

FRACTIONAL SOBOLEV–POINCARÉ AND FRACTIONAL HARDY INEQUALITIES IN UNBOUNDED JOHN DOMAINS

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ABSTRACT. We prove fractional Sobolev–Poincaré inequalities in unbounded John domains and we characterize fractional Hardy inequalities there.

1. INTRODUCTION

Let D be a bounded c -John domain in \mathbb{R}^n , $n \geq 2$. Let numbers $\delta, \tau \in (0, 1)$ and exponents $p, q \in [1, \infty)$ be given such that $1/p - 1/q = \delta/n$. Then there is a constant $C = C(\delta, \tau, p, n, c)$ such that the fractional Sobolev–Poincaré inequality

$$(1) \quad \int_D |u(x) - u_D|^q dx \leq C \left(\int_D \int_{B^n(x, \tau \operatorname{dist}(x, \partial D))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p}$$

holds for all functions $u \in L^1(D)$. For a proof we refer the reader to [11, Theorem 4.10] when $1 < p < n/\delta$ and to [4] when $p = 1$.

We prove the inequality corresponding to (1) in unbounded John domains, Theorem 5.1. The classical Sobolev–Poincaré inequality for an unbounded c -John domain D has been proved in [9, Theorem 4.1]: there is a finite constant $C(n, p, c)$ such that the inequality

$$\inf_{a \in \mathbb{R}} \int_D |u(x) - a|^{np/(n-p)} dx \leq C(n, p, c) \left(\int_D |\nabla u(x)|^p dx \right)^{n/(n-p)}$$

holds for all $u \in L_p^1(D) = \{u \in \mathcal{D}'(D) : \nabla u \in L^p(D)\}$; here $1 \leq p < n$. We obtain the fractional Sobolev inequalities (10) in unbounded John domains too, Theorem 5.2.

As an application of the fractional Sobolev inequalities we characterize the fractional Hardy inequalities

$$\int_D \frac{|u(x)|^q}{\operatorname{dist}(x, \partial D)^{q(\delta+n(1/q-1/p))}} dx \leq C \left(\int_D \int_D \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p}$$

in unbounded John domains D whenever $\delta \in (0, 1)$ and exponents $p, q \in [1, \infty)$ are given such that $p < n/\delta$ and $0 \leq 1/p - 1/q \leq \delta/n$ and the constant C does not depend on $u \in C_0(D)$, Theorem 6.1. We also give sufficient geometric conditions for the fractional Hardy inequalities in Corollary 6.3.

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2. NOTATION AND PRELIMINARIES

Throughout the paper we assume that D is a domain and G is an open set in the Euclidean n -space \mathbb{R}^n , $n \geq 2$. The open ball centered at $x \in \mathbb{R}^n$ and with radius $r > 0$ is $B^n(x, r)$. The Euclidean distance from $x \in G$ to the boundary of G is written as $\text{dist}(x, \partial G)$. The diameter of a set A in \mathbb{R}^n is $\text{diam}(A)$. The Lebesgue n -measure of a measurable set A is denoted by $|A|$. For a measurable set A with finite and positive measure and for an integrable function u on A the integral average is written as

$$u_A = \frac{1}{|A|} \int_A u(x) dx.$$

We write χ_A for the characteristic function of a set A . For a proper open set G in \mathbb{R}^n we fix a Whitney decomposition $\mathcal{W}(G)$. The construction and the properties of Whitney cubes can be found in [16, VI 1]. The family $C_0(G)$ consists of all continuous functions $u : G \rightarrow \mathbb{R}$ with a compact support in G . We let $C(*, \dots, *)$ denote a constant which depends on the quantities appearing in the parentheses only.

We define the c -John domains so that unbounded domains are allowed, too. For other equivalent definitions we refer the reader to [17] and [9].

Definition 2.1. *A domain D in \mathbb{R}^n with $n \geq 2$ is a c -John domain, $c \geq 1$, if each pair of points $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma : [0, \ell] \rightarrow D$ parametrized by its arc length such that $\text{dist}(\gamma(t), \partial D) \geq \min\{t, \ell - t\}/c$ for every $t \in [0, \ell]$.*

Examples of unbounded John domains are the Euclidean n -space \mathbb{R}^n and the infinite cone

$$\{(x', x_n) \in \mathbb{R}^n : x_n > \|x'\|\}.$$

For more examples we refer the reader to [9, 4.3 Examples].

We recall a useful property of bounded John domains from [17, Theorem 3.6].

Lemma 2.2. *Let D in \mathbb{R}^n be a bounded c -John domain, $n \geq 2$. Then there exists a central point $x_0 \in D$ such that every point x in D can be joined to x_0 by a rectifiable curve $\gamma : [0, \ell] \rightarrow D$, parametrized by its arc length, with $\gamma(0) = x$, $\gamma(\ell) = x_0$, and $\text{dist}(\gamma(t), \partial D) \geq t/4c^2$ for each $t \in [0, \ell]$.*

The following engulfing property is in [17, Theorem 4.6].

Lemma 2.3. *A c -John domain D in \mathbb{R}^n can be written as the union of domains D_1, D_2, \dots such that*

- (1) $\overline{D_i}$ is compact in D_{i+1} for each $i = 1, 2, \dots$,
- (2) D_i is a c_1 -John domain for each $i = 1, 2, \dots$ with $c_1 = c_1(c, n)$.

We define the upper and lower Assouad dimension of a given set $E \neq \emptyset$ in \mathbb{R}^n . The upper Assouad dimension measures how thin a given set is and the lower Assouad dimension measures its fatness. For further discussion on these dimensions we refer to [13, §1].

Definition 2.4. *The upper Assouad dimension of E , written as $\overline{\dim}_A(E)$, is defined as the infimum of all numbers $\lambda \geq 0$ as follows: There exists a constant $C = C(E, \lambda) > 0$ such that for every $x \in E$ and for all $0 < r < R < 2\text{diam}(E)$ the set $E \cap B^n(x, R)$ can be covered by at most $C(R/r)^\lambda$ balls that are centered in E and have radius r .*

Definition 2.5. *The lower Assouad dimension of E , written as $\underline{\dim}_A(E)$, is defined as the supremum of all numbers $\lambda \geq 0$ as follows: There exists a constant $C = C(E, \lambda) > 0$ such that for every $x \in E$ and for all $0 < r < R < 2\text{diam}(E)$ at least $C(R/r)^\lambda$ balls centered in E and with radius r are needed to cover the set $B^n(x, R) \cap E$.*

Let G be an open set in \mathbb{R}^n . Let $0 < p < \infty$ and $0 < \tau, \delta < 1$ be given. We write

$$|u|_{W^{\delta,p}(G)} = \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p}$$

and

$$|u|_{\dot{W}_\tau^{\delta,p}(G)} = \left(\int_G \int_{B^n(x, \tau \text{dist}(x, \partial G))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{1/p}$$

for appropriate measurable functions u on G . When $G = \mathbb{R}^n$ both of the integrals in the latter form are taken over the whole space. The homogeneous fractional Sobolev space $\dot{W}_\tau^{\delta,p}(G)$ consists of all measurable functions $u : G \rightarrow \mathbb{R}$ with $|u|_{\dot{W}_\tau^{\delta,p}(G)} < \infty$.

