INTERSECTIONS OF HOMOGENEOUS CANTOR SETS AND BETA-EXPANSIONS

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ABSTRACT. Let $\Gamma_{\beta,N}$ be the N-part homogeneous Cantor set with $\beta \in (1/(2N-1), 1/N)$. Any string $(j_\ell)_{\ell=1}^\infty$ with $j_\ell \in \{0, \pm1, \dots, \pm(N-1)\}$ such that $t = \sum_{\ell=1}^\infty j_\ell \beta^{\ell-1}(1-\beta)/(N-1)$ is called a code of t. Let $\mathcal{U}_{\beta,\pm N}$ be the set of $t \in [-1,1]$ having a unique code, and let $\mathcal{S}_{\beta,\pm N}$ be the set of $t \in \mathcal{U}_{\beta,\pm N}$ which make the intersection $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ a self-similar set. We characterize the set $\mathcal{U}_{\beta,\pm N}$ in a geometrical and algebraical way, and give a sufficient and necessary condition for $t \in \mathcal{S}_{\beta,\pm N}$. Using techniques from beta-expansions, we show that there is a critical point $\beta_c \in (1/(2N-1), 1/N)$, which is a transcendental number, such that $\mathcal{U}_{\beta,\pm N}$ has positive Hausdorff dimension if $\beta \in (1/(2N-1), \beta_c)$, and contains countably infinite many elements if $\beta \in (\beta_c, 1/N)$. Moreover, there exists a second critical point $\alpha_c = \big[N+1-\sqrt{(N-1)(N+3)}\big]/2 \in (1/(2N-1), \beta_c)$ such that $\mathcal{S}_{\beta,\pm N}$ has positive Hausdorff dimension if $\beta \in (1/(2N-1), \alpha_c)$, and contains countably infinite many elements if $\beta \in [\alpha_c, 1/N)$.

Keywords: Homogeneous Cantor set; self-similarity; iterated function system; critical point; beta-expansion; Thue-Morse sequence.

MSC: 28A80, 28A78

1. Introduction

Let $\{f_i(x) = r_i x + b_i\}_{i=1}^p$ be a family of functions on $\mathbb{R}$ with $0 < |r_i| < 1$. It is well known (cf. [5]) that there exists a unique nonempty compact set $\Gamma \subseteq \mathbb{R}$ such that

$$\Gamma = \bigcup_{i=1}^p f_i(\Gamma).$$

In this case, Γ is called the *self-similar set* generated by the *iterated function system* (IFS) $\{f_i(\cdot)\}_{i=1}^p$.

We will be interested in the self-similar set $\Gamma_{\beta,\Omega}$ generated by an IFS $\{\phi_d(\cdot) : d \in \Omega\}$, where Ω is a finite set of integers, and

$$\phi_d(x) = \beta x + d(1-\beta)/(N-1), \quad x \in \mathbb{R}$$

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for some $N \geq 2$ and $\beta \in (0, 1/N)$. It is well known that one can establish a surjective map $\pi_\Omega : \Omega^\infty \to \Gamma_{\beta,\Omega}$ by letting

$$\pi_\Omega(J) = \sum_{\ell=1}^{\infty} \frac{j_\ell \beta^{\ell-1}(1-\beta)}{N-1} \tag{1}$$

for $J = (j_\ell)_{\ell=1}^\infty \in \Omega^\infty$. The infinite string J is called an Ω-*code* of $\pi_\Omega(J)$. Note that an element $x \in \Gamma_{\beta,\Omega}$ may have multiple Ω-codes. These Ω-codes are closely related to the classical beta-expansions (cf. [4, 7, 12, 17, 18, 19, 20]). A sequence $(s_\ell)_{\ell=1}^\infty \in \Omega^\infty$ is called a β-*expansion of* x *with digit set* Ω if we can write

$$x = \sum_{\ell=1}^{\infty} s_\ell \beta^\ell, \quad s_\ell \in \Omega.$$

Let $\Omega_N := \{0, 1, \ldots, N-1\}$. We simplify the notation Γ_{β,Ω_N} to $\Gamma_{\beta,N}$, so this set satisfies

$$\Gamma_{\beta,N} = \bigcup_{d \in \Omega_N} \phi_d(\Gamma_{\beta,N}).$$

The set $\Gamma_{\beta,N}$ is called the N-*part homogeneous Cantor set*. Thus $\Gamma_{1/3,2}$ is the *classical middle-third Cantor set* and $\Gamma_{\beta,2}$ is the middle-α Cantor set with $\alpha = 1 - 2\beta$.

In terms of (1), let $\pi_N := \pi_{\Omega_N}$. Thus we can rewrite $\Gamma_{\beta,N}$ as

$$\Gamma_{\beta,N} = \pi_N\left(\Omega_N^\infty\right) = \left\{ \sum_{\ell=1}^{\infty} \frac{j_\ell \beta^{\ell-1}(1-\beta)}{N-1} : j_\ell \in \Omega_N \right\}. \tag{2}$$

We consider the intersection of $\Gamma_{\beta,N}$ with its translation by t. It is easy to check that

$$\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) \neq \emptyset \quad \text{if and only if} \quad t \in \Gamma_{\beta,N} - \Gamma_{\beta,N}.$$

Here we denote for a real number a, and sets $A, B \subseteq \mathbb{R}$, $aA := \{ax : x \in A\}$, $A + B := \{x + y : x \in A, y \in B\}$, and $A + a := A + \{a\}$.

It follows from Equation (2) that the difference set $\Gamma_{\beta,N} - \Gamma_{\beta,N}$ can be written as

$$\Gamma_{\beta,N} - \Gamma_{\beta,N} = \left\{ \sum_{k=1}^{\infty} \frac{t_\ell \beta^{\ell-1}(1-\beta)}{N-1} : t_\ell \in \Omega_{\pm N} \right\} = \pi_{\pm N}\left(\Omega_{\pm N}^\infty\right) = \Gamma_{\beta,\Omega_{\pm N}},$$

where $\Omega_{\pm N} := \Omega_N - \Omega_N = \{0, \pm 1, \ldots, \pm(N-1)\}$ and $\pi_{\pm N} := \pi_{\Omega_{\pm N}}$. Since $\Omega_{2N-1} = \{0, 1, \ldots, 2N-2\} = \Omega_{\pm N} + N - 1$, it is easy to see that $(t_\ell)_{\ell=1}^\infty$ is a $\Omega_{\pm N}$-code of $t \in \Gamma_{\beta,N} - \Gamma_{\beta,N}$ if and only if $(t_\ell + N - 1)_{\ell=1}^\infty$ is an β-expansion of $(t+1)\beta(N-1)/(1-\beta)$ with digit set Ω_{2N-1}. Thus some results and techniques from beta-expansions can be used to deal with the difference set $\Gamma_{\beta,N} - \Gamma_{\beta,N}$.

In the past two decades, intersections of Cantor sets have been studied by several authors (cf. [2, 8, 9, 10, 11, 13]). Recently, Deng et al. [3] gave a necessary and sufficient condition for $t \in [-1, 1]$ such that $\Gamma_{1/3,2} \cap (\Gamma_{1/3,2} + t)$ is a self-similar set. Their results were extended to the case $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ with $\beta \in (0, 1/(2N-1)]$ by Li et al. [15], and to the case $\Gamma_{\beta,2} \cap (\Gamma_{\beta,2} + t)$ with $\beta \in (1/3, 1/2)$ and t having a unique $\Omega_{\pm 2}$-code by Zou et al. [21].

In this paper we consider arbitrary $N \geq 2$, and $\beta \in (1/(2N-1), 1/N)$. Then Lebesgue a.a. $t \in \Gamma_{\beta,N} - \Gamma_{\beta,N} = [-1, 1]$ have a continuum of distinct $\Omega_{\pm N}$-codes. This gives the set

$\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ a more complicated structure. We summarize the results in the following. In Section 2, an algebraical and geometrical description of the set

$$\mathcal{U}_{\beta,\pm N} := \{t \in [-1,1] : |\pi_{\pm N}^{-1}(t)| = 1\}$$

(i.e., the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code) is given in Theorem 2.2, where throughout the paper $|A|$ denotes the number of members in the set A. Section 3 is mainly devoted to investigating the self-similar structure of $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$. Let

$$\mathcal{S}_{\beta,\pm N} := \{t \in \mathcal{U}_{\beta,\pm N} : \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) \text{ is a self-similar set}\}.$$

Theorem 3.2 gives a sufficient and necessary condition for $t \in \mathcal{S}_{\beta,\pm N}$. In Section 4, we study the set $\mathcal{U}_{\beta,\pm N}$ for different $\beta \in (1/(2N-1), 1/N)$ culminating in Theorem 4.6. Using techniques from beta-expansions, we obtain a *critical point* $\beta_c \in (1/(2N-1), 1/N)$ such that $\mathcal{U}_{\beta,\pm N}$ has positive Hausdorff dimension if $\beta \in (1/(2N-1), \beta_c)$, and contains countably infinite many elements if $\beta \in (\beta_c, 1/N)$. We point out that the critical point β_c is a transcendental number which is related to the famous Thue-Morse sequence (cf. [12]). In Section 5 we find the second critical point $\alpha_c = [N+1-\sqrt{(N-1)(N+3)}]/2 \in (1/(2N-1), \beta_c)$ (see Theorem 5.1) such that $\mathcal{S}_{\beta,\pm N}$ has positive Hausdorff dimension if $\beta \in (1/(2N-1), \alpha_c)$, and contains countably infinite many elements if $\beta \in [\alpha_c, 1/N)$. In the following table, we give the critical points $\beta_c = \beta_c(N)$ and $\alpha_c = \alpha_c(N)$ calculated for different integers N by means of Mathematica.

N	2	3	4	5	6	7	8	9
$\beta_c \approx$	0.39433	0.27130	0.21004	0.17221	0.14625	0.12722	0.11265	0.10111
$\alpha_c \approx$	0.38197	0.26795	0.20871	0.17157	0.14590	0.12702	0.11252	0.10102

Thus for $\beta \in [\alpha_c, \beta_c)$, the set $\mathcal{U}_{\beta,\pm N}$ (the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code) has positive Hausdorff dimension, but only countably many $t \in \mathcal{U}_{\beta,\pm N}$ make the intersection $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ a self-similar set.

2. Geometrical description of $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$

We say that the IFS $\{f_i(\cdot)\}_{i=1}^p$ satisfies the *open set condition* (OSC) if there exists a nonempty bounded open set $O \subseteq \mathbb{R}$ such that $O \supseteq \bigcup_{i=1}^p f_i(O)$, with a disjoint union on the right side. An IFS $\{f_i(\cdot)\}_{i=1}^p$ is said to satisfy the *strong separation condition* (SSC) if the union $\Gamma = \bigcup_{i=1}^p f_i(\Gamma)$ is disjoint.

