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## Riemannian Geometry on Quantum Spaces

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### **Abstract**

An algebraic formulation of Riemannian geometry on quantum spaces is presented, where Riemannian metric, distance, Laplacian, connection, and curvature have their counterparts. This description is also extended

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to complex manifolds. Examples include the quantum sphere, the complex quantum projective spaces and the two-sheeted space.

## 1 Introduction

In [1] Chamseddine, Felder and Fröhlich developed the notions of Riemannian metric, connection and curvature in the framework of the non-commutative geometry of A. Connes [5]. In their formulation the Hilbert space representation is a prerequisite. The purpose of this paper is to propose a purely algebraic formulation of Riemannian geometry on quantum spaces. It is suitable for physicists to build physical models. The question of mathematical rigor is left for future study.

In Sec.2 we describe this algebraic formulation of Riemannian geometry on quantum spaces and quantum complex manifolds. It is applied to the quantum sphere  $S_q^2$ [2] in Sec.3. In particular the explicit expression for the quantum distance is worked out. A comment on the connection with A. Connes' work is made. The complex projective spaces  $CP_q(N)$ [4] is considered in Sec.4, and the two-sheeted space [1] in Sec.5.

## 2 Riemannian Structure on Quantum Spaces

The general notion of non-commutative differential calculus is reviewed in Sec.2.1.

Those who are familiar with it can start with Sec.2.2.

## 2.1 Differential Calculus on Quantum Spaces

A quantum space is specified by an unital, associative, non-commutative  $*$ -algebra  $\mathcal{A}$  generated by  $\{1, x^a, \mathcal{I}_i\}$  over the field  $k = \mathbf{C}$ , where  $x^a$ 's are the coordinates on the quantum space, and  $\mathcal{I}_i$ 's are non-commutative constants, including for example the generators of the algebra of functions of the quantum group which specifies the quantum symmetry on the quantum space.  $\mathcal{A}$  is called the algebra of functions on the quantum space.

To talk about differential geometry on the quantum space, we should have  $\mathcal{A}$  extended to the algebra of differential calculus  $\Omega(\mathcal{A})$  generated by  $\{1, x^a, \xi^a, \chi_a, \mathcal{I}_i\}$ , where the commutation relations among the generators are given so that one knows how to rewrite a product of elements in  $\Omega(\mathcal{A})$  in any preferred order of the elements. (But the commutation relations between  $\xi^a$  and  $\chi_b$  are not necessary.) The  $\xi^a$ 's are differential one-forms and the  $\chi_a$ 's are the derivations dual to them so that the exterior derivative is  $d = \xi^a \chi_a$ . The  $*$ -involution on functions is also extended to all elements in  $\Omega(\mathcal{A})$  and it always reverses the ordering of a product. All constants, namely the unity or  $\mathcal{I}_i$ 's, should vanish under  $d$ .

All the commutation relations should never mix terms of differential forms of different degrees so that  $\Omega(\mathcal{A})$  is graded. The action of  $d$  is:  $(da) := [d, a] = da - ad$  for forms of even degrees (including elements in  $\mathcal{A}$ ), and  $(da) := \{d, a\} = da + ad$  for forms of odd degrees. Leibniz rule follows this definition. We take

the convention that  $(da)^* = d(a^*)$  for even degrees and  $(da)^* = -d(a^*)$  for odd degrees. That is,  $d$  is anti-self-dual:  $d^* = -d$ . We also require the nilpotency of  $d$ , namely,  $dd \equiv 0$ .

## 2.2 Riemannian Metric, Vector Fields and Tensor Fields

A general coordinate transformation is specified by:  $x^a \rightarrow x'^a$ , where  $x'^a = x'^a(x)$  are elements in  $\mathcal{A}$ . (Einstein's summation convention applies to the whole paper unless otherwise stated.) This transformation induces the transformation of  $\xi^a$ . For example, if

$$\xi^a = \sum f_i^a(x) dg_i^a(x), \quad (1)$$

then

$$\xi^a \rightarrow \xi'^a = \sum f_i^a(x') dg_i^a(x'),$$

where 'a' is not summed over. Re-expressing  $\xi'^a$  in terms of  $\xi^a$  and using commutation relations between  $x^a$  and  $\xi^a$  one can re-write the formula above as:

$$\xi^a \rightarrow \xi'^a = \xi^b M_b^a(x)$$

for a certain matrix  $M_b^a$  of elements in  $\mathcal{A}$ .

Since the transformation is not supposed to change the exterior derivative,  $d = \xi^a \chi_a \rightarrow d' = \xi'^a \chi'_a = d$ , so  $\chi_a \rightarrow M^{-1}{}^b{}_a \chi_b$ .

A Riemannian metric  $g^{ab}(x)$  is an invertible matrix of elements in  $\mathcal{A}$  which

transforms like a rank-two tensor (to be defined later):

$$g^{ab} \rightarrow g'^{ab} = M_c^{*a} g^{cd} M_d^b,$$

and is also Hermitian-symmetric:

$$(g^{ab})^* = g^{ba}. \quad (2)$$

Note that this symmetry is preserved by the transformation. In the classical case there is no need of the  $*$ -involution (complex conjugation) in (2) if all coordinates are real. But if one is allowed to use complex coordinates, for example,  $(x, y) \rightarrow (x + iy, x - iy)$ , then (2) is a reality condition for the Riemannian manifold. The existence of the inverse of  $g^{ab}$  is assumed and it is denoted as  $g_{ab}$  so that:  $g^{ab} g_{bc} = \delta_c^a = g_{cb} g^{ba}$ . The transformation of  $g_{ab}$  follows this definition.

A covariant vector field is a set of elements  $\{\alpha_a\}$  in  $\mathcal{A}$  which transform like  $\{\chi_a\}$ :

$$\alpha_a \rightarrow M_a^{-1b} \alpha_b.$$

Similarly,  $\{\beta^a\}$  is called a contra-variant vector field if it transforms like  $\{\xi^a\}$ :

$$\beta^a \rightarrow \beta^b M_b^a.$$

Note that  $\alpha = \xi^a \alpha_a$  and  $\beta = \beta^a \chi_a$  are both invariant:  $\alpha \rightarrow \alpha$ ,  $\beta \rightarrow \beta$ , so that we can simply use  $\alpha, \beta$  to denote the vector fields  $\{\alpha_a\}$  and  $\{\beta^a\}$  in a coordinate-independent way.

Similarly we can define rank-two tensors of different types according to their transformations:

$$\alpha^{ab} \rightarrow M_c^{*a} \alpha^{cd} M_d^b,$$

$$\alpha_a^b \rightarrow M_a^{-1c} \alpha_c^d M_d^b,$$

$$\alpha^a_b \rightarrow M_c^{*a} \alpha_c^d (M_b^{-1d})^*,$$

$$\alpha_{ab} \rightarrow M_a^{-1c} \alpha_{cd} (M_b^{-1d})^*.$$

Just like in the commutative case, the positions of indices of a tensor tell you the way it transforms. In these formulas, the ordinary contraction of indices is of this type:  $\searrow$ , When indices are contracted as  $\nearrow$ , \*-involution is involved.

