ITEP-M-7/93
hep-th/9311142
November 1993

BOSONISATION OF COORDINATE RING OF $U_q(SL(N))$ THE CASES OF $N = 2$ and $N = 3$

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ABSTRACT

Non-abelian coordinate ring of $U_q(SL(N))$ (quantum deformation of the algebra of functions) for $N = 2, 3$ is represented in terms of conventional creation and annihilation operators. This allows to construct explicitly representations of this algebra, which were earlier described in somewhat more abstract algebraic fashion. Generalizations to $N > 3$ and Kac-Moody algebras are not discussed but look straightforward.

1 Free-field representations of Lie algebras

All algebras, relevant for applicatons in theoretical physics usually possess the "free-field representations" which allow to express all the generators of the algebra through creation and annihilation operators, i.e. to embed the original algbera into the universal envelopping of the (several copies of the) Heisenberg algebra. This kind of representations for the 1-loop (Kac-Moody, Virasoro, W- etc) algebras [1, 2, 3] is the basis of the modern conformal field theory. Their analogues for the ordinary (0-loop) Lie algebras is also well known and widely used.

For the simplest case of $SL(2)$ this is the familiar representation:

$$\begin{aligned} J^- &= \frac{d}{dx}, \\ J^0 &= x\frac{d}{dx} - j, \\ J^+ &= x^2\frac{d}{dx} - 2jx, \end{aligned} \tag{1}$$

and it can be considered as the "zero-mode" part of the Wakimoto representation [2, 3] for $\hat{SL_k}(2)$,

$$\begin{aligned} J^-(z) &= W(z), \\ J^0(z) &= \chi W(z) - q\partial\phi(z), \\ J^+(z) &= \chi^2 W(z) - 2q\chi\partial\phi(z) - k\partial\chi(z), \\ q = \sqrt{k+2}, &\quad W(z) = \frac{\delta}{\delta\chi(z)}, \end{aligned} \tag{2}$$

which for $k = 1$ turns into the standard Frenkel-Kac representation [1] for $\hat{SL_1}(2)$ [4],

$$\begin{aligned} J^\pm(z) &= \mathcal{C}_\pm e^{\pm\Phi(z)}, \\ J^0(z) &= \partial\Phi(z). \end{aligned} \tag{3}$$

The free-field representations should and can be generalized to include quantum groups. The analogue of 3 for $k = 1$ and $q \neq 1$ is known as the Frenkel-Jing representation [5] (see also [6]), generalizations of (2) for any k, though easy to derive, are explicitly available only in the non-transparent terms of three scalar fields (instead of one scalar and β, γ system), see [7] for a brief review. The analogue of (1) can be easily obtained in terms of the finite-difference operators [8], e.g.

$$
\begin{aligned}
J_q^- &= D^+, \\
q^{\pm J_q^0} &= q^{\mp j} M^\pm, \\
J_q^+ &= \frac{2q^{j+1}}{q+1} z^{2j+2} D^+ z^{-2j} M^-.
\end{aligned} \tag{4}
$$

Here

$$
\begin{aligned}
D^\pm f(x) &= \frac{f(x) - f(q^{\pm 1}x)}{(1-q^{\pm 1})x}, \\
M^\pm f(x) &= f(q^{\pm 1}x), \\
M^\pm &= I + (q^{\pm 1} - 1) x D^\pm.
\end{aligned} \tag{5}
$$

All these formulas (and their analogues for any N) can be easily deduced from the commutation relations for the generators T_α of any Lie algebra by the following procedure. Introduce

$$
\mathcal{F}(\mathbf{x}) = \langle \mathbf{j} | \prod_{\alpha > \mathbf{0}} \hat{e}_q \left(x_\alpha T_{-\alpha} \right) = \langle \mathbf{j} | \hat{e}_q \left(\sum_{\alpha > \mathbf{0}} f_q^\alpha(\mathbf{x}) T_{-\alpha} \right), \tag{6}
$$

where α are somehow ordered labels of all the generators, and the bra-vacuum $\langle \mathbf{j}|$ is annihilated by all the "positive" generators T_α, $\alpha > 0$ and is the eigenvector of all the "Cartanian" (mutualy commuting) ones, T_α, $\alpha \in \{\mathbf{0}\}$, eigenvalues being defined by the set $\mathbf{j}$. For non-quantum groups $q = 1$ and the q-exponent[1] $\hat{e}_q$ in (6) is substituted by the ordinary exponential function.