When $\beta \in (0, 1/(2N-1))$ the IFS $\{\phi_d(\cdot) : d \in \Omega_{\pm N}\}$ satisfies the SSC, so each point in $\Gamma_{\beta,\Omega_{\pm N}}$ has a unique $\Omega_{\pm N}$-code. In case $\beta = 1/(2N-1)$, the IFS $\{\phi_d(\cdot) : d \in \Omega_{\pm N}\}$ fails to satisfy the SSC but satisfies the OSC, so each point has a unique $\Omega_{\pm N}$-code except for countably many points having two $\Omega_{\pm N}$-codes. However, for the case $\beta \in (1/(2N-1), 1/N)$ the IFS $\{\phi_d(\cdot) : d \in \Omega_{\pm N}\}$ fails to satisfy the OSC and $\Gamma_{\beta,\Omega_{\pm N}} = [-1,1]$. In this case, Lebesgue a.a. $t \in [-1,1]$ have a continuum of distinct $\Omega_{\pm N}$-codes (cf. [19]). This gives

$\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)$ a more complicated structure, since it follows ([13]) that for $t\in\Gamma_{\beta,\Omega_{\pm N}}$

$$\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)=\bigcup_{\tilde t}\pi_N\left(\prod_{\ell=1}^{\infty}D_{\ell,\tilde t}\right) \tag{3}$$

where the union is taken over all $\Omega_{\pm N}$-codes of t, and for each code $\tilde t=(t_\ell)_{\ell=1}^\infty\in\Omega_{\pm N}^\infty$

$$D_{\ell,\tilde t}=\Omega_N\cap(\Omega_N+t_\ell)=\{0,1,\ldots,N-1\}\cap(\{0,1,\ldots,N-1\}+t_\ell).$$

Moreover, $\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)$ has the following properties:

(P1) the union on the right side of (3) consists of pairwise disjoint sets;

(P2) for each $\Omega_{\pm N}$-code $\tilde t=(t_\ell)_{\ell=1}^\infty$ of t, we have

$$1+t-\pi_N\left(\prod_{\ell=1}^{\infty}D_{\ell,\tilde t}\right)=\pi_N\left(\prod_{\ell=1}^{\infty}D_{\ell,\tilde t}\right),$$

i.e., $\pi_N(\prod_{\ell=1}^\infty D_{\ell,\tilde t})$ is centrally symmetric. Furthermore, $1+t-\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)=\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)$.

These properties can be obtained as follows. Let $(t_\ell)_{\ell=1}^\infty$ be a $\Omega_{\pm N}$-code of t and let $J=(j_\ell)_{\ell=1}^\infty\in\Omega_N^\infty$. If

$$\pi_N(J)=\sum_{\ell=1}^{\infty}\frac{j_\ell\beta^{\ell-1}(1-\beta)}{N-1}\in\pi_N\left(\prod_{\ell=1}^{\infty}\Omega_N\cap(\Omega_N+t_\ell)\right),$$

then $(j_\ell-t_\ell)_{\ell=1}^\infty\in\Omega_N^\infty$. Note that the IFS $\{\phi_d(\cdot):d\in\Omega_N\}$ satisfies the SSC (since $\beta<1/N$). This implies that each point $x\in\Gamma_{\beta,N}$ has a unique Ω_N-code. Thus $(j_\ell-t_\ell)_{\ell=1}^\infty$ is the unique Ω_N-code of $\pi_N(J)-t$, implying (P1). In addition, one can check that for each $\ell\ge 1$,

$$N-1+t_\ell-\Omega_N\cap(\Omega_N+t_\ell)=\Omega_N\cap(\Omega_N+t_\ell),$$

implying (P2).

Let Ω be a nonempty finite subset of $\mathbb{Z}$. Denote by ε the empty word and put $\Omega^0=\{\varepsilon\}$. For $I\in\bigcup_{\ell=0}^\infty\Omega^\ell$ and $J\in\Omega^\infty\cup\bigcup_{\ell=0}^\infty\Omega^\ell$, let $IJ\in\Omega^\infty\cup\bigcup_{\ell=0}^\infty\Omega^\ell$ be the concatenation of I and J. So in particular $\varepsilon J=J$. For a nonnegative integer k and a finite string $I\in\bigcup_{\ell=1}^\infty\Omega^\ell$, let $I^k:=\overbrace{I\ldots I}^{k}$ be the k times repeating of I and $I^\infty:=III\cdots\in\Omega^\infty$ be the infinite repeating of I. In particular, $I^0=\varepsilon$. For $J=(j_\ell)_{\ell=1}^\infty\in\Omega^\infty$ and $k\in\mathbb{N}$, let $J|_k=(j_\ell)_{\ell=1}^k\in\Omega^k$. We define the algebraic difference between two infinite strings $I=(i_\ell)_{\ell=1}^\infty, J=(j_\ell)_{\ell=1}^\infty\in\Omega^\infty$ by $I-J=(i_\ell-j_\ell)_{\ell=1}^\infty$, and for a positive integer k let $I|_k-J|_k=(I-J)|_k=(i_\ell-j_\ell)_{\ell=1}^k$.

Given $\beta\in(1/(2N-1),1/N)$ and $t\in[-1,1]$, for an integer $d\in\mathbb{Z}$, let

$$\psi_d(x)=\beta x+d(1-\beta)/(N-1)+t(1-\beta),\quad x\in\mathbb{R}.$$

Then

$$\Gamma_{\beta,N}+t=\bigcup_{d\in\Omega_N}\psi_d(\Gamma_{\beta,N}+t).$$

For $J=(j_\ell)_{\ell=1}^k\in\Omega_N^k$ with $k\in\mathbb{N}$, let $\psi_J:=\psi_{j_1}\circ\cdots\circ\psi_{j_k}$ (the same for ϕ_J). For a real number x, it is easy to see that $\psi_d(t+x)=\phi_d(x)+t$ for all $d\in\Omega_N$. Thus by induction we obtain

$$\psi_J(t+x)=\phi_J(x)+t \quad \text{for all } J\in\bigcup_{\ell=1}^{\infty}\Omega_N^\ell,\ x\in\mathbb{R}. \tag{4}$$

The sets $\Gamma_{\beta,N}$ and $\Gamma_{\beta,N}+t$ can be represented in a geometrical way as (cf. [5])

$$\Gamma_{\beta,N}=\bigcap_{k=1}^{\infty}\bigcup_{J\in\Omega_N^k}\phi_J([0,1]) \quad \text{and} \quad \Gamma_{\beta,N}+t=\bigcap_{k=1}^{\infty}\bigcup_{J\in\Omega_N^k}\psi_J([t,1+t]).$$

We call $\phi_J([0,1]),\psi_J([t,1+t])$ with $J\in\Omega_N^k$ the *k-level components* of $\Gamma_{\beta,N}$ and $\Gamma_{\beta,N}+t$, respectively. The 1-level components of $\Gamma_{\beta,N}$ are $\phi_0([0,1]),\phi_1([0,1]),\ldots,\phi_{N-1}([0,1])$ of length β. All gaps between them have the same length $(1-\beta)/(N-1)-\beta$. The left endpoint of $\phi_0([0,1])$ is 0 and the right endpoint of $\phi_{N-1}([0,1])$ is 1. For a ℓ-level component $\phi_J([0,1]),J\in\Omega_N^\ell$, the $(\ell+1)$-level components $\phi_{J0}([0,1]),\phi_{J1}([0,1]),\ldots,\phi_{J(N-1)}([0,1])$ have the same length $\beta^{\ell+1}$ and all gaps (called $(\ell+1)$-level gaps) between them have the same length $\beta^\ell(1-\beta)/(N-1)-\beta^{\ell+1}$. The left endpoint of $\phi_{J0}([0,1])$ coincides with the left endpoint of $\phi_J([0,1])$ and the right endpoint of $\phi_{J(N-1)}([0,1])$ coincides with the right endpoint of $\phi_J([0,1])$. The requirement $\beta\in(1/(2N-1),1/N)$ implies the following simple properties:

(P3) the length of a k-level gap is less than the length of a k-level component, i.e.,

$$\beta^{k-1}(1-\beta)/(N-1)-\beta^k<\beta^k;$$

(P4) if $\phi_I([0,1])\cap\psi_J([t,t+1])\neq\emptyset$ for $I,J\in\Omega_N^k$ with $k\in\mathbb{N}$, then

$$\phi_I([0,1])\cap\psi_J([t,1+t])\cap\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)\neq\emptyset.$$

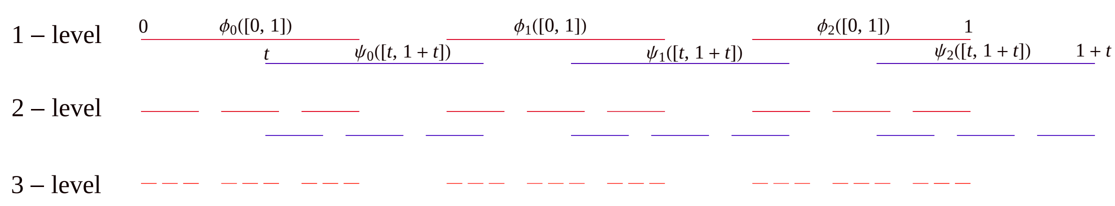

FIGURE 1. $N=3$, $\beta=0.28$, $t=0.19$. The 1-level components of $\Gamma_{\beta,N}$ are $\phi_0([0,1])$, $\phi_1([0,1])$ and $\phi_2([0,1])$. The 1-level components of $\Gamma_{\beta,N}+t$ are $\psi_0([t,1+t])$, $\psi_1([t,1+t])$ and $\psi_2([t,1+t])$. Here $\mathcal{N}_t(0)=\{\psi_0([t,1+t])\}$, $\mathcal{N}_t(1)=\{\psi_0([t,1+t]),\ \psi_1([t,1+t])\}$ and $\mathcal{N}_t(2)=\{\psi_1([t,1+t]),\ \psi_2([t,1+t])\}$.

