BLOCKS OF HOMOGENEOUS EFFECT ALGEBRAS

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ABSTRACT. Effect algebras, introduced by Foulis and Bennett in 1994, are partial algebras which generalize some well known classes of algebraic structures (for example orthomodular lattices, MV algebras, orthoalgebras etc.). In the present paper, we introduce a new class of effect algebras, called *homogeneous effect algebras*. This class includes orthoalgebras, lattice ordered effect algebras and effect algebras satisfying Riesz decomposition property. We prove that every homogeneous effect algebra is a union of its blocks, which we define as maximal sub-effect algebras satisfying Riesz decomposition property. This generalizes a recent result by Riečanová, in which lattice ordered effect algebras were considered. Moreover, the notion of a block of a homogeneous effect algebra is a generalization of the notion of a block of an orthoalgebra. We prove that the set of all sharp elements in a homogeneous effect algebra E forms an orthoalgebra E_S. Every block of E_S is the center of a block of E. The set of all sharp elements in the compatibility center of E coincides with the center of E. Finally, we present some examples of homogeneous effect algebras and we prove that for a Hilbert space $\mathbb{H}$ with $\dim(\mathbb{H}) > 1$, the standard effect algebra $\mathcal{E}(\mathbb{H})$ of all effects in $\mathbb{H}$ is not homogeneous.

1. Introduction

Effect algebras (or D-posets) have recently been introduced by Foulis and Bennett in [8] for study of foundations of quantum mechanics. (See also [15], [10].) The prototype effect algebra is $(\mathcal{E}(\mathbb{H}), \oplus, 0, I)$, where $\mathbb{H}$ is a Hilbert space and $\mathcal{E}(\mathbb{H})$ consists of all self-adjoint operators A of $\mathbb{H}$ such that $0 \leq A \leq I$. For $A, B \in \mathcal{E}(\mathbb{H})$, $A \oplus B$ is defined iff $A + B \leq 1$ and then $A \oplus B = A + B$. $\mathcal{E}(\mathbb{H})$ plays an important role in the foundations of quantum mechanics [16], [3].

The class of effect algebras includes orthoalgebras [9] and a subclass (called MV-effect algebras or Boolean D-posets or Boolean effect algebras), which is essentially equivalent to MV-algebras, introduced by Chang in [4] (cf. e.g. [6], [1] for results on MV-algebras in the context of effect algebras). The class of orthoalgebras includes other classes of well-known sharp structures, like orthomodular posets [17] and orthomodular lattices [14],[2].

One of the most important results in the theory of effect algebras was proved by Riečanová in her paper [20]. She proved that every lattice ordered effect algebra is a union of maximal mutually compatible sub-effect algebras, called blocks. This result generalizes the well-known fact that an orthomodular lattice is a union of its maximal Boolean subalgebras. Moreover, as proved in [13], in every lattice ordered effect algebra E the set of all sharp elements forms a sub-effect algebra E_S, which is a sub-lattice of E; E_S is then an orthomodular lattice, and every block of

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E_S is the center of some block of E. On the other hand, every orthoalgebra is a union of maximal Boolean sub-orthoalgebras. Thus, although the classes of lattice ordered effect algebras and orthoalgebras are independent, both lattice ordered effect algebras and orthoalegebras are covered by their blocks. This observation leads us to a natural question:

Question 1.1. Is there a class of effect algebras, say $\mathbb{X}$, with the following properties?

- $\mathbb{X}$ includes orthoalgebras and lattice ordered effect algebras.
- Every $E \in \mathbb{X}$ is a union of (some sort of) blocks.

In the present paper, we answer this question in the affirmative. We introduce a new class of effect algebras, called homogeneous effect algebras. This class includes lattice ordered effect algebras, orthoalgebras and effect algebras satisfying Riesz decomposition property (cf. e.g. [18]). The blocks in homogeneous algebras are maximal sub-effect algebras satisfying Riesz decomposition property. We prove that the set of all sharp elements E_S in a homogeneous effect algebra E forms a sub-effect algebra (of course, E_S is an orthoalgebra) and every block of E_S is the center of a block of E. In the last section we present some examples of homogeneous effect algebras and we prove that $\mathcal{E}(\mathbb{H})$ is not homogeneous unless $\dim(\mathbb{H}) \leq 1$.

2. Definitions and basic relationships

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions.

- (E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
- (E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
- (E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.
- (E4) If $a \oplus 1$ exists, then $a = 0$

Effect algebras were introduced by Foulis and Bennett in their paper [8]. Independently, Kôpka and Chovanec introduced an essentially equivalent structure called *D-poset* (see [15]). Another equivalent structure, called *weak orthoalgebras* was introduced by Giuntini and Greuling in [10].

For brevity, we denote the effect algebra $(E, \oplus, 0, 1)$ by E. In an effect algebra E, we write $a \leq b$ iff there is $c \in E$ such that $a \oplus c = b$. It is easy to check that every effect algebra is cancellative, thus $\leq$ is a partial order on E. In this partial order, 0 is the least and 1 is the greatest element of E. Moreover, it is possible to introduce a new partial operation $\ominus$; $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. It can be proved that $a \oplus b$ is defined iff $a \leq b'$ iff $b \leq a'$. Therefore, it is usual to denote the domain of $\oplus$ by $\perp$. If $a \perp b$, we say that a and b are *orthogonal*. Let $E_0 \subseteq E$ be such that $1 \in E_0$ and, for all $a, b \in E_0$ with $a \geq b$, $a \ominus b \in E_0$. Since $a' = 1 \ominus a$ and $a \oplus b = (a' \ominus b)'$, E_0 is closed with respect to $\oplus$ and $'$. We then say that $(E_0, \oplus, 0, 1)$ is a *sub-effect algebra of* E. Another possibility to construct a substructure of an effect algebra E is to restrict $\oplus$ to an interval $[0, a]$, where $a \in E$, letting a act as the unit element. We denote such effect algebra by $[0, a]_E$.

Remark. For our purposes, it is natural to consider orthomodular lattices, orthomodular posets, MV-algebras, and Boolean algebras as special types of effect algebras. In the present paper, we will write shortly "orthomodular lattice" instead of "effect algebra associated with an orthomodular lattice" and similarly for orthomodular posets, MV-algebras, and Boolean algebras.

An effect algebra satisfying $a \perp a \implies a = 0$ is called an *orthoalgebra* (cf. [9]). An effect algebra E is an *orthomodular poset* iff, for all $a, b, c \in E$, $a \perp b \perp c \perp a$ implies that $a \oplus b \oplus c$ exists (cf. [8]). An orthoalgebra is an *orthomodular lattice* iff it is lattice ordered.