Furthermore the Riemannian metric  $g^{ab}$  can be used to raise or lower indices:

$$\alpha^a = \alpha_b^* g^{ba}, \tag{3}$$

$$\alpha_a = g_{ab} (\alpha^b)^*, \tag{4}$$

$$\alpha^{ab} = g^{ac} \alpha_c^b,$$

$$\alpha_{ab} = \alpha_a^c g_{cb},$$

and so on. Because of eq.(2), if we raise and then lower an index we will get back to the original object. It can also be checked that  $(\alpha^a)^* \beta^b$  and  $\alpha_a \beta^b$  are tensors if  $\alpha$  and  $\beta$  are vectors. The contraction of indices of two tensors can make a new tensor. But sometimes the contraction of indices has to be accompanied by

appropriate \*-involution.

As  $\alpha^a\beta_a$  is invariant for any two vector fields  $\alpha$  and  $\beta$ , we can always use the Riemannian metric to define the inner product  $\langle \cdot, \cdot \rangle$  between vector fields  $\alpha$ ,  $\beta$  as in the classical case. For example,

$$\langle \alpha, \beta \rangle = \langle \xi^a \alpha_a, \xi^b \beta_b \rangle = \alpha_a^* \langle \xi^a, \xi^b \rangle \beta_b = \alpha_a^* g^{ab} \beta_b,$$

$$\langle \alpha, \beta \rangle = \langle \alpha^a \chi_a, \beta^b \chi_b \rangle = \alpha^a \langle \chi_a, \chi_b \rangle \beta^{b*} = \alpha^a g_{ab} \beta^{b*}.$$

In both cases,  $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle^*$ .

The magnitude of a vector  $|\alpha|^2 := \langle \alpha, \alpha \rangle$  is real:  $|\alpha|^{2*} = |\alpha|^2$ , due to eq.(2).

The invariant operator

$$\nabla^2 := \chi^a \chi_a = \chi_a^* g^{ab} \chi_b \tag{5}$$

is called the Laplacian. It can be used to define the equation of motion for a scalar field  $\Phi$  with mass  $m$ :

$$(\nabla^2 + m^2)\Phi = 0.$$

*The non-commutativity forbids any tensor of rank higher than two.* Therefore physical laws, if written as equations of motion, can only be written in terms of scalars, vectors, and rank-two tensors. Fortunately all major classical physical laws are governed by tensor equations of rank less than or equal to two.

Given a Hilbert space representation of the algebra  $\mathcal{A}$  on  $\mathcal{H}$ , one can define



the “distance” between two states  $s, s'$  (as generalized points) as [5]:

$$D(s, s') := \sup\{|s(f) - s'(f)|; \quad \|\|df|^2\| \leq 1, f \in \mathcal{A}\}. \quad (6)$$

The definition of  $|df|^2$  is based on the metric  $g^{ab}$  and therefore the metric possesses the classical geometrical meaning. The norm  $\|\cdot\|$  is defined by:

$$\|f\|^2 := \sup\{\frac{\langle \phi | f^* f | \phi \rangle}{\langle \phi | \phi \rangle}; \quad |\phi\rangle \in \mathcal{H}\}.$$

An algebraic version of quantum distance can also be given without mentioning Hilbert space representations. An example is given in Sec.3.

## 2.3 Connection

As it is defined in [1], a connection  $\nabla$  acts on  $f \in \mathcal{A}$  as  $d: \nabla f = df$ , on  $\xi^a$  by the connection one-forms  $\omega_b^a$ :

$$\nabla \xi^a := \xi^b \otimes_{\mathcal{A}} \omega_b^a, \quad (7)$$

and on a one-form  $\alpha = \xi^a \alpha_a$  by Leibniz rule:

$$\begin{aligned} \nabla \alpha &= (\nabla \xi^a) \alpha_a - \xi^a \otimes_{\mathcal{A}} (d\alpha_a) \\ &= -\xi^a \otimes_{\mathcal{A}} (\nabla \alpha)_a, \end{aligned}$$

where  $(\nabla \alpha)_a := d\alpha_a - \omega_a^b \alpha_b$ .

For  $(\nabla \alpha)_a$  to be a covariant vector the connection one-form has the transformation:

$$\omega_a^b \rightarrow M_a^{-1c} \omega_c^d M_d^b - M_a^{-1c} dM_c^b. \quad (8)$$

For the Leibniz rule to hold on the inner product:

$$d\langle\alpha, \beta\rangle = \nabla\langle\alpha, \beta\rangle = (\nabla\alpha)^a\beta_a + \alpha^a(\nabla\beta)_a,$$

we define:

$$(\nabla\alpha)^a := d\alpha^a + \alpha^b\omega_b^a,$$

which also ensures that  $(\nabla\alpha)^a$  is a contra-variant vector.

The covariant derivation of  $\alpha$  in the direction of the vector field  $\beta$  is:

$$(\nabla_\beta\alpha)^a = \langle\beta, (\nabla\alpha)^a\rangle.$$

The equation of geodesic flows is therefore:

$$(\nabla_\alpha\alpha)^a = 0.$$

To define the action of  $\nabla$  on rank-two tensor fields we consider the scalar  $f = \alpha_a^*\gamma^{ab}\beta_b$ . Because  $f$  is a scalar field, we have the equation:  $\nabla f = df$ . By the undeformed Leibnitz rules of  $\nabla$ , we should have for the left hand side (omitting the symbols  $\otimes_{\mathcal{A}}$ ):

$$\begin{aligned} \nabla(\alpha_a^*\gamma^{ab}\beta_b) &= (\nabla\alpha)_a^*\gamma^{ab}\beta_b + \alpha_a^*(\nabla\gamma)^{ab}\beta_b + \alpha_a^*\gamma^{ab}(\nabla\beta)_b \\ &= (d\alpha_a - \omega_a^c\alpha_c)^*\gamma^{ab}\beta_b + \alpha_a^*(\nabla\gamma)^{ab}\beta_b + \alpha_a^*\gamma^{ab}(d\beta_b - \omega_b^c\beta_c), \end{aligned}$$

and for the right hand side:

$$d(\alpha_a^*\gamma^{ab}\beta_b) = d(\alpha_a^*)\gamma^{ab}\beta_b + \alpha_a^*d\gamma^{ab}\beta_b + \alpha_a^*\gamma^{ab}d\beta_b.$$

Identifying them we find:

$$(\nabla\gamma)^{ab} := d\gamma^{ab} + \gamma^{ac}\omega_c^b + (\omega_c^a)^*\gamma^{cb},$$

which also ensures  $(\nabla\gamma)^{ab}$  to be a rank-two tensor.

Suppose one has the physical law  $(\nabla\alpha)^a = \beta^a$ , it is equivalent to  $(\nabla\alpha)_a = \beta_a$

if

$$(\nabla g)^{ab} = dg^{ab} + \omega^{ab} + (\omega^{ba})^* = 0, \quad (9)$$

which is called the metricity condition.

If  $dg^{ab} = 0$ , we have from the metricity condition  $(\omega^{ab})^* = -\omega^{ba}$ , where  $\omega^{ab} := g^{ac}\omega_c^b$ .

The torsion  $T^a$  is the covariant vector defined by [1]:

$$T^a := (d - m \circ \nabla)\xi^a = d\xi^a - \xi^b\omega_b^a,$$

where  $m$  is the multiplication map  $m(\alpha \otimes \beta) := \alpha\beta$ .

In the classical case eq.(9) and

$$T^a = 0, \quad (10)$$

plus the reality conditions imply that the connection one-form  $\omega_a^b$  is uniquely fixed by the metric  $g^{ab}$ . The general expression for the analogous reality conditions in the quantum case is so far unknown. The difficulty is that the general transformation will spoil the reality of a non-invariant quantity. One has to

invent appropriate conditions for each particular case according to its algebraic properties.