For $J\in\Omega_N^k$ with $k\in\mathbb{N}$, the *neighborhood* of $\phi_J([0,1])$ with respect to the $k-$level components of $\Gamma_{\beta,N}+t$ is defined as (see Figure 1)

$$\mathcal{N}_t(J):=\Big\{\psi_I([t,1+t]):I\in\Omega_N^k,\ \phi_J([0,1])\cap\psi_I([t,1+t])\neq\emptyset\Big\}.$$

The set $\mathcal{N}_t(J)$ may be empty and $|\mathcal{N}_t(J)| \in \{0,1,2\}$. For $k \geq 1$ let

$$\Lambda_k := \Big\{ J \in \Omega_N^k : |\mathcal{N}_t(J)| \geq 1 \Big\} \quad \text{and} \quad \Lambda := \Big\{ J \in \Omega_N^\infty : J|_k \in \Lambda_k \text{ for all } k \in \mathbb{N} \Big\}.$$

Then $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ can be rewritten in a geometrical way as

$$\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) = \pi_N(\Lambda) = \bigcap_{k=1}^{\infty} \bigcup_{J \in \Lambda_k} \phi_J([0,1]).$$

A set $D \subseteq \Omega_N$ is said to be *consecutive* if $D = \Omega_N \cap (\Omega_N + d)$ for some $d \in \Omega_{\pm N}$.

Proposition 2.1. *Given $N \geq 2$ and $\beta \in (1/(2N-1), 1/N)$, let $t \in [-1,1]$. If $|\mathcal{N}_t(J)| \leq 1$ for all $J \in \bigcup_{\ell=1}^\infty \Omega_N^\ell$, then*

$$\Lambda = \prod_{\ell=1}^{\infty} D_\ell$$

with each D_ℓ consecutive.

Proof. The condition $\beta \in (1/(2N-1), 1/N)$ implies (P3), i.e., all gaps between the intervals $\phi_d([0,1]), d \in \Omega_N$ have the same length strictly less than β, the length of $\phi_d([0,1])$ (see Figure 1). Thus since $t \in [-1,1]$, either $|\mathcal{N}_t(0)| = 1$ or $|\mathcal{N}_t(N-1)| = 1$, which implies that

$$D_1 := \Big\{ d \in \Omega_N : |\mathcal{N}_t(d)| = 1 \Big\} \neq \emptyset.$$

It follows from $|\mathcal{N}_t(d)| \leq 1$ for all $d \in \Omega_N$ that D_1 is consecutive and $\Lambda_1 = D_1$.

Now for $k \in \mathbb{N}$ let the consecutive sets $D_1, \ldots, D_k$ be chosen such that $\Lambda_k = \prod_{\ell=1}^k D_\ell$. Fix a $J \in \Lambda_k$ and take

$$D_{k+1} := \Big\{ d \in \Omega_N : |\mathcal{N}_t(Jd)| = 1 \Big\}.$$

Then D_{k+1} is nonempty by (P3), and is consecutive by the same argument as above. Note that D_{k+1} is independent of the choice of $J \in \Lambda_k$. Thus $\Lambda_{k+1} = \prod_{\ell=1}^{k+1} D_\ell$ which implies $\Lambda = \prod_{\ell=1}^\infty D_\ell$ by induction. $\square$

The following theorem characterizes the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code from a geometrical and an algebraical aspect.

Theorem 2.2. *Given $N \geq 2$ and $\beta \in (1/(2N-1), 1/N)$, let $\mathcal{U}_{\beta,\pm N}$ be the set of $t \in [-1,1]$ which have a unique $\Omega_{\pm N}$-code. Then the following conditions are equivalent.*

(A) $t \in \mathcal{U}_{\beta,\pm N}$;

(B) $|\mathcal{N}_t(J)| \leq 1$ *for all* $J \in \bigcup_{\ell=1}^\infty \Omega_N^\ell$;

(C) t *has a* $\Omega_{\pm N}$*-code* $(t_\ell)_{\ell=1}^\infty$ *such that for all* $k \geq 1$

$$\begin{cases} \sum_{\ell=1}^\infty t_{k+\ell}\beta^\ell < \frac{1-N\beta}{1-\beta}, & \textit{if} \quad t_k < N-1 \\ \sum_{\ell=1}^\infty t_{k+\ell}\beta^\ell > -\frac{1-N\beta}{1-\beta}, & \textit{if} \quad t_k > 1-N. \end{cases} \tag{5}$$

Proof. (A) $\Rightarrow$ (B). Suppose that $|\mathcal{N}_t(J)| = 2$ for some $J = (j_\ell)_{\ell=1}^k \in \Omega_N^k$ with $k \geq 1$. Then either $|\mathcal{N}_t(J|_{k-1}0)| = 2$ or $|\mathcal{N}_t(J|_{k-1}(N-1))| = 2$. Without loss of generality, let $|\mathcal{N}_t(J|_{k-1}0)| = 2$. Then there exists $d \in \Omega_N$ such that $|\mathcal{N}_t(J|_{k-1}d)| = 1$ by the geometric structure of $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ (see Figure 2).

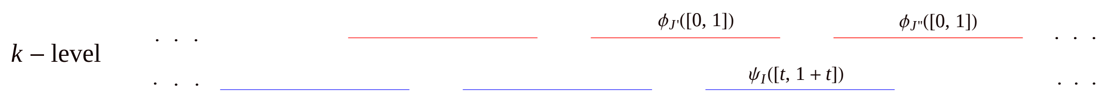

FIGURE 2. $N = 3$. Here $J' = J|_{k-1}1$, $J'' = J|_{k-1}2$ and $\mathcal{N}_t(J') \cap \mathcal{N}_t(J'') = \{\psi_I([t, 1+t])\}$.

Let $J' = J|_{k-1}(d-1)$ and $J'' = J|_{k-1}d$. Then

$$\mathcal{N}_t(J') \cap \mathcal{N}_t(J'') = \big\{\psi_I([t, 1+t])\big\}$$

for some $I = i_1 i_2 \cdots i_{k-1}(N-1) \in \Omega_N^k$. By (P4) we can pick

$$x \in \phi_{J'}([0,1]) \cap \psi_I([t, 1+t]) \cap \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$$

and

$$y \in \phi_{J''}([0,1]) \cap \psi_I([t, 1+t]) \cap \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t).$$

Let $(x_\ell)_{\ell=1}^\infty$ and $(y_\ell)_{\ell=1}^\infty$ be the unique Ω_N-code of x and y, respectively. Then $x_k = d-1$ and $y_k = d$. On the other hand, $x - t, y - t \in \Gamma_{\beta,N}$ and by $(x_\ell^*)_{\ell=1}^\infty, (y_\ell^*)_{\ell=1}^\infty$ we denote their unique Ω_N-code, respectively. It follows from (4) that

$$x \in \psi_I([t, 1+t]) = \phi_I([0,1]) + t \text{ and } y \in \psi_I([t, 1+t]) = \phi_I([0,1]) + t,$$

which imply $x - t, y - t \in \phi_I([0,1])$. Thus $x_k^* = y_k^* = N-1$. Hence $t = x - (x-t) = y - (y-t)$ has two distinct $\Omega_{\pm N}$-codes: $(x_\ell - x_\ell^*)_{\ell=1}^\infty$ and $(y_\ell - y_\ell^*)_{\ell=1}^\infty$.

(B) $\Rightarrow$ (A). By Proposition 2.1, we have $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) = \pi_N(\prod_{\ell=1}^\infty D_\ell)$ with D_ℓ consecutive. Thus, it follows from (3) that t has a unique $\Omega_{\pm N}$-code $(t_\ell)_{\ell=1}^\infty$ with each t_ℓ determined by $D_\ell = \Omega_N \cap (\Omega_N + t_\ell)$.

(B) $\Rightarrow$ (C). It follows from Proposition 2.1 that $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) = \pi_N(\prod_{\ell=1}^\infty D_\ell)$ with each D_ℓ consecutive. Take $J = (j_\ell)_{\ell=1}^\infty \in \prod_{\ell=1}^\infty D_\ell$. Then $\pi_N(J) \in \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$. Let $J^* = (j_\ell^*)_{\ell=1}^\infty$ be the unique Ω_N-code of $\pi_N(J) - t \in \Gamma_{\beta,N}$. Thus it follows by (4) that for each $k \geq 1$

$$\pi_N(J) \in \phi_{J|_k}([0,1]) \cap (\phi_{J^*|_k}([0,1]) + t) = \phi_{J|_k}([0,1]) \cap \psi_{J^*|_k}([t, 1+t]),$$

and

$$J - J^* = (j_\ell - j_\ell^*)_{\ell=1}^\infty = (t_\ell)_{\ell=1}^\infty$$

is the unique $\Omega_{\pm N}$-code of t (the uniqueness is given by $(B) \Rightarrow (A)$). We shall prove $(t_\ell)_{\ell=1}^\infty$ satisfies (5) in the following.

Case I. $t_k \neq \pm(N-1)$.

In this case, $(j_k, j_k^*) \notin \{(N-1, 0), (0, N-1)\}$. This together with the requirements in (B) imply that the distance between the left endpoints of $\phi_{J|_k}([0,1])$ and $\psi_{J^*|_k}([t, t+1])$ must be less than the length of the k-th gap (see Figure 3), i.e., $|\psi_{J^*|_k}(t) - \phi_{J|_k}(0)| < \beta^{k-1}(1-\beta)/(N-1) - \beta^k$.

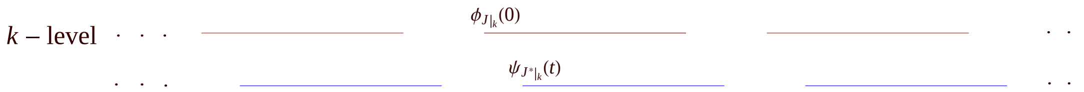

FIGURE 3. $N=3$. Here $\phi_{J|_k}(0)$ is the left endpoint of the k-level component $\phi_{J|_k}([0,1])$ of $\Gamma_{\beta,N}$, and $\psi_{J^*|_k}(t)$ is the left endpoint of k-level component $\phi_{J^*|_k}([t,1+t])$ of $\Gamma_{\beta,N}+t$.

Thus (5) follows by the following computation.

$$\begin{aligned}\left|\sum_{\ell=1}^{\infty}\frac{t_{k+\ell}\beta^{\ell-1}(1-\beta)}{N-1}\right| &= \beta^{-k}\left|\sum_{\ell=k+1}^{\infty}\frac{t_\ell\beta^{\ell-1}(1-\beta)}{N-1}\right| = \beta^{-k}\left|t-\sum_{\ell=1}^{k}\frac{t_\ell\beta^{\ell-1}(1-\beta)}{N-1}\right| \\ &= \beta^{-k}\left|t-\left(\sum_{\ell=1}^{k}\frac{j_\ell\beta^{\ell-1}(1-\beta)}{N-1}-\sum_{\ell=1}^{k}\frac{j_\ell^*\beta^{\ell-1}(1-\beta)}{N-1}\right)\right| \\ &= \beta^{-k}|t-(\phi_{J|_k}(0)-\phi_{J^*|_k}(0))| = \beta^{-k}|\psi_{J^*|_k}(t)-\phi_{J|_k}(0)| \\ &< \frac{1-N\beta}{\beta(N-1)}.\end{aligned}$$

Case II. $t_k = N-1$.