The following lemma tells that the functions $u \in \dot{W}_\tau^{\delta,p}(G)$ are locally L^p -integrable in G , that is $u \in L^p_{\text{loc}}(G)$. We improve this for John domains in Corollary 5.3.

Lemma 2.6. *Suppose that G is an open set in \mathbb{R}^n . Let $0 < p < \infty$ and $0 < \tau, \delta < 1$ be given. Let K be a compact set in G . If $u \in \dot{W}_\tau^{\delta,p}(G)$ then $u \in L^p(K)$.*

Proof. We may assume that $G \neq \mathbb{R}^n$. If $G = \mathbb{R}^n$, then we just remove one point from $G \setminus K$. By covering K with a finite number of balls B such that $\overline{B} \subset G$ we may assume that K is the closure of such a ball. Let us fix $\varepsilon > 0$ such that $\varepsilon\tau/(1 - \varepsilon\tau) < \tau$. We obtain

$$\begin{aligned} & \int_K \int_{K \cap B^n(z, \tau \text{dist}(z, \partial G))} |u(z) - u(y)|^p dy dz \\ (2) \quad & \leq \text{diam}(K)^{n+\delta p} \int_K \int_{K \cap B^n(z, \tau \text{dist}(z, \partial G))} \frac{|u(z) - u(y)|^p}{|z - y|^{n+\delta p}} dy dz \\ & \leq \text{diam}(K)^{n+\delta p} |u|_{\dot{W}_\tau^{\delta,p}(G)}^p < \infty. \end{aligned}$$

Let us fix $x \in K$ and $0 < r_x < \varepsilon\tau \text{dist}(x, \partial G)$. Since K is the closure of some ball, we have the inequality $|K \cap B^n(x, r_x)| > 0$. By our estimates in (2) there is a point $z_x \in K \cap B^n(x, r_x)$ so that

$$(3) \quad \int_{K \cap B^n(z_x, \tau \text{dist}(z_x, \partial G))} |u(z_x) - u(y)|^p dy < \infty.$$

By the choice of ε we have $x \in B^n(z_x, \tau \text{dist}(z_x, \partial G))$ for each $x \in K$. Thus,

$$K \subset \bigcup_{x \in K} B^n(z_x, \tau \text{dist}(z_x, \partial G)).$$

By the compactness of the set K there are points x_1, \dots, x_N in K such that K is contained in the union of the balls $B^n(z_i, \tau \operatorname{dist}(z_i, \partial G))$, where $z_i = z_{x_i}$ for each i . Hence, by inequality (3) we obtain

$$\begin{aligned} \int_K |u(y)|^p dy &\leq \sum_{i=1}^N \int_{K \cap B^n(z_i, \tau \operatorname{dist}(z_i, \partial G))} |u(y)|^p dy \\ &\leq 2^p \sum_{i=1}^N \int_{K \cap B^n(z_i, \tau \operatorname{dist}(z_i, \partial G))} |u(z_i)|^p + |u(z_i) - u(y)|^p dy < \infty. \end{aligned}$$

This concludes the proof. \square

The following definition is from [8, §1]. It arises from generalized Poincaré inequalities that are studied in [7, §7]. Let us fix $\kappa \geq 1$ and an open set G in \mathbb{R}^n . For $\delta \in [0, 1]$, $0 < p \leq \infty$, and $u \in L^1_{\operatorname{loc}}(G)$ we write

$$|u|_{A_{\kappa}^{\delta, p}(G)} = \sup_{\mathcal{Q}_{\kappa}(G)} \left\| \sum_{Q \in \mathcal{Q}_{\kappa}(G)} \left(\frac{1}{|Q|^{1+\delta/n}} \int_Q |u(x) - u_Q| dx \right) \chi_Q \right\|_{L^p(G)},$$

where the supremum is taken over all families of cubes $\mathcal{Q}_{\kappa}(G)$ such that $\kappa Q \subset G$ for every $Q \in \mathcal{Q}_{\kappa}(G)$ and $Q \cap R = \emptyset$ if Q and R belong to $\mathcal{Q}_{\kappa}(G)$ and $Q \neq R$.

Lemma 2.7. *Suppose that G is an open set in \mathbb{R}^n . Let $0 < \tau, \delta < 1$ and $1 \leq p < \infty$ be given. Then there is a constant $\kappa = \kappa(n, \tau) \geq 1$ such that inequality*

$$(4) \quad |u|_{A_{\kappa}^{\delta, p}(G)} \leq (\sqrt{n})^{n/p+\delta} |u|_{W_{\tau}^{\delta, p}(G)}$$

holds for every $u \in L^1(G)$.

Proof. Let us choose $\kappa = \kappa(n, \tau) \geq 1$ such that $Q \subset B^n(x, \tau \operatorname{dist}(x, \partial G))$ whenever $x \in Q \in \mathcal{Q}_{\kappa}(G)$. Then we fix a family of cubes $\mathcal{Q} := \mathcal{Q}_{\kappa}(G)$. By Jensen's inequality we obtain

$$\sum_{Q \in \mathcal{Q}} |Q| \left(\frac{1}{|Q|^{1+\delta/n}} \int_Q |u(x) - u_Q| dx \right)^p \leq \sum_{Q \in \mathcal{Q}} |Q|^{-\delta p/n} \int_Q |u(x) - u_Q|^p dx.$$

By using Jensen's inequality again

$$\begin{aligned} &\sum_{Q \in \mathcal{Q}} |Q|^{-\delta p/n} \int_Q |u(x) - u_Q|^p dx \\ &\leq (\sqrt{n})^{n+\delta p} \sum_{Q \in \mathcal{Q}} \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \leq (\sqrt{n})^{n+\delta p} |u|_{W_{\tau}^{\delta, p}(G)}^p. \end{aligned}$$

Taking supremum over all families $\mathcal{Q}_k(G)$ gives inequality (4). \square

3. INEQUALITIES IN BOUNDED JOHN DOMAINS

We give the following fractional Sobolev–Poincaré inequality in bounded John domains. The inequality for $p > 1$ is already in [11, Theorem 4.10], but we need a better control over the dependencies of the constant C .

Theorem 3.1. *Suppose that D is a bounded c -John domain in \mathbb{R}^n , $n \geq 2$. Let $\tau, \delta \in (0, 1)$ and $1 \leq p < n/\delta$ be given. Then there is a constant $C = C(\delta, \tau, p, n, c) > 0$ such that the fractional Sobolev–Poincaré inequality*

$$(5) \quad \int_D |u(x) - u_D|^{np/(n-\delta p)} dx \leq C |u|_{W_{\tau}^{\delta, p}(D)}^{np/(n-\delta p)}$$

holds for every $u \in L^1(D)$.