Let E be an effect algebra. Let $C = (c_1, \dots, c_n)$ be a finite family of elements of E. We say that C is *orthogonal* iff the sum $c_1 \oplus \dots \oplus c_n$ exists. We then write $\bigoplus C = c_1 \oplus \dots \oplus c_n$. For $n = 0$, we put $\bigoplus C = 0$. We say that $Ran(C) = \{c_1, \dots, c_n\}$ is *the range of* C. Let $C = (c_1, \dots, c_n), D = (d_1, \dots, d_k)$ be orthogonal families of elements. We say that D is a *refinement of* C iff there is a partition $P = \{P_1, \dots, P_n\}$ of $\{1, \dots, k\}$ such that, for all $1 \leq i \leq n$, $c_i = \bigoplus_{j \in P_i} d_j$. Note that if D is a refinement of C, then $\bigoplus C = \bigoplus D$.

A finite subset M_F of an effect algebra E is called *compatible with cover in* $X \subseteq E$ iff there is a finite orthogonal family $C = (c_1, \dots, c_n)$ with $Ran(C) \subseteq X$ such that for every $a \in M_F$ there is a set $A \subseteq \{1, \dots, n\}$ with $a = \bigoplus_{i \in A} c_i$. C is then called an *orthogonal cover* of M_F. A subset M of E is called *compatible with covers in* $X \subseteq E$ iff every finite subset of M is compatible with covers in X. A subset M of E is called *internally compatible* iff M is compatible with covers in M. A subset M of E is called *compatible* iff M is compatible with covers in E. An effect algebra E is said to be *compatible* if E is a compatible subset of E. If $\{a, b\}$ is a compatible set, we write $a \leftrightarrow b$. It is easy to check that $a \leftrightarrow b$ iff there are $a_1, b_1, c \in E$ such that $a_1 \oplus c = a$, $b_1 \oplus c = b$, and $a_1 \oplus b_1 \oplus c$ exists. A subset M of E is called *mutually compatible* iff, for all $a, b \in M$, $a \leftrightarrow b$. Obviously, every compatible subset of an effect algebra is mutually compatible. In the class of lattice ordered effect algebras, the converse also holds. It is well known that in an orthomodular poset, a mutually compatible set need not to be compatible (cf. e.g. [17]).

A lattice ordered effect algebra E is called an *MV-algebra* iff E is compatible (cf. [6]). An MV-algebra which is an orthoalgebra is a *Boolean algebra*. Recently, Z. Riečanová proved in her paper [20] that every lattice ordered effect algebra is a union of MV-algebras, which are maximal mutually compatible subsets. These are called *blocks*. She proved that every block of a lattice ordered effect algebra E is a sub-effect algebra and a sublattice of E. Note that Riečanová's results imply that every mutually compatible subset of a lattice ordered effect algebra is compatible. Indeed, let M be a mutually compatible set. Then M can be embedded into a block B, which is an MV-algebra and hence compatible. Since B is compatible and $M \subseteq B$, M is compatible.

On the other hand, it is easy to prove that every element of an orthoalgebra can be embedded into a maximal sub-orthoalgebra, which is a Boolean algebra.

We say that an effect algebra E satisfies *Riesz decomposition property* iff, for all $u, v_1, \dots, v_n \in E$ such that $v_1 \oplus \dots \oplus v_n$ exists and $u \leq v_1 \oplus \dots \oplus v_n$, there are $u_1, \dots, u_n \in E$ such that,for all $1 \leq i \leq n$, $u_i \leq v_i$ and $u = u_1 \oplus \dots \oplus u_n$. It is easy to check that an effect algebra E satisfies Riesz decomposition property iff E satisfies Riesz decomposition property with fixed $n = 2$. A lattice ordered effect algebra E satisfies Riesz decomposition property iff E is an MV-algebra. An orthoalgebra E satisfies Riesz decomposition property iff E is a Boolean algebra.

Let E_1, E_2 be effect algebras. A map $\phi : E_1 \mapsto E_2$ is called a *morphism* iff $\phi(1) = 1$ and $a \perp b$ implies that $\phi(a) \perp \phi(b)$ and then $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. A morphism ϕ is an *isomorphism* iff ϕ is bijective and ϕ^{-1} is a morphism.

Definition 2.1. An effect algebra E is called *homogeneous* iff, for all $u, v_1, v_2 \in E$ such that $v_1 \perp v_2$, $u \leq v_1 \oplus v_2$, $u \leq (v_1 \oplus v_2)'$, there are u_1, u_2 such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

Proposition 2.2.

(a) *Every orthoalgebra is homogeneous.*
(b) *Every effect algebra satisfying Riesz decomposition property is homogeneous.*
(c) *Every lattice ordered effect algebra is homogeneous.*

Proof. For the proof of (a), observe that $u \leq v_1 \oplus v_2$ and $u \leq (v_1 \oplus v_2)'$ imply that $u \perp u$ and thus $u = 0$. (b) is obvious. For the proof of (c), let E be a lattice ordered effect algebra. Note that $v_1 \perp v_2$, $u \leq (v_1 \oplus v_2)'$ imply that the set $\{u, v_1, v_2\}$ is mutually orthogonal and thus mutually compatible. Therefore, by [20], $\{u, v_1, v_2\}$ can be embedded into a block B. Since B is an MV-algebra, B satisfies Riesz decomposition property, hence E is homogeneous. □

Proposition 2.3. *Let E be a homogeneous effect algebra. Let $u, v_1, \dots, v_n \in E$ be such that $v_1 \oplus \dots \oplus v_n$ exists, $u \leq v_1 \oplus \dots \oplus v_n$ and $u \leq (v_1 \oplus \dots \oplus v_n)'$. Then there are $v_1, \dots, v_n$ such that, for all $1 \leq i \leq n$, $v_i \leq u_i$ and $u = u_1 \oplus \dots u_n$.*

Proof. (By induction.) For $n = 1$, it suffices to put $u_1 = u$. Assume that the proposition holds for $n = k$. Let $u, v_1, \dots, v_{k+1}$ be such that $v_1 \oplus \dots \oplus v_{k+1}$ exists, $u \leq v_1 \oplus v_2 \oplus \dots \oplus v_{k+1}$ and $u \leq (v_1 \oplus v_2 \oplus \dots \oplus v_{k+1})'$. Since E is homogeneous, there are $u_1 \leq v_1$ and $z \leq v_2 \oplus \dots \oplus v_{k+1}$ such that $u = u_1 \oplus z$. Since

$$z \leq u \leq (v_1 \oplus \dots \oplus v_{k+1})' \leq (v_2 \oplus \dots \oplus v_{k+1})',$$

we see that $z \leq (v_2 \oplus \dots \oplus v_{k+1})'$. Thus, we may apply induction hypothesis. The rest is trivial. □

3. Blocks of homogeneous effect algebras

Let E be an effect algebra. We say that a sub-effect algebra B of E is a *block of* E iff B is a maximal sub-effect algebra satisfying the Riesz decomposition property. This definition of a block is consistent with the definition of a block of the theory of orthoalgebras (maximal Boolean sub-orthoalgebra) and also in the theory of lattice ordered effect algebras (maximal mutually compatible subset).

