

TROPICALIZING VS COMPACTIFYING THE TORELLI MORPHISM

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ABSTRACT. In this paper, we compare the compactified Torelli morphism \bar{t}_g (as defined by V. Alexeev) and the tropical Torelli map t_g^{tr} (as defined by the author in a joint work with S. Brannetti and M. Melo, and furthered studied by M. Chan). Our aim is twofold: on one hand, we will review the construction and main properties of \bar{t}_g and t_g^{tr} , focusing in particular on the description of their fibers achieved by the author in joint works with L. Caporaso; on the other hand, we will clarify the relationship between \bar{t}_g and t_g^{tr} via the introduction of the reduction maps and the tropicalization maps.

1. INTRODUCTION

The Torelli morphism (or map) $t_g : \mathcal{M}_g \rightarrow \mathcal{A}_g$ (for $g \geq 2$) is the morphism from the moduli stack \mathcal{M}_g of connected smooth projective curves of genus g to the moduli stack \mathcal{A}_g of principally polarized (or p.p. for short) abelian varieties of dimension g , sending a curve into its Jacobian. The Torelli morphism t_g has played a central role since the early developments of classical algebraic geometry since it establishes a bridge between the two most studied moduli stacks, namely \mathcal{M}_g and \mathcal{A}_g . We just want to mention two classical results on the Torelli morphism: the Torelli theorem which says that t_g is injective (on geometric points); the Schottky problem which asks for a characterization of the image of t_g .

In this paper, we are interested in two other maps that have been recently defined starting from the Torelli morphism: the **compactified Torelli morphism** \bar{t}_g (defined in [Ale04] based upon the work [Ale02]) and the **tropical Torelli map** t_g^{tr} (defined in [BMV11] based upon the work [CV10], and furthered studied in [Cha12]). Our aim is twofold: on one hand, we will review the definitions and main properties of the above maps, focusing in particular on the description of their fibers achieved in [CV10] and [CV11]; on the other hand, we will explain and better clarify the relationship between \bar{t}_g and t_g^{tr} . Before we can state the results of this paper, we need to briefly recall the definitions of \bar{t}_g and t_g^{tr} .

The moduli stacks \mathcal{M}_g and \mathcal{A}_g are irreducible and separated but not proper. However, they both admit a modular compactification: \mathcal{M}_g is an open and dense substack of the Deligne-Mumford [DM69] moduli stack $\overline{\mathcal{M}}_g$ of stable curves (which we review in Subsection 2.1); \mathcal{A}_g is an open and dense substack of the main irreducible component $\overline{\mathcal{A}}_g$ of the Alexeev [Ale02] moduli stack of p.p. stable semi-abelic pairs (which we review in Subsection 3.2).

V. Alexeev has shown in [Ale04] that the Torelli morphism t_g extends to a modular morphism $\bar{t}_g : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{A}}_g$, called the *compactified Torelli morphism*, which sends a stable curve into its compactified Jacobian of degree $g - 1$; see Subsection 4.2 for more details. The fibers of the compactified Torelli morphism \bar{t}_g have been described by Caporaso-Viviani in [CV11]. In particular, it turns out that \bar{t}_g is injective on the open subset of $\overline{\mathcal{M}}_g$ consisting of stable curves that do not have separating edges nor separating pairs of edges or, in other words, stable curves whose dual graph is 3-edge-connected (see Corollary 4.2.11). We recall the precise

description of the fibers of \bar{t}_g in Subsection 4.2.4, where we also point out a new interesting relationship with the image of the canonical morphism (see Theorem 4.2.13).

On the tropical side of the picture, Brannetti-Melo-Viviani [BMV11] (based on the work of Caporaso-Viviani in [CV10]) constructed the moduli space M_g^{tr} of tropical curves of genus g and the moduli space A_g^{tr} of tropical p.p. abelian varieties of dimension g . The spaces M_g^{tr} and A_g^{tr} are constructed in loc. cit. as stacky fans, i.e. connected topological spaces obtained by gluing in a suitable way cones quotiented out by finite automorphism groups. In particular, M_g^{tr} and A_g^{tr} are endowed with a natural topology, which we call the euclidean topology, with respect to which they are Hausdorff spaces, as proved by L. Caporaso [Cap12] for M_g^{tr} and by M. Chan [Cha12] for A_g^{tr} . We refer the reader to Subsection 2.2 for more details on M_g^{tr} and to Subsection 3.1 for more details on A_g^{tr} .

In [BMV11], the authors also construct a map $t_g^{\text{tr}} : M_g^{\text{tr}} \rightarrow A_g^{\text{tr}}$, called the *tropical Torelli map*, which sends a tropical curve into its tropical Jacobian. In loc. cit., it is shown that t_g^{tr} is a map of stacky fans; in particular, it is a continuous map. The fibers of the tropical Torelli map t_g^{tr} have been described by Caporaso-Viviani in [CV10]. In particular, it turns out that t_g^{tr} is injective on the locally closed subset of M_g^{tr} consisting of tropical curves whose underlying graph is 3-vertex-connected and has genus g (see Corollary 4.1.16). We recall the precise description of the fibers of t_g^{tr} in Subsection 4.1.8 (see Fact 4.1.15).

The main motivation of this work is the following natural

Question: *What is the relationship between the compactified Torelli morphism \bar{t}_g and the tropical Torelli map t_g^{tr} ?*

In order to answer the above question, let us fix a complete DVR (= discrete valuation ring) R with maximal ideal \mathfrak{m} and assume that its residue field $k := R/\mathfrak{m}$ is algebraically closed. Let K be the quotient field of R and $\text{val} : K \rightarrow \mathbb{Z} \cup \{\infty\}$ the associated valuation. Note that the valuation val induces a topology on K , which is called the non-archimedean topology (see Notations 1.0.2). The sets $\mathcal{M}_g(K)$ and $\mathcal{A}_g(K)$ of K -valued points of, respectively, \mathcal{M}_g and \mathcal{A}_g inherit a topology from the topology on K , which we also call non-archimedean topology. The classical Torelli morphism $t_g : \mathcal{M}_g \rightarrow \mathcal{A}_g$ induces a continuous map $\mathcal{M}_g(K) \rightarrow \mathcal{A}_g(K)$ which, by a slight abuse of notation, we also denote by t_g .

On the other hand, the compactified Torelli morphism \bar{t}_g induces a map $\overline{\mathcal{M}}_g(k) \rightarrow \overline{\mathcal{A}}_g(k)$ between the k -valued points of, respectively, $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{A}}_g$ (which we also denote by \bar{t}_g by a slight abuse of notation); moreover this map is continuous with respect to the Zariski topologies on $\overline{\mathcal{M}}_g(k)$ and on $\overline{\mathcal{A}}_g(k)$.

After these preliminaries, we can state the first main result of this note, which can be seen as an answer to the above Question.

Theorem A. *There is a commutative diagram of sets*

$$(1.1) \quad \begin{array}{ccccc} \overline{\mathcal{M}}_g(k) & \xleftarrow{\text{red}} & \mathcal{M}_g(K) & \xrightarrow{\text{trop}} & M_g^{\text{tr}} \\ \downarrow \bar{t}_g & & \downarrow t_g & & \downarrow t_g^{\text{tr}} \\ \overline{\mathcal{A}}_g(k) & \xleftarrow{\text{red}} & \mathcal{A}_g(K) & \xrightarrow{\text{trop}} & A_g^{\text{tr}} \end{array}$$

where the vertical maps are continuous if we put the Zariski topology on the spaces on the left hand side, the non-archimedean topology on the spaces in the middle and the euclidean topology on the spaces on the right hand side.

The spaces appearing on the left and on the right hand sides of the above diagram (1.1) admit natural stratifications into locally closed subsets and our second main result concerns the compatibility of these stratifications with respect to the above reduction maps red and tropicalization maps trop . Let us briefly review how these stratifications are defined.

On one hand, to every stable weighted graph (Γ, w) of genus g , we associate two locally closed subsets

$$\begin{aligned}\overline{\mathcal{M}}_g(\Gamma, w) &\subset \overline{\mathcal{M}}_g(k), \\ M_g^{\text{tr}}(\Gamma, w) &\subset M_g^{\text{tr}},\end{aligned}$$

where $\overline{\mathcal{M}}_g(\Gamma, w)$ consists of all stable curves $X \in \overline{\mathcal{M}}_g(k)$ whose dual graph is (Γ, w) and $M_g^{\text{tr}}(\Gamma, w)$ consists of all tropical curves $C \in M_g^{\text{tr}}$ whose underlying combinatorial type is (Γ, w) . As observed already in [BMV11, Sec. 6.3], the above stratifications of $\overline{\mathcal{M}}_g(k)$ and of M_g^{tr} enjoy the following duality property with respect to the inclusions among the closures of strata:

$$(1.2) \quad \overline{\mathcal{M}}_g(\Gamma, w) \subseteq \overline{\overline{\mathcal{M}}_g(\Gamma', w')} \iff \overline{M_g^{\text{tr}}(\Gamma, w)} \supseteq M_g^{\text{tr}}(\Gamma', w').$$

We refer the reader to Section 2 for more details.

On the other hand, to every equivalence class of Delaunay decompositions $[\Delta]$ of \mathbb{R}^g , we associate two locally closed subsets

$$\begin{aligned}\overline{\mathcal{A}}_g([\Delta]) &\subset \overline{\mathcal{A}}_g(k), \\ A_g^{\text{tr}}([\Delta]) &\subset A_g^{\text{tr}},\end{aligned}$$

where $\overline{\mathcal{A}}_g([\Delta])$ consists of all p.p. stable semi-abelic pairs whose associated Delaunay decomposition is $[\Delta]$ and $A_g^{\text{tr}}([\Delta])$ consists of all tropical p.p. abelian varieties whose associated Delaunay decomposition is $[\Delta]$. Also the stratifications of $\overline{\mathcal{A}}_g(k)$ and of A_g^{tr} enjoy a similar duality property:

$$(1.3) \quad \overline{\mathcal{A}}_g([\Delta]) \subseteq \overline{\overline{\mathcal{A}}_g([\Delta'])} \iff \overline{A_g^{\text{tr}}([\Delta])} \supseteq A_g^{\text{tr}}([\Delta']).$$

We refer the reader to Section 3 for more details.

Theorem B.

(i) For any stable weighted graph (Γ, w) of genus g , it holds that

$$\text{red}^{-1}(\overline{\mathcal{M}}_g(\Gamma, w)) = \text{trop}^{-1}(M_g^{\text{tr}}(\Gamma, w)).$$

(ii) For any equivalence class of Delaunay decompositions $[\Delta]$ of \mathbb{R}^g , it holds that

$$\text{red}^{-1}(\overline{\mathcal{A}}_g([\Delta])) = \text{trop}^{-1}(A_g^{\text{tr}}([\Delta])).$$

We have observed in Theorem A that the vertical maps appearing in diagram (1.1) are continuous with respect to the topologies specified in loc. cit. What about the continuity properties of the reduction maps red and of the tropicalization maps trop ? For the reduction maps red , we can prove the following

Theorem C. *In the diagram (1.1), the reduction maps red are anticontinuous (i.e. the inverse image of a closed subset is open).*

Indeed, we prove more generally that for any stack \mathcal{X} proper over $\text{Spec } R$ the natural reduction map $\mathcal{X}(K) \rightarrow \mathcal{X}(k)$ is anticontinuous (see Corollary 5.0.17).

As far as the tropicalization maps trop are concerned, we make the following

Conjecture. *In the diagram (1.1), the tropicalization maps trop are continuous.*

Note that a positive answer to the above Conjecture, together with Theorem C, could be regarded as a conceptual explanation of the duality (1.2) among the

stratifications of M_g^{tr} and of $\overline{\mathcal{M}}_g(k)$ as well as of the duality (1.3) among the stratifications of A_g^{tr} and of $\overline{\mathcal{A}}_g(k)$.

While this work was been completed, we heard from [Rab] of a work in progress of M. Baker and J. Rabinoff, where they will prove the commutativity of the right square of diagram (1.1) in greater generality, namely working over an arbitrary non-archimedean (not necessarily discrete) valued field K and replacing the topological spaces in the middle with the (bigger) Berkovich analytifications of \mathcal{M}_g and of \mathcal{A}_g .

Moreover, while this paper was under the refereeing process, the interesting preprint [ACP] by D. Abramovich, L. Caporaso and S. Payne was posted on arXiv. In [ACP, Thm. 1.2.1(1)], the authors prove that the compactification $\overline{M}_g^{\text{tr}}$ of M_g^{tr} constructed by L. Caporaso in [Cap12, §3.3] is isomorphic to the skeleton of the Berkovich analytification $\overline{\mathcal{M}}_g^{\text{an}}$ of $\overline{\mathcal{M}}_g$. Moreover, they show in [ACP, Thm. 1.2.2(2)] that the tropicalization map $\text{trop} : \mathcal{M}_g(K) \rightarrow M_g^{\text{tr}}$ of Theorem A extends to a continuous, proper and surjective map $\text{Trop} : \overline{\mathcal{M}}_g^{\text{an}} \rightarrow \overline{M}_g^{\text{tr}}$. As a corollary, one gets that the tropical map $\text{trop} : \mathcal{M}_g(K) \rightarrow M_g^{\text{tr}}$ is continuous, thus providing a positive partial answer to the above Conjecture.

We conclude this introduction with an outline of the paper and with the notations we are going to use throughout the paper.