[1] Let us remind that the q-exponent $e_q(x) = 1/E_q(-x)$ is characterized by the following set of properties:

1. $D^+ \hat{e}_q(x) = \hat{e}_q(x)$; $\hat{e}_q(x) \equiv e_q\left((1-q)x\right)$ and $\lim_{q\to 1} \hat{e}_q(x) = e^x$;
2. $e_q(x) = \sum_{k\geq 0} \frac{x^k}{(q,q)_k}$, $E_q(x) = \sum_{k\geq 0} \frac{q^{k(k-1)/2} x^k}{(q,q)_k}$,

where $(a,q)_k \equiv \prod_{i=0}^{k-1} (1 - aq^i) = \frac{(a,q)_\infty}{(aq^k,q)_\infty}$;

3. $e_q(x) = \frac{1}{(x,q)_\infty}$, thus $E_q(x) = (-x,q)_\infty$ and $\theta_{00}(x) \equiv \sum_{k=-\infty}^{\infty} q^{k^2/2} x^k = (q,q)_\infty E_q(q^{1/2}x) E_q(q^{1/2}x^{-1})$;
4. $E_q(x)E_q(y) = E_q(x+y)$ and $e_q(y)e_q(x) = e_q(x+y)$, provided $xy = qyx$;
5. $E_q(y)E_q(x) = E_q(x+y+yx)$ and $e_q(x)e_q(y) = e_q(x+y-yx)$, provided $xy = qyx$.

Functions $f_1^\alpha(\mathbf{x})$ are polinomials in $\mathbf{x}$-variables, degree of the polinomial being equal to the number of items in decomposition of α in the sum of the simple roots. Representation of original algebra is now defined from the relation

$$J_\alpha \mathcal{F}(\mathbf{x}) = \mathcal{F}(\mathbf{x}) T_\alpha. \tag{7}$$

where T_α at the r.h.s. is "carried" through the exponential operator to act on the vacuum, and terms arising from commutation of operators can be imitated by taking x_β derivatines. Then

$$J_\alpha J_\beta \mathcal{F}(\mathbf{x}) = J_\alpha \mathcal{F}(\mathbf{x}) T_\beta = \mathcal{F}(\mathbf{x}) T_\alpha T_\beta. \tag{8}$$

In this way it is easy to derive not only (1-4), but also all the other formulas from ref.[3] and further papers on free-field representations.

2 Coordinate ring of the quantum group

The purpose of this letter is to discuss the free-field representation of the somewhat new object: the coordinate ring of the quantum group, which is an essential piece of the theory but has a trivial classical limit as $q = 1$, where it becomes just a free abelian algebra. For $q \neq 1$ this algebra is no longer abelian and provides a solution to the basic equation [9] $\mathcal{R}(T \otimes T) = (T \otimes T)\mathcal{R}$, where $\mathcal{R}$ is the R-matrix, i.e. solution of the Yang-Baxter equation.

In the case of $U_q(SL(2))$ $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where the elements a, b, c, d (the $q \neq 1$ analogues of matrix elements) are no longer c-numbers, but operators with the following commutation relations:

$$\begin{aligned} ab &= qba, \\ ac &= qca, \\ ad - da &= (q - q^{-1})bc, \\ bc &= cb, \end{aligned}$$

The first three properties explain the relevance of the q-special functions as solutions to finite difference equations (i.e. to various periodicity constraints), while the last two are crucial for occurence of the same functions in the study of non-commutative algebras and problems of quantum mechanics and quantum field theory.

$$bd = qdb,$$
$$cd = qdc. \tag{9}$$

This algebra can be also considered as that of the linear automorphisms of the "quantum phase plane" [10], parametrized by the non-commuting "coordinates"

$$u_1 u_2 = q u_2 u_1 \tag{10}$$

(they can be considered as exponentials of the coordinate and momentum operators, $u_1 = e^Q,\ u_2 = e^P,\ q = e^{ih},\ P = -ih\frac{d}{dQ}$).

Commutation relations for the entries of T-matrices, associated with $U_q(SL(N))$ can be easily described in terms of those for $U_q(SL(2))$. If $T = (A_{ij}),\ i,j = 1\ldots N$, then for any fixed $i < k,\ j < l$ the 2×2 matrix $\begin{pmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl} \end{pmatrix}$ has exactly the same properties as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in (9) (i.e. $A_{ij}A_{il} = qA_{il}A_{ij}$, $A_{ij}A_{kl} - A_{kl}A_{ij} = (q - q^{-1})A_{il}A_{jk}$ etc).

