Pressure Rigidity of Three Dimensional Contact Anosov Flows

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Abstract — *Let ϕ_t be a three dimensional contact Anosov flow. Then we prove that its cohomological pressure coincides with its metric entropy if and only if ϕ_t is C^∞ flow equivalent to a special time change of a three dimensional algebraic contact Anosov flow.*

1. Introduction

Let M be a C^∞-closed manifold. A C^∞-flow, ϕ_t, generated by a non-singular vector field X on M is called *Anosov*, if there exists a ϕ_t-invariant continuous splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^+ \oplus E^-,$$

a Riemannian metric on M and two positive numbers a and b, such that

$$\| D\phi_{-t}(u^+) \| \leq a \cdot e^{-bt} \| u^+ \|, \ \forall\ u^+ \in E^+, \ \forall\ t > 0,$$

and

$$\| D\phi_t(u^-) \| \leq a \cdot e^{-bt} \| u^- \|, \ \forall\ u^- \in E^-, \ \forall\ t > 0.$$

The continuous distributions E^+ and E^- are called respectively the strong unstable and strong stable distributions of ϕ_t. They are both integrable to continuous foliations with C^∞ leaves (see [**HK**]).

The *canonical* 1-*form* of ϕ_t is by definition the continuous 1-form on M such that $\lambda(X) = 1$ and $\lambda(E^\pm) = 0$. It is easily seen that λ is ϕ_t-invariant. By definition, ϕ_t is said to be a *contact Anosov* flow if λ is C^∞ and there exists $n \in \mathbb{N}$ such that $\lambda \wedge (\wedge^n d\lambda)$ is a volume form on M. It is easy to see that contact Anosov flows are contact in the classical sense (see [**Pa**]).

Let ϕ_t be a contact Anosov flow on a closed manifold M of dimension $2n+1$. Then $\lambda \wedge (\wedge^n d\lambda)$ is a ϕ_t-invariant volume form. Denote by ν the ϕ_t-invariant probability measure determined by this volume form. Then the measure-theoretic entropy of ϕ_t with respect to ν is said to be the metric entropy of ϕ_t and is denoted by $h_\nu(\phi_t)$.

Denote by $\mathcal{M}(\phi_t)$ the set of ϕ_t-invariant probability measures. For each C^∞ function f on M, the *topological pressure* of ϕ_t with respect to f is dy definition the following number

$$P(\phi_t, f) = sup_{\mu \in \mathcal{M}(\phi_t)} \{h_\mu(\phi_t) + \int_M f d\mu\},$$

where $h_\mu(\phi_t)$ denotes the metric entropy of ϕ_t with respect to μ. It is well known (see **[HK]** and **[BR]**) that there exists a unique ϕ_t-invariant probability measure μ such that

$$P(\phi_t, f) = h_\mu(\phi_t) + \int_M f d\mu.$$

This measure μ is said to be the *Gibbs measure* of ϕ_t with respect to f. For example, the classical Bowen-Margulis of ϕ_t is just the Gibbs measure of ϕ_t with respect to the zero function. Recall also that $P(\phi_t, 0)$ is just the topological entropy of ϕ_t denoted by $h_{top}(\phi_t)$.

Let f' be another C^∞ function on M with Gibbs measure μ'. Then we can prove (see **[HK]**) that $\mu = \mu'$ if and only if there exist a C^∞ funcion H and a constant c such that

$$f = f' + X(H) + c.$$

Denote by $H^1(M, \mathbb{R})$ the first cohomology group of M. For $[\beta] \in H^1(M, \mathbb{R})$ we denote by $P(\phi_t, [\beta])$ the topological pressure of ϕ_t with respect to $\beta(X)$. For each smooth function g and $\mu \in \mathcal{M}(\phi_t)$ we have

$$\int_M X(g) d\mu = 0.$$

So $P(\phi_t, [\beta])$ is independent of the closed 1-form chosen in the cohomological class $[\beta]$. We call the Gibbs measure of ϕ_t with respect to $\beta(X)$ that of ϕ_t with respect to $[\beta]$. We define

$$P(\phi_t) = inf_{[\beta] \in H^1(M, \mathbb{R})} \{P(\phi_t, [\beta])\},$$

which is said to be the *cohomological pressure* of ϕ_t. This notion was firstly defined by R. Sharp in **[Sh]**. By the equivalence of $(ii)'$ and (iii) of Theorem one in **[Sh]**, we know that there exists a unique element $[\alpha]$ in $H^1(M,\mathbb{R})$ such that

$$P(\phi_t,[\alpha]) = P(\phi_t).$$

We call this cohomology class $[\alpha]$ the *Gibbs class* of ϕ_t.