1.0.1. Outline of the paper

In Section 2, we review the definition and main properties of the moduli stack $\overline{\mathcal{M}}_g$ of Deligne–Mumford stable curves (Subsection 2.1) and of the moduli space M_g^{tr} of tropical curves (Subsection 2.2). Moreover, we define the reduction map for curves in 2.1.9, the tropicalization map for curves in 2.2.3, and we prove Theorem B(i) at the end of the section.

In Section 3, we review the definition and main properties of the moduli space A_g^{tr} of tropical p.p. abelian varieties (Subsection 3.1) and of the main component $\overline{\mathcal{A}}_g$ of the moduli stack of Alexeev p.p. stable semiabelic pairs (Subsection 3.2). Moreover, we define the tropicalization map for abelian varieties in 3.1.7, the reduction map for abelian varieties in 3.2.6 and we prove Theorem B(ii) at the end of the section.

Section 4 is devoted to the two Torelli maps t_g^{tr} and \overline{t}_g . In Subsection 4.1, we review the definition of the tropical Torelli map t_g^{tr} and we prove the second half of Theorem A, i.e. that the tropicalization maps commute with the Torelli maps (see Theorem 4.1.7). Moreover, we recall the description obtained in [CV10] of the fibers of t_g^{tr} in 4.1.8. In Subsection 4.2, we review the definition of the compactified Torelli morphism \overline{t}_g and we prove the first half of Theorem A, i.e. that the reduction maps commute with the Torelli maps (see Theorem 4.2.3). Moreover, we recall the description obtained in [CV11] of the fibers of \overline{t}_g in 4.2.4. Section 4 ends with a new description of the fibers of \overline{t}_g on the locus of curves free from separating nodes and not hyperelliptic in terms of their canonical morphisms (see Theorem 4.2.13).

Finally, in Section 5, we prove Theorem C, i.e. the anticontinuity of the reduction maps. Indeed, we show that the same result is true for any stack proper over $\text{Spec } R$ (see Corollary 5.0.17).

1.0.2. Notations

- Throughout the paper, we fix an integer $g \geq 2$.
- We fix a complete¹ discrete valuation ring (DVR for short) R with maximal ideal \mathfrak{m} and we assume that its residue field $k := R/\mathfrak{m}$ is algebraically

¹Indeed, everything that we are going to say in this paper can be extended to a strictly Henselian discrete valuation ring R . However, for simplicity, we assume that R is complete.

closed. Given an element $x \in R$, we denote by $\bar{x} \in k$ its reduction modulo the maximal ideal \mathfrak{m} . Let K be the quotient field of R .

- We denote by s the closed (or special) point of $\text{Spec } R$ and by η its generic point. In particular, the residue field of s is equal to k while the residue field of η is equal to K .
- Let $\text{val} : K \rightarrow \mathbb{Z} \cup \{\infty\}$ be the valuation associated to the discrete valuation ring R . The valuation val induces a non-archimedean norm $|\cdot|$ on K defined as

$$|x| := e^{-\text{val}(x)},$$

where e is the Euler number (indeed for the purpose of what follows we can replace e by any positive real number). The norm $|\cdot|$ induces a metric d on K defined by

$$d(x, y) = |x - y|.$$

The topology on K induced by this metric d is called the *non-archimedean topology* on K . We endow $R \subset K$ with the subspace topology, which is called the non-archimedean topology on R . Note that the maximal ideal $\mathfrak{m} \subset R$ coincides with the open ball of radius 1 centered at 0:

$$\mathfrak{m} = \{x \in R : |x| < 1\} = \{x \in R : d(0, 1) < 1\}.$$

Similarly, the product topologies on R^n and K^n are called non-archimedean topologies.

- Given any finite extension of fields $K \subseteq K'$, the valuation val on K can be extended in a unique way to a valuation val' on K' (using the fact that K is complete with respect to val). The valuation ring of val' , also called the valuation ring of K' and denoted by R' , is also equal to the integral closure of R in the field K' . Note that the valuation ring R' is also a complete DVR. With a slight abuse of notation, we denote by s the special point of $\text{Spec } R'$ and by η the generic point of $\text{Spec } R'$.
- A map $f : X \rightarrow Y$ between topological spaces is said to be *anticontinuous* if the inverse image of any closed subset of Y is open in X , or equivalently if the inverse image of any open subset of Y is closed in X .

2. MODULI SPACES OF CURVES

2.1. The moduli stack $\overline{\mathcal{M}}_g$ of stable curves. The moduli stack \mathcal{M}_g of connected smooth projective curves of genus $g \geq 2$ can be compactified by adding stable curves.

Definition 2.1.1. A *stable curve* X of genus g over a field k is a connected projective nodal curve over k of arithmetical genus g whose canonical sheaf ω_X is ample.

The following celebrated result is due to Deligne-Mumford [DM69].

Fact 2.1.2 (Deligne-Mumford). *The stack $\overline{\mathcal{M}}_g$ of stable curves of genus g is proper and smooth over $\text{Spec } \mathbb{Z}$. Moreover, $\overline{\mathcal{M}}_g$ is irreducible of dimension $3g - 3$ and it contains \mathcal{M}_g as a dense open substack.*

2.1.3. *The stratification of $\overline{\mathcal{M}}_g(k)$*

The set $\overline{\mathcal{M}}_g(k)$ of all stable curves of genus g defined over k endowed with its Zariski topology admits a stratification into locally closed subspaces, parametrized by stable weighted graphs of genus g , whose definition we recall below.

Definition 2.1.4. A *weighted graph* is a couple (Γ, w) consisting of a finite connected graph Γ (possibly with loops or parallel edges) and a function $w : V(\Gamma) \rightarrow \mathbb{N}$, called the weight function. A weighted graph is called *stable* if any vertex v of weight

zero (i.e. such that $w(v) = 0$) has valence $\text{val}(v) \geq 3$. The total weight of a weighted graph (Γ, w) is

$$|w| := \sum_{v \in V(\Gamma)} w(v),$$

and the genus of (Γ, w) is equal to

$$g(\Gamma, w) := g(\Gamma) + |w|.$$

Given a weighted graph (Γ, w) , the automorphism group $\text{Aut}(\Gamma, w)$ of (Γ, w) consists of all the pairs (σ, ψ) where σ is a permutation of the vertices $V(\Gamma)$ of Γ and ψ is a permutation of the edges $E(\Gamma)$ of Γ such that:

- $w(\sigma(v)) = w(v)$ for any $v \in V(\Gamma)$;
- if an edge $e \in E(\Gamma)$ is incident to a vertex $v \in V(\Gamma)$ then $\psi(e)$ is incident to $\sigma(v)$.

To every stable curve X of genus g it is naturally associated a stable weighted graph of genus g , called its dual weighted graph, which captures the combinatorics of the stable curve.

Definition 2.1.5. The *dual weighted graph* of a stable curve X of genus g is the weighted graph (Γ_X, w_X) defined as it follows.

The vertices $V(\Gamma_X)$ of the graph Γ_X are in bijection with the irreducible components of X while the edges $E(\Gamma_X)$ of X are in bijection with the nodes of X . An edge $e \in E(\Gamma_X)$ corresponding to a node n_e of X links the (possibly equal) vertices v_1 and v_2 corresponding to the (possibly equal) irreducible components C_{v_1} and C_{v_2} which contain the node n_e .

The weight function $w_X : V(\Gamma_X) \rightarrow \mathbb{N}$ assigns to every vertex v of Γ_X the geometric genus of the irreducible component C_v corresponding to the vertex v .

It is easy to check that the dual weighted graph (Γ_X, w_X) of a stable curve X of genus g is stable and of genus g .

To every stable weighted graph (Γ, w) of genus g , we associate the following subset of $\overline{\mathcal{M}}_g(k)$:

$$(2.1) \quad \overline{\mathcal{M}}_g(\Gamma, w) := \{X \in \overline{\mathcal{M}}_g(k) : (\Gamma_X, w_X) = (\Gamma, w)\}.$$

As (Γ, w) varies among all stable weighted graphs of genus g , we get a stratification of $\overline{\mathcal{M}}_g(k)$ into disjoint locally closed subsets. In order to describe the inclusion relations between the closures of these strata, we introduce the following order relation among all stable weighted graphs of genus g .

Definition 2.1.6. Given two weighted graphs (Γ, w) and (Γ', w') , we say that (Γ, w) *dominates* (Γ', w') , and we write $(\Gamma, w) \geq (\Gamma', w')$, if Γ' is obtained from Γ by contracting some of its edges and the weight function w' is obtained from the weight function w by an iteration of the following rule:

- If Γ' is obtained from Γ by contracting an edge e that joins two distinct vertices v_1 and v_2 , then the vertex \tilde{v} of Γ' which is the image of the two vertices v_1 and v_2 has weight $w'(\tilde{v}) = w(v_1) + w(v_2)$.
- If Γ' is obtained from Γ by contracting a loop e around the vertex v , then the vertex \tilde{v} of Γ' which is the image of v has weight $w'(\tilde{v}) = w(v) + 1$.

It is easy to see that if $(\Gamma, w) \geq (\Gamma', w')$ then $g(\Gamma, w) = g(\Gamma', w')$ and moreover, if (Γ, w) is stable, then (Γ', w') is stable.

We can now describe the inclusion relation among the closures of the strata of $\overline{\mathcal{M}}_g$.

Fact 2.1.7. *The space $\overline{\mathcal{M}}_g(k)$ admits a stratification into disjoint locally closed subsets*

$$\overline{\mathcal{M}}_g(k) = \coprod_{(\Gamma, w)} \overline{\mathcal{M}}_g(\Gamma, w),$$

as (Γ, w) varies among all stable weighted graphs of genus g .

Given two stable weighted graphs (Γ, w) and (Γ', w') of genus g , we have that

$$\overline{\mathcal{M}}_g(\Gamma, w) \subseteq \overline{\mathcal{M}}_g(\Gamma', w') \Leftrightarrow (\Gamma, w) \geq (\Gamma', w').$$

Proof. This is well-known, see e.g. [GAC2, Chap. XII, Sec. 10] or [Cap12, §4.2]. \square

Remark 2.1.8. Indeed, the stratification of the topological space $\overline{\mathcal{M}}_g(k)$ described in Fact 2.1.7 is induced by a stratification of the stack $\overline{\mathcal{M}}_g$ into locally closed substacks. We refer to [GAC2, Chap. XII] for more details.

2.1.9. *The reduction map $\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$*

We are now ready to define the reduction map $\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$ appearing in the diagram (1.1). Since the stack $\overline{\mathcal{M}}_g$ is proper, the valuative criterion of properness for stacks gives that for any map $f : \text{Spec } K \rightarrow \mathcal{M}_g \subseteq \overline{\mathcal{M}}_g$ there exists a finite extension K' of K with valuation ring R' and a unique map $\phi : \text{Spec } R' \rightarrow \overline{\mathcal{M}}_g$ such that the following diagram is commutative

$$\begin{array}{ccccc} \text{Spec } R' & & & & \\ \uparrow & \searrow \phi & & & \\ \text{Spec } K' & \longrightarrow & \text{Spec } K & \xrightarrow{f} & \mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g. \end{array}$$

In other words, given a connected smooth projective curve $X \in \mathcal{M}_g(K)$, up to a finite extension $K \subseteq K'$ with valuation ring R' , there exists a unique family of stable curves $\mathcal{X}' \rightarrow \text{Spec } R'$, called the *stable reduction* of X with respect to the extension $K \subseteq K'$, such that its generic fiber $\mathcal{X}'_{\eta'} := \mathcal{X}' \times_{\text{Spec } R'} \text{Spec } K'$ is isomorphic to $X_{K'} := X \times_K K'$. Note that the residue field of R' is equal to k , since k was assumed to be algebraically closed.

Lemma - Definition 2.1.10. *The reduction map*

$$\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$$

is defined by sending $X \in \mathcal{M}_g(K)$ to the central fiber $\mathcal{X}'_s \in \overline{\mathcal{M}}_g(k)$ of a stable reduction $\mathcal{X}' \rightarrow \text{Spec } R'$ of X with respect to some finite field extension $K \subseteq K'$ with valuation ring R' . The isomorphism class of $\mathcal{X}'_s \in \overline{\mathcal{M}}_g(k)$ does not depend on the chosen field extension $K \subseteq K'$ and is denoted by $\text{red}(X)$.

Proof. Let K' and K'' two finite field extensions of K , with valuation rings respectively R' and R'' , such that X admits a stable reduction $\mathcal{X}' \rightarrow \text{Spec } R'$ with respect to K' and a stable reduction $\mathcal{X}'' \rightarrow \text{Spec } R''$ with respect to K'' . Choose an algebraic closure \overline{K} of K that contains K' and K'' and consider, inside \overline{K} , the smallest field extension $K \subseteq L$ that contains K' and K'' . Clearly, L is a finite field extension of K and we denote by S its valuation ring. The base change of each of the two families $\mathcal{X}' \rightarrow \text{Spec } R'$ and $\mathcal{X}'' \rightarrow \text{Spec } R''$ to $\text{Spec } S$ is clearly a stable reduction with respect to the extension $K \subseteq L$. By the uniqueness of the stable reduction, these two pull-backs must be isomorphic and in particular their central fibers must be isomorphic. However, since k is assumed to be algebraically closed, the central fibers of these two pull-backs are equal to \mathcal{X}'_s and \mathcal{X}''_s ; hence we must have that $\mathcal{X}'_s \cong \mathcal{X}''_s$, q.e.d. \square

2.2. The moduli space M_g^{tr} of tropical curves. Recall the definition of tropical curves introduced in [BMV11], generalizing slightly the original definition of Mikhalkin-Zharkov in [MZ07].