Below we describe representation of (9) and its analogue for $U_q(SL(3))$ in terms of annihilation and creation operators. The case of arbitrary N and Kac-Moody algebras will be discussed elsewhere, as well as their relation to the more sophisticated representations of the quantum group themselves (these are not the same, of course: compare (9) and (4)). Instead we discuss briefly representations of the coordinate ring, which were earlier found in pure algebraic terms in [11, 12] and can be now constructed explicitly in terms of functions of commuting variables.

3 Oscillator representation of the basic algebras

Before addressing the question of bosonization of the algebra (9), we consider the even simpler "quantum hyperplane" algebras (compare with (10). These will provide us with the building blocks for further constructions. We shall need two such algebras: the "chain" (or "quantum phase space") one,

$$w_i w_j = q w_j w_i \ \text{ for } j = i+1,$$
$$w_i w_j = w_j w_i \ \text{ for } |j-i| > 1, \tag{11}$$

and the "huperplane" one

$$u_i u_j = q u_j u_i \text{ for any } j > i. \tag{12}$$

Free-field representation (bosonization) expresses these generators through those of Heisenberg algebra,

$$[\alpha_i, \alpha_j^\dagger] = \delta_{ij} \log q. \tag{13}$$

Such representation for (11) is straightforward:

$$w_i = e^{\alpha_{i-1}^\dagger + \alpha_i}. \tag{14}$$

That for (12) can be obtained from (14):

$$u_i = :\prod_{k\le i} w_k: = :\exp\left(\sum_{k<i}\alpha_k^\dagger + \sum_{k\le i}\alpha_k\right): \tag{15}$$

All the operators involved are exponentials of linear combinations of creation and annihilation operators and normal ordering can be defined by just requesting that whenever the Wick theorem is applied for evaluation of correlation functions and/or commutation relations, no contractions are included of operators standing under the normal ordering signs. For any such operators

$$\mathcal{O} = :\exp\left(\sum_k A_k\alpha_k + \sum_k B_k\alpha_k^\dagger\right): \tag{16}$$

we have:

$$\mathcal{O}_1 \cdot \mathcal{O}_2 = \sqrt{\epsilon_{12}} :\mathcal{O}_1\mathcal{O}_2: = \epsilon_{12}\mathcal{O}_2 \cdot \mathcal{O}_1 \tag{17}$$

where the c-number $\epsilon_{12} = q^{\sum_k\left(A_k^{(1)}B_k^{(2)} - B_k^{(1)}A_k^{(2)}\right)}$. Since all the operators below will be of the form (16), in what follows we use (17) without special reference.

4 The case of $U_q(SL(2)$

.

We can now describe bosonization formulas for the algebra (9). For this purpose we need two mutualy commuting copies of algebra (12), their generators will be denoted by $\{u_i\}$ and $\{v_i\}$, $u_i v_j = v_j u_i$. Then (9) is immediately satisfied, if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}. \tag{18}$$

The only relation which is a little non-trivial is $ad - da = (q - q^{-1})bc$. It is at this place that (17) plays the crucial role. Indeed,

$$\begin{aligned} ad &= : u_1 v_1 : \; \cdot \; : u_2 v_2 : = q : u_1 v_1 u_2 v_2 :, \\ da &= : u_2 v_2 : \; \cdot \; : u_1 v_1 : = q^{-1} : u_1 v_1 u_2 v_2 :, \\ bc &= : u_2 v_1 : \; \cdot \; : u_1 v_2 : =: u_1 v_1 u_2 v_2 := cb. \end{aligned} \tag{19}$$

Representation (18) has an obvious generalization for $U_q(SL(N))$ with any N: it is enough to take

$$T = (A_{ij}) = (u_i v_j)\,. \tag{20}$$

However, both (18) and (20) are *non-generic* representations: they are actually degenerate. This is clear, because the c-number $D = \det_q T = ad - qbc = da - q^{-1}bc$ in the case of (18) is identically vanishing: $D = 0$. In the case of (20) the situation is even worse: not only the full determinant of the matrix T, but also all its minors are identiically vanishing. We now proceed to description of generic, non-degenerate representations.

For the case of $U_q(SL(2))$ it is very simple to introduce the necessary correction: instead of (18) one can take

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 + D_2 \frac{1}{u_1 v_1} \end{pmatrix}. \tag{21}$$

Here $D_2 = \det_q T$ is a c-number, commuting with all the operators u and v. Since u's and v's are of the form (16) there are no problems with the definition of their negative powers. Using (12) along with its obvious corollary,

$$u_i \frac{1}{u_j} = q \frac{1}{u_j} u_i, \text{ for } i > j, \tag{22}$$

it is easy to check that all the relations (9) are still true for representation (21).

5 The case of $U_q(SL(3))$

.