For the quantum complex Hermitian manifolds defined later, the situation is much simpler. Just like their classical counterparts, Eq.(9) alone determines the connection uniquely.

## 2.4 Curvature

The curvature two-form is a rank-two tensor defined by:

$$R_a{}^b := d\omega_a{}^b - \omega_a{}^c \omega_c{}^b. \quad (11)$$

Using (9), one can show that

$$(R_{ab})^* = R_{ba}.$$

It is easy to check that the Bianchi identity and consistency condition are satisfied:

$$dR_a{}^b - \omega_a{}^c R_c{}^b + R_a{}^c \omega_c{}^b = 0,$$

$$dT^a = \xi^b R_b{}^a.$$

Classically, in order to have the scalar curvature and Ricci tensor one usually just strips the differential forms from the curvature two-form  $R_a{}^b$  to get  $R_a{}^b{}_{cd}$  and then contracts  $b, d$  for Ricci tensor, and contracts in addition  $a, c$  for the

scalar curvature. However in the deformed case this kind of operation is not covariant under general transformations.

Another more elegant way of defining the classical scalar curvature and the Ricci tensor is to use the Hodge-\*. In the quantum case, The Hodge-\* is required to satisfy

$$*(f\alpha g) = f(*\alpha)g \quad \forall f, g \in \mathcal{A}, \alpha \in \Omega(\mathcal{A}) \quad (12)$$

and

$$(*\alpha)^* = *(\alpha^*) \quad \forall \alpha \in \Omega(\mathcal{A}), \quad (13)$$

so that the scalar curvature defined as

$$\mathcal{R} := (-1)^{D+1} * (\xi^a (*R_a^b) \xi_b) \quad (14)$$

is invariant under general transformations ( $D$  is the dimension of the space) and real ( $\mathcal{R}^* = \mathcal{R}$ ). The integral

$$\int \xi^a (*R_a^b) \xi_b \quad (15)$$

is a candidate for the action of a gravitational theory on the quantum space.

Similarly one can try to define the Ricci tensor as

$$\mathcal{R}_a^b := *((*R_a^c) \xi_c \xi^b).$$

There are, however, many other inequivalent expressions that are covariant under the general transformations. For example, it is equally justified to define

the Ricci tensor as

$$\mathcal{R}_a{}^b := *(\xi_a \xi^c (*R_c{}^b)).$$

This ambiguity in the Ricci tensor makes the scalar curvature better for physical applications.

One can define the operator  $\delta := -*d*$  for a quantum space, and naturally one will define the Laplacian by  $\nabla^2 := -(d + \delta)^2$ , which is equivalent to  $-\delta d$  when acting only on functions if  $\delta\delta = 0$ . In such cases the metric is determined by the Hodge-\* according to (5).

## 2.5 Complex Manifolds

We define quantum complex manifolds (more precisely, Hermitian manifolds) to be an associative \*-algebra  $\mathcal{A}$  generated by  $\{1, z^a, \bar{z}^{\bar{a}}\}$  together with its differential calculus  $\Omega(\mathcal{A})$  generated by  $\{1, z^a, \bar{z}^{\bar{a}}, dz^a, d\bar{z}^{\bar{a}}, \partial_a, \bar{\partial}_{\bar{a}}\}$  with the following properties:

1.  $d = \delta + \bar{\delta}$ , where  $\delta = dz^a \partial_a$  and  $\bar{\delta} = \bar{\partial}_{\bar{a}} d\bar{z}^{\bar{a}}$  with  $\delta\delta = \bar{\delta}\bar{\delta} = 0$  and  $\delta\bar{\delta} = -\bar{\delta}\delta$ .  
 $\delta$  and  $\bar{\delta}$  should observe Leibniz rule separately, and  $(\delta\alpha)^* = (-1)^p \bar{\delta}(\alpha^*)$  for any form  $\alpha$  of degree  $p$ .
2. The generators of the algebra are divided into the holomorphic part  $\{z^a\}$  and the anti-holomorphic part  $\{\bar{z}^{\bar{a}} = (z^a)^*\}$ . The one-forms  $\{dz^a\}$  and  $\{d\bar{z}^{\bar{a}}\}$  are all independent.

3. Denote  $\Omega^+(\mathcal{A}) := \{dz^a \alpha_a : \alpha_a \in \mathcal{A}\}$  and  $\Omega^-(\mathcal{A}) := \{\alpha_{\bar{a}} d\bar{z}^{\bar{a}} : \alpha_{\bar{a}} \in \mathcal{A}\}$ . The commutation relations between one-forms and functions in  $\mathcal{A}$  are such that  $\Omega^\pm(\mathcal{A})\mathcal{A} = \mathcal{A}\Omega^\pm(\mathcal{A})$ . That is,  $\Omega^\pm(\mathcal{A})$  do not get mixed by commutation.
4. The metric  $g^{\bar{a}b} = (g^{\bar{b}a})^*$  is given. The connection one-forms  $\omega_a^b$  and  $\omega_{\bar{a}}^{\bar{b}} := (\omega_a^b)^*$  are also given such that  $\omega_a^b \in \Omega^+(\mathcal{A})$  and  $\omega_{\bar{a}}^{\bar{b}} \in \Omega^-(\mathcal{A})$ . Leibniz rule holds for the connection  $\nabla$  and we have:  $\nabla dz^a = dz^b \otimes_{\mathcal{A}} \omega_b^a$  and  $\nabla d\bar{z}^{\bar{a}} = -d\bar{z}^{\bar{b}} \otimes_{\mathcal{A}} \omega_{\bar{b}}^{\bar{a}}$ .

The coordinate transformations are restricted to the holomorphic transformations only. Holomorphic transformations are defined as those which maps  $z^a$  to holomorphic functions  $f^a(z)$  and  $\bar{z}^{\bar{a}} = (z^a)^*$  to  $(f^a(z))^*$ . The properties of a complex manifold imply that this transformation induces the map  $dz^a \rightarrow dz^b M_b^a$  where  $M_a^b$  is holomorphic, namely,  $\bar{\delta} M_a^b = 0$ . Similarly, we have  $d\bar{z}^{\bar{a}} \rightarrow (dz^b M_b^a)^* = (M_a^b)^* d\bar{z}^{\bar{b}}$  where  $(M_a^b)^*$  is anti-holomorphic,  $\delta(M_a^b)^* = 0$ .

All the formulas we had before for Riemannian manifolds can be easily modified for a complex manifold with the understanding that the indices are only summed over the holomorphic or anti-holomorphic part.

With the fourth property of a complex manifold eq.(9) says:

$$dg^{\bar{a}b} + \omega_{\bar{c}}^{\bar{a}} g^{\bar{c}b} + g^{\bar{a}c} \omega_c^b = 0.$$

Since  $d = \delta + \bar{\delta}$  we can separate the equation into  $\Omega^+(\mathcal{A})$  and  $\Omega^-(\mathcal{A})$ , and so the connection can be directly solved:

$$\omega_a{}^b = -g_{a\bar{c}}(\delta g^{\bar{c}b}). \quad (16)$$

Only the holomorphic transformations will be consistent with this solution. (That is, the transformation of the connection induced from the transformation of the metric by these expressions will be the same as (8) only for holomorphic transformations.)