Definition 2.2.1. A *tropical curve* C is the datum of a triple (Γ, w, l) consisting of a stable weighted graph (Γ, w) , called the combinatorial type of C , and a function $l : E(\Gamma) \rightarrow \mathbb{R}_{>0}$, called the length function. The genus $g(C)$ of C is the genus of its combinatorial type.

Given a stable weighted graph (Γ, w) of genus g , we define $M_g^{\text{tr}}(\Gamma, w)$ to be the set of tropical curves of combinatorial type equal to (Γ, w) . Note that a tropical curve $C \in M_g^{\text{tr}}(\Gamma, w)$ is determined by a length function $l : E(\Gamma) \rightarrow \mathbb{R}_{>0}$. However, different length functions can give rise to the same tropical curve if they differ by an automorphism of the weighted graph (Γ, w) . Therefore, we have a natural identification

$$(2.2) \quad M_g^{\text{tr}}(\Gamma, w) = \mathbb{R}_{>0}^{E(\Gamma)} / \text{Aut}(\Gamma, w).$$

These spaces can indeed be glued together along their boundaries in order to obtain a topological space M_g^{tr} , called the moduli space of tropical curves of genus g , whose points are in bijection with tropical curves of genus g (see [BMV11]). In loc. cit., the space M_g^{tr} is endowed with the structure of a stacky fan. Here, for simplicity, we treat it simply as a topological space. We summarize all the known properties of M_g^{tr} in the following

Fact 2.2.2.

- (i) *There exists a topological space M_g^{tr} whose points are in natural bijection with tropical curves of genus g . Moreover, the topological space M_g^{tr} is normal (hence Hausdorff), locally compact, paracompact, locally contractible, metrizable and second countable.*
- (ii) *The topological space M_g^{tr} admits a stratification into disjoint locally closed subsets*

$$M_g^{\text{tr}} = \coprod_{(\Gamma, w)} M_g^{\text{tr}}(\Gamma, w),$$

as (Γ, w) varies among all stable weighted graphs of genus g .

- (iii) *Given two stable weighted graphs (Γ, w) and (Γ', w') of genus g , we have that*

$$\overline{M_g^{\text{tr}}(\Gamma, w)} \supseteq M_g^{\text{tr}}(\Gamma', w') \Leftrightarrow (\Gamma, w) \geq (\Gamma', w').$$

Proof. The topological space M_g^{tr} has been constructed in [BMV11] and further studied in [Cap12]. Properties (ii) and (iii) follows from [BMV11, §3]. The topological properties of M_g^{tr} stated in (i) are proved in [CMV, §2]. \square

2.2.3. The tropicalization map $\text{trop} : \overline{\mathcal{M}}_g(K) \rightarrow M_g^{\text{tr}}$

We are now ready to define the tropicalization map $\text{trop} : \mathcal{M}_g(K) \rightarrow M_g^{\text{tr}}$ appearing in the diagram (1.1). Given a connected projective smooth curve X over K , consider a finite field extension $K \subseteq K'$ with valuation ring R' such that the base change $X_{K'}$ of X to K' admits a stable reduction $\mathcal{X}' \rightarrow \text{Spec } R'$, in the sense of 2.1.9. Consider now a node n of the central fiber \mathcal{X}'_s of $\mathcal{X}' \rightarrow \text{Spec } R'$. Since the generic fiber of \mathcal{X}' is smooth, by the deformation theory of nodal singularities, it follows easily that a local equation of the surface \mathcal{X}' at n can be chosen to be $xy = (t')^{w_n}$, where t' is some fixed uniformizer of R' (i.e. a generator of the maximal ideal \mathfrak{m}' of R') and $w_n \in \mathbb{Z}_{>0}$ is some uniquely determined natural number, which we call the *width of the node* n .

Lemma - Definition 2.2.4. *The tropicalization map*

$$\text{trop} : \mathcal{M}_g(K) \rightarrow M_g^{\text{tr}}$$

is defined by sending $X \in \mathcal{M}_g(K)$ into the tropical curve $C' \in M_g^{\text{tr}}$ such that:

- the combinatorial type of C' is given by the dual weighted graph $(\Gamma_{\mathcal{X}'_s}, w_{\mathcal{X}'_s})$ of the special fiber \mathcal{X}'_s of the stable reduction of X with respect to some finite field extension $K \subseteq K'$;
- the length of an edge $e \in E(\Gamma_{\mathcal{X}'_s})$ is equal to $l'(e) := \frac{w_{n_e}}{[K' : K]}$, where w_{n_e} is the width of the node n_e of \mathcal{X}'_s corresponding to the edge e and $[K' : K]$ is the degree of the finite field extension $K \subseteq K'$.

The so defined tropical curve $C' \in M_g^{\text{tr}}$ does not depend on the chosen field extension $K \subseteq K'$ and is denoted by $\text{trop}(X)$.

Proof. Let K' and K'' two finite field extensions of K , with valuation rings respectively R' and R'' , such that X admits a stable reduction $\mathcal{X}' \rightarrow \text{Spec } R'$ with respect to K' and a stable reduction $\mathcal{X}'' \rightarrow \text{Spec } R''$ with respect to K'' . Denote by C' and C'' the tropical curves associated to the stable reductions \mathcal{X}' and \mathcal{X}'' according to the above described procedure.

As in the proof of Lemma-Definition 2.1.10, we can find a finite field extension $K \subseteq L$, with valuation ring S , that contains K' and K'' as subfields. In the proof of loc. cit., we have shown that the special fibers \mathcal{X}'_s and \mathcal{X}''_s of the two stable reductions \mathcal{X}' and \mathcal{X}'' are isomorphic. This implies that the combinatorial types of C' and of C'' are the same. It remains to show that the length function l' on C' coincides with the length function l'' on C'' . Consider now an edge $e \in E(\Gamma_{\mathcal{X}'_s}) = E(\Gamma_{\mathcal{X}''_s})$ and its corresponding node $n := n_e \in \mathcal{X}'_s = \mathcal{X}''_s$. If we choose a uniformizer z for S , then $t' := z^{[L:K']}$ is a uniformizer for R' and $t'' := z^{[L:K']}$ is a uniformizer for R'' . Therefore, if the local equation of \mathcal{X}' (resp. \mathcal{X}'') at n is given by $xy = (t')^{w'_n}$ (resp. $xy = (t'')^{w''_n}$) then the local equation of the surface $\mathcal{X}' \times_{\text{Spec } R'} \text{Spec } S$ (resp. $\mathcal{X}'' \times_{\text{Spec } R''} \text{Spec } S$) at n is given by $xy = z^{w'_n [L:K']}$ (resp. $xy = z^{w''_n [L:K']}$). Since $\mathcal{X}' \times_{\text{Spec } R'} \text{Spec } S \cong \mathcal{X}'' \times_{\text{Spec } R''} \text{Spec } S$ by the uniqueness of the stable reduction, we get that $w'_n [L:K'] = w''_n [L:K'']$. This implies that

$$l'(e) = \frac{w'_n}{[K' : K]} = \frac{w'_n [L:K']}{[L:K]} = \frac{w''_n [L:K'']}{[L:K]} = \frac{w''_n}{[K'' : K]} = l''(e),$$

which shows that l' is equal to l'' , q.e.d. □

Remark 2.2.5. Given a curve X over K , the metrized graph underlying the tropical curve $\text{trop}(X)$ associated to X in Lemma-Definition 2.2.4 is the reduction (metrized) graph of X as defined in [CR93, p. 9-10]. Moreover, $\text{trop}(X)$ is the minimal skeleton in the Berkovich analytification X^{an} of X , see [BPR, Cor. 5.50].

Now that we have defined the reduction map $\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$ and the tropicalization map $\text{trop} : \mathcal{M}_g(K) \rightarrow M_g^{\text{tr}}$, it is easy to prove the first half of Theorem B.

Proof of Theorem B(i). By comparing Lemma-Definition 2.1.10 and Lemma-Definition 2.2.4, one easily realizes that, for a smooth curve $X \in \mathcal{M}_g(K)$, the combinatorial type of the tropical curve $\text{trop}(X) \in M_g^{\text{tr}}$ is equal to the dual weighted graph of the stable curve $\text{red}(X) \in \overline{\mathcal{M}}_g(k)$. The conclusion now follows. □

3. MODULI SPACES OF ABELIAN VARIETIES

3.1. The moduli space A_g^{tr} of tropical p.p. abelian varieties. Recall the definition of a tropical principally polarized abelian variety introduced in [BMV11], generalizing slightly the original definition of Mikhalkin-Zharkov in [MZ07].

Definition 3.1.1. A *tropical p.p.* (= principally polarized) *abelian variety* A of dimension g is a pair $(\mathbb{R}^g/\Lambda, Q)$ consisting of a g -dimensional real torus \mathbb{R}^g/Λ (for a rank- g lattice $\Lambda \subset \mathbb{R}^g$) and Q is a positive semi-definite quadratic form on \mathbb{R}^g such that the null space $\text{Null}(Q)$ of Q is defined over $\Lambda \otimes \mathbb{Q}$, i.e. it admits a basis with elements in $\Lambda \otimes \mathbb{Q}$. Two tropical p.p. abelian varieties $A = (\mathbb{R}^g/\Lambda, Q)$ and $A' = (\mathbb{R}^g/\Lambda', Q')$ are isomorphic if there exists $h \in GL(g, \mathbb{R})$ such that $h(\Lambda) = \Lambda'$ and $hQh^t = Q'$.

Indeed, tropical p.p. abelian varieties up to isomorphism are the same thing as positive semi-definite quadratic forms up to arithmetic equivalence, as shown in the following

Remark 3.1.2.

- (i) Every tropical p.p. abelian variety $A = (\mathbb{R}^g/\Lambda, Q)$ can be written in the form $(\mathbb{R}^g/\mathbb{Z}^g, Q')$. In fact, it is enough to consider $Q' = hQh^t$, where $h \in GL(g, \mathbb{R})$ is such that $h(\Lambda) = \mathbb{Z}^g$.
- (ii) $(\mathbb{R}^g/\mathbb{Z}^g, Q) \cong (\mathbb{R}^g/\mathbb{Z}^g, Q')$ if and only if there exists $h \in GL_g(\mathbb{Z})$ such that $Q' = hQh^t$, i.e., if and only if Q and Q' are arithmetically equivalent.

Before stating the main properties of the moduli space A_g^{tr} of tropical p.p. abelian varieties, we need a digression into Delaunay decompositions of \mathbb{R}^g .

Definition 3.1.3.

- (i) A \mathbb{Z}^g -periodic integral *paving* (or face-fitting decomposition) of \mathbb{R}^g of maximal rank g is a set Δ of integral polytopes $\omega \subset \mathbb{R}^g$ satisfying:
 - (a) $\mathbb{R}^g = \cup_{\omega \in \Delta} \omega$;
 - (b) Any face of $\omega \in \Delta$ belongs to Δ ;
 - (c) For any $\omega, \omega' \in \Delta$, the intersection $\omega \cap \omega'$ is either empty or a common face of ω and ω' ;
 - (d) Δ is invariant by translation of \mathbb{Z}^g , i.e. for any $\omega \in \Delta$ and any $h \in \mathbb{Z}^g$ the translate $\omega + h$ belongs to Δ ;
 - (e) $\#\{\omega \bmod \mathbb{Z}^g\}$ is finite.

A \mathbb{Z}^g -periodic integral paving of \mathbb{R}^g of rank $0 \leq r \leq g$ is a set Δ of polyhedra obtained as inverse images via a linear integral projection $\pi : \mathbb{R}^g \rightarrow \mathbb{R}^r$ of the polytopes of a \mathbb{Z}^r -periodic integral paving Δ' of \mathbb{R}^r of maximal rank.

- (ii) Two \mathbb{Z}^g -periodic integral pavings Δ and Δ' of \mathbb{R}^g are *equivalent* if there exists $h \in GL_g(\mathbb{Z})$ such that Δ' is equal to

$$h \cdot \Delta := \{h \cdot \omega : \omega \in \Delta\}.$$

We denote by $[\Delta]$ the equivalence class of a paving Δ of \mathbb{R}^g .

- (iii) Given two \mathbb{Z}^g -periodic integral pavings Δ and Δ' of \mathbb{R}^g , we say that Δ is a *refinement* of Δ' , and we write $\Delta \geq \Delta'$, if every polyhedron of Δ is contained in some polyhedron of Δ' .

Similarly, we say that $[\Delta]$ is a refinement of $[\Delta']$, and we write $[\Delta] \geq [\Delta']$, if there exist $h, h' \in GL_g(\mathbb{Z})$ such that $h \cdot \Delta \geq h' \cdot \Delta'$.

Among the \mathbb{Z}^g -periodic integral pavings of \mathbb{R}^g , a special place is occupied by the Delaunay decompositions associated to a positive semi-definite quadratic forms in \mathbb{R}^g , whose null space is defined over \mathbb{Q}^g .