In this case, $(j_k, j_k^*) = (N-1, 0)$. This together with the requirements in (B) imply that $\phi_{J|_k}(0)-\psi_{J^*|_k}(t) < \beta^{k-1}(1-\beta)/(N-1)-\beta^k$. By a similar argument as in Case I, we have

$$\sum_{\ell=1}^{\infty}\frac{t_{k+\ell}\beta^{\ell-1}(1-\beta)}{N-1} = \beta^{-k}(\psi_{J^*|_k}(t)-\phi_{J|_k}(0)) > -\frac{1-N\beta}{\beta(N-1)},$$

leading to (5).

The final case $t_k = 1-N$ can be done in the same way as above.

(C) $\Rightarrow$ (B). We will prove by induction that for any $k\ge 1$ and $J\in\Omega_N^k$

$$\mathcal{N}_t(J) = \begin{cases} \{\psi_{J-(t_\ell)_{\ell=1}^k}([t,1+t])\}, & \text{if } J\in\prod_{\ell=1}^k\big(\Omega_N\cap(\Omega_N+t_\ell)\big) \\ \emptyset, & \text{otherwise.}\end{cases} \tag{6}$$

For $k=1$, let $J\in\Omega_N\cap(\Omega_N+t_1)$. In view of the proof of $(B)\Rightarrow(C)$, (5) becomes

$$\begin{cases} \psi_{J-t_1}(t)-\phi_J(0) < (1-\beta)/(N-1)-\beta, & \text{if } t_1 < N-1 \\ \phi_J(0)-\psi_{J-t_1}(t) < (1-\beta)/(N-1)-\beta, & \text{if } t_1 > 1-N.\end{cases}$$

This implies (6) from the geometrical structure of $\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)$.

Suppose that (6) is true for $k=n$. Let $J=(j_\ell)_{\ell=1}^{n+1}\in\Omega_N^{n+1}$. Then $\mathcal{N}_t(J)=\emptyset$ if $J|_n\notin\prod_{\ell=1}^n\big(\Omega_N\cap(\Omega_N+t_\ell)\big)$. Thus we assume $J|_n\in\prod_{\ell=1}^n\big(\Omega_N\cap(\Omega_N+t_\ell)\big)$. For $j_{n+1}\in\Omega_N\cap(\Omega_N+t_{n+1})$, (5) becomes

$$\begin{cases} \psi_{J-(t_\ell)_{\ell=1}^{n+1}}(t)-\phi_J(0) < \beta^n(1-\beta)/(N-1)-\beta^{n+1}, & \text{if } t_{n+1} < N-1 \\ \phi_J(0)-\psi_{J-(t_\ell)_{\ell=1}^{n+1}}(t) < \beta^n(1-\beta)/(N-1)-\beta^{n+1}, & \text{if } t_{n+1} > 1-N,\end{cases}$$

which implies (6) for $k=n+1$. $\square$

3. The Self-similar structure of $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$

Let Ω be a nonempty finite subset of $\mathbb{Z}$. An infinite string $K \in \Omega^\infty$ is called *strongly periodic with period* q (or simply, strongly periodic) if there exist two finite strings $I = (i_\ell)_{\ell=1}^q, J = (j_\ell)_{\ell=1}^q \in \Omega^q$ with $q \geq 1$ such that $K = I\, J^\infty$ and $I \preccurlyeq J$, where $I \preccurlyeq J$ means $i_\ell \leq j_\ell, 1 \leq \ell \leq q$. For two infinite strings $I, J \in \Omega^\infty$, we say $I \preccurlyeq J$ if $I|_k \preccurlyeq J|_k$ for all $k \in \mathbb{N}$. The following lemma (cf. [15, Lemma 3.1]) gives a description of strongly periodic infinite strings.

Lemma 3.1. *Let $(j_\ell)_{\ell=1}^\infty \in \Omega_N^\infty$. If there exists a positive integer q such that $j_{\ell+q} \geq j_\ell$ for all $\ell \in \mathbb{N}$, then $(j_\ell)_{\ell=1}^\infty$ is strongly periodic with period q.*

When t has a unique $\Omega_{\pm N}$-code $(t_\ell)_{\ell=1}^\infty$, from the proof of Theorem 2.2 it follows that there exists a sequence of consecutive subsets $\Omega_N \cap (\Omega_N + t_\ell)$ such that

$$\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) = \pi_N\left(\prod_{\ell=1}^\infty \Omega_N \cap (\Omega_N + t_\ell)\right).$$

Let γ_* be the smallest member of $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$. It is easy to check that

$$\Gamma_t := \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) - \gamma_* = \pi_N\left(\prod_{\ell=1}^\infty \{0, \ldots, N-1-|t_\ell|\}\right). \tag{7}$$

Thus the Hausdorff and packing dimensions of $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ are given by (cf. [14])

$$\begin{aligned}
\dim_H \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) \quad &= \dim_H \Gamma_t = -\frac{1}{\log\beta} \underline{\lim}_{k\to\infty} \frac{\sum_{\ell=1}^k (N - |t_\ell|)}{k};\\
\dim_P \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) \quad &= \dim_P \Gamma_t = -\frac{1}{\log\beta} \overline{\lim}_{k\to\infty} \frac{\sum_{\ell=1}^k (N - |t_\ell|)}{k}.
\end{aligned}$$

The following properties make it easier to deal with Γ_t.

(P5) For $I, J \in \Omega_N^\infty$, if $I \preccurlyeq J$ and $\pi_N(J) \in \Gamma_t$, then $\pi_N(I) \in \Gamma_t$;

(P6) $\Gamma_t = \gamma^* - \Gamma_t$ where $\gamma^* = \pi_N\big((N-1-|t_\ell|)_{\ell=1}^\infty\big)$ is the largest member in Γ_t.

Thus, when Γ_t is generated by an IFS, say $\{f_i(x) = r_i x + b_i\}_{i=1}^p$, we can require all $r_i > 0$: if $r_i < 0$ we can replace $f_i(x)$ by $f_i^*(x) = -r_i x + b_i + r_i\gamma^*$. This follows from a simple computation (cf. [3, 15])

$$f_i^*(\Gamma_t) = -r_i\Gamma_t + b_i + r_i\gamma^* = r_i(\gamma^* - \Gamma_t) + b_i = r_i\Gamma_t + b_i = f_i(\Gamma_t).$$

Furthermore, we can assume $0 = b_1 \leq b_2 \leq \cdots \leq b_p$ since $0 = \pi_N(0^\infty) \in \Gamma_t$ by (P5).

The following theorem gives a sufficient and necessary condition for $t \in \mathcal{S}_{\beta,\pm N}$, i.e., the set of $t \in [-1,1]$ which have a unique $\Omega_{\pm N}$-code and at the same time make the intersection $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ a self-similar set.

Theorem 3.2. *Given $N \geq 2$ and $\beta \in (1/(2N-1), 1/N)$, let $(t_\ell)_{\ell=1}^\infty$ be the unique $\Omega_{\pm N}$-code of $t \in \mathcal{U}_{\beta,\pm N}$. Then $t \in \mathcal{S}_{\beta,\pm N}$ if and only if $(N-1-|t_\ell|)_{\ell=1}^\infty$ is strongly periodic.*

Proof. It suffices to prove that Γ_t, given by (7), is a self-similar set if and only if $(N-1-|t_\ell|)_{\ell=1}^\infty$ is strongly periodic. Firstly, we prove the sufficiency. If $(N-1-|t_\ell|)_{\ell=1}^\infty \in \Omega_N^\infty$ is strongly periodic, it can be written as $(N-1-|t_\ell|)_{\ell=1}^\infty = \sigma\,(\sigma+\tau)^\infty \in \Omega_N^\infty$ where $\sigma = (\sigma_\ell)_{\ell=1}^q, \tau = (\tau_\ell)_{\ell=1}^q \in \Omega_N^q$ for some $q \in \mathbb{N}$ and $\sigma + \tau = (\sigma_\ell + \tau_\ell)_{\ell=1}^q \in \Omega_N^q$. Let

$$\mathcal{S} := \left\{ \beta^{-q} \sum_{\ell=1}^{2q} \frac{j_\ell \beta^{\ell-1}(1-\beta)}{N-1} : \Omega_N^{2q} \ni (j_\ell)_{\ell=1}^{2q} \preccurlyeq \sigma\tau \right\}.$$

One can check that Γ_t can be generated by the IFS $\{f_s(x) = \beta^q(x+s) : s \in \mathcal{S}\}$ (cf. [15]).

Next, we will prove the necessity. By (P6), we can assume that Γ_t is generated by an IFS $\{f_i(x) = r_i x + b_i\}_{i=1}^p$ with $r_i \in (0,1)$ and $0 = b_1 \le b_2 \le \cdots \le b_p$. Note that the union $(0,1) = \bigcup_{q=0}^\infty [\beta^{q+1}, \beta^q)$ is disjoint, there exist some $q \ge 0$ such that $r_1 \in [\beta^{q+1}, \beta^q)$.

Case I. $r_1 = \beta^{q+1}$. Then for each $\ell \ge 1$, it follows from (P5) that

$$\frac{(N-1-|t_\ell|)\beta^{\ell-1}(1-\beta)}{N-1} = \pi_N\big(0^{\ell-1}(N-1-|t_\ell|)0^\infty\big) \in \Gamma_t.$$

Thus

$$f_1\left(\frac{(N-1-|t_\ell|)\beta^{\ell-1}(1-\beta)}{N-1}\right) = \frac{(N-1-|t_\ell|)\beta^{\ell+q}(1-\beta)}{N-1} \in \Gamma_t$$

which implies that $N-1-|t_\ell| \le N-1-|t_{\ell+q+1}|$ for each $\ell \ge 1$. So $(N-1-|t_\ell|)_{\ell=1}^\infty$ is strongly periodic with period $q+1$ by Lemma 3.1.

Case II. $\beta^{q+1} < r_1 < \beta^q$. Let $r_1 = \beta^{q+\gamma}$ with $0 < \gamma < 1$.

(IIa) γ is rational. Take $k \in \mathbb{N}$ such that $k\gamma \in \mathbb{N}$. Note that the IFS $\{f_0(x) = r_1^k x, f_i(x) = r_i x + b_i, 1 \le i \le p\}$ generates Γ_t. Thus the conclusion can be proved in the same way as that in Case I.