From eq.(11) the curvature two-form can now be expressed directly in terms of the metric as:

$$R_a{}^b = \bar{\delta}\omega_a{}^b = -\bar{\delta}(g_{a\bar{c}}\delta g^{\bar{c}b}). \quad (17)$$

The curvature two-form gives the scalar curvature according to (14) with the indices restricted to the holomorphic or anti-holomorphic part, and so it is in general not invariant under general coordinate transformations. The scalar curvature for a quantum complex manifold is

$$\mathcal{R} = (-1)^{D+1} * (dz^a(*R_{a\bar{b}})d\bar{z}^b). \quad (18)$$

The definition of the Ricci tensor is not unique.

The condition (12) for the Hodge  $*$  can be weakened to be no more than an ordering prescription:

$$*(f(z)\alpha g(\bar{z})) = f(z)(* \alpha)g(\bar{z}).$$



Due to the extensive use of the  $*$ -involution in the quantum case, we see that the complex structure helps to admit a Riemannian structure.

### 3 The Quantum Sphere $S_q^2$

The Riemannian structure on a quantum space with a quantum group symmetry should respect its quantum symmetry.

In this section we describe one particular quantum sphere  $S_q^2$  in the family of quantum spheres of Podleś [7] with the  $SU_q(2)$  symmetry (the one with  $c = 0$  in his notation) in terms of the stereographic projection coordinates [2] as an example of both Riemannian manifolds and complex manifolds. For the convenience of the reader we review the stereographic projection of  $S_q^2$  [2] in the following section.

#### 3.1 The Stereographic Projection of $S_q^2$

$S_q^2$  is the homogeneous space of  $SU_q(2)/U(1)$  for  $0 \leq q \leq 1$ . The algebra and differential calculus of  $S_q^2$  in terms of the complex coordinates  $(z, \bar{z})$  can be induced from those of  $SU_q(2)$  by the identification:

$$z := \alpha\gamma^{-1}, \quad \bar{z} := -\delta\beta^{-1}, \tag{19}$$

where  $\alpha, \beta, \gamma, \delta$  are the elements of an  $SU_q(2)$ -matrix  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . Classically  $z, \bar{z}$  are the stereographic projection coordinates projected from the north pole onto the tangent plane at the south pole.

The commutation relation between  $z, \bar{z}$  is therefore:

$$q(1 + z\bar{z}) = q^{-1}(1 + \bar{z}z),$$

and the differential calculus induced from the left-covariant 3D calculus on  $SU_q(2)$  [9], which is equivalent to one of the possible differential structures on  $S_q^2$  studied by Podleś [8], is specified by the following list of commutation relations:

$$\begin{aligned} z dz &= q^{-2} dz z, & \bar{z} dz &= q^2 dz \bar{z}, \\ z d\bar{z} &= q^{-2} d\bar{z} z, & \bar{z} d\bar{z} &= q^2 d\bar{z} \bar{z}, \\ dz dz &= d\bar{z} d\bar{z} = 0, & dz d\bar{z} &= -q^{-2} d\bar{z} dz, \\ \partial z &= 1 + q^{-2} z \partial, & \partial \bar{z} &= q^2 \bar{z} \partial, \\ \bar{\partial} z &= q^{-2} z \bar{\partial}, & \bar{\partial} \bar{z} &= 1 + q^2 \bar{z} \bar{\partial}, \\ \partial \bar{\partial} &= q^{-2} \bar{\partial} \partial, \end{aligned} \tag{20}$$

where  $d = dz\partial + d\bar{z}\bar{\partial}$  when acting on functions

The  $*$ -involution on this whole algebra also follows from that of  $SU_q(2)$ :

$$z^* = \bar{z},$$

$$(dz)^* = d\bar{z},$$

$$\partial^* = -q^{-2}(1 + \bar{z}z)^2 \bar{\partial}(1 + \bar{z}z)^{-2},$$

$$\bar{\partial}^* = -q^2(1 + \bar{z}z)^2 \partial(1 + \bar{z}z)^{-2}.$$

The left  $SU_q(2)$  transformation on  $SU_q(2)$  induces an  $SU_q(2)$  transformation on  $S_q^2$ :

$$z \rightarrow (az + b)(cz + d)^{-1},$$

$$\bar{z} \rightarrow (-c + d\bar{z})(a - b\bar{z})^{-1},$$

$$dz \rightarrow dz(q^{-1}cz + d)^{-1}(cz + d)^{-1},$$

$$d\bar{z} \rightarrow d\bar{z}(a - qb\bar{z})^{-1}(a - b\bar{z})^{-1},$$

$$\partial \rightarrow (cz + d)(q^{-1}cz + d)\partial,$$

$$\bar{\partial} \rightarrow (a - b\bar{z})(a - qb\bar{z})\bar{\partial},$$

where  $a, b, c, d$  are elements of an  $SU_q(2)$ -matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  commuting with  $z, \bar{z}, dz, d\bar{z}, \partial, \bar{\partial}$ . The above is a simpler notation of the equivalent one for left-coaction, e.g.,  $\Delta_L(z) = (a \otimes z + b \otimes 1)(c \otimes z + d \otimes 1)^{-1}$ .

The integration on  $S_q^2$ , which is denoted by  $\langle \cdot \rangle_{S_q^2}$ , can be defined as the integration on  $SU_q(2)$  by restricting the integrand to be an element in  $S_q^2$  and re-expressing it through (19).

It is justified to call  $dzd\bar{z}(1 + \bar{z}z)^{-2}$  the volume form in the sense that if one treats the same whole algebra as the algebra on a quantum plane so that everything else remains unchanged except  $\partial^* = -q^2\bar{\partial}$ , the translational invariant integration on this plane  $\int dzd\bar{z}$  is related to the integration  $\langle \cdot \rangle_{S_q^2}$  by:

$$\int dzd\bar{z}(1 + \bar{z}z)^{-2}f = \langle f \rangle_{S_q^2} \quad (21)$$

(up to normalization).

Details of everything above in this subsection can be found in [2]. One can check that all the requirements of a complex manifold are met.

## 3.2 $S_q^2$ As a Complex Manifold

In this section we will treat  $S_q^2$  as a quantum complex manifold.

### 3.2.1 Metric And Connection

The notation itself suggests that we take  $\Omega^+(S_q^2) = S_q^2 dz$  and  $\Omega^-(S_q^2) = S_q^2 d\bar{z}$ .

Let the only possible value for an index be 0, that is,  $z^0 := z$  and  $\bar{z}^0 := \bar{z}$ . Also denote  $g := g^{\bar{0}0}$  and  $g^{-1} := g_{0\bar{0}}$ . To define the metric  $g$  we note that for  $S_q^2$  the Laplacian, the  $SU_q(2)$ -invariant derivation of order two, is

$$\nabla^2 = -c(1 + \bar{z}z)^2 \bar{\partial}\partial, \quad (22)$$

where  $c$  is an arbitrary real number. On the other hand, the Laplacian of  $S_q^2$  as a complex manifold, the holomorphic-transformation independent derivation of

order two, is  $\partial^*g\partial$ . Equating these two expressions one gets

$$g = q^2c(1 + \bar{z}z)^2.$$

Because the factor  $(1 + \bar{z}z)$  will appear frequently, we shall denote it in the following by:

$$\rho := 1 + \bar{z}z.$$

Classically any two-dimensional complex manifold is also a Kähler manifold and one can locally find a Kähler potential. Analogy can be made here. Define the Kähler form to be  $K = dzg^{-1}d\bar{z}$ . (It plays a special role in the differential calculus [2].) Obviously  $dK = 0$  for the same reason as in the classical case. The Kähler potential  $V$  defined by  $\delta\bar{\delta}V = K$  therefore exists. One can solve  $V$  in term of the deformed log:

$$\begin{aligned} V &= q^{-4}c^{-1} \sum_{n=0}^{\infty} \log_{q^{-1}}(1 - \rho^{-1}) \\ &= q^{-4}c^{-1} \sum_{n=0}^{\infty} \frac{\rho^{-(n+1)}}{[n+1]_{q^{-1}}}, \end{aligned}$$

where  $[n]_q = \frac{q^{2n}-1}{q^2-1}$ . In fact, the Kähler form  $dzg^{-1}d\bar{z}$  is just the volume form (up to normalization).