Definition 3.1.4. Let Q be a positive semi-definite quadratic form of rank r in \mathbb{R}^g , whose null space $\text{Null}(Q)$ is defined over \mathbb{Q}^g . For any $\alpha \in \mathbb{R}^g$, a lattice element $x \in \mathbb{Z}^g$ is called α -nearest if

$$Q(x - \alpha) = \min\{Q(y - \alpha) : y \in \mathbb{Z}^g\}.$$

A Delaunay cell is defined as the closed convex hull of all elements of \mathbb{Z}^g which are α -nearest for some fixed $\alpha \in \mathbb{R}^g$. Together, all the Delaunay cells form a \mathbb{Z}^g -periodic integral paving of \mathbb{R}^g of rank r , called the *Delaunay decomposition* of Q and denoted Del_Q . We say that a \mathbb{Z}^g -periodic integral paving of \mathbb{R}^g is a Delaunay paving if it is isomorphic to Del_Q for some quadratic form Q as before.

Clearly, if two quadratic forms Q and Q' are arithmetic equivalent in the sense of Remark 3.1.2, then their associated Delaunay decompositions Del_Q and $\text{Del}_{Q'}$ are equivalent in the sense of Definition 3.1.3. This shows that we can associate to every tropical p.p. abelian variety an equivalence class of Delaunay decompositions of \mathbb{R}^g .

Definition 3.1.5. Given a tropical p.p. abelian variety $A \cong (\mathbb{R}^g/\mathbb{Z}^g, Q)$ (see Remark 3.1.2), the Delaunay decomposition $[\text{Del}_A]$ of A is defined to be

$$[\text{Del}_A] := [\text{Del}_Q].$$

We are now ready to summarize the main properties of the moduli space of tropical p.p. abelian varieties.

Fact 3.1.6.

- (i) *There exists a topological space A_g^{tr} whose points are in natural bijection with tropical p.p. abelian varieties of dimension g . Moreover, the topological space A_g^{tr} is normal (hence Hausdorff), locally compact, paracompact, locally contractible, metrizable and second countable.*
- (ii) *The topological space A_g^{tr} admits a stratification into disjoint locally closed subsets*

$$A_g^{\text{tr}} = \coprod_{[\Delta]} A_g^{\text{tr}}([\Delta]),$$

as $[\Delta]$ varies among all equivalence classes of Delaunay decompositions of \mathbb{R}^g and

$$A_g^{\text{tr}}([\Delta]) := \{A \in A_g^{\text{tr}} : [\text{Del}_A] = [\Delta]\}.$$

- (iii) *Given two equivalence classes $[\Delta]$ and $[\Delta']$ of Delaunay decompositions of \mathbb{R}^g , we have that*

$$\overline{A_g^{\text{tr}}([\Delta])} \supseteq A_g^{\text{tr}}([\Delta']) \Leftrightarrow [\Delta] \geq [\Delta'].$$

Proof. The construction of A_g^{tr} and the properties (ii) and (iii) can be found in [BMV11, §4] or [Cha12, §4]. Note that the definition of A_g^{tr} given in [BMV11, Def. 4.2.2] contains a mistake that was corrected in [Cha12, Def. 4.9]. The topological properties of A_g^{tr} stated in (i) are proved in [CMV, §2]. \square

3.1.7. *The tropicalization map $\text{trop} : \mathcal{A}_g(K) \rightarrow A_g^{\text{tr}}$*

We want now to define the tropicalization map $\text{trop} : \mathcal{A}_g(K) \rightarrow A_g^{\text{tr}}$ appearing in the diagram (1.1).

Recall that given an abelian variety A over K there is a canonical way of extending it to a separated group scheme over $\text{Spec } R$, namely via the theory of Néron models.

Definition 3.1.8. Given an abelian variety A over K , a *Néron model* of A over $\text{Spec } R$ is a smooth, separated and finite type group scheme $\mathcal{N}(A) \rightarrow \text{Spec } R$ such that its generic fiber $\mathcal{N}(A)_K$ is isomorphic to A and, moreover, such that it satisfies

the following universal property (called the Néron mapping property): for each smooth morphism $\mathcal{Y} \rightarrow \text{Spec } R$ and any K -morphism $\phi_K : \mathcal{Y}_K \rightarrow \mathcal{N}(A)_K \cong A$ there exists a unique morphism $\phi : \mathcal{Y} \rightarrow \mathcal{N}(A)$ over $\text{Spec } R$ extending the given morphism ϕ_K .

Clearly, the Néron mapping property uniquely characterizes the Néron model of an abelian variety A over K , if it exists at all. Indeed, it is a deep theorem of Néron that such models always exist.

Fact 3.1.9 (Néron). *Any abelian variety over K admits a Néron model $\mathcal{N}(A)$ over $\text{Spec } R$.*

For a proof, we refer the reader to the original paper of Néron [Ner64] or to the book [BLR90] for a modern treatment.

Recall now that given any smooth group scheme $\mathcal{A} \rightarrow \text{Spec } R$ (as for example the Néron model $\mathcal{N}(A)$ of an abelian variety A defined over K), there exists an open subgroup scheme $\mathcal{A}^\circ \subseteq \mathcal{A}$, called the *neutral component* of \mathcal{A} , such that the fibers of $\mathcal{A}^\circ \rightarrow \text{Spec } R$ are the connected components of the fibers of $\mathcal{A} \rightarrow \text{Spec } R$ which contain the identity (see [SGA3I, Exposé VIB, Thm. 3.10]).

It was proved by Grothendieck (see [SGA7I, Exposé IX, Thm. 3.6]) that any abelian variety over K is potentially semiabelian, i.e. that, after a finite extension of K , the neutral component $\mathcal{N}(A)^\circ$ of the Néron model of A is a semiabelian scheme.

Fact 3.1.10 (Grothendieck). *Given an abelian variety A over K , there exists a finite extension $K \subseteq K'$ such that the neutral component $\mathcal{N}(A')^\circ$ of the Néron model $\mathcal{N}(A')$ of $A' := A \times_K K'$ is a semiabelian scheme, i.e. the special fiber $\mathcal{N}(A')_s^\circ$ of $\mathcal{N}(A')^\circ$ fits in a unique extension*

$$(3.1) \quad 0 \rightarrow T' \rightarrow \mathcal{N}(A')_s^\circ \rightarrow B' \rightarrow 0,$$

where T' is a torus over k of dimension r (called the rank of $\mathcal{N}(A')_s^\circ$) and B' is an abelian variety over k of dimension $g - r$.

We call the scheme $\mathcal{N}(A')^\circ$ as above the *semiabelian reduction* of A with respect to the extension $K \subseteq K'$. It is the analogue for abelian varieties of the stable reduction for curves (see 2.1.9). We also say that an abelian variety A over K has *semiabelian reduction* if the neutral component $\mathcal{N}(A)^\circ$ of the Néron model of A is a semiabelian scheme. So Fact 3.1.10 is saying that any abelian variety A over K has potentially semiabelian reduction, i.e. there exists a finite field extension $K \subseteq K'$ such that $A' := A \times_K K'$ has semiabelian reduction.

Moreover, in the case where an abelian variety A over K has semiabelian reduction, Grothendieck has shown (see [SGA7I, Exposé IX, Thm. 10.4]) that any polarization ξ (for example a principal polarization) on A gives rise to a monodromy pairing on the lattice of characters $\Lambda(T) := \text{Hom}(T, \mathbb{G}_m)$ of the maximal torus T of the central fiber of $\mathcal{N}(A)^\circ$, as in (3.1).

Fact 3.1.11 (Grothendieck). *Let A be an abelian variety over K and assume that A has semiabelian reduction. Denote by $\Lambda(T) := \text{Hom}(T, \mathbb{G}_m)$ the lattice of characters of the biggest torus T contained in the special fiber $\mathcal{N}(A)_s^\circ$ of $\mathcal{N}(A)^\circ$ as in (3.1) and by $\Lambda(T)_\mathbb{R} := \Lambda(T) \otimes_{\mathbb{Z}} \mathbb{R}$ the associated real vector space. Then any polarization ξ on A gives rise to a positive definite quadratic form*

$$(3.2) \quad Q_\xi : \Lambda_\mathbb{R} \otimes \Lambda_\mathbb{R} \rightarrow \mathbb{R},$$

which is moreover integral over Λ (i.e. such that $Q_\xi(\Lambda, \Lambda) \subseteq \mathbb{Z}$).

The quadratic form Q_ξ in (3.2) is called the *monodromy pairing* associated to the polarized abelian variety (A, ξ) .

We are now ready to define the tropicalization $\text{trop} : \mathcal{A}_g(K) \rightarrow A_g^{\text{tr}}$ appearing in the diagram (1.1).

Lemma - Definition 3.1.12. *The tropicalization map*

$$\text{trop} : \mathcal{A}_g(K) \rightarrow A_g^{\text{tr}}$$

is defined by sending $(A, \xi) \in \mathcal{A}_g(K)$ into the tropical p.p. abelian variety $(\mathbb{R}^g/\Lambda', Q') \in A_g^{\text{tr}}$ such that:

- Λ' is equal to $\Lambda' := \mathbb{Z}^{g-r} \oplus \Lambda(T')$, where $\Lambda(T')$ is the lattice of characters of the torus T' appearing in the extension (3.1) with respect to some chosen field extension $K \subseteq K'$ such that the neutral component $\mathcal{N}(A')^\circ$ of the Néron model of $A' := A \times_K K'$ is a semiabelian scheme;
- The quadratic form is identically zero on $\mathbb{R}^{g-r} := \mathbb{Z}^{g-r} \otimes_{\mathbb{Z}} \mathbb{R}$ while on $\Lambda(T')_{\mathbb{R}}$ it is equal to

$$(3.3) \quad Q'_{\Lambda(T')_{\mathbb{R}} \otimes \Lambda(T')_{\mathbb{R}}} := \frac{Q_{\xi'}}{[K' : K]},$$

where $Q_{\xi'}$ is the monodromy pairing of Fact 3.1.11 associated to the p.p. abelian variety $(A', \xi') := (A, \xi) \otimes_K K' \in \mathcal{A}_g(K')$.

The so defined tropical p.p. abelian variety $(\mathbb{R}^g/\Lambda', Q') \in A_g^{\text{tr}}$ does not depend on the chosen field extension $K \subseteq K'$ and is denoted by $\text{trop}(A, \xi)$.

Proof. Let K' and K'' two finite field extensions of K such that the neutral components $\mathcal{N}(A')^\circ$ (resp. $\mathcal{N}(A'')^\circ$) of $A' := A \times_K K'$ (resp. $A'' := A \times_K K''$) are semiabelian schemes over the spectrum of the valuation ring R' (resp. R'') of K' (resp. K'').

As in the proof of Lemma-Definition 2.1.10, we can find a finite field extension $K \subseteq L$, with valuation ring S , that contains K' and K'' as subfields. It follows from [SGA7I, Exposé IX, Cor. 3.3] that $\mathcal{N}(\tilde{A})^\circ = \mathcal{N}(A')^\circ \times_{\text{Spec } R'} \text{Spec}(S) = \mathcal{N}(A'')^\circ \times_{\text{Spec } R''} \text{Spec}(S)$, where $\tilde{A} := A \times_K L$. In particular we can canonically identify the lattice of characters $\Lambda(\tilde{T})$ of the maximal torus \tilde{T} of $\mathcal{N}(\tilde{A})^\circ$ with the lattice of characters $\Lambda(T')$ (resp. $\Lambda(T'')$) of the maximal torus T' (resp. T'') of $\mathcal{N}(A')^\circ$ (resp. $\mathcal{N}(A'')^\circ$).

With respect to these canonical identifications, the monodromy pairing $Q_{\tilde{\xi}}$ of the p.p. abelian variety $(\tilde{A}, \tilde{\xi}) = (A, \xi) \times_K L \in \mathcal{A}_g(L)$ is related to the monodromy pairing $Q_{\xi'}$ (resp. $Q_{\xi''}$) of the p.p. abelian variety $(A', \xi') := (A, \xi) \times_K K' \in \mathcal{A}_g(K')$ (resp. $(A'', \xi'') := (A, \xi) \times_K K'' \in \mathcal{A}_g(K'')$) via the formulas (see [SGA7I, Exposé IX, (10.3.5)]):

$$(3.4) \quad Q_{\tilde{\xi}} = [L : K'] Q_{\xi'} = [L : K''] Q_{\xi''}.$$

By combining (3.3) and (3.4), we get that

$$Q' = \frac{Q_{\xi'}}{[K' : K]} = \frac{Q_{\tilde{\xi}}}{[L : K]} = \frac{Q_{\xi''}}{[K'' : K]} = Q'',$$

where Q' (resp. Q'') is the quadratic form associated to the extension $K \subseteq K'$ (resp. $K \subseteq K''$). This shows that the definition of $\text{trop}(X)$ is independent of the chosen field extension $K \subseteq K'$. \square

3.2. The moduli stack $\overline{\mathcal{A}}_g$ of p.p. stable semi-abelic pairs. The moduli stack \mathcal{A}_g of principally polarized (p.p. for short) abelian varieties of dimension g admits a modular compactification via p.p. stable semi-abelic pairs.

Definition 3.2.1 (Alexeev). A *p.p. stable semi-abelic pair* of dimension g over k is a triple $(G \curvearrowright P, \Theta)$ where

- (i) G is a semiabelian variety of dimension g over k , that is an algebraic group which is an extension of an abelian variety A by a torus T :

$$1 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0.$$

- (ii) P is a seminormal, connected, projective variety of pure dimension g .
- (iii) G acts on P with finitely many orbits, and with connected and reduced stabilizers contained in the toric part T of G .
- (iv) Θ is an effective ample Cartier divisor on P which does not contain any G -orbit, and such that $h^0(P, \mathcal{O}_P(\Theta)) = 1$.