In this section, we prove that blocks of homogeneous effect algebras coincide with the maximal internally compatible subsets, which contain 1. As a consequence, every homogeneous effect algebra is a union of its blocks.

The main tool we use is the closure operation $M \mapsto \overline{M}$ which is defined on the system of all subsets of an effect algebra E in the following way. Let M be a subset of an effect algebra E. First we define certain subsets M_n ($n \in \mathbb{N}$) of E as follows : $M_0 = M$ and for $n \in \mathbb{N}$

$$M_{n+1} = \{x : x \leq y, y' \text{ for some } y \in M_n\} \cup \{y \ominus x : x \leq y, y' \text{ for some } y \in M_n\}. \tag{1}$$

Then we put $\overline{M} = \bigcup_{n \in \mathbb{N}} M_n$. Note that, for all $n \in \mathbb{N}$, $M_n \subseteq M_{n+1}$ and that $\overline{\overline{M}} = \overline{M}$. In an orthoalgebra, $M = \overline{M}$ for every set M.

Lemma 3.1. *Let E be an effect algebra. Let M be an compatible subset of E. Then M can be embedded into a maximal compatible subset of E.*

Proof. The proof is an easy application of Zorn's lemma and is left to the reader. □

Proposition 3.2. *Let E be a homogeneous effect algebra. Let $M \subseteq E$ be a finite compatible set, $a, b \in M$, $a \geq b$. Let $C = (c_1, \dots, c_k)$ be an orthogonal cover of M. Let $A, B \subseteq \{1, \dots, k\}$ be such that $a = \bigoplus_{i \in A} c_i$ and $b = \bigoplus_{i \in B} c_i$. Then, there is a refinement of C, say $W = (w_1, \dots, w_n)$ and sets $B_W \subseteq A_W \subseteq \{1, \dots, n\}$ such that $(w_i)_{i \in A_W}$ is a refinement of $(c_i)_{i \in A}$ and $(w_i)_{i \in B_W}$ is a refinement of $(c_i)_{i \in B}$. Moreover, we have $Ran(W) \subseteq \overline{Ran(C_0)}$.*

Proof. If $|B \setminus A| = 0$ then $B \subseteq A$ and there is nothing to prove.

Let $l \in \mathbb{N}$. Assume that Proposition 3.2 holds for all C, A, B with $|B \setminus A| = l$. Let C_0, A_0, B_0 be as in the assumption of Proposition 3.2, with $|B_0 \setminus A_0| = l + 1$.

To avoid double indices, we may safely assume that A_0 and B_0 are such that, for some $0 \leq r, s, t \leq k$, $B_0 \setminus A_0 = \{1, \dots, r\}$, $B_0 \cap A_0 = \{r + 1, \dots, s\}$, $A_0 \setminus B_0 = \{s + 1, \dots, t\}$.

Write $b_1 = c_1 \oplus \ldots \oplus c_{l+1}$, $d = c_{l+2} \oplus \ldots \oplus c_s$, $a_1 = c_{s+1} \oplus \ldots \oplus c_t$. Since $b_1 \oplus d = b \leq a = a_1 \oplus d$, we see that $c_{l+1} \leq b_1 \leq a_1$. Since C_0 is an orthogonal family, $c_{l+1} \leq a_1{}'$. By Proposition 2.3, this implies that there are $v_{s+1}, \dots, v_t$ such that, for all $s + 1 \leq i \leq t$, $v_i \leq c_i$ and $c_{l+1} = v_{s+1} \oplus \ldots \oplus v_t$. Let us construct a refinement of C_0, say $C_1 = (e_i)$, as follows.

$$\begin{aligned} C_1 \quad = \quad & (c_1, \dots, c_l, v_{s+1}, \dots, v_t, c_{l+2}, \dots, c_s, \\ & c_{s+1} \ominus v_{s+1}, \dots, c_t \ominus v_t, c_{t+1}, \dots, c_k, c_{l+1}) \end{aligned}$$

Obviously, C_1 is a refinement of C_0 and $Ran(C_1) \subseteq \overline{Ran(C_0)}$. Moreover, we have

$$b = \bigoplus (c_1, \dots, c_l, v_{s+1}, \dots, v_t, c_{l+2}, \dots, c_s)$$

and

$$a = \bigoplus (v_{s+1}, \dots, v_t, c_{l+2}, \dots, c_s, c_{s+1} \ominus v_{s+1}, \dots, c_t \ominus v_t).$$

By latter equations, we can find sets A_1, B_1 of indices such that $a = \bigoplus_{i \in A_1} e_i$, $b = \bigoplus_{i \in B_1} e_i$ and $B_1 \setminus A_1 = \{1, \dots, l\}$. Moreover, $(e_i)_{i \in A_1}$ is a refinement of $(c_i)_{i \in A_0}$ and $(e_i)_{i \in B_1}$ is a refinement of $(c_i)_{i \in B_0}$. As $|B_1 \setminus A_1| = l$, we may apply the induction hypothesis on C_1, A_1, B_1 to find a refinement $W = (w_1, \dots, w_n)$ of C_1 with $Ran(W) \subseteq \overline{Ran(C_1)}$ and sets $B_W \subseteq A_W \subseteq \{1, \dots, n\}$ such that $(w_i)_{i \in A_W}$ is a refinement of $(e_i)_{i \in A_1}$ and $(w_i)_{i \in B_W}$ is a refinement of $(e_i)_{i \in B_1}$. Obviously, W is a refinement of C_0 and we see that

$$Ran(W) \subseteq \overline{Ran(C_1)} \subseteq \overline{\overline{Ran(C_0)}} = \overline{Ran(C_0)}.$$

Similarly, $(w_i)_{i \in A_W}$ is a refinement of $(c_i)_{i \in A_0}$ and $(w_i)_{i \in B_W}$ is a refinement of $(c_i)_{i \in B_0}$. This concludes the proof. □

Corollary 3.3. *Let M be a finite compatible subset of a homogeneous effect algebra E. Let $a, b \in M$ be such that $a \geq b$. Then $M \cup \{a \ominus b\}$ is a compatible set.*

Proof. Let W, A_W, B_W be as in Proposition 3.2. Then $a \ominus b = \bigoplus_{i \in A_W \setminus B_W} w_i$, so W is an orthogonal cover of $M \cup \{a \ominus b\}$. □