Using (16) and (17), we can immediately find the connection form and the curvature two-form:

$$\omega_0^0 = -q^2(1 + q^2)dz\rho^{-1}\bar{z}, \tag{23}$$

$$R_0^0 = q^4(1 + q^2)dzd\bar{z}\rho^{-2}. \tag{24}$$

It is easy to see that the torsion is zero in this case.

We define the vielbeins on  $S_q^2$  as

$$e := \rho^{-1} dz, \quad \bar{e} := \rho^{-1} d\bar{z}. \quad (25)$$

The commutation relations between  $e, \bar{e}$  and  $z, \bar{z}$  are simply classical:

$$ez = ze, \quad e\bar{z} = \bar{z}e,$$

$$\bar{e}z = z\bar{e}, \quad \bar{e}\bar{z} = \bar{z}\bar{e}.$$

The Hodge-\* map satisfying (13) is given by

$$(*e) = ie, \quad (*\bar{e}) = -i\bar{e},$$

$$(*1) = ic'^{-1}e\bar{e}, \quad (*e\bar{e}) = -ic', \quad (26)$$

where  $c'$  is a constant, and the action of Hodge-\* on any form follows (12).

Now we can define  $\delta := - * d *$  and  $\nabla^2 := -\frac{1}{2}(d + \delta)^2$ . \* When acting on a function  $f$

$$\begin{aligned} \nabla^2 f &= \frac{1}{2} * d * df \\ &= -c' \rho^2 \bar{\delta} \delta f. \end{aligned}$$

Hence  $c'$  should be identified with  $c$  because of (22). This identification further justifies our Hodge-\* structure.

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\*Here we have a factor of  $\frac{1}{2}$  because we want to identify  $\nabla^2$  with (22), which is only the holomorphic part in  $-(d + \delta)^2$ .

The scalar curvature is found to be

$$\mathcal{R} = cq^2(1 + q^2). \quad (27)$$

The Ricci tensor defined by (2.4) is

$$\mathcal{R}_0^0 = cq^4(1 + q^2). \quad (28)$$

### 3.3 $S_q^2$ As a Riemannian Manifold

In this section we treat  $S_q^2$  as a (real) Riemannian manifold. The difference between Riemannian and complex manifolds is that the latter is not invariant under general coordinate transformation. When indices are contracted for complex manifolds they are summed over only half (the holomorphic part) of the possible values for an ordinary Riemannian manifold.

Assuming that  $g^{00} = g^{\bar{0}\bar{0}}$  in the Riemannian case (since we had  $g^{\bar{0}\bar{0}} = g^{00}$  in the complex case), we get from (22)

$$g^{00} = g^{\bar{0}\bar{0}} = \frac{c}{1 + q^{-2}}\rho^2.$$

Note that since the normalization of  $g$  is changed from Sec.(3.2.1), the parameter  $c'$  used in (26) should be changed accordingly.

The equation  $\nabla g^{00} = 0$  is identical to the one solved earlier for complex  $S_q^2$  and we assign  $\omega_0^0$  to be the same as (23). Let

$$\omega_{\bar{0}}^{\bar{0}} = -(1 + q^{-2})d\bar{z}\rho^{-1}z,$$

which is the complex manifold connection form  $\omega_0^0$  had we labelled the coordinates the opposite way:  $\bar{z}^0 := z$ , and  $z^0 := \bar{z}$ . It solves  $\nabla g^{\bar{0}0} = 0$ .

Similarly, the curvature two-form  $R_0^0$  is given by (24) and  $R_{\bar{0}}^{\bar{0}}$  is

$$R_{\bar{0}}^{\bar{0}} = \delta\omega_{\bar{0}}^{\bar{0}} = (1 + q^{-2})d\bar{z}dz\rho^{-2}.$$

A straightforward calculation shows:

$$\mathcal{R}_0^0 = \frac{cq^4}{4}(1 + q^2)^2,$$

$$\mathcal{R}_{\bar{0}}^{\bar{0}} = \frac{cq^{-2}}{4}(1 + q^2)^2,$$

$$\mathcal{R} = \frac{c}{4}(1 + q^2)^3.$$

This is different from (27) and (28) by constant factors. The reason is that the Riemannian structure of  $S_q^2$  as a Riemannian manifold is concerned with the general transformation and that of  $S_q^2$  as a complex manifold is concerned only with the holomorphic transformations. Unlike the situation in the classical case, without leaving the holomorphic description of the complex manifold  $S_q^2$  one will never be able to know its Riemannian structure. The discrepancy is introduced by the non-commutativity of the algebra.

### 3.3.1 Distances on $S_q^2$

In this section we consider the “distance” between “points” on  $S_q^2$ . As mentioned earlier, (6) can be used to define a number called the *distance* between any two



states.

Before finding the distance we display a representation of  $S_q^2$  which shows clearly the correspondence between states and points. The basis of the Hilbert space is labelled as  $|k, \theta\rangle$  for  $k = 0, 1, 2, \dots, \infty$ ,  $\theta \in [0, 2\pi)$ . (It is an irreducible representation given in [7] for  $S_q^2$  supplemented with an additional index  $\theta$ . It is also equivalent by Fourier transform to an irreducible representation given in [9] for  $SU_q(2)$ .) The algebra is represented in the following way:

$$\begin{aligned} z|k, \theta\rangle &= e^{i\theta}(q^{-2k} - 1)^{1/2}|k - 1, \theta\rangle, \\ \bar{z}|k, \theta\rangle &= e^{-i\theta}(q^{-2(k+1)} - 1)^{1/2}|k + 1, \theta\rangle. \end{aligned}$$

So we have

$$\rho|k, \theta\rangle = q^{-2k}|k, \theta\rangle.$$

Roughly speaking,  $\theta$  corresponds to the azimuthal angle on  $S_q^2$  and  $\frac{q^{2k}-1}{q^{2k+1}}$  corresponds to the cosine of the polar angle.

However, for what follows it is not necessary to specify the representation. The only thing we need is  $Sp(\rho)$ , the spectrum of  $\rho$ , which follows the commutation relations

$$z\rho = q^{-2}\rho z, \quad \bar{z}\rho = q^2\rho\bar{z},$$

and that  $\rho = 1 + \bar{z}z \geq 1$ . It is easy to see that  $Sp(\rho) = \{q^{-2k}; k = 0, 1, 2, \dots\}$ .

Hence in the following  $\theta$  is interpreted as the collection of all parameters except  $k$ , which labels the eigenvalue of  $\rho$ .