Recall that a k -variety X is said to have seminormal singularities if any morphism $Y \rightarrow X$ from a k -variety Y which is bijective on k -points is an isomorphism.

Remark 3.2.2. If $(A \curvearrowright P, \Theta)$ is a p.p. stable semi-abelic pair with A being an abelian variety, then P is a A -torsor and the divisor $\Theta \subset P$ gives rise to a well-defined class $[\Theta]$ in the Néron-Severi group of A which is a principal polarization on A . Conversely, every p.p. abelian variety (A, ξ) can be obtained in this way from a unique p.p. stable semi-abelic pair $(A \curvearrowright P, \Theta)$. See [Ale02, Sec. 3] for more details on this correspondence.

The following celebrated result is due to Alexeev [Ale02].

Fact 3.2.3 (Alexeev). *The stack $\overline{\mathcal{A}}_g^{\text{mod}}$ of p.p. stable semi-abelic pairs of dimension g is proper over $\text{Spec } \mathbb{Z}$. The stack \mathcal{A}_g can be identified with the open substack of $\overline{\mathcal{A}}_g^{\text{mod}}$ consisting of the p.p. stable semi-abelic pairs $(G \curvearrowright P, \Theta)$ such that G is an abelian variety.*

Unfortunately, the stack $\overline{\mathcal{A}}_g^{\text{mod}}$ is not irreducible (see [Ale01]). Therefore, only one of its irreducible components, called the *main component* of $\overline{\mathcal{A}}_g^{\text{mod}}$ and denoted by $\overline{\mathcal{A}}_g$, will contain \mathcal{A}_g . Indeed, it is known that the normalization of the main component $\overline{\mathcal{A}}_g$ is isomorphic to the 2nd Voronoi toroidal compactification $\overline{\mathcal{A}}_g^V$ of \mathcal{A}_g (see [AMRT75] and [Nam80]). To the best of our knowledge, it is not known whether the main component $\overline{\mathcal{A}}_g$ is normal (see [Bri07]).

3.2.4. The stratification of $\overline{\mathcal{A}}_g(k)$

According to general theory developed in [Ale02], to every p.p. stable semi-abelic pair $(G \curvearrowright P, \Theta)$ over k , it is naturally associated a \mathbb{Z}^g -period integral paving of \mathbb{R}^g , up to the action of $\text{GL}_g(\mathbb{Z})$, which captures the combinatorics of the G -orbits on P . Moreover, such a paving is a Delaunay decomposition if and if $(G \curvearrowright P, \Theta)$ belongs to the main component $\overline{\mathcal{A}}_g$ (see [Ale01]). In this way we get a stratification of $\overline{\mathcal{A}}_g$ into locally closed subsets parametrized by equivalence classes of Delaunay decompositions of \mathbb{R}^g .

Fact 3.2.5 (Alexeev).

- (i) *A p.p. stable semi-abelic pair $(G \curvearrowright P, \Theta) \in \overline{\mathcal{A}}_g^{\text{mod}}(k)$ determines an equivalence class of a \mathbb{Z}^g -period integral pavings of \mathbb{R}^g , which we denote by $[\Delta(G \curvearrowright P, \Theta)]$.*

Furthermore, $(G \curvearrowright P, \Theta)$ belongs to the main irreducible component $\overline{\mathcal{A}}_g(k)$ if and only if $[\Delta(G \curvearrowright P, \Theta)]$ is an equivalence class of Delaunay decompositions of \mathbb{R}^g .

(ii) The topological space $\overline{\mathcal{A}}_g(k)$ admits a stratification into disjoint locally closed subsets

$$\overline{\mathcal{A}}_g(k) = \coprod_{[\Delta]} \overline{\mathcal{A}}_g([\Delta]),$$

as $[\Delta]$ varies among all equivalence classes of Delaunay decompositions of \mathbb{R}^g and

$$\overline{\mathcal{A}}_g([\Delta]) := \{(G \curvearrowright P, \Theta) \in \overline{\mathcal{A}}_g(k) : [\Delta(G \curvearrowright P, \Theta)] = [\Delta]\}.$$

Given two equivalence classes $[\Delta]$ and $[\Delta']$ of Delaunay decompositions of \mathbb{R}^g , we have that

$$\overline{\mathcal{A}}_g([\Delta]) \subseteq \overline{\overline{\mathcal{A}}_g([\Delta'])} \Leftrightarrow [\Delta] \geq [\Delta'].$$

Proof. Part (i) follows from the general structure theorems on p.p. stable semi-abelic varieties developed in [Ale02] (see also [Ale04, Sec. 2] for a nice discussion).

Part (ii): the strata of $\overline{\mathcal{A}}_g$ are the images of the strata of the 2nd Voronoi toroidal compactification $\overline{\mathcal{A}}_g^V$ under the finite normalization map $\overline{\mathcal{A}}_g^V \rightarrow \overline{\mathcal{A}}_g$ (see [Ale01]) and the required properties are known for the strata of $\overline{\mathcal{A}}_g^V$, as it follows from the general theory of toroidal compactifications of \mathcal{A}_g (see [AMRT75] or [Nam80]). Therefore, the same properties hold for the strata of $\overline{\mathcal{A}}_g(k)$. \square

3.2.6. The reduction map $\text{red} : \mathcal{A}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$

We are now ready to define the reduction map $\text{red} : \mathcal{A}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$ appearing in the diagram (1.1). Since the stack $\overline{\mathcal{A}}_g$ is proper, the valuative criterion of properness for stacks gives that for any map $f : \text{Spec } K \rightarrow \mathcal{A}_g \subseteq \overline{\mathcal{A}}_g$ there exists a finite extension K' of K with valuation ring R' and a unique map $\phi : \text{Spec } R' \rightarrow \overline{\mathcal{A}}_g$ such that the following diagram is commutative

$$\begin{array}{ccccc} \text{Spec } R' & & & & \\ \uparrow & \searrow & \phi & & \\ \text{Spec } K' & \longrightarrow & \text{Spec } K & \xrightarrow{f} & \mathcal{A}_g \hookrightarrow \overline{\mathcal{A}}_g. \end{array}$$

In other words, given a p.p. abelian variety $(A, \xi) \in \mathcal{A}_g(K)$, up to a finite extension $K \subseteq K'$ with valuation ring R' , there exists a unique family of p.p. stable semi-abelic pairs $(\mathcal{G} \curvearrowright \mathcal{P}, \tilde{\Theta})$ over $\text{Spec } R'$, called the *stable semi-abelic reduction* of (A, ξ) with respect to the extension $K \subseteq K'$, such that $(A, \xi) \times_K K'$ is the p.p. abelian variety associated to the generic fiber of $(\mathcal{G} \curvearrowright \mathcal{P}, \tilde{\Theta})$, according to Remark 3.2.2. Note that the residue field of R' is equal to k , since k was assumed to be algebraically closed.

Lemma - Definition 3.2.7. *The reduction map*

$$\text{red} : \mathcal{A}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$$

is defined by sending $(A, \xi) \in \mathcal{A}_g(K)$ to the central fiber $(\mathcal{G} \curvearrowright \mathcal{P}, \tilde{\Theta})_s \in \overline{\mathcal{A}}_g(k)$ of the stable semi-abelic reduction $(\mathcal{G} \curvearrowright \mathcal{P}, \tilde{\Theta})$ of (A, ξ) with respect to some finite field extension $K \subseteq K'$. The isomorphism class of $(\mathcal{G} \curvearrowright \mathcal{P}, \tilde{\Theta})_s \in \overline{\mathcal{A}}_g(k)$ does not depend on the chosen field extension $K \subseteq K'$ and is denoted by $\text{red}(A, \xi)$.

Proof. Same proof as in Lemma-Definition 2.1.10 based on the uniqueness of the stable semi-abelic reduction. \square

Now that we have defined the reduction map $\text{red} : \mathcal{A}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$ and the tropicalization map $\text{trop} : \mathcal{A}_g(K) \rightarrow A_g^{\text{tr}}$, we can prove the second half of Theorem B.

Proof of Theorem B(ii). We have to prove that for any p.p. abelian variety $(A, \xi) \in \mathcal{A}_g(K)$ it holds

$$[\text{Del}_{\text{trop}(A, \xi)}] = [\Delta(\text{red}(A, \xi))],$$

following the notations of Definition 3.1.5 and of Fact 3.2.5. This is simply a restatement in our language of what Alexeev proved in [Ale02, Sec. 5.7]. \square

4. THE TORELLI MAPS

4.1. The tropical Torelli map. The tropical Torelli map $t_g^{\text{tr}} : M_g^{\text{tr}} \rightarrow A_g^{\text{tr}}$ has been constructed in [BMV11] and further studied in [Cha12]. In order to recall the definition of t_g^{tr} , we need first to recall the definition of the tropical Jacobian associated to a tropical curve.

Definition 4.1.1. Let $C = (\Gamma, w, l)$ be a tropical curve of genus g and total weight $|w|$. The *Jacobian* $\text{Jac}(C)$ of C is the tropical p.p. abelian variety of dimension g given by the real torus $(H_1(\Gamma, \mathbb{R}) \oplus \mathbb{R}^{|w|}) / (H_1(\Gamma, \mathbb{Z}) \oplus \mathbb{Z}^{|w|})$ together with the positive semi-definite quadratic form $Q_C = Q_{(\Gamma, w, l)}$ which vanishes identically on $\mathbb{R}^{|w|}$ and is given on $H_1(\Gamma, \mathbb{R})$ as

$$(4.1) \quad Q_C \left(\sum_{e \in E(\Gamma)} \alpha_e \cdot e \right) = \sum_{e \in E(\Gamma)} \alpha_e^2 \cdot l(e).$$

In other words, the value of the quadratic form Q_C on a cycle of Γ , seen as an element of $H_1(\Gamma, \mathbb{R})$, is equal to its length measured with respect to the length function l of the tropical curve C .

Remark 4.1.2. The referee noticed that the quadratic form Q_C defined in (4.1) appears already in the definition of the canonical polarization on the Jacobian of a Mumford curve, see [Ger82] and [vdP83, Prop. 2.2].

Fact 4.1.3 ([BMV11]). *The map (called the tropical Torelli map)*

$$\begin{aligned} t_g^{\text{tr}} : M_g^{\text{tr}} &\longrightarrow A_g^{\text{tr}} \\ C &\longmapsto \text{Jac}(C) \end{aligned}$$

is a continuous map.

Indeed, it is proved in [BMV11, Thm. 5.1.5] that t_g^{tr} is a full map of stacky fans, i.e. that sends each strata of M_g^{tr} surjectively onto some strata of A_g^{tr} via a linear map. In order to make this result more precise, we need to recall the definition of the Delaunay decomposition of \mathbb{R}^g associated to a stable weighted graph of genus g .

Definition 4.1.4. Let (Γ, w) be a stable weighted graph of genus g . Consider the positive semi-definite quadratic form $Q_{(\Gamma, w)}$ on $H_1(\Gamma, \mathbb{R}) \oplus \mathbb{R}^{|w|}$ which is identically zero on $\mathbb{R}^{|w|}$ and is given on $H_1(\Gamma, \mathbb{R})$ by

$$(4.2) \quad Q_{(\Gamma, w)} \left(\sum_{e \in E(\Gamma)} \alpha_e \cdot e \right) = \sum_{e \in E(\Gamma)} \alpha_e^2.$$

By fixing an isomorphism of free abelian groups $\phi : H_1(\Gamma, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^{b_1(\Gamma)}$, we can view $Q_{(\Gamma, w)}$ as a positive semi-definite quadratic form on \mathbb{R}^g . The equivalence class

$[\text{Del}_{Q(\Gamma, w)}]$ of the induced Delaunay decomposition of \mathbb{R}^g (which clearly does not depend upon the chosen isomorphism ϕ) is called the Delaunay decomposition of (Γ, w) and is denoted by $[\text{Del}(\Gamma, w)]$.

Remark 4.1.5. It is well known that an equivalent definition of $[\text{Del}(\Gamma, w)]$ is the following. Each edge e of Γ gives rise to a linear functional e^* on $H_1(\Gamma, \mathbb{R}) \oplus \mathbb{R}^{|w|}$ which is identically zero on $\mathbb{R}^{|w|}$ and it is equal on $H_1(\Gamma, \mathbb{R})$ to

$$e^* \left(\sum_{f \in E(\Gamma)} \alpha_f \cdot f \right) = \alpha_e.$$

After fixing an isomorphism $\phi : H_1(\Gamma, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^{b_1(\Gamma)}$ as before, the Delaunay decomposition $[\text{Del}(\Gamma, w)]$ is the \mathbb{Z}^g -periodic integral paving of \mathbb{R}^g consisting of all polyhedra which are cut out by all hyperplanes of equation $e^* = n$ for $e \in E(\Gamma)$ and $n \in \mathbb{Z}$. We refer the reader to [CV10, Sec. 3.2] for more details on the Delaunay decompositions associated to graphs.

Fact 4.1.6 ([BMV11]). *The tropical Torelli map t_g^{tr} sends the strata $M_g^{\text{tr}}(\Gamma, w) \subset M_g^{\text{tr}}$ surjectively onto the strata $A_g^{\text{tr}}([\text{Del}(\Gamma, w)] \subset A_g^{\text{tr}}$, i.e.*

$$t_g^{\text{tr}}(M_g^{\text{tr}}(\Gamma, w)) = A_g^{\text{tr}}([\text{Del}(\Gamma, w)])$$

for each stable weighted graph (Γ, w) of genus g .