(IIb) γ is irrational. Take $k \in \mathbb{N}$ such that

$$\beta < \beta^{1-k\gamma+[k\gamma]} < \frac{1-\beta}{N-1}. \tag{8}$$

This is possible since the set $\{k\gamma - [k\gamma] : k \in \mathbb{N}\}$ is dense in the interval $(0,1)$. Let $f_0(x) = r_1^k x$. Then for some $\beta^{\ell-1}(1-\beta)/(N-1) \in \Gamma_t$ we have

$$f_0\left(\frac{\beta^{\ell-1}(1-\beta)}{N-1}\right) = \frac{\beta^{kq+k\gamma+\ell-1}(1-\beta)}{N-1} < \xi := \frac{\beta^{kq+[k\gamma]+\ell-1}(1-\beta)}{N-1}.$$

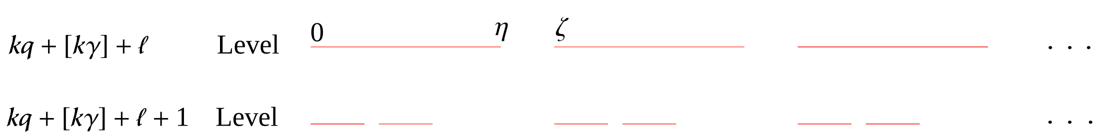

FIGURE 4. $\xi = (\beta^{kq+[k\gamma]+\ell-1}(1-\beta))/(N-1)$, $\eta = \beta^{kq+[k\gamma]+\ell}$. From the geometrical construction of Γ_t, it is easy to see that $(\eta, \xi) \cap \Gamma_t = \emptyset$.

On the other hand, from (8) it follows that

$$\frac{\beta^{kq+k\gamma+\ell-1}(1-\beta)}{N-1} > \eta := \beta^{kq+[k\gamma]+\ell}.$$

Thus $f_0(\frac{\beta^{\ell-1}(1-\beta)}{N-1}) \notin \Gamma_t$ (see Figure 4), leading to a contradiction. $\square$

In fact, the above proof gives a general result on the structure of a class of subsets of the N-part homogeneous Cantor set.

Corollary 3.3. *Given $N \geq 2$ and $\beta \in (0, 1/N)$, let $(i_\ell)_{\ell=1}^\infty, (j_\ell)_{\ell=1}^\infty \in \Omega_N^\infty$ satisfying $(i_\ell)_{\ell=1}^\infty \preccurlyeq (j_\ell)_{\ell=1}^\infty$. Then $\pi_N(\prod_{\ell=1}^\infty \{i_\ell, i_\ell+1, \ldots, j_\ell\})$ is a self-similar set if and only if $(j_\ell - i_\ell)_{\ell=1}^\infty$ is strongly periodic.*

4. The critical point for $\mathcal{U}_{\beta,\pm N}$

According to a result of Sidorov [19, Proposition 3.8] pertaining to the general digit sets, we have that Lebesgue a.a. $t \in [-1, 1]$ have a continuum of distinct $\Omega_{\pm N}$-codes if $\beta \in (1/(2N-1), 1/N)$. However, we will show in this section, for the same set of β's, that there are infinitely many $t \in [-1, 1]$ having a unique $\Omega_{\pm N}$-code. Note that these t form exactly the set $\mathcal{U}_{\beta,\pm N}$ defined earlier. Moreover, there is a critical point $\beta_c \in (1/(2N-1), 1/N)$ such that $\mathcal{U}_{\beta,\pm N}$ has positive Hausdorff dimension if $\beta \in (1/(2N-1), \beta_c)$, and contains countably infinite many elements if $\beta \in (\beta_c, 1/N)$. This can be seen in Theorem 4.6 which is proved by using techniques from beta-expansions.

Given $m \geq 2$ and $\beta \in (1/m, 1)$, let $\Omega_m := \{0, 1, \ldots, m-1\}$. Recall that the sequence $(s_\ell)_{\ell=1}^\infty \in \Omega_m^\infty$ is called a *β-expansion of x with digit set Ω_m* if we can write $x = \sum_{\ell=1}^\infty s_\ell \beta^\ell$ with $s_\ell \in \Omega_m$. The largest number we can obtain in this way is $x_{\max} := (m-1)\beta/(1-\beta)$. Now for any $x \in (0, x_{\max}]$, let us define a sequence $(s_\ell)_{\ell=1}^\infty \in \Omega_m^\infty$ recursively by the *quasi-greedy algorithm* (cf. [20]): let $s_0 = 0$, and if s_ℓ is already defined for all $\ell < n$, then let s_n be the largest element in Ω_m satisfying $\sum_{\ell=1}^n s_\ell \beta^\ell < x$. Obviously, $\sum_{\ell=1}^\infty s_\ell \beta^\ell = x$, and we call $(s_\ell)_{\ell=1}^\infty$ the *quasi-greedy β-expansion of x with digit set Ω_m*. We always call $(s_\ell)_{\ell=1}^\infty$ a quasi-greedy expansion of x if there is no confusion about β and the digit set Ω_m. It is easy to see that $(s_\ell)_{\ell=1}^\infty$ is an *infinite expansion* (i.e., infinitely many s_ℓ are non-zeros).

We use systematically the lexicographical order between sequences: we write $(a_\ell)_{\ell=1}^\infty < (b_\ell)_{\ell=1}^\infty$ or $(b_\ell)_{\ell=1}^\infty > (a_\ell)_{\ell=1}^\infty$ if there exists an $n \in \mathbb{N}$ such that $a_\ell = b_\ell$ for $\ell < n$ and $a_n < b_n$. Furthermore, we write $(a_\ell)_{\ell=1}^\infty \leq (b_\ell)_{\ell=1}^\infty$ or $(b_\ell)_{\ell=1}^\infty \geq (a_\ell)_{\ell=1}^\infty$ if we also allow the equality of the two sequences. Similarly, for two s-blocks $c_1 \ldots c_s$ and $d_1 \ldots d_s$, we write $(c_\ell)_{\ell=1}^s < (d_\ell)_{\ell=1}^s$ if there exists $1 \leq n \leq s$ such that $c_1 \ldots c_{n-1} = d_1 \ldots d_{n-1}$ and $c_n < d_n$. Moreover, we write $(c_\ell)_{\ell=1}^s \leq (d_\ell)_{\ell=1}^s$ if we allow the equality of the two blocks.

Therefore, the quasi-greedy expansion of $x \in (0, x_{\max}]$ is the largest infinite expansion among all the β-expansions of x in the sense of lexicographical order. Note that $1 \in (0, x_{\max}]$ since $\beta > 1/m$. In the remainder of the paper we will reserve the notation $(\delta_\ell)_{\ell=1}^\infty = (\delta_\ell(\beta))_{\ell=1}^\infty$ for the quasi-greedy β-expansion of 1 with digit set Ω_m. The following important properties of the quasi-greedy expansion of 1, will be used in the proof of Theorem 4.6.

Proposition 4.1 (Parry [17])**.** *Given $m \geq 2$, the map $\beta \to (\delta_\ell(\beta))_{\ell=1}^\infty \in \Omega_m^\infty$, with $\beta \in (1/m, 1)$, is strictly decreasing in the sense of lexicographical order. Moreover, the map is continuous w.r.t. the topology in Ω_m^∞ induced by the metric $d\big((a_\ell)_{\ell=1}^\infty, (b_\ell)_{\ell=1}^\infty\big) = 2^{-\min\{j : a_j \neq b_j\}}$.*

Proposition 4.2 (de Vries and Komornik [20])**.** *Given $m \geq 2$ and $\beta \in (1/m, 1)$, let $(\gamma_\ell)_{\ell=1}^\infty$ be an infinite β-expansion of 1 with digit set Ω_m. Then $(\gamma_\ell)_{\ell=1}^\infty$ is the quasi-greedy expansion of 1 if and only if for all $k \geq 1$*

$$\gamma_{k+1}\gamma_{k+2}\cdots \leq \gamma_1\gamma_2\dots \tag{9}$$

in the lexicographical order.

Given $m \geq 2$, let $\overline{d} = m-1-d$ be the *reflection of the digit* $d \in \Omega_m$. For a sequence $(a_\ell)_{\ell=1}^\infty \in \Omega_m^\infty$, let $\overline{(a_\ell)_{\ell=1}^\infty} = (\overline{a_\ell})_{\ell=1}^\infty = (m-1-a_\ell)_{\ell=1}^\infty$ be the *reflection of the sequence* $(a_\ell)_{\ell=1}^\infty \in \Omega_m^\infty$. A sequence $(a_\ell)_{\ell=1}^\infty \in \Omega_m^\infty$ is said to be *admissible* if for all $k \geq 1$

$$\begin{cases} a_{k+1}a_{k+2}\cdots < a_1a_2\dots, & \text{if} \quad a_k < m-1 \\ \overline{a_{k+1}a_{k+2}\dots} < a_1a_2\dots, & \text{if} \quad a_k > 0. \end{cases}$$

Let $(\tau_\ell)_{\ell=0}^\infty \in \Omega_2^\infty$ be the classical Thue-Morse sequence, i.e., $\tau_0 = 0$, and if τ_ℓ is already defined for some $\ell \geq 0$, set $\tau_{2\ell} = \tau_\ell$ and $\tau_{2\ell+1} = \overline{\tau_\ell} = 1 - \tau_\ell$. Then the sequence $(\tau_\ell)_{\ell=0}^\infty$ begins as follows

$$0\ 1101\ 0011\ 0010\ 1101\ 0010\ 1100\ 1101\ 0011\ 0010\ 1100\dots.$$

We construct a sequence $(\lambda_\ell)_{\ell=1}^\infty = (\lambda_\ell(m))_{\ell=1}^\infty \in \Omega_m^\infty$ for the even and odd numbers m respectively.

$$\begin{array}{llll} \text{(I).} & \lambda_\ell = q-1+\tau_\ell \text{ for } \ell \geq 1, & \text{if} & m = 2q \text{ with } q \geq 1; \\ \text{(II).} & \lambda_\ell = q+\tau_\ell-\tau_{\ell-1} \text{ for } \ell \geq 1, & \text{if} & m = 2q+1 \text{ with } q \geq 1. \end{array} \tag{10}$$

Komornik and Loreti [12] showed that $(\lambda_\ell)_{\ell=1}^\infty$ is the smallest admissible sequence in Ω_m^∞ in the sense of lexicographical order. Moreover, they gave the following proposition.

Proposition 4.3 (Komornik and Loreti [12])**.** *Let $(\lambda_\ell)_{\ell=1}^\infty \in \Omega_m^\infty$ be defined in* (10). Then for all $k \geq 1$

$$\lambda_{k+1}\lambda_{k+2}\cdots < \lambda_1\lambda_2\dots, \quad \overline{\lambda_{k+1}\lambda_{k+2}\dots} < \lambda_1\lambda_2\dots.$$

For a more general digit set Ω, there also exist some results on the smallest admissible sequence which is related to the Thue-Morse sequence (cf. [1]).