Now we consider the distance between the two states  $|k, \theta\rangle$  and  $|k', \theta\rangle$ . In the classical limit, it is just the radius ( $\frac{1}{2}$ ) times the difference in their polar angles:

$$D(p, p') = |F(z(p), \bar{z}(p)) - F(z(p'), \bar{z}(p'))|,$$

where  $F(z, \bar{z}) = \frac{1}{2} \cos^{-1} \left( \frac{z\bar{z}-1}{z\bar{z}+1} \right) = \sin^{-1}(\rho^{-1/2})$ . For convenience we shall suppress the index  $\theta$  of a state from now on. It is fixed for all consideration below.

Given the distance function  $F$ , we can always decompose  $F$  as  $F = f(\rho) + h(z, \bar{z})$ , where  $h = \sum_{n=1}^{\infty} f_n(\rho)z^n + g_n(\rho)\bar{z}^n$ . Since  $\langle k|h|k\rangle = 0$  for all  $k$ , if  $F$  gives the distance between states  $|k\rangle$  and  $|k'\rangle$  then  $f$  does, too. But we have to check that the magnitude of  $df$  is not larger than 1. Note that

$$|dF|^2 = |df|^2 + |dh|^2 + (CT),$$

where the cross-terms are

$$(CT) = (\partial f)^* g(\partial h) + (\bar{\partial} f)^* g(\bar{\partial} h) + (\partial h)^* g(\partial f) + (\bar{\partial} h)^* g(\bar{\partial} f).$$

Since  $\langle p|(CT)|p\rangle = 0$ ,

$$\langle p| |df|^2 |p\rangle \leq \langle p| |dF|^2 |p\rangle \leq \| |dF|^2 \| \leq 1, \quad \forall p,$$

which implies that  $\| |df|^2 \| \leq 1$ . Hence we only have to consider functions of  $\rho$  for our purpose.

Assume that the distance function is  $F(\rho)$  with  $|dF|^2 = 1$ . Because  $(\partial \rho^n) =$

$q\lambda^{-1}\rho^{-1}((q^2\rho)^n - \rho^n)\bar{z}$ , where  $\lambda = q - q^{-1}$ , we have

$$(\partial F(\rho)) = q\lambda^{-1}\rho^{-1}(F(q^2\rho) - F(\rho))\bar{z}. \quad (29)$$

Similarly,

$$(\bar{\partial} F(\rho)) = -q\lambda^{-1}\rho^{-1}(F(q^{-2}\rho) - F(\rho))z. \quad (30)$$

Therefore,

$$\begin{aligned} |dF|^2 &= (\partial F)^* g^{00}(\partial F) + (\bar{\partial} F)^* g^{\bar{0}\bar{0}}(\bar{\partial} F) \\ &= \frac{cq^4}{\lambda^2(1+q^{-2})}(W(q^{-2}\rho) + W(\rho)), \end{aligned}$$

where  $W(\rho) = (\rho - 1)(F(q^2\rho) - F(\rho))^2$ .  $|dF|^2 = 1$  implies that  $W(\rho)$  can only be the constant  $\frac{1}{2}c^{-1}q^{-4}\lambda^2(1+q^{-2})$ . Hence,

$$F(q^2\rho) - F(\rho) = -\lambda \left( \frac{2cq^4(\rho - 1)}{1 + q^{-2}} \right)^{-1/2} \quad (31)$$

and so  $F$  can be solved as a power series expansion:

$$F(\rho) = - \left( \frac{1 + q^{-2}}{2cq^2} \right)^{1/2} \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 [n + 1/2]_{q^{-1}}} \rho^{-n-1/2}.$$

This is not the only solution of (31). Any function  $f(\rho)$  satisfying  $f(q^2\rho) = f(\rho)$  can be added to it and (31) still holds. However, due to the structure of  $Sp(\rho)$ , such functions will not contribute to the distance between  $|k\rangle$  and  $|k'\rangle$ . For  $q = 1$ ,  $c = 4$ , this solution is the power series expansion of  $-\sin^{-1}(\rho^{-1/2})$ .

It remains to argue that that the assumption  $|dF|^2 = 1$  is correct. Consider a function  $f(\rho)$  with  $|df|^2 < 1$ . Then  $F' := f + \epsilon F$  has  $|dF'|^2 < 1$  for  $|\epsilon|$

sufficiently small. And an appropriate phase of  $\epsilon$  can make  $|F'(k) - F'(k')| > |f(k) - f(k')|$ . So any  $f$  with  $|df|^2 < 1$  is not the distance function.

Therefore we have the lemma:

**Lemma 1** *The distance between states  $|m, \theta\rangle$  and  $|n, \theta\rangle$ ,  $m \geq n$ , according to (6) is equal to  $F(q^{-2m}) - F(q^{-2n})$ .*

The distance between the north pole and the south pole on  $S_q^2$ , for example, can be expressed as

$$F(\infty) - F(1) = \left( \frac{1 + q^{-2}}{2cq^2} \right)^{1/2} \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 [n + 1/2]_{q^{-1}}},$$

which is the deformed  $\pi/2$ .

The distance between any two points can be obtained, by using the quantum group symmetry of  $SU_q(2)$ , from the distance between the north pole ( $z = \infty$ ) and an arbitrary point, which we have just obtained above.

Using the commutation relations of  $z, \bar{z}$ , one can check that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} z\rho^{-1/2} & -q\rho^{-1/2} \\ \rho^{-1/2} & \rho^{-1/2}\bar{z} \end{pmatrix} \quad (32)$$

is an  $SU_q(2)$ -matrix. This matrix transforms the north pole ( $z = \infty$ ) to  $z$ . It also transforms the point

$$z'' := (\delta z' - q^{-1}\beta)(-q\gamma z' + \alpha)^{-1}$$

to  $z'$ . The quantum group symmetry tells us that the distance between  $z$  and  $z'$  is the same as the distance between the north pole and  $z''$ , which is a function of  $z, z'$ . Therefore we have:

**Proposition 1** *The distance between  $(z, \bar{z})$  and  $(z', \bar{z}')$  on  $S_q^2$  is  $|F(\rho'')|$ , where  $\rho'' = (1 + z''\bar{z}'') = (1 + z\bar{z})(1 + z'\bar{z}')(z - z')^{-1}(\bar{z} - \bar{z}')^{-1}$ .*

Note that as the coordinates of points on the same sphere the commutation relation between  $z$  and  $z'$  should be that of the standard braiding [3],

$$zz' = q^2 z'z - q\lambda z'^2,$$

which is covariant under simultaneous  $SU_q(2)$  transformation on  $z$  and  $z'$ . This implies that  $z$  and  $z''$  simply commute with each other. (The braiding is also formally satisfied by  $(\infty, z'')$ . Divide the braiding relation on both sides by  $z$  we get  $z'z^{-1} = q^2 z^{-1}z' - q\lambda z^{-1}z'^2 z^{-1}$  which is satisfied by  $(z, z') = (\infty, z'')$  but not by  $(z, z') = (z'', \infty)$ .)

A state  $|s\rangle$  in the Hilbert space representation of the braided algebra generated by  $\{z, \bar{z}, z', \bar{z}'\}$  corresponds to two “points” on  $S_q^2$ . So the distance between them is  $\langle s|F(\rho'')|s\rangle$ . This is a modification of A. Connes’ formula (6) requested by the braiding.

### 3.3.2 Connection with Connes' Formulation

Here we make a connection with A. Connes' quantum Riemannian geometry [5] by re-formulating the quantum sphere in a way as close to his as possible.