Theorem 3.1 follows from Proposition 3.2 and Proposition 3.3. The following result from [4], based upon the Maz'ya truncation method [15] adapted to the fractional setting, shows that it is enough to prove a weak fractional Sobolev–Poincaré inequality.

Proposition 3.2. *Suppose that G is an open set in \mathbb{R}^n with $|G| < \infty$. Let $0 < \delta, \tau < 1$ and $0 < p \leq q < \infty$ be given. Then the following conditions are equivalent.*

(A) *There is a constant $C_1 > 0$ such that inequality*

$$\begin{aligned} & \inf_{a \in \mathbb{R}} \sup_{t > 0} |\{x \in G : |u(x) - a| > t\}| t^q \\ & \leq C_1 \left(\int_G \int_{B^n(y, \tau \operatorname{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz dy \right)^{q/p} \end{aligned}$$

holds for every $u \in L^\infty(G)$.

(B) *There is a constant $C_2 > 0$ such that inequality*

$$\inf_{a \in \mathbb{R}} \int_G |u(x) - a|^q dx \leq C_2 \left(\int_G \int_{B^n(y, \tau \operatorname{dist}(y, \partial G))} \frac{|u(y) - u(z)|^p}{|y - z|^{n+\delta p}} dz dy \right)^{q/p}$$

holds for every $u \in L^1(G)$.

In the implication from (A) to (B) $C_2 = C(p, q)C_1$ and from (B) to (A) $C_1 = C_2$.

The weak fractional Sobolev–Poincaré inequalities hold in bounded John domains by the following proposition.

Proposition 3.3. *Suppose that D is a bounded c -John domain in \mathbb{R}^n . Let $\tau, \delta \in (0, 1)$ and $1 \leq p < n/\delta$ be given. Then there is a constant $C = C(\delta, \tau, p, n, c) > 0$ such that the weak fractional Sobolev–Poincaré inequality*

$$\inf_{a \in \mathbb{R}} \sup_{t > 0} |\{x \in D : |u(x) - a| > t\}| t^{np/(n-\delta p)} \leq C |u|_{W_{\tau}^{\delta, p}(D)}^{np/(n-\delta p)}$$

holds for every $u \in L^\infty(D)$.

For a simple proof of Proposition 3.3 we refer to [11, Theorem 4.10]. The dependencies of the constants appearing in [11, Theorem 4.10] can be tracked more explicitly in order to obtain Proposition 3.3. In the present paper, we give a more general argument that might be of independent interest.

The following Theorem 3.4 is the key result for proving Proposition 3.3.

Theorem 3.4. *Suppose that D is a bounded c -John domain in \mathbb{R}^n . Let $\kappa \geq 1$ be fixed. Let $\delta \in [0, 1]$ and $1 \leq p < n/\delta$ be given. Then there exists a constant $C = C(n, \kappa, p, \delta, c)$ such that the inequality*

$$(6) \quad \inf_{a \in \mathbb{R}} \sup_{t > 0} |\{x \in D : |u(x) - a| > t\}| t^{np/(n-\delta p)} \leq C |u|_{A_{\kappa}^{\delta, p}(D)}^{np/(n-\delta p)}$$

holds for every $u \in L^1(D)$.

We give the proof of Theorem 3.4 in Section 4. By using Theorem 3.4 the claim of Proposition 3.3 follows easily.

Proof of Proposition 3.3. By Lemma 2.7 it is enough to prove that there is a constant $C = C(\delta, \tau, p, n, c)$ such that the inequality

$$\inf_{a \in \mathbb{R}} \sup_{t > 0} |\{x \in D : |u(x) - a| > t\}| t^{np/(n-\delta p)} \leq C |u|_{A_{\kappa(n, \tau)}^{\delta, p}(D)}^{np/(n-\delta p)}$$

holds for all $u \in L^\infty(D)$. This inequality follows from Theorem 3.4. \square

4. PROOF OF THEOREM 3.4

We start to build up the proof for Theorem 3.4 by giving auxiliary results. The following lemma gives local inequalities. Similar results are known in metric measure spaces, [8, Theorem 4.1].

Lemma 4.1. *Let $1 \leq p, q < \infty$ be given such that $1/p - 1/q = \delta/n$ with $\delta \in [0, 1]$. Then there is a constant $C = C(n, p, \delta) > 0$ such that inequality*

$$(7) \quad \sup_{t > 0} |\{x \in Q : |u(x) - u_Q| > t\}| t^q \leq C |u|_{A_1^{\delta, p}(Q)}^q$$

holds for every cube Q in \mathbb{R}^n and for all $u \in L_{\text{loc}}^1(\mathbb{R}^n)$.

Proof. Let us fix $u \in L_{\text{loc}}^1(\mathbb{R}^n)$. We write for cubes Q in \mathbb{R}^n

$$\begin{aligned} a(Q) &= |u|_{A_1^{\delta, p}(Q)} \cdot |Q|^{-1/q} \\ &= \left\{ |Q|^{-p/q} \cdot \sup_{Q_1(Q)} \sum_{R \in Q_1(Q)} |R|^{1-\delta p/n} \left(\frac{1}{|R|} \int_R |u(x) - u_R| dx \right)^p \right\}^{1/p}. \end{aligned}$$

Inequality (7) follows from the generalized Poincaré inequality theorem [7, Theorem 7.2(a)] as soon as we prove inequalities (8) and (9). The inequality

$$(8) \quad \frac{1}{|Q|} \int_Q |u(x) - u_Q| dx \leq a(Q)$$

holds for every cube Q in \mathbb{R}^n . Namely,

$$\begin{aligned} \frac{1}{|Q|} \int_Q |u(x) - u_Q| dx &= \left\{ |Q|^{-p/q} \cdot |Q|^{1-\delta p/n} \left(\frac{1}{|Q|} \int_Q |u(x) - u_Q| dx \right)^p \right\}^{1/p} \\ &\leq a(Q), \end{aligned}$$

because $1 - p/q - \delta p/n = 0$. We need to show that the inequality

$$(9) \quad \sum_{P \in \mathcal{Q}_1(Q)} a(P)^q |P| \leq 2^{q/p} a(Q)^q |Q|$$

holds for all cubes Q in \mathbb{R}^n and all families $\mathcal{Q}_1(Q)$ of pairwise disjoint cubes inside Q . In order to prove inequality (9) let us fix a cube Q and its family $\mathcal{Q}_1(Q)$. For each $P \in \mathcal{Q}_1(Q)$ we fix its family $\mathcal{Q}_1(P)$ such that

$$|u|_{A_1^{\delta,p}(P)}^p \leq 2 \sum_{R \in \mathcal{Q}_1(P)} |R|^{1-\delta p/n} \left(\frac{1}{|R|} \int_R |u(x) - u_R| dx \right)^p.$$