The following important theorem on the set

$$\mathcal{A}_{\beta,m} := \Big\{x \in [0, x_{\max}] : \ x = \sum_{\ell=1}^\infty \varepsilon_\ell\beta^\ell, \ \varepsilon_\ell \in \Omega_m \ \text{ has a unique } \beta\text{-expansion}\Big\}$$

is due to Parry [17], Erdös et al. [4], Komornik et al. [12] and de Vries et al. [20].

Theorem 4.4. *Given $m \geq 2$ and $\beta \in (1/m, 1)$, let $(\delta_\ell)_{\ell=1}^\infty$ be the quasi-greedy β-expansion of 1 with digit set Ω_m. Then $\sum_{\ell=1}^\infty \varepsilon_\ell\beta^\ell \in \mathcal{A}_{\beta,m}$ if and only if for all $k \geq 1$*

$$\begin{cases} \varepsilon_{k+1}\varepsilon_{k+2}\cdots < \delta_1\delta_2\dots, & \text{if} \quad \varepsilon_k < m-1 \\ \overline{\varepsilon_{k+1}\varepsilon_{k+2}\dots} < \delta_1\delta_2\dots, & \text{if} \quad \varepsilon_k > 0. \end{cases}$$

For $m \geq 2$, let $\beta_{c,m}$ be the unique positive solution of the following equation

$$1 = \sum_{\ell=1}^{\infty} \lambda_\ell \beta^\ell, \tag{11}$$

where $(\lambda_\ell)_{\ell=1}^\infty = (\lambda_\ell(m))_{\ell=1}^\infty \in \Omega_m^\infty$ is defined in (10). We remark here that $\beta_{c,m}$ is a transcendental number for all $m \geq 2$ (cf. [12]). For $m = 2$, Glendinning and Sidorov [7] have shown that the critical point for $\mathcal{A}_{\beta,2}$ is $\beta_{c,2}$, i.e., $\mathcal{A}_{\beta,2}$ has positive Hausdorff dimension if $\beta < \beta_{c,2}$ and $\mathcal{A}_{\beta,2}$ contains at most countably many elements if $\beta > \beta_{c,2}$. Their results can be generalized to the even number case, i.e., for an even number $m \geq 2$, the critical point for $\mathcal{A}_{\beta,m}$ is $\beta_{c,m}$. However, it is more intricate to find the critical point for $\mathcal{A}_{\beta,m}$ for an odd number m. Inspired by [7] we show that for an odd number $m \geq 3$, the critical point for $\mathcal{A}_{\beta,m}$ is still $\beta_{c,m}$, the unique positive solution of Equation (11).

Given $N \geq 2$ and $\beta \in (1/(2N-1), 1/N)$, we will find the critical point for $\mathcal{U}_{\beta,\pm N}$, which is the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code.

To make the connection with the theory of beta-expansions we shift $\Omega_{\pm N}$ to the set

$$\Omega_{\pm N} + N - 1 = \{0, 1, \ldots, 2N-2\} = \Omega_{2N-1}.$$

Thus from $[-1,1] = \pi_{\pm N}\big(\Omega_{\pm N}^\infty\big)$ it follows that

$$[0,2] = \pi_{2N-1}\big(\Omega_{2N-1}^\infty\big) = \left\{ \sum_{\ell=1}^{\infty} \frac{\varepsilon_\ell \beta^{\ell-1}(1-\beta)}{N-1} : \varepsilon_\ell \in \{0,1,\ldots,2N-2\} \right\},$$

where $\pi_{2N-1} := \pi_{\Omega_{2N-1}}$ is as in (1). Let

$$\mathcal{U}_{\beta,2N-1} := \left\{ t \in [0,2] : |\pi_{2N-1}^{-1}(t)| = 1 \right\},$$

i.e., the set of $t \in [0,2]$ having a unique Ω_{2N-1}-code. Thus, it is easy to see that

$$\mathcal{U}_{\beta,2N-1} = \mathcal{U}_{\beta,\pm N} + 1.$$

For $\beta \in (1/(2N-1), 1/N)$, note that

$$x \in \mathcal{A}_{\beta,2N-1} \quad \Longleftrightarrow \quad \frac{1-\beta}{\beta(N-1)} x \in \mathcal{U}_{\beta,2N-1}.$$

Thus Theorem 4.4 yields the the following important theorem which could also be shown in a different way by using (5).

Theorem 4.5. *Given $N \geq 2$ and $\beta \in (1/(2N-1), 1/N)$, let $(\delta_\ell)_{\ell=1}^\infty$ be the quasi-greedy β-expansion of 1 with digit set Ω_{2N-1}. Then $(\varepsilon_\ell)_{\ell=1}^\infty \in \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$ if and only if for all $k \geq 1$*

$$\begin{cases} \varepsilon_{k+1}\varepsilon_{k+2}\cdots < \delta_1\delta_2\ldots, & \text{if} \quad \varepsilon_k \in \{0,\ldots,2N-3\} \\ \overline{\varepsilon_{k+1}\varepsilon_{k+2}\ldots} < \delta_1\delta_2\ldots, & \text{if} \quad \varepsilon_k \in \{1,\ldots,2N-2\}, \end{cases} \tag{12}$$

where $\overline{\varepsilon_{k+1}\varepsilon_{k+2}\ldots}$ is the reflection of $\varepsilon_{k+1}\varepsilon_{k+2}\cdots \in \Omega_{2N-1}^\infty$.

Therefore, dealing with the set $\mathcal{U}_{\beta,\pm N}$ is equivalent to dealing with the set of sequences $(\varepsilon_\ell)_{\ell=1}^\infty \in \Omega_{2N-1}^\infty$ which satisfy (12). Substituting $m = 2N-1$ in (10), we get the smallest admissible sequence $(\lambda_\ell)_{\ell=1}^\infty \in \Omega_{2N-1}^\infty$ which starts with

$$N(N-1)(N-2)N \quad (N-2)(N-1)N(N-1) \quad (N-2)(N-1)N(N-2)\dots.$$

It is helpful to give another equivalent definition of the sequence $(\lambda_\ell)_{\ell=1}^\infty \in \Omega_{2N-1}^\infty$ (cf. [12]), i.e.,

$$\begin{aligned} &\lambda_1 = N, \quad \lambda_{2^{n+1}} = \overline{\lambda_{2^n}} + 1 = 2N-1-\lambda_{2^n} \quad \text{for } n = 0,1,\dots, \\ &\lambda_{2^n+\ell} = \overline{\lambda_\ell} = 2N-2-\lambda_\ell \qquad \text{for } 1 \le \ell < 2^n,\ n = 1,2,\dots. \end{aligned} \tag{13}$$

So it is easy to see $\lambda_{2^n} = N$ for $n = 0,2,4,\dots$ and $\lambda_{2^n} = N-1$ for $n = 1,3,5,\dots$.

Theorem 4.6. *Given $N \ge 2$, $\beta \in (1/(2N-1), 1/N)$, let $\mathcal{U}_{\beta,\pm N}$ be the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code and $\beta_c \in (1/(2N-1), 1/N)$ be the unique positive solution of Equation* (11) with $(\lambda_\ell)_{\ell=1}^\infty \in \Omega_{2N-1}^\infty$ defined in (13). Then

(1) If $\beta \in (1/(2N-1), \beta_c)$, then $\dim_H \mathcal{U}_{\beta,\pm N} > 0$;
(2) If $\beta = \beta_c$, then $|\mathcal{U}_{\beta_c,\pm N}| = 2^{\aleph_0}$ and $\dim_H \mathcal{U}_{\beta_c,\pm N} = 0$;
(3) If $\beta \in (\beta_c, 1/N)$, then $|\mathcal{U}_{\beta,\pm N}| = \aleph_0$.

Since $\mathcal{U}_{\beta,\pm N} = \mathcal{U}_{\beta,2N-1} - 1$, the critical point of $\mathcal{U}_{\beta,\pm N}$ is equal to the critical point of $\mathcal{U}_{\beta,2N-1}$. Thus we only need to show the corresponding conclusions for the set $\mathcal{U}_{\beta,2N-1}$.

Using Proposition 4.2 and Proposition 4.3, we obtain $(\delta_\ell(\beta_c))_{\ell=1}^\infty = (\lambda_\ell)_{\ell=1}^\infty$, i.e., $(\lambda_\ell)_{\ell=1}^\infty$ is the quasi-greedy β_c-expansion of 1 with digit set Ω_{2N-1}. The proof of Theorem 4.6 will be divided into several lemmas.

Lemma 4.7. *$\lambda_k \dots \lambda_{k+2^n-2} < \lambda_1 \dots \lambda_{2^n-1}$ for any $n \ge 2$ and any $k \in \{2, \dots, 2^n-1\}$; $\overline{\lambda_k \dots \lambda_{k+2^n-2}} < \lambda_1 \dots \lambda_{2^n-1}$ for any $n \ge 2$ and any $k \in \{1, \dots, 2^n-1\}$.*

Proof. Since for $n = 2$ the lemma is quickly checked, let $n \ge 3$ and $k \in \{2, \dots, 2^n-1\}$. Then by Proposition 4.3 $\lambda_k\lambda_{k+1}\dots < \lambda_1\lambda_2\dots$, which implies $\lambda_k \dots \lambda_{k+2^n-2} \le \lambda_1 \dots \lambda_{2^n-1}$. It is easy to check that $\lambda_k \dots \lambda_{k+2^n-2} < \lambda_1 \dots \lambda_{2^n-1}$ for $k < 7$. For all other k we can write $k = 2^s + 2^p + j$ with $1 \le p < s < n$ and $1 \le j < 2^p$. It follows from [12, Lemma 5.4] that

$$\lambda_k \dots \lambda_{k+2^{p+1}-j} < \lambda_j \dots \lambda_{2^{p+1}} \le \lambda_1 \dots \lambda_{2^{p+1}-j+1}$$

which implies $\lambda_k \dots \lambda_{k+2^n-2} < \lambda_1 \dots \lambda_{2^n-1}$, since $n > p+1$.