Corollary 3.4. *Let M be a finite compatible subset of a homogeneous effect algebra E. Let $a, b \in M$ be such that $a \perp b$. Then $M \cup \{a \oplus b\}$ is a compatible set.*

Proof. It is easy to check that, for every compatible set M_0, $M_0 \cup M_0{}' = M_0 \cup \{a' : a \in M_0\}$ is a compatible set. The rest follows from Corollary 3.3 and from the equation $a \oplus b = (a' \ominus b)'$. □

Theorem 3.5. *Let E be an effect algebra. The following are equivalent.*

(a) *E satisfies Riesz decomposition property.*
(b) *E is homogeneous and compatible.*

Proof. (a) implies (b): It is evident that E is homogeneous. It remains to prove that every n-element subset of E is compatible. For $n = 1$, there is nothing to prove. For $n > 1$, let us assume that every $(n-1)$-element subset of E is compatible. Let $X = \{x_1, \dots, x_n\}$ be a subset of E. By induction hypothesis, $X_0 = \{x_1, \dots, x_{n-1}\}$ is compatible. Thus, there is an orthogonal cover of X_0, say $C = (c_1, \dots, c_k)$. Since $x_n \leq (\bigoplus C) \oplus (\bigoplus C)'$ and E satisfies Riesz decomposition property, there exist y_1, y_2 such that $y_1 \leq (\bigoplus C)$, $y_2 \leq (\bigoplus C)'$ and $x_n = y_1 \oplus y_2$. Since $y_1 \leq (\bigoplus C)$, there are $z_1, \dots, z_k$ such that, for all $1 \leq i \leq k$, $z_i \leq c_i$ and $y_1 = z_1 \oplus \ldots \oplus z_k$. Consequently,

$$(z_1, c_1 \ominus z_1, \dots, z_k, c_k \ominus z_k, y_2)$$

is an orthogonal cover of X and X is compatible.

(b) implies (a): Let $u, v_1, v_2 \in E$ be such that $v_1 \perp v_2$, $u \leq v_1 \oplus v_2$. If $v_1 = 0$ or $v_2 = 0$, there is nothing to prove. Thus, let us assume that $v_1, v_2 \neq 0$. By Proposition 3.2, $v_1 \leq v_1 \oplus v_2$ implies that there an orthogonal cover $W = (w_1, \dots, w_m)$ of $\{u, v_1, v_2, v_1 \oplus v_2\}$ such that, for some $V_1 \subseteq V \subseteq \{1, \dots, m\}$, we have $\bigoplus_{i \in V} w_i = v_1 \oplus v_2$ and $\bigoplus_{i \in V_1} w_i = v_1$. This implies that $\bigoplus_{i \in V \setminus V_1} w_i = v_2$. By Proposition 3.2, $u \leq v_1 \oplus v_2$ implies that there is a refinement of W, say $Q = (q_1, \dots, q_n)$, and some $U \subseteq Z \subseteq \{1, \dots, n\}$ such that $\bigoplus_{i \in U} q_i = u$ and $\bigoplus_{i \in Z} q_i = v_1 \oplus v_2$. Moreover, by Proposition 3.2, we may assume that $(q_i)_{i \in Z}$ is a refinement of $(w_i)_{i \in V}$. This implies that there is $Z_1 \subseteq Z$ such that $\bigoplus_{i \in Z_1} q_i = v_1$. Put $u_1 = \bigoplus_{i \in U \cap Z_1} q_i$ and $u_2 = \bigoplus_{i \in U \cap (Z \setminus Z_1)} q_i$. It remains to observe that $u = u_1 \oplus u_2$, $u_1 \leq v_1$ and $u_2 \leq v_2$. □

Example 3.6. Let R_6 be a 6-elements effect algebra with two atoms $\{a, b\}$, satisfying equation $a \oplus a \oplus a = a \oplus b \oplus b = 1$. Since (a, b, b) is an orthogonal cover of R_6, R_6 is a compatible effect algebra. However, R_6 does not satisfy Riesz decomposition property, since $a \leq b \oplus b$ and $a \wedge b = 0$. This example shows that there are compatible effect algebras that do not satisfy Riesz decomposition property.

Proposition 3.7. *Let M be a subset of a homogeneous effect algebra E such that M is compatible with covers in $\overline{M}$. Then $\overline{M}$ is internally compatible.*

Proof. Consider (1). Since each finite subset of $\overline{M}$ can be embedded into some M_n, it suffices to prove that, for all $n \in \mathbb{N}$, M_n is compatible with covers in $\overline{M}$. By assumption, $M = M_0$ is compatible with covers in $\overline{M}$. Assume that, for some $n \in \mathbb{N}$, M_n is compatible with covers in $\overline{M}$. Evidently, every finite subset of M_{n+1} can be embedded into a set of the form

$$\{x_1, y_1 \ominus x_1, \dots, x_k, y_k \ominus x_k\} \subseteq M_{n+1}, \tag{2}$$

where for all $1 \leq i \leq k$ we have $x_i \leq y_i, y_i{}'$ and $y_i \in M_n$. We now prove the following

Claim. Let x_i, y_i be as above. For every cover C_0 of $\{y_1, \dots, y_k\}$, there is a refinement W of C_0 such that W covers $\{x_1, y_1 \ominus x_1, \dots, x_k, y_k \ominus x_k\}$ and $Ran(W) \subseteq \overline{Ran(C_0)}$.

Proof of the Claim. For $k = 0$, we may put $W = C_0$. Assume that the Claim is satisfied for some $k = l \in \mathbb{N}$. Let C_0 be a cover of $\{y_1, \dots, y_{l+1}\} \subseteq M_n$. Since C_0 is a cover of $\{y_1, \dots, y_l\}$ as well, by induction hypothesis there is a refinement of C_0, say C_1, such that C_1 covers $\{x_1, y_1 \ominus x_1, \dots, x_l, y_l \ominus x_l\}$ and $Ran(C_1) \subseteq \overline{Ran(C_0)}$. As C_1 is a refinement of C_0, C_1 covers $\{y_1, \dots, y_{l+1}\}$. Thus, there are $(c_1, \dots, c_m) \subseteq C_1$ such that $y_{l+1} = c_1 \oplus \dots \oplus c_m$. Since $x_{l+1} \leq y_{l+1}, {y_{l+1}}'$, Proposition 2.3 implies that there are $z_1, \dots, z_m$ such that, for all $1 \leq i \leq m$, $z_i \leq c_i$ and $x_{l+1} = z_1 \oplus \dots \oplus z_l$. Let us construct a refinement W of C_1 by replacing each of the c_i's by the pair $(z_i, c_i \ominus z_i)$. Then W is a refinement of C_1 and W covers $\{x_1, y_1 \ominus x_1, \dots, x_{l+1}, y_{l+1} \ominus x_{l+1}\}$. Moreover, for all $1 \leq i \leq m$, $z_i \leq x_{l+1} \leq {y_{l+1}}' \leq {c_i}'$, hence

$$Ran(W) \subseteq \overline{Ran(C_1)} \subseteq \overline{\overline{Ran(C_0)}} = \overline{Ran(C_0)}.$$