Since $q/p \geq 1$,

$$\sum_{P \in \mathcal{Q}_1(Q)} a(P)^q |P| \leq 2^{q/p} \left\{ \sum_{P \in \mathcal{Q}_1(Q)} \sum_{R \in \mathcal{Q}_1(P)} |R|^{1-\delta p/n} \left(\frac{1}{|R|} \int_R |u(x) - u_R| dx \right)^p \right\}^{q/p}.$$

Then writing $\mathcal{Q} := \cup_{P \in \mathcal{Q}_1(Q)} \mathcal{Q}_1(P)$ allows us to estimate

$$\begin{aligned} &\sum_{P \in \mathcal{Q}_1(Q)} a(P)^q |P| \\ &\leq 2^{q/p} \left\{ \sum_{R \in \mathcal{Q}} |R|^{1-\delta p/n} \left(\frac{1}{|R|} \int_R |u(x) - u_R| dx \right)^p \right\}^{q/p} \leq 2^{q/p} a(Q)^q |Q|. \end{aligned}$$

This implies inequality (9). \square

For a bounded c -John domain D we let $\mathcal{W}^\kappa(D)$ be its modified Whitney decomposition with a fixed $\kappa \geq 1$ such that $\kappa Q^* = \kappa \frac{9}{8} Q \subset D$ for each $Q \in \mathcal{W}^\kappa(D)$. This decomposition is obtained by dividing each Whitney cube $Q \in \mathcal{W}(D)$ into sufficiently small dyadic subcubes, their number depending on κ and n only. The family of cubes in $\mathcal{W}^\kappa(D)$ with side length 2^{-j} , $j \in \mathbf{Z}$, is written as $\mathcal{W}_j^\kappa(D)$.

Let Q be in $\mathcal{W}^\kappa(D)$. Let us suppose that we are given a chain $\mathcal{C}(Q) \subset \mathcal{W}^\kappa(D)$ of cubes

$$\mathcal{C}(Q) = (Q_0, \dots, Q_k),$$

joining a fixed cube $Q_0 \in \mathcal{W}^\kappa(D)$ to $Q_k = Q$ such that there exists a constant $C(n, \kappa)$ so that the inequality

$$|u_{Q^*} - u_{Q_0^*}| \leq C(n, \kappa) \sum_{R \in \mathcal{C}(Q)} \frac{1}{|R^*|} \int_{R^*} |u(x) - u_{R^*}| dx$$

holds whenever $u \in L^1_{\text{loc}}(D)$. The family $\{\mathcal{C}(Q) : Q \in \mathcal{W}^\kappa(D)\}$ of chains of cubes is called a chain decomposition of D . The shadow of a given cube $Q \in \mathcal{W}^\kappa(D)$ is the family

$$\mathcal{S}(R) = \{Q \in \mathcal{W}^\kappa(D) : R \in \mathcal{C}(Q)\}.$$

The following key lemma is a straightforward modification of [10, Proposition 2.5] once we have Lemma 2.2.

Lemma 4.2. *Let D be a bounded c -John domain in \mathbb{R}^n . Let $\kappa \geq 1$ and $1 \leq q < \infty$ be given. Then there exist a chain decomposition of D and constants $\sigma, \rho \in \mathbf{N}$ such that*

- (1) $\ell(Q) \leq 2^\rho \ell(R)$ for each $R \in \mathcal{C}(Q)$ and $Q \in \mathcal{W}^\kappa(D)$,
- (2) $\#\{R \in \mathcal{W}_j^\kappa(D) : R \in \mathcal{C}(Q)\} \leq 2^\rho$ for each $Q \in \mathcal{W}^\kappa(D)$ and $j \in \mathbf{Z}$,
- (3) the inequality

$$\sup_{j \in \mathbf{Z}} \sup_{R \in \mathcal{W}_j^\kappa(D)} \frac{1}{|R|} \sum_{k=j-\rho}^{\infty} \sum_{\substack{Q \in \mathcal{W}_k^\kappa(D) \\ Q \in \mathcal{S}(R)}} |Q| (\rho + 1 + k - j)^q < \sigma$$

holds.

The constants σ and ρ depend on κ , n , q , and the John constant c only.

We are ready for the proof of Theorem 3.4.

Proof of Theorem 3.4. Let us denote $q = np/(n - \delta p)$. We need to show that there is a constant $C(n, \kappa, p, \delta, c)$ such that the inequality

$$\inf_{a \in \mathbb{R}} \sup_{t > 0} |\{x \in D : |u(x) - a| > t\}| t^q \leq C(n, \kappa, p, \delta, c) |u|_{A_{\kappa}^{\delta, p}(D)}^q$$

holds for each $u \in L^1(D)$. Let Q_0 be the fixed cube in the chain decomposition of D given by Lemma 4.2. By the triangle inequality we obtain

$$|u(x) - u_{Q_0^*}| \leq \left| u(x) - \sum_{Q \in \mathcal{W}^\kappa(D)} u_{Q^*} \chi_Q(x) \right| + \left| \sum_{Q \in \mathcal{W}^\kappa(D)} u_{Q^*} \chi_Q(x) - u_{Q_0^*} \right|$$

for almost every $x \in D$. We write

$$\left| u(x) - \sum_{Q \in \mathcal{W}^\kappa(D)} u_{Q^*} \chi_Q(x) \right| =: g_1(x)$$

and

$$\left| \sum_{Q \in \mathcal{W}^\kappa(D)} u_{Q^*} \chi_Q(x) - u_{Q_0^*} \right| =: g_2(x)$$

for $x \in D$. For a fixed $t > 0$ we estimate

$$\begin{aligned} & t^q |\{x \in D : |u(x) - u_{Q_0^*}| > t\}| \\ & \leq t^q |\{x \in D : g_1(x) > t/2\}| + t^q |\{x \in D : g_2(x) > t/2\}|. \end{aligned}$$

The local term g_1 is estimated by Lemma 4.1 and the inequality $p \leq q$:

$$\begin{aligned} t^q |\{x \in D : g_1(x) > t/2\}| &= \sum_{Q \in \mathcal{W}^\kappa(D)} t^q |\{x \in \text{int}(Q) : |u(x) - u_{Q^*}| > t/2\}| \\ &\leq C 2^q \left(\sum_{Q \in \mathcal{W}^\kappa(D)} |u|_{A_1^{\delta,p}(Q^*)}^p \right)^{q/p}. \end{aligned}$$