We can now prove the second half of Theorem A.

Theorem 4.1.7. *The following diagram is commutative*

$$(4.3) \quad \begin{array}{ccc} \mathcal{M}_g(K) & \xrightarrow{\text{trop}} & M_g^{\text{tr}} \\ \downarrow t_g & & \downarrow t_g^{\text{tr}} \\ \mathcal{A}_g(K) & \xrightarrow{\text{trop}} & A_g^{\text{tr}} \end{array}$$

Proof. Let X be an element of $\mathcal{M}_g(K)$, i.e. a connected smooth projective curve of genus g over K .

Assume first that X has a stable model over $\text{Spec } R$, i.e. there exists a family $\mathcal{X} \rightarrow \text{Spec } R$ of stable curves of genus g such that the generic fiber \mathcal{X}_η is isomorphic to X .

According to Lemma-Definition 2.2.4, $\text{trop}(X)$ has combinatorial type equal to the dual weighted graph $(\Gamma_{\mathcal{X}_s}, w_{\mathcal{X}_s})$ of the special fiber \mathcal{X}_s of \mathcal{X} and its length function $l : E(\Gamma_{\mathcal{X}_s}) \rightarrow \mathbb{R}_{>0}$ is such that, for every $e \in E(\Gamma_{\mathcal{X}_s})$:

$$(4.4) \quad l(e) = w_{n_e}$$

where w_{n_e} is the width of the node $n_e \in \mathcal{X}_s$ corresponding to e (see 2.2.3).

By blowing up each node n of the central fiber \mathcal{X}_s a number of times equal to $(w_n - 1)$, we get that a new family of nodal curves $\mathcal{Y} \rightarrow \text{Spec } R$ such that $\mathcal{Y}_\eta \cong X$ and \mathcal{Y} is regular. The central fiber \mathcal{Y}_s of \mathcal{Y} is a nodal (non stable, in general) curve which is obtained from \mathcal{X}_s by inserting at each node n of \mathcal{X}_s a chain of smooth rational curves of length equal to $(w_n - 1)$. This implies that the dual graph $\Gamma_{\mathcal{Y}_s}$ of \mathcal{Y}_s is obtained from the dual graph $\Gamma_{\mathcal{X}_s}$ of \mathcal{X}_s by subdividing each edge $e \in E(\Gamma_{\mathcal{X}_s})$ a number of times equal to $(w_{n_e} - 1)$. In particular, we have a canonical isomorphism $H_1(\Gamma_{\mathcal{X}_s}, \mathbb{Z}) \cong H_1(\Gamma_{\mathcal{Y}_s}, \mathbb{Z})$. Moreover, the pull-back map induces a canonical isomorphism $J(\mathcal{X}_s) \xrightarrow{\cong} J(\mathcal{Y}_s)$ between the generalized Jacobians of \mathcal{X}_s and of \mathcal{Y}_s .

According to [BLR90, Sec. 9.3, Thm. 7], there exists a scheme $J(\mathcal{Y})$ smooth and separated over $\text{Spec } R$, called the *relative Jacobian* of the family $\mathcal{Y} \rightarrow \text{Spec } R$,

such that its generic fiber $J(\mathcal{Y})_\eta$ is isomorphic to the Jacobian $J(\mathcal{Y}_\eta) = J(X)$ of the generic fiber and its special fiber $J(\mathcal{Y})_s$ is isomorphic to the generalized Jacobian $J(\mathcal{Y}_s)$ of the special fiber. Moreover, a well-known result of Raynaud (see [BLR90, Sec. 9.5, Thm. 4]) says that, since \mathcal{Y} is regular, the relative Jacobian $J(\mathcal{Y})$ of $\mathcal{Y} \rightarrow \text{Spec } R$ is isomorphic to the neutral component $\mathcal{N}(J(X))^\circ$ of the Néron model of the Jacobian $J(X)$ of $X \cong \mathcal{Y}_\eta$. In particular, since $J(\mathcal{Y})$ is a semiabelian scheme over $\text{Spec } R$, the Jacobian $J(X)$ of X has semiabelian reduction over K . Note that the lattice of characters of the maximal subtorus of $J(\mathcal{Y})_s$ is canonically isomorphic to $H_1(\Gamma_{\mathcal{Y}_s}, \mathbb{Z}) \cong H_1(\Gamma_{\mathcal{X}_s}, \mathbb{Z})$.

Now the Picard-Lefschetz formula (see [SGA7I, Exposé IX, Thm. 12.5]) says that the monodromy pairing Q_ξ on $H_1(\Gamma_{\mathcal{Y}_s}, \mathbb{R})$ associated to the principal polarization $[\Theta_X]$ on $J(X)$ induced by the theta divisor $\Theta_X \subset \text{Pic}^{g-1}(X) \cong J(X)$ (see Fact 3.1.11) is equal to

$$(4.5) \quad Q_{[\Theta_X]} \left(\sum_{e \in E(\Gamma_{\mathcal{Y}_s})} \alpha_e \cdot e \right) = \sum_{e \in E(\Gamma_{\mathcal{Y}_s})} \alpha_e^2$$

Using the canonical isomorphism $H_1(\mathcal{Y}_s, \mathbb{Z}) \cong H_1(\mathcal{X}_s, \mathbb{Z})$, it is immediate to check that the above monodromy pairing Q_ξ on $H_1(\mathcal{Y}_s, \mathbb{Z})$ becomes isomorphic to the quadratic form $Q_{\text{trop}(X)}$ on $H_1(\Gamma_{\mathcal{X}_s}, \mathbb{R})$ defined by (4.1). By comparing Lemma-Definition 3.1.12 with Definition 4.1.1, we see that $\text{trop}(J(X), [\Theta_X]) = \text{Jac}(\text{trop}(X))$, which shows the commutativity of the diagram (4.3).

In the general case (when X does not have a stable reduction over K), we can find a finite field extension $K \subseteq K'$ with valuation ring R' such that the base change $X_{K'}$ of X to K' admits a stable reduction $\mathcal{X}' \rightarrow \text{Spec } R'$. We can repeat the above argument working with the family $\mathcal{X}' \rightarrow \text{Spec } R'$ with the following two modifications: in defining the length of the tropical curve $\text{trop}(X)$ we have to divide the right hand side of (4.4) by $[K' : K]$ and in defining the quadratic form giving $\text{trop}(X, [\Theta_X])$ we have to divide the monodromy pairing (4.5) by $[K' : K]$. Clearly, with these two modifications, the equality $\text{trop}(J(X), [\Theta_X]) = \text{Jac}(\text{trop}(X))$ continues to hold, and the commutativity of the diagram (4.3) in the general case follows. \square

4.1.8. The fibers of the tropical Torelli map t_g^{tr}

The aim of this subsection is to recall the description of the fibers of t_g^{tr} obtained in [CV10].

A first step is to describe the strata $M_g^{\text{tr}}(\Gamma, w)$ of M_g^{tr} that are mapped to the same stratum $A_g^{\text{tr}}([\Delta])$ of A_g^{tr} . To this aim, we recall the following classical definition, due to Whitney.

Definition 4.1.9 (Whitney). Two graphs Γ_1 and Γ_2 are said to be *cyclically equivalent* (or 2-isomorphic), and we write $\Gamma_1 \equiv_{\text{cyc}} \Gamma_2$, if there exists a bijection $\phi : E(\Gamma_1) \rightarrow E(\Gamma_2)$ inducing a bijection between cycles of Γ_1 and cycles of Γ_2 . We denote by $[\Gamma]_{\text{cyc}}$ the cyclic isomorphism class of a graph Γ .

In the sequel, graphs with edge-connectivity at least 3 will play an important role. Here it is the standard definition.

Definition 4.1.10. Let Γ be a connected graph.

- (i) An edge e of Γ is called a *separating edge* (or a coloop or a bridge) if the graph obtained by removing e is disconnected. Two edges e and f are said to be *coparallel* if neither of them is a separating edge and the graph obtained by removing e and f is disconnected.

- (ii) Γ is said to be *3-edge-connected* if Γ does not have separating edges nor pairs of coparallel edges.

It is easy to see that the property of being coparallel defines an equivalence relation on the set of non separating edges of Γ . The equivalence classes with respect to this equivalence relation are called *coparallel classes*².

There is a canonical way of obtaining a 3-edge-connected graph, up to cyclic isomorphism, starting from any graph.

Definition 4.1.11. Let Γ be a connected graph.

A *3-edge-connectivization* of Γ is a graph, denoted by Γ^3 , obtained from Γ by contracting all the separating edges and all but one among the edges of each coparallel class of Γ . The cyclic isomorphism class of Γ^3 (which is well-defined and it does not depend on the choice of Γ^3) is called the *3-edge-connectivization class* of Γ and is denoted by $[\Gamma^3]_{\text{cyc}}$.

After these preliminary definitions, we can now recall the following result (proved in [CV10, Sec. 3.2]) which characterizes the stable weighted graphs that have the same associated Delaunay decomposition.

Fact 4.1.12 (Caporaso-Viviani). *Let (Γ_1, w_1) and (Γ_2, w_2) two stable weighted graphs of genus g . Then*

$$[\text{Del}(\Gamma_1, w_1)] = [\text{Del}(\Gamma_2, w_2)] \Leftrightarrow [\Gamma_1^3]_{\text{cyc}} = [\Gamma_2^3]_{\text{cyc}}.$$

We turn now to the following natural question: for which tropical curves $C, C' \in M_g^{\text{tr}}$ it holds that $t_g^{\text{tr}}(C) = t_g^{\text{tr}}(C')$? We first need a couple of definitions.

Definition 4.1.13. Two tropical curves $C = (\Gamma, w, l)$ and $C' = (\Gamma', w', l')$ are cyclic isomorphic, and we write $C \equiv_{\text{cyc}} C'$, if there exists a bijection $\phi : E(\Gamma) \rightarrow E(\Gamma')$, commuting with the length functions l and l' , that induces a cyclic isomorphism between Γ and Γ' . We denote by $[C]_{\text{cyc}}$ the cyclic isomorphism equivalence class of a tropical curve C .

Similarly to definition 4.1.10, we have the following

Lemma - Definition 4.1.14. *Let $C = (\Gamma, l, w)$ a tropical curve. A 3-edge-connectivization of C is a tropical curve $C^3 = (\Gamma^3, l^3, w^3)$ obtained in the following manner:*

- (i) Γ^3 is a 3-edge-connectivization of Γ in the sense of Definition 4.1.11, i.e. Γ^3 is obtained from Γ by contracting all the separating edges of Γ and, for each coparallel class S of Γ , all but one the edges of S , which we denote by e_S ;
- (ii) w^3 is the weight function on Γ^3 induced by the weight function w on Γ in the following way: at each contraction of some edge of Γ , the new vertex has weight with respect to w^3 equal to the sum of the weights with respect to w of the two vertices mapping to it;
- (iii) l^3 is the length function on Γ^3 given by

$$l^3(e_S) = \sum_{e \in S} l(e),$$

for each coparallel class S of Γ .

The cyclic isomorphism class of C^3 is well-defined; it will be called the *3-edge-connectivization class* of C and denoted by $[C^3]_{\text{cyc}}$.

²These equivalence classes were called C1-sets in [CV10], with a terminology coming from algebraic geometry (see [CV11]). Here we choose to use the more graph-theoretic terminology of coparallel in order to suggest that the coparallel equivalence relation is the dual notion (in the sense of matroid theory) of the parallel equivalence relation.

The following result was proved by Caporaso-Viviani in [CV10, Thm. 4.1.9] in the case when the total weights of the tropical curves are zero and then the proof was easily adapted to the general case by Brannetti-Melo-Viviani in [BMV11, Thm. 5.3.3].

Fact 4.1.15 (Caporaso-Viviani). *Let $C_1, C_2 \in M_g^{\text{tr}}$. Then*

$$t_g^{\text{tr}}(C_1) = t_g^{\text{tr}}(C_2) \Leftrightarrow [C_1^3]_{\text{cyc}} = [C_2^3]_{\text{cyc}}.$$

The previous Fact allows us to describe a locally closed subset of M_g^{tr} where the tropical Torelli map t_g^{tr} is injective.

Recall that a connected graph Γ is said to be *3-vertex-connected* if, for any pair $\{v_1, v_2\}$ of (possibly equal) vertices of Γ , the graph $\Gamma \setminus \{v_1, v_2\}$ obtained from Γ by removing v_1, v_2 together with all the edges that are adjacent to them is connected. It is easily seen that a 3-vertex-connected graph is also 3-edge-connected in the sense of Definition 4.1.10.

Corollary 4.1.16. *The tropical Torelli map $t_g^{\text{tr}} : M_g^{\text{tr}} \rightarrow A_g^{\text{tr}}$ is injective on the locally closed subset F of M_g^{tr} consisting of tropical curves C whose combinatorial type (Γ, w) is such that Γ is 3-vertex-connected and $g(\Gamma) = g$.*

Proof. Since F is the union of strata of M_g^{tr} , it is clear that F is locally closed. Now it follows from a classical result of Whitney (see [CV10, Thm. 2.2.4] and the references therein) that if $C_1, C_2 \in F$ then

$$[C_1^3]_{\text{cyc}} = [C_1]_{\text{cyc}} = [C_2]_{\text{cyc}} = [C_2^3]_{\text{cyc}} \Leftrightarrow C_1 = C_2,$$

which, together with Fact 4.1.15, finishes the proof. \square

4.2. The compactified Torelli morphism. The Torelli morphism $t_g : \mathcal{M}_g \rightarrow \mathcal{A}_g$ can be extended to a modular morphism $\bar{t}_g : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{A}}_g$, as shown by Alexeev in [Ale04]. Before recalling his result, we need the following definitions.