The non-degenerate representation now looks as follows:

$$T = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 \cdot W & u_2v_3 \cdot W \\ u_3v_1 & u_3v_2 \cdot W & u_3v_3 \cdot W + D_3 : \frac{u_4v_4}{u_3v_3} : \end{pmatrix};$$

$$W = 1 + \frac{1}{q} : \frac{u_3v_3}{u_1v_1u_2v_2u_4v_4} : \tag{23}$$

or

$$T = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 + : \frac{u_3v_3}{u_1v_1u_4v_4} : & u_2v_3 + : \frac{u_3v_3^2}{u_1v_1v_2u_4v_4} : \\ u_3v_1 & u_3v_2 + : \frac{u_3^2v_3}{u_1v_1u_2u_4v_4} : & u_3v_3 + : \frac{u_3^2v_3^2}{u_1v_1u_2v_2u_4v_4} : + D_3 : \frac{u_4v_4}{u_3v_3} : \end{pmatrix} \tag{24}$$

The q-determinant of the entire matrix T is equal to $\det_q T = D_3$ and commutes with all the entries of T. The extra terms, appering in expressions for d, f, h, k are important to make the 2×2 minors of T non-degenerate, for example:

$$\Delta_{33} \equiv ad - qbc = : \frac{u_3v_3}{u_4v_4} : \tag{25}$$

It is clear that Δ_{33} commutes with all the other contributions to the elements a, b, c, d of the $U_q(SL(2))$ subalgebra. Representation of this subalgebra reduces to (21) if Δ_{33} is identified with D_2. However, in $U_q(SL(3))$ Δ_{33} is no longer a c-number, but it still has simple (multiplicative-like) commutation relation with k:

$$\Delta_{33}k - q^2k\Delta_{33} = (1 - q^2)D_3 \tag{26}$$

Such simple relations arise as a rule for q-commutators of the minors for all the $U_q(SL(N))$ algebras [12].

The check that all the commutation relations of $U_q(SL(3)$ are satisfied for representation (24) is a somewhat tedious but straightforward exercise. Again one should repeatedly make use of the relation (17). As an example, let us check that $fk = qkf$. This is true since

$$fk = \left(u_2v_3 + :\frac{u_3v_3^2}{u_1v_1v_2u_4v_4}:\right)\cdot\left(u_3v_3 + :\frac{u_3^2v_3^2}{u_1v_1u_2v_2u_4v_4}: + D_3 :\frac{u_4v_4}{u_3v_3}:\right) =$$

$$= q^{1/2} : u_2u_3v_3^2 : \ + (q^{3/2}+q^{-1/2}) : \frac{u_3^2v_3^3}{u_1v_1v_2u_4v_4} : \ +$$

$$+ \ q^{1/2} : \frac{u_3^3v_3^4}{u_1^2v_1^2u_2v_2^2u_4^2v_4^2} : + \ q^{1/2}D_3\left(: \frac{u_2u_4v_4}{u_3} : + : \frac{v_3}{u_1v_1v_2} : \right) \tag{27}$$

while

$$kf = q^{-1/2} : u_2u_3v_3^2 : \ + (q^{-3/2}+q^{1/2}) : \frac{u_3^2v_3^3}{u_1v_1v_2u_4v_4} : \ +$$

$$+ \ q^{-1/2} : \frac{u_3^3v_3^4}{u_1^2v_1^2u_2v_2^2u_4^2v_4^2} : + \ q^{-1/2}D_3\left(: \frac{u_2u_4v_4}{u_3} : + : \frac{v_3}{u_1v_1v_2} : \right) \tag{28}$$

6 On representation theory of coordinate rings

According to [11, 12] representations of the $U_q(SL(N))$ can be described in terms of vectors, obtained by the action of certain "creation operators"[2] on the "vacuum". The "vacuum" is defined as the common eigenvector of the maximum set of commuting generators of coordinate ring, while "creation operators" are certain minors of the matrix T [12]. Realization of T in terms of the generators of Heisenberg algebra allows to construct all these objects explicitly. We present here only the example of $U_q(SL(2))$.

The maximum set of commuting generators for generic q consists of b and c. Thus "vacuum" state is defined to satisfy

$$b|vac\rangle = \mu|vac\rangle, \quad c|vac\rangle = \nu|vac\rangle. \tag{29}$$

In order to obtain a highest weight representation one also requires that

$$d|vac\rangle = 0, \tag{30}$$

and the entire representation is formed by the vectors

$$a^n|vac\rangle. \tag{31}$$

[2]Not to be mixed with the Heisenberg operators $\alpha^\dagger, \ \beta^\dagger$.