Let us note that $\kappa R \subset \kappa Q^* \subset D$ if $R \in \mathcal{Q}_1(Q^*)$ and $Q \in \mathcal{W}^\kappa(D)$. We divide the family $\{Q^* : Q \in \mathcal{W}^\kappa(D)\}$ of cubes into $C(n, \kappa)$ families so that each of them consists of pairwise disjoint cubes. As in the proof of Lemma 4.1 we obtain

$$t^q |\{x \in D : g_1(x) > t/2\}| \leq C |u|_{A_\kappa^{\delta,p}(D)}^q.$$

We start to estimate the chaining term g_2 :

$$\begin{aligned} t^q |\{x \in D : g_2(x) > t/2\}| &= t^q \sum_{Q \in \mathcal{W}^\kappa(D)} |\{x \in \text{int}(Q) : |u_{Q^*} - u_{Q_0^*}| > t/2\}| \\ &\leq 2^q \sum_{Q \in \mathcal{W}^\kappa(D)} |Q| |u_{Q^*} - u_{Q_0^*}|^q =: \Sigma. \end{aligned}$$

By property (1) of the chain decomposition in Lemma 4.2 we obtain

$$\begin{aligned} \Sigma &\leq C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k^\kappa(D)} |Q| \left(\underbrace{\sum_{j=-\infty}^{k+\rho} \sum_{\substack{R \in \mathcal{W}_j^\kappa(D) \\ R \in \mathcal{C}(Q)}} \frac{1}{|R^*|} \int_{R^*} |u(x) - u_{R^*}| dx}_{=: \Sigma_{j,Q}} \right)^q \\ &= C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k^\kappa(D)} |Q| \left(\sum_{j=-\infty}^{k+\rho} \underbrace{(\rho+1+k-j)^{-1} (\rho+1+k-j)}_{=1} \Sigma_{j,Q} \right)^q. \end{aligned}$$

Property (2) in Lemma 4.2 and the equation $1/p - 1/q = \delta/n$ give

$$\begin{aligned} \Sigma_{j,Q}^q &= \left(\sum_{\substack{R \in \mathcal{W}_j^\kappa(D) \\ R \in \mathcal{C}(Q)}} \frac{1}{|R^*|} \int_{R^*} |u(x) - u_{R^*}| dx \right)^q \\ &\leq C \sum_{\substack{R \in \mathcal{W}_j^\kappa(D) \\ R \in \mathcal{C}(Q)}} \left(\frac{1}{|R^*|} \int_{R^*} |u(x) - u_{R^*}| dx \right)^q \leq C \sum_{\substack{R \in \mathcal{W}_j^\kappa(D) \\ R \in \mathcal{C}(Q)}} \frac{|u|_{A_1^{\delta,p}(R^*)}^q}{|R^*|}. \end{aligned}$$

Thus, Hölder's inequality and property (3) in Lemma 4.2 imply that

$$\begin{aligned}
\Sigma &\leq C \sum_{k=-\infty}^{\infty} \sum_{Q \in \mathcal{W}_k^{\kappa}(D)} |Q| \sum_{j=-\infty}^{k+\rho} (\rho+1+k-j)^q \sum_{\substack{R \in \mathcal{W}_j^{\kappa}(D) \\ R \in \mathcal{C}(Q)}} \frac{|u|_{A_1^{\delta,p}(R^*)}^q}{|R^*|} \\
&= C \sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j^{\kappa}(D)} |u|_{A_1^{\delta,p}(R^*)}^q \cdot \frac{1}{|R|} \sum_{k=j-\rho}^{\infty} \sum_{\substack{Q \in \mathcal{W}_k^{\kappa}(D) \\ Q \in \mathcal{S}(R)}} |Q| (\rho+1+k-j)^q \\
&\leq C \left(\sum_{j=-\infty}^{\infty} \sum_{R \in \mathcal{W}_j^{\kappa}(D)} |u|_{A_1^{\delta,p}(R^*)}^p \right)^{q/p} \leq C |u|_{A_{\kappa}^{\delta,p}(D)}^q.
\end{aligned}$$

The theorem is proved. \square

5. SOBOLEV–POINCARÉ INEQUALITIES IN UNBOUNDED JOHN DOMAINS

We prove a fractional Sobolev–Poincaré inequality in unbounded John domains.

Theorem 5.1. *Suppose that D in \mathbb{R}^n is an unbounded c -John domain and that $\tau, \delta \in (0, 1)$ are given. Let $1 \leq p < n/\delta$. Then there is a constant $C = C(\delta, \tau, p, n, c) > 0$ such that the fractional Sobolev–Poincaré inequality*

$$\inf_{a \in \mathbb{R}} \int_D |u(x) - a|^{np/(n-\delta p)} dx \leq C |u|_{W_{\tau}^{\delta,p}(D)}^{np/(n-\delta p)}$$

holds for each $u \in \dot{W}_{\tau}^{\delta,p}(D)$.

The proof is similar to the proof of [9, Theorem 4.1] where the classical Sobolev–Poincaré inequality has been proved in unbounded domains which have an engulfing property. The proof is based on an idea from [12].

Proof of Theorem 5.1. By Lemma 2.3 the c -John domain D has an engulfing property. That is, there are bounded c_1 -John domains D_i with $c_1 = c_1(c, n)$ such that

$$D_i \subset \overline{D_i} \subset D_{i+1}, \quad i = 1, 2, \dots,$$

and

$$D = \bigcup_{i=1}^{\infty} D_i.$$

Let us fix $u \in \dot{W}_{\tau}^{\delta,p}(D)$. By Lemma 2.6 with $K = \overline{D_i}$ we obtain that $u \in L^p(D_i)$ and hence $u \in L^1(D_i)$ for each i . Therefore we may write

$$u_i = u_{D_i} = \frac{1}{|D_i|} \int_{D_i} u(x) dx, \quad i = 1, 2, \dots$$

The sequence (u_i) is bounded. Namely, by the triangle inequality

$$|u_i| = \frac{1}{|D_1|} \int_{D_1} |u_i| dx \leq \frac{1}{|D_1|} \left(\int_{D_1} |u(x) - u_i| dx + \int_{D_1} |u(x)| dx \right).$$

By Hölder's inequality with exponents $(np/(np - n + \delta p), np/(n - \delta p))$ and by Theorem 3.1 applied in a bounded c_1 -John domain D_i we obtain

$$\begin{aligned} \int_{D_1} |u(x) - u_i| dx &\leq |D_1|^{1-1/p+\delta/n} \|u - u_{D_i}\|_{L^{np/(n-\delta p)}(D_1)} \\ &\leq |D_1|^{1-1/p+\delta/n} \|u - u_{D_i}\|_{L^{np/(n-\delta p)}(D_i)} \leq |D_1|^{1-1/p+\delta/n} C |u|_{W_\tau^{\delta,p}(D)} < \infty \end{aligned}$$

with a constant $C = C(\delta, \tau, p, n, c_1)$.