For the second inequality, ignoring the trivial cases $k = 1$ and 2, suppose $k = 2^q + j$ with $1 \le j < 2^q$ and $1 \le q < n$. Then it again follows from [12, Lemma 5.5] that

$$\overline{\lambda_k \dots \lambda_{k+2^q-j}} < \lambda_j \dots \lambda_{2^q} \le \lambda_1 \dots \lambda_{2^q-j+1}.$$

which implies that $\overline{\lambda_k \dots \lambda_{k+2^n-2}} < \lambda_1 \dots \lambda_{2^n-1}$, since $n > q$. □

Lemma 4.8. *Let $n \ge 3$ be an odd integer. If $\overline{\lambda_k \dots \lambda_{2^n-1}} = \lambda_1 \dots \lambda_{2^n-k}$ for some $k \in \{1, \dots, 2^n-1\}$, then $\lambda_{2^n-k+1} = N$.*

Proof. Suppose $\overline{\lambda_k \dots \lambda_{2^n-1}} = \lambda_1 \dots \lambda_{2^n-k}$. It can not happen that $k < 2^{n-1}$ since then we will obtain that $\overline{\lambda_k \dots \lambda_{k+2^{n-1}-2}} = \lambda_1 \dots \lambda_{2^{n-1}-1}$ which contradicts Lemma 4.7. It is also impossible that $k = 2^{n-1}$ since then $N-2 = \overline{\lambda_{2^{n-1}}} = \lambda_1 = N$. Thus we must have $k > 2^{n-1}$. From the definition of $(\lambda_\ell)_{\ell=0}^\infty$ in (13) it follows that

$$\lambda_{k-2^{n-1}} \dots \lambda_{2^{n-1}-1} = \overline{\lambda_k \dots \lambda_{2^n-1}} = \lambda_1 \dots \lambda_{2^n-k},$$

which implies $N \geq \lambda_{2^n-k+1} \geq \lambda_{2^{n-1}} = N$ by Proposition 4.3. □

We want to approximate $(\lambda_\ell)_{\ell=1}^\infty$ by eventually periodic sequences which satisfy (9). This does not work for the obvious choice $(\lambda_1 \dots \lambda_{2^n})^\infty$. Thus we define for $n \geq 0$

$$C_n^\infty = \lambda_1 \dots \lambda_{2^n} (\lambda_{2^n+1} \dots \lambda_{2^{n+1}})^\infty.$$

Since for all $n \geq 0$ we have $\lambda_{2^{n+1}} > \overline{\lambda_{2^n}}$, we obtain that

$$\lambda_1 \dots \lambda_{2^n} (\lambda_{2^n+1} \dots \lambda_{2^{n+1}})^3 > \lambda_1 \dots \lambda_{2^{n+1}} \lambda_{2^{n+1}+1} \dots \lambda_{2^{n+2}},$$

which implies

(P7) $C_0^\infty > C_1^\infty > \dots > C_n^\infty > \dots > (\lambda_\ell)_{\ell=1}^\infty$ in the lexicographical order.

Lemma 4.9. *Let $n \geq 3$ be an odd number. Then for any $k \geq 1$ we have $\sigma^k(C_n^\infty) < C_n^\infty$, where σ is the left-shift map.*

Proof. Since C_n^∞ is an eventually periodic sequence in Ω_{2N-1}^∞, we only have to check the lemma for $k \in \{1, \dots, 2^{n+1}-1\}$. For $k = 2^n - 1$ or $2^{n+1} - 1$, it is easy to check that $\sigma^k(C_n^\infty) < C_n^\infty$. Then we only need to consider the following two cases.

(I) $k \in \{1, \dots, 2^n - 2\}$. It follows from Lemma 4.7 that

$$\sigma^k(C_n^\infty) = \lambda_{k+1} \dots \lambda_{2^n+k-1} \dots < \lambda_1 \dots \lambda_{2^n-1} \lambda_{2^n} (\lambda_{2^n+1} \dots \lambda_{2^{n+1}})^\infty = C_n^\infty.$$

(II) $k \in \{2^n, \dots, 2^{n+1} - 2\}$. Write $k = 2^n + \ell$. Then, by the definition of $(\lambda_\ell)_{\ell=1}^\infty$,

$$\begin{aligned} \sigma^k(C_n^\infty) &= \lambda_{k+1} \dots \lambda_{2^{n+1}-1} \lambda_{2^{n+1}} (\lambda_{2^n+1} \dots \lambda_{2^{n+1}})^\infty \\ &= \overline{\lambda_{\ell+1} \dots \lambda_{2^n-1}} \lambda_{2^{n+1}} (\lambda_{2^n+1} \dots \lambda_{2^{n+1}})^\infty. \end{aligned}$$

If $\overline{\lambda_{\ell+1} \dots \lambda_{2^n-1}} < \lambda_1 \dots \lambda_{2^n-\ell-1}$, we have shown that $\sigma^k(C_n^\infty) < C_n^\infty$. Otherwise, $\ell \geq 2$ and we have by Proposition 4.3 that $\overline{\lambda_{\ell+1} \dots \lambda_{2^n-1}} = \lambda_1 \dots \lambda_{2^n-\ell-1}$. Using Lemma 4.8 we obtain that also $\lambda_{2^{n+1}} = N = \lambda_{2^n-\ell}$. Thus it is enough to show

$$\lambda_{2^n+1} \dots \lambda_{2^{n+1}-1} < \lambda_{2^n-\ell+1} \dots \lambda_{2^{n+1}-\ell-1}.$$

Taking reflections on both sides, this is equivalent to showing $\lambda_1 \dots \lambda_{2^n-1} > \overline{\lambda_{2^n-\ell+1} \dots \lambda_{2^{n+1}-\ell-1}}$, which is true by Lemma 4.7 since $\ell \geq 2$. □

Lemma 4.10. *Let $n \geq 3$ be an odd integer and $\xi_n = (N-1)\lambda_1 \dots \lambda_{2^n-1}, \eta_n = (N-2)\lambda_1 \dots \lambda_{2^n-1}$. Then for any $k \in \{0, \dots, 2^n - 1\}$*

$$\sigma^k(\xi_n \eta_n) < \lambda_1 \dots \lambda_{2^{n+1}-k}, \quad \sigma^k(\overline{\xi_n \eta_n}) < \lambda_1 \dots \lambda_{2^{n+1}-k}, \quad \sigma^k(\eta_n \overline{\xi_n}) \leq \lambda_1 \dots \lambda_{2^{n+1}-k},$$
$$\sigma^k(\overline{\eta_n} \xi_n) < \lambda_1 \dots \lambda_{2^{n+1}-k}, \quad \sigma^k(\xi_n \overline{\xi_n}) \leq \lambda_1 \dots \lambda_{2^{n+1}-k}, \quad \sigma^k(\overline{\xi_n} \xi_n) < \lambda_1 \dots \lambda_{2^{n+1}-k}.$$

Proof. Since the lemma is quickly checked for $k = 0$ and 1, we can assume $k \in \{2, \dots, 2^n - 1\}$. It follows by $\lambda_{2^n} = N - 1$ (since n is odd) that

$$\sigma^k(\xi_n\eta_n) = \lambda_k \dots \lambda_{2^n-1}(N-2)\lambda_1 \dots \lambda_{2^n-1} < \lambda_k \dots \lambda_{2^n-1}\lambda_{2^n} \dots \lambda_{2^{n+1}-1} \le \lambda_1 \dots \lambda_{2^{n+1}-k}.$$

For the second inequality, note that $\sigma^k(\overline{\xi_n\eta_n}) = \overline{\lambda_k \dots \lambda_{2^n-1}}N\overline{\lambda_1 \dots \lambda_{2^n-1}}$. If $\overline{\lambda_k \dots \lambda_{2^n-1}} < \lambda_1 \dots \lambda_{2^n-k}$, we have shown $\sigma^k(\overline{\xi_n\eta_n}) < \lambda_1 \dots \lambda_{2^{n+1}-k}$. Otherwise, it follows by Proposition 4.3 that $\overline{\lambda_k \dots \lambda_{2^n-1}} = \lambda_1 \dots \lambda_{2^n-k}$ which implies $k > 2$. Thus we obtain by Lemma 4.8 that $\lambda_{2^n-k+1} = N$. Hence we only have to show $\overline{\lambda_1 \dots \lambda_{2^n-1}} < \lambda_{2^n-k+2} \dots \lambda_{2^{n+1}-k}$ which is equivalent to showing $\lambda_1 \dots \lambda_{2^n-1} > \overline{\lambda_{2^n-k+2} \dots \lambda_{2^{n+1}-k}}$. This is true by Lemma 4.7 since $k > 2$. Therefore, $\sigma^k(\overline{\xi_n\eta_n}) < \lambda_1 \dots \lambda_{2^{n+1}-k}$ for $k \in \{2, \dots, 2^n - 1\}$. The remaining four inequalities follow from Lemma 4.7 and the fact that for $k \in \{2, \dots, 2^n - 1\}$

$$\sigma^k(\eta_n\overline{\xi_n}) = \sigma^k(\xi_n\overline{\xi_n}) = \lambda_k \dots \lambda_{2^n-1}\overline{(N-1)\lambda_1 \dots \lambda_{2^n-1}} = \lambda_k \dots \lambda_{2^{n+1}-1},$$
$$\sigma^k(\overline{\eta_n}\xi_n) = \sigma^k(\overline{\xi_n}\xi_n) = \overline{\lambda_k \dots \lambda_{2^n-1}}(N-1)\lambda_1 \dots \lambda_{2^n-1} = \overline{\lambda_k \dots \lambda_{2^{n+1}-1}}.$$

□

From Lemma 4.9 and Proposition 4.2 it follows that C_n^∞ is the quasi-greedy expansion of 1 for some base β_n, i.e., $(\delta_\ell(\beta_n))_{\ell=1}^\infty = C_n^\infty$. Then we obtain from (P7) and Proposition 4.1 that β_n increases to β_c as $n \to \infty$. Thus for $\beta < \beta_c$ there exists a large odd number $n \ge 3$ such that $\beta < \beta_n < \beta_c$, which together with Proposition 4.1 imply that

$$(\delta_\ell(\beta))_{\ell=1}^\infty > (\delta_\ell(\beta_n))_{\ell=1}^\infty = C_n^\infty = \lambda_1 \dots \lambda_{2^n}(\lambda_{2^n+1} \dots \lambda_{2^{n+1}})^\infty.$$

It follows from Lemma 4.10 and Theorem 4.5 that

$$X_A^{(n)} \subseteq \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}),$$

where $X_A^{(n)}$ is a subshift of finite type $X_A^{(n)} := \{(e_\ell)_{\ell=1}^\infty \in \mathfrak{A}^\infty : A(e_\ell, e_{\ell+1}) = 1\}$ over the alphabet $\mathfrak{A} = \{\xi_n, \eta_n, \overline{\xi_n}, \overline{\eta_n}\}$ defined by the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to obtain that $r(A)$, the spectral radius of A, equals $\frac{1+\sqrt{5}}{2}$. Since $\pi_{2N-1}(X_A^{(n)})$ is a graph-directed set satisfying the OSC for large n, we conclude from [16] that

$$\dim_H \mathcal{U}_{\beta,2N-1} \ge \dim_H \pi_{2N-1}(X_A^{(n)}) = \frac{\log r(A)}{-2^n \log \beta} = \frac{\log \frac{1+\sqrt{5}}{2}}{-2^n \log \beta} > 0,$$

which establishes Part (1) of Theorem 4.6.