For each element $[\beta]$ in $H^1(M,\mathbb{R})$, it is easy to see that

$$\int_M \beta(X)\lambda \wedge (\wedge^n d\lambda) = \int_M \beta \wedge (\wedge^n d\lambda) = 0.$$

So $\int_M \beta(X)d\nu = 0$. Thus we have

$$h_{top}(\phi_t) \geq P(\phi_t) \geq h_\nu(\phi_t).$$

In general these inequalities are strict.

Let N be a C^∞ closed negatively curved manifold. Denote by ϕ_t its geodesic flow and by μ the Bowen-Margulis measure of ϕ_t. Since μ is invariant under the flip map, then for each $[\alpha]$ in $H^1(SN,\mathbb{R})$ we have

$$\int_{SN} \alpha(X)d\mu = 0,$$

where SN denotes the unitary bundle of N (see **[Pa]**). So we get

$$P(\phi_t) \geq inf_{[\alpha]\in H^1(SN,\mathbb{R})}\{h_\mu(\phi_t) + \int_{SN} \alpha(X)d\mu\} = h_\mu(\phi_t) = h_{top}(\phi_t).$$

We deduce that $P(\phi_t) = h_{top}(\phi_t)$. So for the geodesic flows of negatively curved manifolds the cohomological pressure coincides with the topological entropy.

2. Rigidity in the case of dimension three

The classical examples of three dimensional contact Anosov flows are constructed as following. Denote by Γ a uniform lattice in $\widetilde{SL(2,\mathbb{R})}$. Then on the quotient manifold $\Gamma\diagdown\widetilde{SL(2,\mathbb{R})}$, the following flow is contact Anosov and is said to be *algebraic*.

$$\phi_t : \Gamma\diagdown\widetilde{SL(2,\mathbb{R})} \to \Gamma\diagdown\widetilde{SL(2,\mathbb{R})},$$

$$\Gamma \cdot A \to \Gamma \cdot (A \cdot \exp(t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})).$$

Up to a constant change of time scale and finite covers, such a flow is just the geodesic flow of a certain closed hyperbolic surface.

Quite recently, P. Foulon constructed in **[Fo]** plenty of surgerical three dimensional contact Anosov flows. These examples make the study of three dimensional contact Anosov flows very interesting.

Now suppose that ϕ_t is a three dimensional contact Anosov flow on a closed manifold M. A *special time change* of ϕ_t is by definition the flow of $\frac{X}{a-\alpha(X)}$, where $a > 0$ and α denotes a C^∞ closed 1-form on M such that $a - \alpha(X) > 0$.

Lemma 2.1. *Let ϕ_t be a three dimensional contact Anosov flow and ψ_t be a smooth time change of ϕ_t. Then ψ_t is contact iff it is a special time change of ϕ_t.*

Proof. Suppose that ψ_t is contact. Then its canonical 1-form $\bar{\lambda}$ is C^∞. Denote by Y the generator of ψ_t and suppose that $Y = fX$. Then we have

$$f \cdot \bar{\lambda}(X) = \bar{\lambda}(Y) = 1.$$

So we have $Y = \frac{X}{\bar{\lambda}(X)}$. Since $\bar{\lambda}$ is ψ_t-invariant, then

$$i_Y d\bar{\lambda} = 0.$$

We deduce that

$$\mathcal{L}_X d\bar{\lambda} = di_X d\bar{\lambda} + i_X dd\bar{\lambda} = 0.$$

So $d\bar{\lambda}$ is ϕ_t-invariant. Denote by λ the canonical 1-form of ϕ_t. Then by the ergodicity of ϕ_t (see **[An]**) there exists a constant b such that

$$\lambda \wedge d\bar{\lambda} = b \cdot \lambda \wedge d\lambda.$$

So there exists a C^∞ closed 1-form α such that

$$\bar{\lambda} = b \cdot \lambda + \alpha.$$

If b is non-positive, then $\alpha(X) > 0$ on M. Denote by Z the field $\frac{X}{\alpha(Z)}$. Then α is easily seen to be the canonical 1-form of ϕ_t^Z. So by **[Pl]**, ϕ_t admits a global section, which is absurd for a contact flow. We deduce that $b > 0$. Thus ψ_t is a special time change of ϕ_t.