To do so we consider the Hilbert space realization of  $\Omega(S_q^2)$ . The Hilbert space representation presented here is composed of two parts. The first part  $\{|\psi\rangle\}$  is the Hilbert space representing the algebra generated by  $z, \bar{z}$ . An example is given in Sec.3.3.1. Another example is the GNS construction using the integration  $\langle \cdot \rangle$ . The second part  $V$  is a vector space of, say, 2-component column vectors representing the differential forms. The differential calculus can then be represented in terms of the representation  $\pi$  of  $S_q^2$  as (for  $v \in V$ ):

$$\pi(dz)|\psi\rangle \otimes v = \sqrt{c(1+q^2)}\pi(\rho)|\psi\rangle \otimes \tau v,$$

$$\pi(d\bar{z})|\psi\rangle \otimes v = \sqrt{c(1+q^2)}\pi(\rho)|\psi\rangle \otimes \tau^\dagger v,$$

where  $\tau := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\tau^\dagger := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ , and they satisfy:  $q\tau\tau^\dagger + q^{-1}\tau^\dagger\tau = \mathcal{I}$  for

$\mathcal{I} := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ . The vielbeins  $e$  and  $\bar{e}$  (25) are represented by the  $\gamma$ -matrices

$\tau$  and  $\tau^\dagger$ , which satisfy a deformed Clifford algebra. The column  $v$  is used to specify the direction of a cotangent vector at a "point" on  $S_q^2$ .

Let the Dirac operator be

$$\mathcal{D} := k \begin{pmatrix} i & \bar{z} \\ -z & -i \end{pmatrix},$$

where  $k = q\lambda^{-1}\sqrt{c(1+q^2)}$ . It is chosen such that  $dz = [D, z]$  and  $d\bar{z} = [D, \bar{z}]$ .

The goal is that the exterior derivative is realized by  $D$ .

Since  $D^2$  is not central,  $(d^2\alpha) = [D^2, \alpha]$  for a form  $\alpha$  is non-zero. The nilpotency is achieved by taking the quotient of the algebra over the ideal called the auxiliary fields. They are the differential forms  $\{a[D^2, b]c; a, b, c \in \mathcal{A}\}$ . For our case the auxiliary fields are found to be

$$a \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

for all functions  $a$  in  $S_q^2$ .

The calculus is  $\mathbf{Z}_2$ -graded by

$$\gamma := k^{-2}(dzd\bar{z} - d\bar{z}dz)\rho^{-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The  $SU_q(2)$ -invariant integration on two-forms can be defined by the trace:

$$\int \alpha := Tr(\gamma\alpha|\mathcal{D}|^{-2}),$$

where  $Tr$  is the trace over the extended Hilbert space  $\{|\psi\rangle \otimes v\}$ . Although this formula resembles that of A. Connes for 2-dimensional integration [5], according

to him the power of  $|\mathcal{D}|^{-1}$  in the trace is determined by the spectrum of  $\mathcal{D}$ . In the classical case one gets the classical dimension, but we get zero in this particular case. Therefore unlike Connes' integration this one does not have the cyclic property  $\int \alpha\beta = \int \beta\alpha$ . Nevertheless, it can be directly checked that the integration satisfies the consistency condition

$$\int Aux = 0 \tag{33}$$

for auxiliary fields, and the Stoke's theorem

$$\int d\alpha = 0 \tag{34}$$

for any one-form  $\alpha$ . Stoke's theorem can be used to derive recursion relations for the integration of two-forms. Equations (33) and (34) vanish already on the trace over the  $2 \times 2$  matrices and hence remain so for any representation  $|\psi\rangle$  of the algebra  $S_q^2$ . They determine up to normalization the integration of two-forms. Hence it agrees with the integration introduced before (21).

## 4 The Complex Quantum Projective Spaces $CP_q(N)$

The results in Sec.3 can be generalized to the quantum projective spaces  $CP_q(N)$  [4]. ( $S_q^2 \sim CP_q(1)$ .) In Sec.4.1 we consider the Hodge  $*$  map. In Sec.4.2 we find the Riemannian structure on  $CP_q(N)$ .



## 4.1 The Construction of the Hodge $*$ Map

A prerequisite of the Riemannian structure is the Hodge  $*$  map. In general, if there exists for a quantum complex manifold a Kähler form  $K = dz^a g_{a\bar{b}} d\bar{z}^{\bar{b}}$  which is real and central, a Hodge  $*$  satisfying (12) and (13) can always be constructed. Let

$$*(dz^{a_1} \cdots dz^{a_p}) := dz^{a_1} \cdots dz^{a_p} K^{N-p},$$

$$*(d\bar{z}^{\bar{a}_r} \cdots d\bar{z}^{\bar{a}_1}) := K^{N-r} d\bar{z}^{\bar{a}_r} \cdots d\bar{z}^{\bar{a}_1}.$$

Since  $K$  is central, the property (12) is satisfied. Since  $K$  is real, (13) is also satisfied. Now we consider the Hodge  $*$  of a differential form which is not purely holomorphic and antiholomorphic. The idea is to “patch” the holomorphic part and the antiholomorphic part together.

Denote

$$\xi_a := g_{a\bar{b}} d\bar{z}^{\bar{b}}.$$

Because  $K = dz^a \xi_a$  is central,

$$K^p = (dz^{a_1} \cdots dz^{a_p})(\xi_{a_p} \cdots \xi_{a_1}).$$

So we have

$$*(dz^{a_1} \cdots dz^{a_p}) = (dz^{a_1} \cdots dz^{a_N})(\epsilon_{a_1 \cdots a_N} \xi_{a_N} \cdots \xi_{a_{p+1}}),$$

where  $\epsilon_{a_1 \cdots a_N}$  is defined by

$$dz^{a_1} \cdots dz^{a_p} = \epsilon_{a_1 \cdots a_N} dz^{a_1} \cdots dz^{a_N}.$$

Similarly,

$$*(d\bar{z}^{\bar{a}_r} \cdots d\bar{z}^{\bar{a}_1}) = (\epsilon_{a_1 \cdots a_N} \eta_{\bar{a}_{r+1}} \cdots \eta_{\bar{a}_N})(d\bar{z}^{\bar{N}} \cdots d\bar{z}^{\bar{1}}),$$

where

$$\eta_{\bar{a}} := (\xi_a)^* = dz^b g_{b\bar{a}}.$$

Let  $\mu(z, \bar{z})$  be the real function defined by the volume form:

$$K^N = dz^1 \cdots dz^N \mu(z, \bar{z}) d\bar{z}^{\bar{N}} \cdots d\bar{z}^{\bar{1}}. \quad (35)$$

Then we define

$$*(d\bar{z}^{\bar{b}_1} \cdots d\bar{z}^{\bar{b}_r} dz^{a_1} \cdots dz^{a_p}) := (\epsilon_{b_1 \cdots b_N} \eta_{\bar{b}_{r+1}} \cdots \eta_{\bar{b}_N}) \mu^{-1}(\epsilon_{a_1 \cdots a_N} \xi_{a_N} \cdots \xi_{a_{p+1}}), \quad (36)$$

Roughly speaking, we put the Hodge  $*$  of the antiholomorphic part and that of the holomorphic part together, and then take out from the middle the volume form (35). It can be shown that the properties (12) and (13) hold.

The commutativity of the Hodge  $*$  with all functions (12) is not necessary for the invariance of the scalar curvature. As mentioned in Sec.2.5, simply a prescription of ordering  $*(f(z)\alpha g(\bar{z})) = f(z)(* \alpha)g(\bar{z})$  is sufficient. Its significance is that any other prescription of ordering, say,  $*(f(\bar{z})\alpha g(z)) = f(\bar{z})(* \alpha)g(z)$ , gives the same result.