Now, let M_F be a finite subset of M_{n+1}. We may assume that M_F is of the form (2). By the outer induction hypothesis, M_n is compatible with covers in $\overline{M}$, thus $\{y_1, \dots, y_k\}$ is compatible with cover in $\overline{M}$. Let C be an orthogonal cover of $\{y_1, \dots, y_k\}$ with $Ran(C) \subseteq \overline{M}$. By Claim, there is a refinement W of C, such that W covers M_F and $Ran(W) \subseteq \overline{Ran(C)} \subseteq \overline{\overline{M}} = \overline{M}$. Thus, M_F is compatible with covers in $\overline{M}$ and we see that $\overline{M}$ is internally compatible. □

The following are immediate consequences of Proposition 3.7.

Corollary 3.8.

(a) *Let M be an internally compatible subset of a homogeneous effect algebra E. Then $\overline{M}$ is an internally compatible set.*

(b) *Let M be a maximal internally compatible subset of a homogeneous effect algebra E. Then $M = \overline{M}$.*

Proposition 3.9. *Let E be a homogeneous effect algebra, let M be an internally compatible set with $M = \overline{M}$. Let $a, b \in M$, $a \geq b$. Then $M \cup \{a \ominus b\}$ is an internally compatible set.*

Proof. Let M_F be a finite subset of M. Since M is internally compatible, there is an orthogonal cover C of $M_F \cup \{a, b\}$ with $Ran(C) \subseteq M$. By Corollary 3.3, $M_F \cup \{a, b, a \ominus b\}$ is then compatible with cover in $\overline{Ran(C)}$. Therefore, $M_F \cup \{a \ominus b\}$ is compatible with cover in $\overline{Ran(C)}$. Since $\overline{Ran(C)} \subseteq \overline{M} = M$, $M \cup \{a \ominus b\}$ is an internally compatible set. □

As we will show later in Example 5.6, a sub-effect algebra of a homogeneous effect algebra need not to be homogeneous. However, we have the following relationship on the positive side.

Proposition 3.10. *Let E be a homogeneous effect algebra. Let F be a sub-effect algebra of E such that $F = \overline{F}$, where the closure is taken in E. Then F is homogeneous.*

Proof. Let $u, v_1, v_2 \in F$ be such that $u \leq v_1 \oplus v_2$ and $u \leq (v_1 \oplus v_2)'$. Since E is homogeneous, there are $u_1, u_2 \in E$ such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

For $i \in \{1,2\}$, we have $u_i \leq v_1 \oplus v_2$ and $u_i \leq (v_1 \oplus v_2)'$. Thus, $u_1, u_2 \in \overline{F} = F$ and F is homogeneous. □

Theorem 3.11. *Let E be a homogeneous effect algebra, let $B \subseteq E$. The following are equivalent.*

(a) *B is a maximal internally compatible set with $1 \in B$.*
(b) *B is a block.*

Proof. Assume that (a) is satisfied. By Corollary 3.8, part (b), $B = \overline{B}$. By Proposition 3.9, this implies that for all $a, b \in B$ such that $a \geq b$, $B \cup \{a \ominus b\}$ is an internally compatible set. Therefore, by maximality of B, B is closed with respect to $\ominus$. Since $1 \in B$, B is a sub-effect algebra of E. Since B is an internally compatible set, B is a compatible effect algebra. By Corollary 3.8(b), $B = \overline{B}$. By Proposition 3.10, this implies that B is homogeneous. Since B is homogeneous and compatible, Theorem 3.5 implies that B satisfies Riesz decomposition property.

Assume that (b) is satisfied. By Theorem 3.5, B is an internally compatible subset. By Lemma 3.1, B can be embedded into a maximal internally compatible subset B_{max} of E. By above part of the proof, $1 \in B \subseteq B_{max}$ implies that B_{max} is a block. Therefore, $B = B_{max}$ and (a) is satisfied. □

Corollary 3.12. *Let E be a homogeneous effect algebra. Every finite compatible subset of E can be embedded into a block.*

Proof. Let M_F be a finite compatible subset of E. Let $C = (c_1, \ldots, c_n)$ be an orthogonal cover of M_F. Then $M_F \cup \{1\}$ is compatible set, with cover $C^+ = (c_1, \ldots, c_n, (\bigoplus C)')$. Thus, $M_F \cup \{1\} \cup Ran(C^+)$ is an internally compatible set containing 1. Therefore, by Lemma 3.1, $M_F \cup \{1\} \cup Ran(C^+)$ can be embedded into a maximal compatible subset B with $1 \in B$. By Theorem 3.11, B is a block. □

Corollary 3.13. *Let E be a homogeneous effect algebra. Then*

$$E = \cup\{B : B \text{ is a block of } E\}.$$

Proof. By Corollary 3.12. □

Corollary 3.14. *For an effect algebra E, the following are equivalent.*

(a) *E is homogeneous.*
(b) *Every finite compatible subset can be embedded into a block.*
(c) *Every finite compatible subset can be embedded into a sub-effect algebra of E satisfying Riesz decomposition property.*
(d) *The range of every finite orthogonal family can be embedded into a block.*
(e) *The range of every finite orthogonal family can be embedded into a sub-effect algebra satisfying Riesz decomposition property.*
(f) *The range of every orthogonal family with three elements can be embedded into a block.*
(g) *The range of every orthogonal family with three elements can be embedded into a sub-effect algebra satisfying Riesz decomposition property.*

Proof. (a) $\implies$ (b) is Corollary 3.12. The implication chains (b) $\implies$ (c) $\implies$ (e) $\implies$ (g) and (b) $\implies$ (d) $\implies$ (f) $\implies$ (g) are obvious. To prove that (h) $\implies$ (a), assume that E is an effect algebra satisfying (g), and let $u, v_1, v_2 \in E$ be such that $u \leq v_1 \oplus v_2$, $u \leq (v_1 \oplus v_2)'$. Then (u, v_1, v_2) is an orthogonal family with three elements. By (g), $\{u, v_1, v_2\}$ can be embedded into a sub-effect algebra R satisfying

Riesz decomposition property. Thus, there are $u_1, u_2 \in R \subseteq E$ such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$. Hence, E is homogeneous. □

Question 3.15. Can every compatible subset of a homogeneous effect algebra E be embedded into a block ? This is true for orthomodular posets (cf. e.g. [17]) and for lattice ordered effect algebras. By Theorem 3.11 and Lemma 3.1, this question reduces to the question, whether a compatible subset can be embedded into an internally compatible subset containing 1.