Suppose that ψ_t is a special time change of ϕ_t, i.e. $Y = \frac{X}{a-\alpha(X)}$. It is easily seen that $a\lambda - \alpha$ is the canonical 1-form of Y. We have

$$(a\lambda - \alpha) \wedge d(a\lambda - \alpha) = a^2 \cdot \lambda \wedge d\lambda - a \cdot \alpha \wedge d\lambda.$$

By integrating this form, we see that $(a\lambda - \alpha) \wedge d(a\lambda - \alpha) \not\equiv 0$. Then we deduce by [**HuK**] that this three form is nowhere zero, i.e. ψ_t is contact. □

In the quite elegant paper [**Ka**], A. Katok proved the following

Theorem 2.1. (A. Katok) Let Σ be a C^∞ closed surface of negative curvature. Then its topological entropy coincides with its metric entropy, if and only if Σ is of constant negative curvature.

Then in [**Fo1**], the following generalization was established by using geometric constructions.

Theorem 2.2. (P. Foulon) *Let ϕ_t be a three dimensional contact Anosov flow. Then its topological entropy coincides with its metric entropy if and only if it is, up to a constant change of time scale, C^∞ flow equivalent to a three dimensional algebraic contact Anosov flow.*

Now we generalize this Theorem to the case of cohomological pressure. Let us prove firstly the following

Lemma 2.2. *Let ϕ_t be a three dimensional algebraic contact Anosov flow. Then for each special time change ψ_t of ϕ_t, there exists an element $[\alpha]$ in $H^1(M, \mathbb{R})$ such that the Gibbs measure of ψ_t with respect to $[\alpha]$ is Lebesgue. In particular the cohomological pressure of ψ_t coincides with its metric entropy.*

Proof. Denote by h the topological entropy of ϕ_t. Since the Anosov splitting of ϕ_t is C^∞, then we can find a C^∞ nonwhere vanishing section Y^+ of E^+ and define for any $x \in M$ and any $t \in \mathbb{R}$,

$$\lambda_t(x) = \frac{(\phi_t)_* Y_x^+}{Y^+_{\phi_t(x)}}.$$

Define also

$$\phi^+ = -\frac{\partial}{\partial t} \mid_{t=0} \lambda_t(\cdot).$$

Then it is easy to see that the Gibbs measure of ϕ_t with respect to ϕ^+ is ν (see [**BR**]). In addition, we have

$$P(\phi_t, \phi^+) = 0.$$

Since the Bowen-Margulis measure of ϕ_t is Lebesgue, then there exists a smooth function f and a contant c such that

$$\phi^+ = X(f) + c.$$

So we have

$$0 = P(\phi_t, \phi^+) = P(\phi_t, X(f) + c) = h + c.$$

We deduce that $\phi^+ = X(f) - h$.

Suppose that ψ_t is generated by the field $\bar{X} = \frac{X}{a-\alpha(X)}$. Then by **[LMM]**, it is easy to see that

$$\bar{E}^+ = \{u^+ + \frac{\alpha(u^+)}{a - \alpha(X)} \cdot X \mid u^+ \in E^+\}.$$

Suppose that $\psi_t(\cdot) = \phi_{\beta(t,\cdot)}(\cdot)$ and define $\bar{Y}^+ = Y^+ + \frac{\alpha(Y^+)}{a-\alpha(X)} X$. Then

$$\bar{\lambda}_t(x) = \frac{(\psi_t)_* \bar{Y}^+_x}{\bar{Y}^+_{\psi_t(x)}} = \lambda_{\beta(t,x)}(x).$$

Since $\dot{\beta}(0,\cdot) = \frac{1}{a-\alpha(X)}$, then

$$\bar{\phi}^+ = \frac{\phi^+}{a - \alpha(X)}.$$

So we get

$$\bar{\phi}^+ = \bar{X}(f) - \frac{h}{a - \alpha(X)}.$$

In addition, we observe easily that

$$(-\frac{h}{a} \cdot \alpha)(\bar{X}) = \frac{h}{a} - \frac{h}{a - \alpha(X)}.$$

So the Gibbs measure of ψ_t with respect to $[-\frac{h}{a} \cdot \alpha]$ is Lebesgue. □

Now we establish the following extension of Theorem 2.2.

