrums and eigenstates. For several mass functions, the corresponding effective potentials are given. In our discussions, we restrict the lowering and rising operators to obey the commutation relation (6) which is same as that for the creation and destruction operators of the harmonic oscillator. With this kind of construction, we have restrict the choice of the parameters  $\alpha, \beta, \gamma$  in the Hamiltonian (2). If we further require that the Hamiltonian (2) can be factorized into the product of the lowering and rising operators, then the potential  $V(x)$  is determined by the mass function, which is consistent with that from the SUSY quantum mechanics method. Different from the SUSY method, the key of our discussions lies in that we can generalize a lot of investigations of the harmonic oscillator to GHO. For example, the coherent state in section II for the system with PDM is studied with this kind of consideration, and it indeed shows some differences from that for the system of constant mass. Also, the orthonormality of the wavefunctions for GHO puts the restriction on the choice of the mass functions.

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