4. Compatibility center and sharp elements

For a homogeneous effect algebra E, we write

$$K(E) = \bigcap\{B : B \text{ is a block of } E\}.$$

We say that $K(E)$ is the *compatibility center* of E. Note that $K(E) = \overline{K(E)}$ and hence, by Proposition 3.10, $K(E)$ is homogeneous.

An element a of an effect algebra is called *sharp* iff $a \wedge a' = 0$. We denote the set of all sharp elements of an effect algebra E by E_S. It is obvious that an effect algebra E is an orthoalgebra iff $E = E_S$. An element a of an effect algebra E is called *principal* iff the interval $[0, a]$ is closed with respect to $\oplus$. Evidently, every principal element in an effect algebra is sharp. A principal element a of an effect algebra is called *central* iff for all $b \in E$ there is a unique decomposition $b = b_1 \oplus b_2$ with $b_1 \leq a$, $b_2 \leq a'$. The set of all central elements of an effect algebra E is called *the center of* E and is denoted by $C(E)$. In [11], the center of an effect algebra was introduced and the following properties of $C(E)$ were proved.

Proposition 4.1. *Let E be an effect algebra. Then*

- *$C(E)$ is a sub-effect algebra of E.*
- *$C(E)$ is a Boolean algebra. Moreover, for all $a \in C(E)$ and $x \in E$, $a \wedge x$ exists.*
- *For all $a \in C(E)$, the map $\phi : E \mapsto [0, a]_E$ given by $\phi(x) = a \wedge x$ is a morphism.*
- *For all $a \in C(E)$, E is naturally isomorphic to $[0, a]_E \times [0, a']_E$. Moreover, for all effect algebras E_1, E_2 such that there is an isomorphism $\phi : E \mapsto E_1 \times E_2$, $\phi^{-1}(1, 0)$ and $\phi^{-1}(0, 1)$ are central in E.*

A subset I of an effect algebra E is called an *ideal* iff the following condition is satisfied : $a, b \in I$, $a \perp b$ is equivalent to $a \oplus b \in I$. An ideal I is called *Riesz ideal* iff, for all i, a, b such that $i \in I$, $a \perp b$ and $i \leq a \oplus b$, there are i_1, i_2 such that $i_1 \leq a$, $i_2 \leq b$ and $i \leq i_1 \oplus i_2$. Riesz ideals were introduced in [12].

For a lattice ordered effect algebra E, it was proved in [19], that $C(E) = K(E) \cap E_S$. Moreover, as proved in [13], for a lattice ordered effect algebra E, E_S is a sublattice of E, a sub-effect algebra of E, and every block of E_S is the center of a block of E. In the remainder of this section, we will extend some of these results to the class of homogeneous effect algebras.

Proposition 4.2. *Let a be an element of a homogeneous effect algebra E. The following are equivalent.*

(a) *$a \in E_S$.*
(b) *a is central in every block of E which contains a.*
(c) *a is central in some block of E.*

Proof. (a) implies (b): Assume that $a \in E$ is sharp, let B be a block of E such that $a \in B$. Since a is sharp in E, a is sharp in B. We will prove that a is principal in B. Let $x_1, x_2 \in B$ be such that $x_1, x_2 \leq a$, $x_1 \perp x_2$. Since B is a sub-effect algebra of E, $x_1 \oplus x_2 \in B$. Since B is internally compatible, $x_1 \oplus x_2 \leftrightarrow a$ in B. By [5], Lemma 2, $x_1 \oplus x_2 \leftrightarrow a$ in B implies that there are $y_1, y_2 \in B$ such that $y_1 \leq a$, $y_2 \leq a'$ and $x_1 \oplus x_2 = y_1 \oplus y_2$. Since B satisfies Riesz decomposition property, $y_2 \leq x_1 \oplus x_2$ implies that there are $t_1, t_2 \in B$ such that $t_1 \leq x_1$, $t_2 \leq x_2$ and $y_2 = t_1 \oplus t_2$. For $i \in \{1, 2\}$, $t_i \leq a, a'$. Since a is sharp in B, this implies that $t_1 = t_2 = 0$. Thus, $x_1 \oplus x_2 = y_1 \leq a$ and a is principal in B and hence $[0, a] \cap B$ is an ideal in B. Since B satisfies Riesz decomposition property, every ideal in B is a Riesz ideal. By [5], an element a of an effect algebra is central iff $[0, a]$ is a Riesz ideal. Therefore, a is central in B.

(b) implies (c): By Corollary 3.13, every element of E is in some block.

(c) implies (a): Let $a \in C(B)$ for some block B, let $b \leq a, a'$. Since $B = \overline{B}$, $b \in B$. Thus, $b = 0$ and a is sharp. □

Corollary 4.3. *Let a be an element of an effect algebra E satisfying Riesz decomposition property. The following are equivalent.*

(a) $a \in E_S$.
(b) $a \in C(E)$.
(c) *a is principal.*

Proof. By Proposition 4.2, (a) is equivalent to (b). In every effect algebra, all principal elements are sharp. Every central element is principal. □

Corollary 4.4. *For a homogeneous effect algebra E, E_S is a sub-effect algebra of E. Moreover, E_S is an orthoalgebra.*

Proof. Obviously, $0, 1 \in E_S$ and E_S is closed with respect to $'$. Assume $a, b \in E_S$, $a \perp b$. Then $\{a, b\}$ is a finite compatible set. Thus, by Corollary 3.12, $\{a, b\}$ can be embedded into a block B. By Proposition 4.2, $a, b \in C(B)$. Since $C(B)$ is a sub-effect algebra of B, $a \oplus b \in C(B)$. By Proposition 4.2, $C(B) \subseteq E_S$, thus $a \oplus b \in E_S$.

Obviously, E_S is an orthoalgebra. □

Since, for a homogeneous effect algebra E, E_S is an orthoalgebra, every compatible subset of E_S can be embedded into a block of E_S, which is a Boolean algebra.

Proposition 4.5. *Let E be a homogeneous effect algebra. For every block B^0 in E_S and for every block B of E such that $B^0 \subseteq B$, $B^0 = C(B)$.*

Proof. Let B^0 be a block of E_S. Let B be a block of E with $B^0 \subseteq B$. By Proposition 4.2, $B^0 \subseteq C(B)$. Since B^0 is a block of E_S and $C(B)$ is a Boolean algebra, $B^0 \subseteq C(B)$ implies that $B^0 = C(B)$. □

Question 4.6. Let B be a block of a homogeneous effect algebra E. Is it true that $C(B)$ is a block of E_S?