Definition 4.2.1. Let X be a stable curve of arithmetic genus g over k .

- (i) The *generalized Jacobian* $J(X)$ of X is the semiabelian variety parametrizing line bundles on X of multidegree 0, i.e. having degree 0 on each irreducible component of X .
- (ii) The *degree $g-1$ canonical compactified Jacobian* $\overline{P_X^{g-1}}$ of X is the moduli space of torsion-free, multirank 1 (i.e. having rank 1 on each irreducible component of X) and degree $g-1$ sheaves \mathcal{I} on X that are ω_X -semistable.
- (iii) The *theta divisor* Θ_X of X is the closed reduced subscheme of $\overline{P_X^{g-1}}$ defined by

$$\Theta_X := \{\mathcal{I} \in \overline{P_X^{g-1}} : h^0(X, \mathcal{I}) > 0\}.$$

It is well known that $J(X)$ is the extension of the Jacobian $J(\tilde{X})$ of the normalization \tilde{X} of X (which has dimension equal to the geometric genus of X) by a torus whose lattice of characters can be naturally identified with the first homology group $H_1(\Gamma_X, \mathbb{Z})$ of the dual graph Γ_X of X . Note also that the generalized Jacobian $J(X)$ acts naturally on $\overline{P_X^{g-1}}$ by tensor product.

The following result was proved by Alexeev in [Ale04].

Fact 4.2.2 (Alexeev).

- (i) *For any stable curve X of genus g , the triple $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta_X)$ is a p - p . stable semi-abelic pair of dimension g .*
- (ii) *The Torelli morphism $t_g : \mathcal{M}_g \rightarrow \mathcal{A}_g$ extends to a morphism $\bar{t}_g : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{A}}_g$, called the compactified Torelli morphism, which sends a stable curve X into the p - p . stable semi-abelic pair $(J(X) \curvearrowright \overline{P_X^{g-1}}, \Theta_X)$.*

(iii) The compactified Torelli morphism \bar{t}_g sends the stratum $\overline{\mathcal{M}}_g(\Gamma, w) \subset \overline{\mathcal{M}}_g(k)$ into the stratum $\overline{\mathcal{A}}_g([\text{Del}(\Gamma, w)]) \subseteq \overline{\mathcal{A}}_g(k)$.

Using the above result, we can now easily prove the first half of Theorem A.

Theorem 4.2.3. *The following diagram is commutative*

$$(4.6) \quad \begin{array}{ccc} \overline{\mathcal{M}}_g(k) & \xleftarrow{\text{red}} & \mathcal{M}_g(K) \\ \downarrow \bar{t}_g & & \downarrow t_g \\ \overline{\mathcal{A}}_g(k) & \xleftarrow{\text{red}} & \mathcal{A}_g(K) \end{array}$$

Proof. Consider an element of $\mathcal{M}_g(K)$, i.e. a morphism $f : \text{Spec } K \rightarrow \mathcal{M}_g \subset \overline{\mathcal{M}}_g$. By applying the valuative criterion of properness to the stack $\overline{\mathcal{M}}_g$, we get that, up to a finite extension $K \subseteq K'$ with valuation ring R' , we can extend the morphism f to a morphism $\phi : \text{Spec } R' \rightarrow \overline{\mathcal{M}}_g$. In this way we get a commutative diagram

$$\begin{array}{ccccc} & & & \text{Spec } R' & \xleftarrow{s} & \text{Spec } k \\ & & & \uparrow \eta & & \\ & & & \text{Spec } K' & \longleftarrow & \text{Spec } K \\ & \swarrow \phi & \xleftarrow{f} & & & \\ \overline{\mathcal{M}}_g & \longleftarrow & \mathcal{M}_g & & & \\ \downarrow \bar{t}_g & & \downarrow t_g & & & \\ \overline{\mathcal{A}}_g & \longleftarrow & \mathcal{A}_g & & & \end{array}$$

where the upper triangle is commutative by construction and the bottom left square is commutative by Fact 4.2.2. As explained in 2.1.9, the first reduction map $\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$ sends the morphism $f : \text{Spec } K \rightarrow \mathcal{M}_g$ into the morphism $\phi \circ s : \text{Spec } k \rightarrow \overline{\mathcal{M}}_g$. Analogously, as explained in 3.2.6, the second reduction map $\text{red} : \mathcal{A}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$ sends the morphism $t_g \circ f : \text{Spec } K \rightarrow \mathcal{A}_g$ into the morphism $\bar{t}_g \circ \phi \circ s : \text{Spec } k \rightarrow \overline{\mathcal{A}}_g$. The commutativity of the diagram (4.6) now follows. \square

4.2.4. The fibers of the compactified Torelli morphism \bar{t}_g

The aim of this subsection is to recall the description of the fibers of the compactified Torelli morphism $\bar{t}_g : \overline{\mathcal{M}}_g(k) \rightarrow \overline{\mathcal{A}}_g(k)$ obtained in [CV11]. Before doing that, we need to recall some definitions.

Definition 4.2.5. Given a stable curve X of genus g , consider its partial normalization $\tau : \widehat{X} \rightarrow X$ at the separating nodes of X , i.e. the nodes n of X such that the partial normalization of X at n is disconnected. Write

$$\widehat{X} := X_1 \coprod \cdots \coprod X_s,$$

where $s \in \mathbb{N}$ and each X_i is a connected nodal curve. We call the curves $\{X_1, \dots, X_s\}$ the *separating blocks* of X .

Note that the separating blocks of X are connected nodal curves free from separating nodes, which are however not stable in general. Consider one of the separating blocks X_i . If X_i has arithmetic genus $p_a(X_i)$ equal to zero, then $X_i \cong \mathbb{P}^1$. Otherwise, X_i is *semistable*, i.e. it is a connected nodal curves such that its canonical line ω_{X_i} has non-negative (possibly zero) degree on each irreducible component of X_i . If, moreover, $p_a(X_i) \geq 2$, then we can consider its *stabilization*, denoted by \overline{X}_i , which is the image of X_i under the map given by $|\omega_{X_i}^m|$ for m sufficiently large (indeed any $m \geq 3$ suffices). It is easy to see that \overline{X}_i is obtained from X_i by contracting to a node all the exceptional subcurves $E \subset X_i$, i.e. subcurves $E \cong \mathbb{P}^1$ such that E intersect the complementary subcurve $\overline{X}_i \setminus E$ in two points. We can

extend the definition of the stabilization \overline{X}_i to the case where $p_a(X_i) = 1$ as it follows: if X_i is smooth then we set $\overline{X}_i = X_i$; if X_i is not smooth (which happens exactly when X_i is a cycle of rational smooth curves) then we set \overline{X}_i be equal to the rational irreducible curve with one node.

For a nodal curve X without separating nodes, we can partition the set X_{sing} of nodes of X into C1-sets as in [CV11, Lemma-Definition 2.1.1].

Definition 4.2.6. Let X be a connected nodal curve free from separating nodes. A *separating pair* $\{n_1, n_2\}$ of nodes of X is a pair consisting of two nodes n_1 and n_2 of X such that the partial normalization of X at n_1 and n_2 is disconnected.

Being a separating pair of nodes is an equivalence relation on the set of nodes X_{sing} of X and we call the associated equivalence classes the *C1-sets* of X . We denote by $\text{Set}^1 X$ the collection of all C1-sets of X .

Note that the C1-sets of X correspond exactly to the coparallel classes (see Definition 4.1.10) of edges in the dual graph Γ_X of X .

We now recall the definition of C1-equivalence introduced in [CV11, Def. 2.1.5].

Definition 4.2.7 (C1-equivalence). Let X and X' be connected nodal curves free from separating nodes; denote by $\nu : X^\nu \rightarrow X$ and $\nu' : X'^\nu \rightarrow X'$ their normalizations. X and X' are *C1-equivalent* if the following conditions hold

- (A) There exists an isomorphism $\phi : X^\nu \xrightarrow{\cong} X'^\nu$.
- (B) There exists a bijection between their C1-sets

$$\psi : \text{Set}^1 X \xrightarrow{\cong} \text{Set}^1 X'$$

such that $\phi(\nu^{-1}(S)) = \nu'^{-1}(\psi(S))$.

The above C1-equivalence relation can be realized via a sequence of twisting operations at pairs of separating nodes.

Definition 4.2.8 (Twist-equivalence). Let X be a connected nodal curve free from separating nodes. Consider a separating pair $\{n_1, n_2\}$ of nodes of X . Denote by $\epsilon : Y \rightarrow X$ the partial normalization of X at n_1 and n_2 , denote by Y_1 and Y_2 the connected components of Y and let $\nu^{-1}(n_1) = \{p^1, p^2\}$ and $\nu^{-1}(n_2) = \{q^1, q^2\}$ with $p^1, q^1 \in Y_1$ and $p^2, q^2 \in Y_2$. In particular X is obtained from Y by gluing p_1 with p_2 and q_1 with q_2 ; or in symbols

$$X = \frac{Y}{\{p^1 \sim p^2, q^1 \sim q^2\}}.$$

The *twist* of X at $\{n_1, n_2\}$ is the curve X' obtained from Y by gluing p^1 with q^2 and q^1 with p^2 , or in symbols:

$$X' = \frac{Y}{\{p^1 \sim q^2, q^1 \sim p^2\}}.$$

We say that two connected nodal curves free from separating nodes X and X' are *twist-equivalent* if X' can be obtained from X via a sequence of twisting at separating pairs of nodes.

Lemma 4.2.9. *Let X and X' be connected nodal curves free from separating nodes. Then X and X' are C1-equivalent if and only if they are twist-equivalent.*

Proof. This follows from the discussion in [CV11, Sec. 2.3.2]. \square

With this definitions, we can now recall the description of the fibers of \overline{t}_g obtained in [CV11].

Fact 4.2.10 (Caporaso-Viviani). *Let $X, X' \in \overline{\mathcal{M}}_g(k)$ two stable curves of genus g . Denote by $\{X_1, \dots, X_r\}$ (resp. $\{X'_1, \dots, X'_{r'}\}$) the separating blocks of X (resp. X') that have arithmetic genus greater than zero.*

The following are equivalent:

- (i) $\bar{t}_g(X) = \bar{t}_g(X')$.
- (ii) *We have that $r = r'$ and, up to reordering the separating blocks, we have that \overline{X}_i is C1-equivalent to \overline{X}'_i for each $1 \leq i \leq r = r'$.*
- (iii) *We have that $r = r'$ and, up to reordering the separating blocks, we have that \overline{X}_i is twist-equivalent to \overline{X}'_i for each $1 \leq i \leq r = r'$.*

Proof. The equivalence (i) \Leftrightarrow (ii) is a restatement of [CV11, Thm. 2.1.7]. The equivalence (ii) \Leftrightarrow (iii) follows from Lemma 4.2.9. \square

Corollary 4.2.11. *The compactified Torelli morphism $\bar{t}_g : \overline{\mathcal{M}}_g(k) \rightarrow \overline{\mathcal{A}}_g(k)$ is injective on the open subset of $\overline{\mathcal{M}}_g(k)$ consisting of stable curves without separating nodes nor separating pairs of nodes.*

We end this subsection with a reformulation of Fact 4.2.10 in the case of curves X free from separating nodes and not hyperelliptic (in the sense of [Cat82, Def. 3.9]), i.e. such that there does not exist two smooth points $p, q \in X$ with $h^0(X, \mathcal{O}_X(p+q)) = 2$. Note that if X is a stable curve of genus $g \geq 2$ free from separating nodes, then ω_X is base point free by [Cat82, Thm. D] and hence the complete linear system $|\omega_X|$ defines a morphism $\phi_{|\omega_X|} : X \rightarrow \mathbb{P}^{g-1}$ (well-defined only up to composing with a projectivity of \mathbb{P}^{g-1}), called the *canonical morphism*. Using results of Catanese [Cat82] and Catanese-Franciosi-Hulek-Reid [CFHR99], the image of the canonical morphism is described as it follows.

Theorem 4.2.12. *Let X be a stable curve X of genus $g \geq 2$ free from separating nodes and not hyperelliptic. Then $\phi_{|\omega_X|}(X)$ is the curve obtained from X by identifying all the nodes belonging to the same C1-set S into a unique point which is moreover a seminormal singularity of multiplicity $2|S|$, i.e. analytically isomorphic to the origin in the union of the coordinates axes of \mathbb{A}^n with $n = 2|S|$.*

Proof. The canonical morphism $\phi_{|\omega_X|}$ is an isomorphism away from the separating nodes of X , as it follows from the proof of [CFHR99, Thm. 3.6]. Moreover from [Cat82, Thm. E] it follows that $\phi_{|\omega_X|}$ sends all the nodes belonging to a C1-set S into the same point p_S and, moreover, that $p_S \neq p_{S'}$ if S and S' are two distinct C1-sets. Finally, from [Cat82, Rmk. 3.8], it follows that each point p_S is analytically isomorphic to the origin in the union of the coordinates axes of \mathbb{A}^n with $n = 2|S|$. \square

Theorem 4.2.13. *Let $X, X' \in \overline{\mathcal{M}}_g(k)$ two stable curves of genus g free from separating nodes and not hyperelliptic. Then*

$$\bar{t}_g(X) = \bar{t}_g(X') \Leftrightarrow \phi_{|\omega_X|}(X) \cong \phi_{|\omega_{X'}|}(X').$$

Proof. According to the above Fact 4.2.12, the curve $\phi_{|\omega_X|}(X)$ can be constructed from the normalization X^ν of X by gluing together the points $\nu^{-1}(S)$, for each C1-set S , into a seminormal singular point (note there is a unique way of performing this gluing, i.e. seminormal curve singularities do not have local moduli). Therefore, $\phi_{|\omega_X|}(X)$ depends only on the C1-equivalence class of X and, conversely, we can recover X up to C1-equivalence from the curve $\phi_{|\omega_X|}(X)$. The Theorem now follows from Fact 4.2.10. \square

5. THE ANTICONTINUITY OF THE REDUCTION MAPS

The aim of this Section is to prove the anticontinuity of the reduction maps appearing in the diagram (1.1). Indeed, this will follow from Corollary 5.0.17 which says that the same is true for any proper stack.