The bounded sequence (u_i) has a convergent subsequence (u_{i_j}) and hence there is a constant $a \in \mathbb{R}$ such that $\lim_{j \rightarrow \infty} u_{i_j} = a$. By Fatou's lemma and Theorem 3.1 applied with the function $u \in L^p(D_j)$ we obtain

$$\begin{aligned} \int_D |u(x) - a|^{np/(n-\delta p)} dx &= \int_D \liminf_{j \rightarrow \infty} \chi_{D_{i_j}}(x) |u(x) - u_{i_j}|^{np/(n-\delta p)} dx \\ &\leq \liminf_{j \rightarrow \infty} \int_{D_{i_j}} |u(x) - u_{i_j}|^{np/(n-\delta p)} dx \\ &\leq C \liminf_{j \rightarrow \infty} |u|_{W_\tau^{\delta,p}(D_{i_j})}^{np/(n-\delta p)} \leq C |u|_{W_\tau^{\delta,p}(D)}^{np/(n-\delta p)}, \end{aligned}$$

where $C = C(\delta, \tau, p, n, c_1)$. The claim follows. \square

A fractional Sobolev inequality holds in unbounded John domains.

Theorem 5.2. *Suppose that D in \mathbb{R}^n is an unbounded c -John domain and that $\tau, \delta \in (0, 1)$ are given. Let $1 \leq p < n/\delta$. Then there is a constant $C = C(\delta, \tau, p, n, c) > 0$ such that the fractional Sobolev inequality*

$$(10) \quad \int_D |u(x)|^{np/(n-\delta p)} dx \leq C |u|_{W_\tau^{\delta,p}(D)}^{np/(n-\delta p)}$$

holds for each $u \in \dot{W}_\tau^{\delta,p}(D)$ with a compact support in D .

Proof. We write $D = \cup_{i=1}^\infty D_i$ as in the proof of Theorem 5.1. By Lemma 2.6 and the fact that the numbers $|D_i|$ converge to $|D| = \infty$ as $i \rightarrow \infty$ we obtain that

$$|u_i| = |u_{D_i}| \leq \left(\frac{1}{|D_i|} \int_{D_i} |u(x)|^p dx \right)^{1/p} \leq |D_i|^{-1/p} \|u\|_{L^p(D)} \xrightarrow{i \rightarrow \infty} 0$$

for $u \in \dot{W}_\tau^{\delta,p}(D)$ with a compact support in D . Therefore, the proof follows as the proof of Theorem 5.1 with $a = 0$. \square

The following is a corollary of Theorem 5.1 and Theorem 5.2. It shows that $\dot{W}_\tau^{\delta,p}(D)$ is embedded to $L^{np/(n-\delta p)}(D)$ if we identify any two functions in $\dot{W}_\tau^{\delta,p}(D)$ whose difference is a constant almost everywhere. This identification is usually included already in the definition of homogeneous spaces of smoothness $0 < \delta < 1$.

Corollary 5.3. *Suppose that D in \mathbb{R}^n is a c -John domain and that $\tau, \delta \in (0, 1)$ are given. Let $1 \leq p < n/\delta$ and $q = np/(n - \delta p)$. Then there is a nonlinear bounded operator*

$$\mathbf{E} : \dot{W}_\tau^{\delta,p}(D) \rightarrow L^q(D), \quad \mathbf{E}(u) = u - a_u,$$

whose norm is bounded by a constant $C = C(\delta, \tau, p, n, c)$; here $a_u \in \mathbb{R}$ for each $u \in \dot{W}_\tau^{\delta,p}(D)$. If D is an unbounded c -John domain, then $\mathbf{E}(u) = u$ for each $u \in \dot{W}_\tau^{\delta,p}(D)$ whose support is a compact set in D .

6. FRACTIONAL HARDY INEQUALITIES IN UNBOUNDED JOHN DOMAINS

We characterize certain fractional Hardy inequalities in unbounded John domains as an application of Theorem 5.2. The following definition is motivated by the fractional Hardy inequalities from [6]. The classical (q, p) -Hardy inequalities are studied in [5].

We say that a fractional (δ, q, p) -Hardy inequality with $0 < \delta < 1$ and $0 < p, q < \infty$ holds in a proper open set G in \mathbb{R}^n , if there is a constant $C > 0$ such that the inequality

$$(11) \quad \int_G \frac{|u(x)|^q}{\text{dist}(x, \partial G)^{q(\delta+n(1/q-1/p))}} dx \leq C \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p}$$

holds for all functions $u \in C_0(G)$. The fractional Sobolev inequality (10) is obtained when $1/p - 1/q = \delta/n$. The usual fractional (δ, p, p) -Hardy inequality is obtained when $q = p$.

Our characterization of fractional Hardy inequalities is given in terms of Whitney cubes from $\mathcal{W}(G)$ and the (δ, p) -capacities

$$\text{cap}_{\delta,p}(K, G) = \inf_u |u|_{W^{\delta,p}(G)}^p$$

of compact sets K in G , where the infimum is taken over all $u \in C_0(G)$ such that $u(x) \geq 1$ for each point $x \in K$.

Theorem 6.1. *Let D be an unbounded c -John domain in \mathbb{R}^n , $D \neq \mathbb{R}^n$. Let $\delta \in (0, 1)$ and $1 \leq p, q < \infty$ be given such that $p < n/\delta$ and $0 \leq 1/p - 1/q \leq \delta/n$. Then the following conditions are equivalent.*

- (A) *A fractional (δ, q, p) -Hardy inequality holds in D .*
- (B) *There exists a positive constant $N > 0$ such that inequality*

$$\left(\sum_{Q \in \mathcal{W}(D)} \text{cap}_{\delta,p}(K \cap Q, D)^{q/p} \right)^{p/q} \leq N \text{cap}_{\delta,p}(K, D)$$

holds for every compact set K in D .

The proof of Theorem 6.1 is based on the fractional Sobolev inequalities and the Maz'ya type characterization for the validity of a fractional (δ, q, p) -Hardy inequality, Theorem 6.5. Before the proof of Theorem 6.1 we give some remarks, corollaries and auxiliary results.

Remark 6.2. *There exist unbounded John domains where a fractional (δ, p, p) -Hardy inequality fails for some values of δ and p . As an example let us define $D = \mathbb{R}^2 \setminus L$, where L is a closed line-segment in \mathbb{R}^2 . Then, the fractional (δ, p, p) -Hardy inequality fails whenever $1 < p < \infty$ and $\delta = 1/p$. This example is a modification of [3, Theorem 9].*

Sufficient geometric conditions for the fractional Hardy inequalities are given in the following corollary. For the relevant notation we refer to Section 2.

Corollary 6.3. *Let D be an unbounded c -John domain in \mathbb{R}^n , $D \neq \mathbb{R}^n$. Let $0 < \delta < 1$ and $1 \leq p, q < \infty$ be given such that $p < n/\delta$ and $0 \leq 1/p - 1/q \leq \delta/n$. Then the fractional (δ, q, p) -Hardy inequality (11) holds in D if either condition (A) or condition (B) holds.*

- (A) $\overline{\dim}_A(\partial D) < n - \delta p$;
- (B) $\underline{\dim}_A(\partial D) > n - \delta p$ and ∂D is unbounded.