Proposition 4.7. *In a homogeneous effect algebra, $C(E) = C(K(E)) = K(E)_S$.*

Proof. It is evident that $C(E) \subseteq C(K(E)) \subseteq K(E)_S$. Let $a \in K(E)_S$. We shall prove that $[0, a]$ is a Riesz ideal. By Lemma 2 of [5], this implies that $a \in C(E)$.

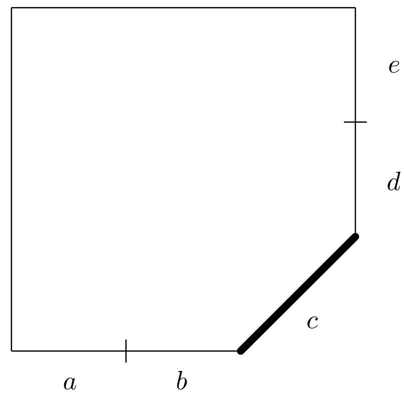

FIGURE 1.

Suppose $x_1, x_2 \leq a$, $x_1 \perp x_2$. Then $\{x_1, x_2\}$ can be embedded into a block B of E. Since $a \in K(E)$, $a \in B$. Since a is sharp, a is central in B. Thus, a is principal in B and hence $x_1 \oplus x_2 \leq a$. Therefore, a is principal in E. Let $i \in [0, a]$, $x \perp y$, $i \leq x \oplus y$. Similarly as above, $\{a, x, y\}$ can be embedded into a block B of E, such that $a \in C(B)$. Obviously, $i \leq (x \oplus y) \wedge a$ and, since a is central in B, $(x \oplus y) \wedge a = (x \wedge a) \oplus (y \wedge a)$. Thus, $[0, a]$ is a Riesz ideal. □

Question 4.8. Let E be a homogeneous effect algebra. Does $K(E)$ satisfy Riesz decomposition property ? This is true for orthoalgebras and for lattice ordered effect algebras.

5. Examples and counterexamples

It is easy to check, that a direct product of a finite number of homogeneous effect algebras is a homogeneous effect algebra.

Example 5.1. Let E_1 be an orthoalgebra. Let E_2 be an effect algebra satisfying Riesz decomposition property, which is not an orthoalgebra. If any of E_1, E_2 is not lattice ordered, then $E_1 \times E_2$ is an example of a homogeneous effect algebra which is not lattice ordered. Moreover, since E_2 is not an orthoalgebra, $E_1 \times E_2$ is not an orthoalgebra.

Another possibility to construct new homogeneous effect algebras from old is to make *horizontal sums* (sometimes called $0,1$-*pastings)*, which means simply identifying the zeroes and ones of the summands.

As shown in the next example, it is possible to construct a lattice ordered (and hence homogeneous) effect algebra by pasting of two MV-algebras in a central element.

Example 5.2. We borrowed the basic idea for this example from Cohen [7]. Consider a system consisting of a firefly in a box pictured in a Figure 1. The box has five windows, separated by thin lines. We shall consider two experiments on this system :

(A) Look at the windows a, b, c.
(B) Look at the windows c, d, e.

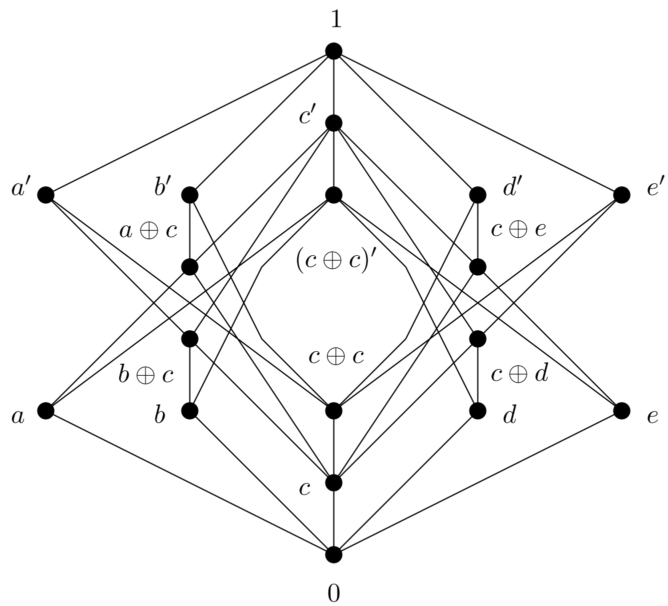

Figure 2. An eighteen elements lattice ordered effect algebra

Suppose that the window c is covered with a grey filter. Unless the firefly is shining very brightly at the moment we are performing the experiment, we cannot be sure that we see the firefly in the c window. The outcomes of experiment (A) are

- (a) We see the firefly in window a.
- (b) We see the firefly in window b.
- (c) We see the firefly in window c, with the level of (un)certainity $\frac{1}{2}$.
- $(c \oplus c)$ We see the firefly in window c.

The outcomes of (B) are similar. The unsharp quantum logic of our experiment is an eighteen elements lattice ordered effect algebra E with five atoms a, b, c, d, e, satisfying

$$a \oplus b \oplus c \oplus c = c \oplus c \oplus d \oplus e = 1.$$

The Hasse diagram of E is given by Figure 2. This effect algebra is constructed by pasting of two MV-algebras

$$A = \{0, a, b, c, a \oplus c, b \oplus c, c \oplus c, a', b', (c \oplus c)', c', 1\}$$

and

$$B = \{0, c, d, e, c \oplus c, c \oplus d, c \oplus e, d', e', (c \oplus c)', c', 1\}.$$

A and B are then blocks of E. The compatibility center of E is the MV-algebra

$$K(E) = \{0, c, c \oplus c, (c \oplus c)', c', 1\}$$

and the center of E is $\{0, c \oplus c, (c \oplus c)', 1\}$. E_S forms a twelve-elements orthomodular lattice with two blocks; each of them is isomorphic to the Boolean algebra 2^3 and they are pasted in one of their atoms (namely $c \oplus c$).