Theorem 2.3. *Let ϕ_t be a three dimensional contact Anosov flow defined on a closed manifold M. Then its cohomological pressure coincides with its metric entropy if and only if ϕ_t is C^∞ flow equivalent to a special time change of a three dimensional algebraic contact Anosov flow.*

Proof. Denote by $[\alpha]$ the Gibbs class of ϕ_t. Since $\int_M \alpha(X)d\nu = 0$ and by assumption

$$h_\nu(\phi_t) = P(\phi_t) = P(\phi_t, [\alpha]),$$

then

$$h_\nu(\phi_t) + \int_M \alpha(X)d\nu = P(\phi_t, [\alpha]),$$

i.e. ν is the Gibbs measure of ϕ_t with respect to $[\alpha]$. Denote $h_\nu(\phi_t)$ by h. Then by the variational principle, we have for $\forall\ \mu \in \mathcal{M}(\phi_t)$ and $\mu \neq \nu$,

$$h > h_\mu(\phi_t) + \int_M \alpha(X)d\mu \geq \int_M \alpha(X)d\mu.$$

Then by **[Gh1]**, there exists a smooth function f such that

$$h > (\alpha + df)(X).$$

Without loss of generality, we replace α by $\alpha + df$.

Up to finite covers, we suppose that E^+ and E^- are both orientable. In **[HuK]**, it is proved that the strong stable and instable distributions of a three dimensional contact Anosov flow are $C^{1,Zyg}$. So we can find a $C^{1,Zyg}$ nonwhere vanishing section Y^+ of E^+ and define the functions $\lambda_t(\cdot)$ and ϕ^+ as in the proof of Lemma 2.3. Then the Gibbs measure of ϕ_t with respect to ϕ^+ is lebesgue.

Define $\bar{X} = \frac{X}{h-\alpha(X)}$ and denote by ψ_t its flow. Then we define as before $\bar{Y}^+$ and $\bar{\phi}^+$. So by the same arguments,

$$\bar{\phi}^+ = \frac{\phi^+}{h - \alpha(X)}.$$

Since the Gibbs measure of ϕ_t with respect to $\alpha(X)$ is Lebesgue, then there exists a smooth function g and a constant c such that

$$\phi^+ = \alpha(X) + X(g) + c.$$

So we have

$$0 = P(\phi_t, \phi^+) = P(\phi_t, [\alpha]) + c = h + c.$$

We deduce that

$$\bar{\phi}^+ = -1 + \bar{X}(g),$$

i.e. the Bowen-Margulis measure of ψ_t is Lebesgue. Then by Theorem 2.2, ψ_t is C^∞ flow equivalent to a three dimensional algebraic contact Anosov flow. We deduce from Lemma 2.2 that ϕ_t is C^∞ flow equivalent to a special time change of such a flow. □

As mentioned above, we know by **[HuK]** that the Anosov splitting of a three dimensional contact Anosov flow is always $C^{1,Zyg}$. In addition by Theorem 4.6 of **[Gh2]**, we know that up to finite covers, a three dimensional contact Anosov flow with $C^{1,lip}$ splitting is C^∞ flow equivalent to a special time change of the geodesic flow of a closed hyperbolic surface (see also **[Gh]**, **[BFL]** and **[HuK]**). Thus by combining these classical results with our previous result, we obtain the following

Theorem 2.4. *Let ϕ be a three dimensional contact Anosov flow. Then its Anosov splitting is $C^{1,Zyg}$ and the following conditions are equivalent :*
(1) *The cohomological pressure of ϕ_t is equal to its metric entropy.*
(2) *The Anosov splitting of ϕ_t is $C^{1,lip}$.*
(3) *Up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to a special time change of the geodesic flow of a closed hyperbolic surface.*

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