This construction of the Hodge  $*$  map is, however, not unique. When one patches the holomorphic and antiholomorphic parts as in (36), one can choose

to put the holomorphic part before or after the antiholomorphic part. They are in general inequivalent. There is also the freedom to normalize (36) by different constant factors for each pair of  $(p, r)$ .

## 4.2 The Riemannian Structure on $CP_q(N)$

The algebra of  $CP_q(N)$  [4] is given by the commutation relations:

$$\begin{aligned} z^a z^b &= q^{-1} \hat{R}_{cd}^{ab} z^c z^d, \\ \bar{z}^a z^b &= q^{-1} (\hat{R}^{-1})_{ac}^{bd} z^c \bar{z}^d - q^{-1} \lambda \delta_a^b, \\ z^a dz^b &= q \hat{R}_{cd}^{ab} dz^c z^d, \\ \bar{z}^a dz^b &= q^{-1} (\hat{R}^{-1})_{ac}^{bd} dz^c \bar{z}^d, \end{aligned}$$

where  $\hat{R}$  is the  $\hat{R}$ -matrix of  $GL_q(N)$  [6]. The  $*$ -involution is  $z^{a*} = \bar{z}^a$ .

The Kähler form  $K = dz^a g_{a\bar{b}} d\bar{z}^b$  for  $CP_q(N)$  [4] is given by the deformed Fubini-Study metric

$$g_{a\bar{b}} = q^{-1} \rho^{-2} (\rho \delta_{ab} - q^2 \bar{z}^a z^b),$$

where  $\rho = 1 + \sum_{a=1}^N z^a \bar{z}^a$ . The inverse of the metric is  $g^{\bar{a}b} = q\rho(\delta_{ab} + \bar{z}^a z^b)$ . This Kähler form is not only real and central, but also invariant under the quantum group transformation.

$$z^a \rightarrow (T_0^0 + z^b T_b^0)^{-1} (T_0^a + z^c T_c^a),$$

where  $T_a^b$  is an  $SU_q(N + 1)$ -matrix. Consequently its corresponding Hodge  $*$  defined as above is also commutative with this quantum group transformation.

The deformed Fubini-Study metric implies that the connection one-form is

$$\omega_a^b = C_{ac}^{bd} \bar{z}^c \rho^{-1} dz^d,$$

where  $C_{ac}^{bd} = \delta_{ac} \delta_{bd} + q^{N-d} \delta_{ab} \delta_{cd}$ . The curvature two form is

$$R_a^b = -C_{ac}^{bd} g_{c\bar{e}} d\bar{z}^{\bar{e}} dz^d.$$

The scalar curvature and Ricci tensor are, up to normalization,

$$\mathcal{R} \propto 1, \quad \mathcal{R}_a^b \propto \delta_a^b. \tag{37}$$

As in the classical case, this result can also be obtained by arguments based on the quantum group symmetry.

## 5 The Two-Sheeted Space

Using the algebraic formulation of Riemannian geometry, we reproduce in this section the theory of gravity for the two-sheeted space which was first described in Ref.[1]. In that paper the Riemannian geometry on quantum spaces is formulated in terms of A. Connes' non-commutative geometry.

The two-sheeted space is the product of a classical 4-dimensional manifold  $\mathcal{M}_4$  and a space of two discrete points  $\mathbf{Z}_2$ . Denote the two points in  $\mathbf{Z}_2$  as  $a$  and

b. The algebra of functions on  $Z_2$  is generated by 1 and  $e$ , where  $1(a) = 1(b) = 1$  and  $e(a) = -e(b) = 1$ . It follows that  $e$  is real and

$$e^2 = 1. \quad (38)$$

Let the exterior derivative to act on (38) we find

$$ede = -dee.$$

We also define  $dede = 0$ . In addition,  $e$  commutes with the coordinates  $\{x^\mu\}$  on  $\mathcal{M}_4$  and  $\{dx^\mu\}$ , and  $de$  commutes with  $\{x^\mu\}$  and anti-commutes with  $\{dx^\mu\}$ .

To obtain the results in Ref.[1] we assume that the vielbeins and connection one-forms can be written as

$$E^a = dx^\mu e_\mu^a, \quad E^5 = de\lambda$$

and

$$\Omega^{AB} = dx^\mu(\omega_\mu^{AB} + ev_\mu^{AB}) + de(l^{AB} + ek^{AB}),$$

where  $e_\mu^a$ ,  $\lambda$ ,  $\omega_\mu^{AB}$ ,  $v_\mu^{AB}$ ,  $l^{AB}$  and  $k^{AB}$  are all real functions of  $x$ . The indices  $A, B$  take values in  $\{1, 2, 3, 4, 5\}$ , where  $\{1, 2, 3, 4\}$  correspond to  $dx^\mu$  or  $E^a$ , and  $\{5\}$  corresponds to  $de$  or  $E^5$ .

The Hodge  $*$  map defined on  $E^A$  is the classical one. For example,  $*(E^A E^B) = \frac{1}{3!} \epsilon^{ABCDEF} E^C E^D E^E E^F$ . This map does not have the property (12), hence the Lagrangian (15) is invariant only under the coordinate transformation restricted to  $\mathcal{M}_4$ , i.e.,  $x^\mu \rightarrow x'^\mu(x)$ ,  $e \rightarrow e$ .

The integration over the whole space can be decomposed into the usual integration over the four-dimensional manifold followed by the integration

$$\int_{Z_2} E^5(a + be) = a$$

for arbitrary numbers  $a, b$ . (The requirement that  $\int_{Z_2} dee$  vanishes implies the cyclic property of Connes' integration in this case:  $\int \alpha\beta = \int \beta\alpha$ .)

Using the metricity condition (9) and the torsion-free condition (10) one can partially solve for the connection. But many components of the connection are still free. They should be viewed as independent fields. It turns out that they are not dynamical fields because in the Lagrangian (the scalar curvature) they do not have time derivatives. Their equations of motion are simply constraints which are solved by their vanishing.

The action for the gravity on this two-sheeted space defined by

$$I = \int_{\mathcal{M}_4 \times Z_2} E^A (*R^{AB}) E^B$$

is, after taking out all non-dynamical fields, the same as [1]

$$I = - \int_{\mathcal{M}_4} (\mathcal{R}_4 - 2\lambda^{-1} \nabla_\mu \partial^\mu \lambda) \sqrt{g_4} d^4x,$$

where  $\mathcal{R}_4$  is the usual scalar curvature of  $\mathcal{M}_4$ ,  $\nabla_\mu$  is the usual covariant derivative on  $\mathcal{M}_4$  and  $g_4$  is the determinant of the metric on  $\mathcal{M}_4$ . The only new dynamical field introduced by the  $Z_2$  structure in spacetime is  $\lambda(x)$ . By changing variable

$\lambda = \exp(\sigma)$  [1] we get

$$I = - \int_{\mathcal{M}_4} (\mathcal{R}_4 - 2\partial_\mu \sigma \partial^\mu \sigma) \sqrt{g_4} d^4x.$$

## 6 Conclusion

In this paper we proposed a straightforward formulation of Riemannian geometry on quantum spaces with a  $*$ -involution and a Hodge  $*$  map, and we showed several examples. In addition to the possibility of applying it to describe physics at the Planck scale, this formulation can be used for Kaluza-Klein theories to build models with the extra dimensions corresponding to quantum spaces.

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