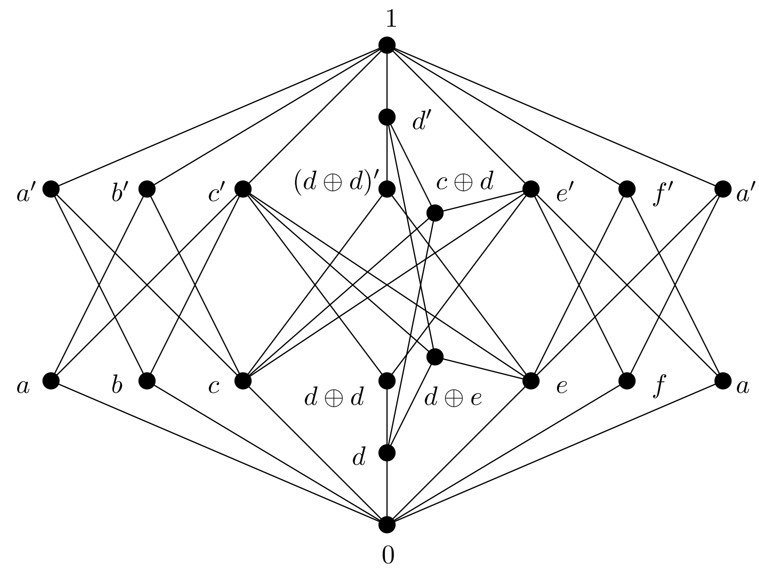

FIGURE 3. A non-lattice ordered homogeneous effect algebra

Example 5.3. Let E be an eighteen elements effect algebra with six atoms a, b, c, d, e, f, satisfying

$$a \oplus b \oplus c = c \oplus d \oplus d \oplus e = e \oplus f \oplus a = 1. \tag{3}$$

The Hasse diagram of E is given by Figure 3. This effect algebra is constructed by pasting of three blocks : two Boolean algebras

$$\begin{aligned} B_1 &= \{0, a, b, c, a', b', c', 1\} \\ B_2 &= \{0, e, f, a, e', f', a', 1\} \end{aligned}$$

and an MV-algebra

$$B_3 = \{0, c, d, e, d \oplus d, d \oplus e, c \oplus d, (d \oplus d)', c', d', e', 1\}.$$

By (3), it is easy to see that the range of every orthogonal family with three elements can be embedded into a block. Thus, by Corollary 3.14, E is homogeneous. All elements except for $d, d', c \oplus d, d \oplus e$ are sharp and E_S is an orthoalgebra with fourteen elements, called the *Wright triangle*, which is not an orthomodular poset.

Proposition 5.4. *Let E be a homogeneous effect algebra. Assume that there is an element $a \in E$ with $a \le a'$, such that E is isomorphic to $[0, a]_E$. Then E satisfies Riesz decomposition property.*

Proof. Let B be a block containing a. Since B is a maximal internally compatible subset of E, Corollary 3.8(b) implies that $[0, a] = \{x \in E : x \le a, a'\} \subseteq B$. This implies that $[0, a]_E$ satisfies Riesz decomposition property. Therefore, E satisfies Riesz decomposition property. □

Corollary 5.5. *For a Hilbert space $\mathbb{H}$, $\mathcal{E}(\mathbb{H})$ is homogeneous iff* $\dim(\mathbb{H}) \le 1$.

Proof. The map $\phi : \mathcal{E}(\mathbb{H}) \mapsto [0, \frac{1}{2}I]$ given by $\phi(A) = \frac{1}{2}A$ is obviously an isomorphism and $\frac{1}{2}I \le (\frac{1}{2}I)'$. Therefore, by Proposition 5.4, every homogeneous $\mathcal{E}(\mathbb{H})$

satisfies Riesz decomposition property. However, it is well known that $\mathcal{E}(\mathbb{H})$ satisfies Riesz decomposition property iff $\dim(\mathbb{H}) \leq 1$. □

The following example shows that a sub-effect algebra of a homogeneous effect algebra need not to be homogeneous.

Example 5.6. Let $E = [0,1] \times [0,1]$, where $[0,1] \subseteq \mathbb{R}$ denotes the unit interval of the real line. Equip E with a partial operation $\oplus$ with domain given by $(a_1, a_2) \perp (b_1, b_2)$ iff $a_1+b_1 \leq 1$ and $a_2+b_2 \leq 1$; then define $(a_1, a_2) \oplus (b_1, b_2) = (a_1+b_1, a_2+b_2)$. Then $(E, \oplus_E, (0,0), (1,1))$ is a homogeneous effect algebra (in fact, it is even an MV-algebra). Let

$$F = \{(x_1, x_2) \in E : x_1 + x_2 \in \mathbb{Q}\}$$

Since $(1,1) \in F$ and F is closed with respect to $\ominus$, F is a sub-effect algebra of E.

It is easy to see that the map $\phi : F \mapsto [(0,0), (\frac{1}{2}, \frac{1}{2})]_F$, given by $\phi(x_1, x_2) = (\frac{1}{2}x_1, \frac{1}{2}x_2)$ is an isomorphism. Note that F is not a compatible effect algebra: for example, $\{(1,0), (\frac{1}{\pi}, 1 - \frac{1}{\pi})\}$ is not compatible in F. Consequently, F does not satisfy Riesz decomposition property and hence, by Proposition 5.4, F is not homogeneous.

Example 5.7. Let μ be the Lebesgue measure on $[0,1]$. Let $E \subseteq [0,1]^{[0,1]}$ be such that, for all $f \in E$,

(a) f is measurable with respect to μ
(b) $\mu(\mathrm{supp}(f)) \in \mathbb{Q}$
(c) $\mu(\{x \in [0,1] : f(x) \notin \{0,1\}\}) = 0$,

where $\mathrm{supp}(f)$ denotes the support of f. It is easy to check that E is a sub-effect algebra of $[0,1]^{[0,1]}$. Obviously, E is not an orthoalgebra. We will show that E is a homogeneous, non-lattice ordered effect algebra and that E does not satisfy Riesz decomposition property. Note that, for all $u \in E$, $u \perp u$ iff $Ran(u) \subseteq [0, \frac{1}{2}]$ and $\mu(\mathrm{supp}(u)) = 0$. Thus, for all $u \in E$ and $u_0 \in [0,1]^{[0,1]}$ such that $u_0 \leq u$ and $u \perp u$, we have $u_0 \in E$.

Let $u, v_1, v_2 \in E$ be such that $u \leq v_1 \oplus v_2$, $u \leq (v_1 \oplus v_2)'$. Since $[0,1]^{[0,1]}$ is an MV-algebra, there are $u_1, u_2 \in [0,1]^X$ such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$. By above paragraph, $u \perp u$ and $u_1, u_2 \leq u \in E$ imply that $u_1, u_2 \in E$. Therefore, E is homogeneous. Let f, g be the characteristic functions of intervals $[0, \frac{2}{3}]$, $[\frac{1}{\pi}, \frac{1}{\pi} + \frac{1}{2}]$, respectively. Then $f \wedge g$ does not exist in E_S. Therefore, E_S is not lattice ordered and hence, by Theorem 3.3 of [13], E is not lattice ordered. Moreover, E does not satisfy Riesz decomposition property. Indeed, assume the contrary. Then, by Proposition 4.3, $E_S = C(E)$. In particular, E_S is then a Boolean algebra. However, this is a contradiction, since E_S is not lattice ordered.

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