Proof. By Theorem 6.1 it is enough to prove a (δ, p, p) -Hardy inequality which is a consequence of [2, Theorem 2]. The plumpness condition required there follows from the John condition in Definition 2.1. \square

Now we start to build up our proof for Theorem 6.1. First we give a characterization which is an extension of [3, Proposition 5] where the special case of $p = q$ is considered. This type of characterizations go back to Vl. Maz'ya, [15].

Theorem 6.4. *Suppose that G is an open set in \mathbb{R}^n and $\omega : G \rightarrow [0, \infty)$ is measurable. Then the following conditions are equivalent whenever $0 < \delta < 1$ and $0 < p \leq q < \infty$.*

- (A) *There is a constant $C_1 > 0$ such that the inequality*

$$\int_G |u(x)|^q \omega(x) dx \leq C_1 |u|_{W^{\delta,p}(G)}^q$$

holds for every $u \in C_0(G)$.

- (B) *There is a constant $C_2 > 0$ such that the inequality*

$$\int_K \omega(x) dx \leq C_2 \text{cap}_{\delta,p}(K, G)^{q/p}$$

holds for every compact set K in G .

In the implication from (A) to (B) $C_2 = C_1$ and from (B) to (A) $C_1 = \frac{C_2 2^{3q+2q/p}}{(1-2^{-p})^{q/p}}$.

As a corollary of Theorem 6.4 we obtain Theorem 6.5 when we choose

$$\omega = \text{dist}(\cdot, \partial G)^{-q(\delta+n(1/q-1/p))}.$$

Theorem 6.5. *Let $0 < \delta < 1$ and $0 < p \leq q < \infty$. Then a (δ, q, p) -Hardy inequality (11) holds in a proper open set G in \mathbb{R}^n if and only if there is a constant $C > 0$ such that the inequality*

$$(12) \quad \int_K \text{dist}(x, \partial G)^{-q(\delta+n(1/q-1/p))} dx \leq C \text{cap}_{\delta,p}(K, G)^{q/p}$$

holds for every compact set K in G .

The proof for Theorem 6.4 is a simple modification of the proof of [3, Proposition 5], but we give the proof in the general case for the convenience of the reader.

Proof of Theorem 6.4. Let us first assume that condition (A) holds. Let $u \in C_0(G)$ be such that $u(x) \geq 1$ for every point $x \in K$. By condition (A) we obtain

$$\int_K \omega(x) dx \leq \int_G |u(x)|^q \omega(x) dx \leq C_1 \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p}.$$

Taking infimum over all such functions u we obtain condition (B) with $C_2 = C_1$.

Now let us assume that condition (B) holds and let $u \in C_0(G)$. We write

$$E_k = \{x \in G : |u(x)| > 2^k\} \quad \text{and} \quad A_k = E_k \setminus E_{k+1}, k \in \mathbb{Z}.$$

Let us note that

$$(13) \quad G = \{x \in G : 0 \leq |u(x)| < \infty\} = \{x \in G : u(x) = 0\} \cup \bigcup_{i \in \mathbb{Z}} A_i.$$

By condition (B)

$$\begin{aligned} \int_G |u(x)|^q \omega(x) dx &\leq \sum_{k \in \mathbb{Z}} 2^{(k+2)q} \int_{A_{k+1}} \omega(x) dx \\ &\leq C_2 2^{2q} \sum_{k \in \mathbb{Z}} 2^{kq} \text{cap}_{\delta,p}(\bar{A}_{k+1}, G)^{q/p}. \end{aligned}$$

Let us define $u_k : G \rightarrow [0, 1]$ by

$$u_k(x) = \begin{cases} 1, & \text{if } |u(x)| \geq 2^{k+1}, \\ \frac{|u(x)|}{2^k} - 1 & \text{if } 2^k < |u(x)| < 2^{k+1}, \\ 0, & \text{if } |u(x)| \leq 2^k. \end{cases}$$

Then $u_k \in C_0(G)$ and $u_k(x) = 1$ if $x \in \bar{E}_{k+1}$. We note that $\bar{A}_{k+1} \subset \bar{E}_{k+1}$. Thus we may take u_k as a test function for the capacity. Let us write $F = \{x \in G : u(x) = 0\}$. By (13),

$$\begin{aligned} \text{cap}_{\delta,p}(\bar{A}_{k+1}, G) &\leq \int_G \int_G \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+\delta p}} dy dx \\ &\leq 2 \sum_{i \leq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+\delta p}} dy dx + 2 \sum_{j \geq k} \int_F \int_{A_j} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+\delta p}} dy dx. \end{aligned}$$

The inequality

$$|u_k(x) - u_k(y)| \leq 2 \cdot 2^{-j} |u(x) - u(y)|$$

holds whenever $(x, y) \in A_i \times A_j$ and $i \leq k \leq j$. Namely: $|u_k(x) - u_k(y)| \leq 2^{-k} |u(x) - u(y)|$ if $x, y \in G$. If $x \in A_i$ and $y \in A_j$ with $i + 2 \leq j$, then $|u(x) - u(y)| \geq |u(y)| - |u(x)| \geq 2^{j-1}$. Hence $|u_k(x) - u_k(y)| \leq 1 \leq 2 \cdot 2^{-j} |u(x) - u(y)|$. Thus, since $q \geq p$,

$$\begin{aligned} \sum_{k \in \mathbb{Z}} 2^{kq} \left(\sum_{i \leq k} \sum_{j \geq k} \int_{A_i} \int_{A_j} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \\ \leq 2^q \left(\sum_{k \in \mathbb{Z}} \sum_{i \leq k} \sum_{j \geq k} 2^{(k-j)p} \int_{A_i} \int_{A_j} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right). \end{aligned}$$

By proceeding in a similar way as before we obtain that

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} 2^{kq} \left(\sum_{j \geq k} \int_F \int_{A_j} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \\ & \leq 2^q \left(\sum_{k \in \mathbb{Z}} \sum_{j \geq k} 2^{(k-j)p} \int_F \int_{A_j} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p}. \end{aligned}$$

Using the sum of the geometric series $\sum_{k=i}^j 2^{(k-j)p} < \sum_{k=-\infty}^j 2^{(k-j)p} \leq \frac{1}{1-2^{-p}}$ and changing the order of summations gives

$$\int_G |u(x)|^q \omega(x) dx \leq \frac{C_2 2^{3q+2q/p}}{(1-2^{-p})^{q/p}} \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p}.$$

Thus condition (A) is true with $C_1 = C_2 2^{3q+2q/p} (1-2^{-p})^{-q/p}$. \square

The following lemma is an extension of [3, Proposition 6].

Lemma 6.6. *Let $0 < \delta < 1$ and $0 < p \leq q < \infty$ be given. Suppose that the fractional (δ, q, p) -Hardy inequality (11) holds in a proper open set G in \mathbb{R}^n with a constant $C_1 > 0$. Then there is a constant $N = N(C_1, n, \delta, q, p) > 0$ such that the inequality*

$$(14) \quad \sum_{Q \in \mathcal{W}(G)} \text{cap}_{\delta,p}(K \cap Q, G)^{q/p} \leq N^{q/p} \text{cap}_{\delta,p}(K, G)^{q/p}$$

holds for every compact set K in G .