For special values of $q = e^{2\pi i/p}$ with integer p the set of commuting operators is actually bigger: d^p commutes with both b and c and instead of (30) "vacuum" can be defined to be also an eigenstate of d^p:

$$d^p|vac\rangle = \delta|vac\rangle, \quad \text{for } q^p = 1, \ p \in Z. \tag{32}$$

Using (21) we can now represent a, b, c, d through a pair of Heisenberg creation and annihilation operators:

$$\begin{aligned} a = e^{\alpha+\beta}, \quad b = e^{\alpha^\dagger+\beta}, \quad c = e^{\beta^\dagger+\alpha}, \\ d = e^{\alpha^\dagger+\beta^\dagger} + D_2 e^{-\alpha-\beta}. \end{aligned} \tag{33}$$

It is now possible to represent Heisenberg generators in terms of differential operators:

$$\begin{aligned} \alpha^\dagger = x\log q, \quad \alpha = \frac{d}{dx}, \\ \beta^\dagger = y\log q, \quad \beta = \frac{d}{dy}. \end{aligned} \tag{34}$$

Then a, b, c, d acquire the form of finite-difference operators:

$$\begin{aligned} a = m_x^+ m_y^+, \quad b = q^y m_x^+, \quad c = q^x m_y^+, \quad d = q^{x+y} + D_2 m_x^- m_y^-, \\ m_x^\pm \equiv e^{d/dx}, \quad m_x^\pm f(x) = f(x\pm 1). \end{aligned} \tag{35}$$

Therefore as long as we deal with coordinate ring only and not with the Heisenberg algebra itself x and y can be considered as variables on the integer lattice: $x, y \in Z$. Solution to eqs.(29) is now given by

$$|vac\rangle \sim q^{-xy}\mu^x\nu^y \equiv |\mu,\nu\rangle. \tag{36}$$

Conditions (30) or (32) can be now considered as definitions of D_2 in (35) in terms of $\mu, \ \nu$ and δ: (30) implies that

$$D_2 = -q\mu\nu, \tag{37}$$

while (32) means that

$$\delta = \prod_{k=1}^{p}\left(1 + \frac{D_2}{q^{2k+1}\mu\nu}\right). \tag{38}$$

Representation itself consists of the states

$$|n \gg \equiv a^n|vac >\sim q^{-(x+n)(y+n)}\mu^{x+n}\nu^{y+n} = q^{-n^2}\mu^n\nu^n|q^{-n}\mu, q^{-n}\nu\rangle. \qquad (39)$$

For $q = e^{2\pi i/p}$ there are at most p linearly independent states in this representation. Moreover, for even p there are actualy irreducible representations of the size $p/2$. This is clear from the fact that all the states (39) are eigenstates of b and c, and action of operators a and d in (35) does not change parity of the integer-valued combination $x+y$. Thus representation (39) can be defined on the sublattice $x, y \in Z$, $x + y \in 2Z$ and $| - \mu, -\nu\rangle$ can be identified with $|\mu, \nu\rangle$.

7 Acknowledgements

I am indebted to A.Bougourzi, V.Karimipour and L.Vinet for explanations and discussions which stimulated my interest to this subject.

References

[1] I.Frenkel and V.Kac *Invent.Math.* **62** (1980) 23.

[2] M.Wakimoto *Comm.Math.Phys.* **104** (1986) 605.

[3] A.Gerasimov et al. *Int.J.Mod.Phys.* **A5** (1990) 2495.

[4] A.Gerasimov et al. *Nucl.Phys.* **B328** (1989) 664; *Phys.Lett.* **236B** (1990) 269.

[5] I.Frenkel and N.Jing *Proc.Nat'l.Acad.Sci.(USA)* **85** (1988) 9373.

[6] L.Vinet et al. *q-Hypergeometric Functions in the Formalism of Free Fields*, preprint CRM-1884, June 1993.

[7] A.Bougourzi, *Uniqueness of the Bosonization of the $U_q(su(2)_k)$ Quantum Current Algebra*, preprint CRM-1852, January 1993.

[8] R.Floreanini and L.Vinet, *to appear*

[9] L.Faddeev, N.Reshetikhin and L.Takhtadjan, *Algebra and Analiz* **1** (1989) 178.

[10] Yu.Manin, preprint CRM, 1988.

[11] M.Ge, X.Liu and C.Sun *J.Math.Phys.* **38** (1992) 7.

[12] V.Karimipour, *Lett.Math.Phys.* **28** (1993) 207; *Representations of the Coordinate Ring of* $GL_q(n)$, preprint ITPM, Tehran, 1993.