Recall that, as usual (see 1.0.2), we fix a complete DVR R with maximal ideal \mathfrak{m} and we assume that its residue field $k := R/\mathfrak{m}$ is algebraically closed. Given an element $x \in R$, we denote by $\bar{x} \in k$ its reduction modulo the maximal ideal \mathfrak{m} .

The following well-known lemma is the key result for what follows.

Lemma 5.0.14. *For any positive integer n consider the reduction map*

$$\begin{aligned} \text{red} : R^n &\longrightarrow k^n \\ \underline{x} = (x_1, \dots, x_n) &\mapsto \text{red}(\underline{x}) := (\bar{x}_1, \dots, \bar{x}_n). \end{aligned}$$

If we put the non-archimedean topology on R^n and the Zariski topology on k^n , then the reduction map red is anticontinuous, i.e. the inverse image of a closed subset is an open subset or, equivalently, the inverse image of an open subset is a closed subset.

Proof. Consider a Zariski closed subset $C \subseteq k^n$. By definition of the Zariski topology on k^n , this means that there exists a finite number of polynomials $F_1, \dots, F_r \in k[t_1, \dots, t_n]$ such that

$$(5.1) \quad C = \bigcap_{i=1}^r V(F_i) := \bigcap_{i=1}^r \{\underline{z} \in k^n : F_i(\underline{z}) = 0\}.$$

For any $1 \leq i \leq r$, we choose a polynomial $\tilde{F}_i \in R[t_1, \dots, t_n]$ whose reduction $\text{red}(\tilde{F}_i)$ is equal to F_i , where the reduction of a polynomial with coefficients in R is the polynomial with coefficients in k obtained by reducing modulo \mathfrak{m} each of its coefficients. For an element $\underline{x} \in R^n$, we see that

$$(5.2) \quad 0 = F_i(\text{red}(\underline{x})) = \text{red}(\tilde{F}_i(\underline{x})) \Leftrightarrow |\tilde{F}_i(\underline{x})| < 1.$$

For any polynomial $F \in R[t_1, \dots, t_n]$, consider the evaluation function

$$\begin{aligned} \Phi_F : R^n &\longrightarrow R \\ \underline{x} &\mapsto \Phi_F(\underline{x}) := F(\underline{x}). \end{aligned}$$

Clearly, the function Φ_F is continuous with respect to the non-archimedean topology on the domain and the codomain. Using the evaluation functions, the equivalence (5.2) can be rewritten as

$$(5.3) \quad \text{red}^{-1}(V(F_i)) = \Phi_{\tilde{F}_i}^{-1}(\mathfrak{m}).$$

Therefore the inverse image of each $V(F_i)$ under the reduction map red is open (recall that $\mathfrak{m} \subset R$ is open in the non-archimedean topology being equal to the open ball centered at 0 and of radius 1, see 1.0.2); the same is true for $\text{red}^{-1}(C)$ because of the representation as in (5.1), which concludes the proof. \square

We can now define the non-archimedean topology on the set $\mathcal{X}(R) := \text{Hom}(\text{Spec } R, \mathcal{X})$ (resp. $\mathcal{X}(K) := \text{Hom}(\text{Spec } K, \mathcal{X})$) of R -valued (resp. K -points) points of any stack \mathcal{X} of finite type over $\text{Spec } R$.

Definition 5.0.15 (Non-archimedean topology).

- (i) Let $X \rightarrow \text{Spec } R$ be an affine scheme of finite type over $\text{Spec } R$ and let $j : X \hookrightarrow \mathbb{A}_R^N$ be a closed embedding into the N -dimensional affine space over R for some N . The non-archimedean topology on the set of R -valued points of $X(R)$ is the subspace topology with respect to the natural inclusion $j_R : X(R) \subseteq \mathbb{A}_R^N(R) = R^N$ and the non-archimedean topology on R^N .

In a similar way, we define the non-archimedean topology on $X(K)$.

- (ii) Let $\mathcal{X} \rightarrow \text{Spec } R$ be a (Artin) stack of finite type over $\text{Spec } R$ and choose an atlas $f : X \rightarrow \mathcal{X}$ (i.e. f a surjective and smooth morphism and X is a scheme over $\text{Spec } R$) of \mathcal{X} with X affine and of finite type over $\text{Spec } R$. The non-archimedean topology on the set $\mathcal{X}(R)$ of R -valued points of \mathcal{X} is the quotient topology with respect to the natural surjective map $f_R : X(R) \rightarrow \mathcal{X}(R)$ and the non-archimedean topology on $X(R)$.

In a similar way, we define the non-archimedean topology on $\mathcal{X}(K)$.

We leave to the reader the straightforward verification that the above definitions do not depend on the choices made, i.e. the embedding j in (i) and the atlas f in (ii).

We can now prove the main result of this section. The result is certainly well-known to the experts (see [Tem, §3.4.1] for the case of strictly K -affinoid spaces and [Tem, §5.2.4] for the case of formal schemes over $\text{Spec } R$) but we include a proof for the lack of a suitable reference in the case of stacks of finite type over $\text{Spec } R$.

Theorem 5.0.16. *Let \mathcal{X} a stack of finite type over $\text{Spec } R$ and consider the reduction map*

$$\text{red}_{\mathcal{X}} : \mathcal{X}(R) := \text{Hom}(\text{Spec } R, \mathcal{X}) \rightarrow \mathcal{X}(k) := \text{Hom}(\text{Spec } k, \mathcal{X})$$

induced by composing with the map $s : \text{Spec } k \rightarrow \text{Spec } R$. If we put the non-archimedean topology on $\mathcal{X}(R)$ and the Zariski topology on $\mathcal{X}(k)$, then $\text{red}_{\mathcal{X}}$ becomes an anticontinuous map.

Proof. We first prove the theorem in two special cases.

Case I: assume that $\mathcal{X} = \mathbb{A}^n$ for some n .

In this case, the theorem reduces to Lemma 5.0.14.

Case II: assume that \mathcal{X} is an affine scheme of finite type over $\text{Spec } R$.

Choose a closed embedding $j : \mathcal{X} \hookrightarrow \mathbb{A}^N$ for some N as in Definition 5.0.15(i).

This induces a commutative diagram

$$(5.4) \quad \begin{array}{ccc} \mathcal{X}(R) & \xrightarrow{\text{red}_{\mathcal{X}}} & \mathcal{X}(k) \\ j_R \downarrow & & \downarrow j_k \\ \mathbb{A}^N(R) & \xrightarrow{\text{red}_{\mathbb{A}^N}} & \mathbb{A}^N(k), \end{array}$$

where the vertical arrows are injective. If we put the non-archimedean topology on the sets on the left of the diagram and the Zariski topology on the sets on the right of the diagram, then we have that:

- $\text{red}_{\mathbb{A}^N}$ is anticontinuous by Case I;
- j_R is continuous by Definition 5.0.15(i);
- j_k is a closed continuous map since j is an embedding.

Now, using the above facts, an easy diagram chase in (5.4) shows that $\text{red}_{\mathcal{X}}$ is anticontinuous and Case II is proved.

Let us now consider an arbitrary stack \mathcal{X} of finite type over $\text{Spec } R$. Choose an atlas $f : X \rightarrow \mathcal{X}$ with X affine and of finite type over $\text{Spec } R$ as in Definition 5.0.15(ii). This induces a commutative diagram

$$(5.5) \quad \begin{array}{ccc} X(R) & \xrightarrow{\text{red}_X} & X(k), \\ f_R \downarrow & & \downarrow f_k \\ \mathcal{X}(R) & \xrightarrow{\text{red}_{\mathcal{X}}} & \mathcal{X}(k) \end{array}$$

where the vertical arrows are surjective. If we put the non-archimedean topology on the sets on the left of the diagram and the Zariski topology on the sets on the right of the diagram, then we have that:

- $\text{red}_{\mathcal{X}}$ is anticontinuous by Case II;
- f_R is a quotient map by Definition 5.0.15(ii);
- f_k is a continuous map because it is induced by the morphism of stacks f .

Now, using the above facts, an easy diagram chase in (5.5) shows that $\text{red}_{\mathcal{X}}$ is anticontinuous, q.e.d. □

In the case of proper stacks \mathcal{X} over $\text{Spec } R$, we can extend the reduction map to the set of K -valued points.

Corollary 5.0.17. *Let \mathcal{X} be a proper stack over $\text{Spec } R$. Then the reduction map $\text{red}_{\mathcal{X}}$ can be extended to a map $\widetilde{\text{red}}_{\mathcal{X}} : \mathcal{X}(K) \rightarrow \mathcal{X}(k)$ as in the following diagram*

$$\begin{array}{ccc} \mathcal{X}(R) & \xrightarrow{\text{red}_{\mathcal{X}}} & \mathcal{X}(k) \\ \downarrow & \nearrow \widetilde{\text{red}}_{\mathcal{X}} & \\ \mathcal{X}(K) & & \end{array}$$

Moreover, $\widetilde{\text{red}}_{\mathcal{X}}$ is anticontinuous with respect to the non-archimedean topology on $\mathcal{X}(K)$ and the Zariski topology on $\mathcal{X}(k)$.

Proof. Let us denote by \mathcal{E} the set of all the finite degree extensions $K \subseteq L$. For each $L \in \mathcal{E}$, we denote by R_L the valuation ring of L with respect to the unique extension of the valuation val on K to a valuation val_L on L (see 1.0.2).

Since K is complete with respect to the valuation val , for each $L \in \mathcal{E}$ there is a unique extension of the valuation val on K to a valuation val_L on L , which is moreover still complete. We denote by $R_L \subset L$ the associated valuation ring. Since k is algebraically closed, the residue field of each of the rings R_L (for $L \in \mathcal{E}$) is equal to k . Therefore, we get a diagram

$$(5.6) \quad \bigcup_{L \in \mathcal{E}} \mathcal{X}(L) \xleftarrow{\eta} \bigcup_{L \in \mathcal{E}} \mathcal{X}(R_L) \xrightarrow{\widetilde{\text{red}}_{\mathcal{X}}} \mathcal{X}(k).$$

We endow the sets appearing in (5.6) with the following topologies: on $\mathcal{X}(k)$ we put the Zariski topology; on $\bigcup_{L \in \mathcal{E}} \mathcal{X}(L)$ we put the finest topology for which all the inclusions $\mathcal{X}(L) \hookrightarrow \bigcup_{L \in \mathcal{E}} \mathcal{X}(L)$ are continuous with respect to the non-archimedean topology on $\mathcal{X}(L)$; the topology on $\bigcup_{L \in \mathcal{E}} \mathcal{X}(R_L)$ is defined in a similar way. With respect to these topologies, the map η is clearly continuous while the map $\widetilde{\text{red}}_{\mathcal{X}}$ is anticontinuous by Theorem 5.0.16. Moreover the valuative criterion for properness of stacks applied to \mathcal{X} implies that η is an homeomorphism. We define a map $\widetilde{\text{red}}_{\mathcal{X}} : \mathcal{X}(K) \rightarrow \mathcal{X}(k)$ by composing the injection $\mathcal{X}(K) \hookrightarrow \bigcup_{L \in \mathcal{E}} \mathcal{X}(L)$, the homeomorphism η^{-1} and the reduction map $\widetilde{\text{red}}_{\mathcal{X}}$. It is now clear the map $\widetilde{\text{red}}_{\mathcal{X}}$ satisfies all the required properties. □

Proof of Theorem C. It is easily checked that the reduction map $\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$ constructed in Lemma-Definition 2.1.10 is the restriction of the reduction map $\widetilde{\text{red}}_{\overline{\mathcal{M}}_g} : \overline{\mathcal{M}}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$ constructed in Corollary 5.0.17 to the open subset $\mathcal{M}_g(K) \subset \overline{\mathcal{M}}_g(K)$. Therefore the anticontinuity of $\text{red} : \mathcal{M}_g(K) \rightarrow \overline{\mathcal{M}}_g(k)$ follows the anticontinuity of the reduction map $\widetilde{\text{red}}_{\overline{\mathcal{M}}_g}$ proved in Corollary 5.0.17.

A similar argument applies to the reduction map $\text{red} : \mathcal{A}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$ using the anticontinuity of the reduction map $\widetilde{\text{red}}_{\overline{\mathcal{A}}_g} : \overline{\mathcal{A}}_g(K) \rightarrow \overline{\mathcal{A}}_g(k)$ (again from Corollary 5.0.17).

□

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