Proof. If $Q \in \mathcal{W}(G)$ we write $\widehat{Q} = \frac{17}{16}Q$ and $Q^* = \frac{9}{8}Q$. We recall that the side lengths of these cubes are comparable to their distances from ∂G .

Let us fix a compact set K in G and $u \in C_0(G)$ such that $u(x) \geq 1$ for each $x \in K$. For every $Q \in \mathcal{W}(G)$ we let φ_Q be a smooth function such that $|\nabla \varphi_Q| \leq C\ell(Q)^{-1}$ and $\chi_Q \leq \varphi_Q \leq \chi_{\widehat{Q}}$. Then, $u_Q := u\varphi_Q$ is an admissible test function for $\text{cap}_{\delta,p}(K \cap Q, G)$. Hence, we can estimate the left hand side of inequality (14) by

$$\begin{aligned} & \sum_{Q \in \mathcal{W}(G)} \left(\int_G \int_G \frac{|u_Q(x) - u_Q(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \\ & \leq C \sum_{Q \in \mathcal{W}(G)} \left(\int_{\widehat{Q}} \frac{|u_Q(x)|^p}{\text{dist}(x, \partial G)^{\delta p}} dx + \int_{Q^*} \int_{Q^*} \frac{|u_Q(x) - u_Q(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \\ & \leq C \sum_{Q \in \mathcal{W}(G)} \left\{ \left(\int_{\widehat{Q}} \frac{|u_Q(x)|^p}{\text{dist}(x, \partial G)^{\delta p}} dx \right)^{q/p} + \left(\int_{Q^*} \int_{Q^*} \frac{|u_Q(x) - u_Q(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \right\}. \end{aligned}$$

Since $|u_Q| \leq |u|$ and $\sum_Q \chi_{\widehat{Q}} \leq C$, we may apply Hölder's inequality with $(q/p, q/(q-p))$ and the (δ, q, p) -Hardy inequality (11) to obtain

$$(15) \quad \begin{aligned} \sum_{Q \in \mathcal{W}(G)} \left(\int_{\widehat{Q}} \frac{|u_Q(x)|^p}{\text{dist}(x, \partial G)^{\delta p}} dx \right)^{q/p} &\leq C \sum_{Q \in \mathcal{W}(G)} \int_{\widehat{Q}} \frac{|u(x)|^q}{\text{dist}(x, \partial G)^{q(\delta+n(1/q-1/p))}} dx \\ &\leq C \left(\int_G \int_G \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} = C |u|_{W^{\delta,p}(G)}^q. \end{aligned}$$

We fix $x, y \in G$ to estimate the remaining series. The following pointwise estimates will be useful,

$$\begin{aligned} |u_Q(x) - u_Q(y)| &\leq |u(x)| |\varphi_Q(x) - \varphi_Q(y)| + |u(x) - u(y)| \varphi_Q(y) \\ &\leq C \cdot \ell(Q)^{-1} \cdot |u(x)| \cdot |x - y| + |u(x) - u(y)|. \end{aligned}$$

Namely, since $\sum_{Q \in \mathcal{W}(G)} \chi_{Q^*} \leq C \chi_G$ and $q \geq p$, we find that

$$\sum_{Q \in \mathcal{W}(G)} \left(\int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \leq C |u|_{W^{\delta,p}(G)}^q.$$

By recalling that $0 < \delta < 1$ and by estimating as in (15) we obtain

$$\begin{aligned} &\sum_{Q \in \mathcal{W}(G)} \left(\ell(Q)^{-p} \int_{Q^*} |u(x)|^p \int_{Q^*} \frac{|x - y|^p}{|x - y|^{n+\delta p}} dy dx \right)^{q/p} \\ &\leq C \sum_{Q \in \mathcal{W}(G)} \left(\ell(Q)^{-\delta p} \int_{Q^*} |u(x)|^p dx \right)^{q/p} \\ &\leq C \sum_{Q \in \mathcal{W}(G)} \left(\int_{Q^*} \frac{|u(x)|^p}{\text{dist}(x, \partial G)^{\delta p}} dx \right)^{q/p} \leq C |u|_{W^{\delta,p}(G)}^q. \end{aligned}$$

By collecting the estimates and taking the infimum over all admissible u for $\text{cap}_{\delta,p}(K, G)$ we complete the proof. \square

Now we are able to give the proof for Theorem 6.1.

Proof of Theorem 6.1. The implication from (A) to (B) is a consequence of Lemma 6.6. Let us then assume that condition (B) holds. In order to have inequality (11) in D , by Theorem 6.5 it is enough to prove that there is a constant $C = C(\delta, p, n, c, N) > 0$ such that inequality

$$(16) \quad \int_K \text{dist}(x, \partial D)^{-q(\delta+n(1/q-1/p))} dx \leq C \text{cap}_{\delta,p}(K, D)^{q/p}$$

holds for every compact set K in D . Let us fix a compact set K in D . We consider a variation of inequality (16): there is a constant $C = C(\delta, p, n, c) > 0$ such that the inequality

$$(17) \quad \left(\int_{K \cap Q} \text{dist}(x, \partial D)^{-q(\delta+n(1/q-1/p))} dx \right)^{1/q} \leq C \text{cap}_{\delta,p}(K \cap Q, D)^{1/p}$$

holds for every Whitney cube $Q \in \mathcal{W}(D)$. To prove inequality (17) we let $u \in C_0(D)$ be a test function such that $u(x) \geq 1$ for every $x \in K \cap Q$. By the properties of Whitney cubes and Theorem 5.2 we estimate the left hand side of inequality (17)

$$\begin{aligned} C|K \cap Q|^{1/q}|Q|^{-(\delta+n(1/q-1/p))/n} &\leq C|K \cap Q|^{1/q-(\delta+n(1/q-1/p))/n} \\ &\leq C\|u\|_{L^{np/(n-\delta p)}(D)} \\ &\leq C|u|_{W^{\delta,p}(D)}. \end{aligned}$$

Inequality (17) follows when we take the infimum over all admissible functions u for the capacity $\text{cap}_{\delta,p}(K \cap Q, D)$.

We may now finish the proof by using inequality (17) and condition (B)

$$\begin{aligned} \int_K \text{dist}(x, \partial D)^{-q(\delta+n(1/q-1/p))} dx &= \sum_{Q \in \mathcal{W}(D)} \int_{K \cap Q} \text{dist}(x, \partial D)^{-q(\delta+n(1/q-1/p))} dx \\ &\leq C \sum_{Q \in \mathcal{W}(D)} \text{cap}_{\delta,p}(K \cap Q, D)^{q/p} \\ &\leq CN^{q/p} \text{cap}_{\delta,p}(K, D)^{q/p}, \end{aligned}$$

where $C = C(\delta, p, n, c)$. The proof is complete. \square

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