In the following we will show Part (2) and (3) simultaneously. Let

$$w_n := \lambda_1 \dots \lambda_{2^n}.$$

Then by the definition of $(\lambda_\ell)_{\ell=1}^\infty$ in (13) it is easy to check that $w_n\overline{w_n} < w_{n+1}$, which implies

(P8) $(w_0\overline{w_0})^\infty < (w_1\overline{w_1})^\infty < \cdots < (w_n\overline{w_n})^\infty < \cdots < (\lambda_\ell)_{\ell=1}^\infty$ in the lexicographical order.

Lemma 4.11. *Given* $N \geq 2$, $\beta \geq \beta_c$ *and* $(\varepsilon_\ell)_{\ell=1}^\infty \in \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$, *if* $\varepsilon_k < 2N-2$ *and* $\varepsilon_{k+1}\cdots\varepsilon_{k+2^n} = w_n$ *for some* $k, n \geq 0$, *then* $\varepsilon_{k+1}\ldots\varepsilon_{k+2^{n+1}} = w_n\overline{w_n}$ *or* w_{n+1}. *Similarly, if* $\varepsilon_k > 0$ *and* $\varepsilon_{k+1}\cdots\varepsilon_{k+2^n} = \overline{w_n}$ *for some* $k, n \geq 0$, *then* $\varepsilon_{k+1}\ldots\varepsilon_{k+2^{n+1}} = \overline{w_n}w_n$ *or* $\overline{w_{n+1}}$.

Proof. Let $(\delta_\ell)_{\ell=1}^\infty := (\delta_\ell(\beta))_{\ell=1}^\infty$. It follows from $\beta \geq \beta_c$ and Proposition 4.1 that

$$(\delta_\ell)_{\ell=1}^\infty \leq (\delta_\ell(\beta_c))_{\ell=1}^\infty = (\lambda_\ell)_{\ell=1}^\infty.$$

Using (12) and the assumption $\varepsilon_k < 2N-2$, we obtain that $\varepsilon_{k+1}\ldots\varepsilon_{k+2^{n+1}} \leq \delta_1\ldots\delta_{2^{n+1}} \leq \lambda_1\ldots\lambda_{2^{n+1}}$. Note that $\varepsilon_{k+1}\cdots\varepsilon_{k+2^n} = w_n = \lambda_1\ldots\lambda_{2^n}$, then $\varepsilon_{k+2^n+1}\ldots\varepsilon_{k+2^{n+1}} \leq \lambda_{2^n+1}\cdots\lambda_{2^{n+1}}$. On the other hand, from (12) and the fact $\varepsilon_{k+2^n} = \lambda_{2^n} > 0$ it follows that $\overline{\varepsilon_{k+2^n+1}\ldots\varepsilon_{k+2^{n+1}}} \leq \delta_1\ldots\delta_{2^n} \leq \lambda_1\ldots\lambda_{2^n}$. Thus by the definition of $(\lambda_\ell)_{\ell=1}^\infty$ in (13), we obtain

$$\lambda_{2^n+1}\cdots\lambda_{2^{n+1}-1}\lambda_{2^{n+1}} \geq \varepsilon_{k+2^n+1}\ldots\varepsilon_{k+2^{n+1}} \geq \overline{\lambda_1\ldots\lambda_{2^n}} = \lambda_{2^n+1}\ldots\lambda_{2^{n+1}-1}(\lambda_{2^{n+1}}-1),$$

which implies $\varepsilon_{k+1}\cdots\varepsilon_{k+2^{n+1}} = w_n\overline{w_n}$ or w_{n+1}.

The result for $\varepsilon_k > 0$ and $\varepsilon_{k+1}\ldots\varepsilon_{k+2^n} = \overline{\lambda_1\ldots\lambda_{2^n}}$ follows similarly. □

Lemma 4.12. *Let* $N \geq 2$ *and* $\beta \in (\beta_c, 1/N)$. *Then there exists some integer* $n^* = n^*(\beta) \geq 0$ *such that* $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1} \setminus \{0,2\})$ *contains only eventually periodic sequences, either with period* 1 *and period block* $N-1$ *or with period* 2^{n+1} *and period block* $w_n\overline{w_n}$ *for some* $n \leq n^*$.

Proof. For $\beta \in (\beta_c, 1/N)$, let $(\delta_\ell)_{\ell=1}^\infty := (\delta_\ell(\beta))_{\ell=1}^\infty$. The proof will be split into two cases: Case I treats $(\delta_\ell)_{\ell=1}^\infty > (w_0\overline{w_0})^\infty$, and Case II treats $(\delta_\ell)_{\ell=1}^\infty \leq (w_0\overline{w_0})^\infty$.

Fix a sequence $(\varepsilon_\ell)_{\ell=1}^\infty \in \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$. In terms of Theorem 4.5, it is easy to see that $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$ is *reflection invariant*, i.e., it contains $(\varepsilon_\ell)_{\ell=1}^\infty$ if and only if it contains $(\overline{\varepsilon_\ell})_{\ell=1}^\infty = (2N-2-\varepsilon_\ell)_{\ell=1}^\infty$. Note that $\overline{N-1} = N-1$ and that the existence of a period block $\overline{w_n}w_n$ implies the existence of a period block $w_n\overline{w_n}$. So we can assume by reflection that $\varepsilon_1 \in \{0, \ldots, N-1\}$. Ignoring the trivial case $(\varepsilon_\ell)_{\ell=1}^\infty = 0^\infty$, let $j \geq 1$ be the least integer such that $\varepsilon_j > 0$. By Proposition 4.1, it follows from $\beta_c < \beta < 1/N$ that

$$(N-1)^\infty = (\delta_\ell(1/N))_{\ell=1}^\infty < (\delta_\ell)_{\ell=1}^\infty < (\delta_\ell(\beta_c))_{\ell=1}^\infty = (\lambda_\ell)_{\ell=1}^\infty,$$

which together with (12) imply $\varepsilon_j \in \{1, \ldots, N\}$. Moreover, we obtain from this with (12) that

$$\varepsilon_{j+1}\varepsilon_{j+2}\cdots \in \prod_1^\infty \{N-2, N-1, N\}.$$

Case I. $(w_0\overline{w_0})^\infty < (\delta_\ell)_{\ell=1}^\infty < (\lambda_\ell)_{\ell=1}^\infty$.

It then follows from (P8) that there exists an integer $n^* \geq 0$ such that $(w_{n^*}\overline{w_{n^*}})^\infty < (\delta_\ell)_{\ell=1}^\infty \leq (w_{n^*+1}\overline{w_{n^*+1}})^\infty$.

(Ia) $\varepsilon_j \in \{1, \ldots, N-1\}$. One case is that $\varepsilon_{j+1}\varepsilon_{j+2}\cdots = (N-1)^\infty$, otherwise, let first $s \geq j$ be the least integer such that $\varepsilon_{s+1} \in \{N, N-2\} = \{w_0, \overline{w_0}\}$, and then let $p = p(s) \geq 0$ be the largest integer such that $\varepsilon_{s+1}\ldots\varepsilon_{s+2^p} = w_p$ or $\overline{w_p}$. Note that when $s > j$, then $0 < \varepsilon_s = N-1 < 2N-2$ or when $s = j$, then $0 < 1 \leq \varepsilon_s \leq N-1 < 2N-2$. Thus substituting $k = s$ and $n = p$ in Lemma 4.11 we obtain $\varepsilon_{s+1}\ldots\varepsilon_{s+2^{p+1}} \in \{w_p\overline{w_p}, \overline{w_p}w_p, w_{p+1}, \overline{w_{p+1}}\}$.

If $\varepsilon_{s+1}\dots\varepsilon_{s+2^{p+1}}=w_{p+1}$ or $\overline{w_{p+1}}$, substituting $k=s$ and $n=p+1$ in Lemma 4.11, we can determine the next 2^{p+1} terms as above. Otherwise, using that $\varepsilon_{s+2^p}=\lambda_{2^p}$ or $\overline{\lambda_{2^p}}$, and then substituting $k=s+2^p$ and $n=p$ in Lemma 4.11 we can determine the next 2^p terms. This procedure can be continued.

Note that $\varepsilon_{s+1}\varepsilon_{s+2}\dots$ can not have block w_{n^*+1}, otherwise, it follows from (P8) that for some $\ell\geq s$, either

$$\varepsilon_{\ell+1}\varepsilon_{\ell+2}\cdots\geq(w_{n^*+1}\overline{w_{n^*+1}})^\infty\geq(\delta_\ell)_{\ell=1}^\infty$$

with $\varepsilon_\ell<N\leq 2N-2$, or

$$\overline{\varepsilon_{\ell+1}\varepsilon_{\ell+2}\dots}\geq(w_{n^*+1}\overline{w_{n^*+1}})^\infty\geq(\delta_\ell)_{\ell=1}^\infty$$

with $\varepsilon_\ell>N-2\geq 0$. This is in contradiction with (12).

Therefore, $(\varepsilon_\ell)_{\ell=1}^\infty$ must be eventually periodic either with period block $N-1$ or with period block $w_n\overline{w_n}$ for some $n\leq n^*$.

(Ib) $\varepsilon_j=N$. Let $s=j-1$ in (Ia) and then the result follows by the same argument.

Case II. $(N-1)^\infty<(\delta_\ell)_{\ell=1}^\infty\leq(w_0\overline{w_0})^\infty$.

We conclude in this case that $\varepsilon_{j+1}\varepsilon_{j+2}\cdots=(N-1)^\infty$. Otherwise, there exists a $s\geq j$ such that $\varepsilon_{s+1}=w_0$ or $\overline{w_0}$. Thus by the same argument as in Case I, we obtain for some integer $\ell\geq s$ that either $\varepsilon_{\ell+1}\varepsilon_{\ell+2}\cdots\geq(w_0\overline{w_0})^\infty\geq(\delta_\ell)_{\ell=1}^\infty$ with $\varepsilon_\ell<2N-2$, or $\overline{\varepsilon_{\ell+1}\varepsilon_{\ell+2}\dots}\geq(w_0\overline{w_0})^\infty\geq(\delta_\ell)_{\ell=1}^\infty$ with $\varepsilon_\ell>0$, leading to a contradiction with (12). □

Lemma 4.12 yields Part (3) of Theorem 4.6 directly. Let $\mathcal{G}$ be the set of sequences in Ω_{2N-1}^∞ which are eventually periodic with period block $N-1$ or $w_n\overline{w_n}$ for some integer $n\geq 0$. Then the set $\mathcal{G}$ is countable. When $\beta=\beta_c$, it follows from Lemma 4.11 and the proof of Lemma 4.12 that $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta_c,2N-1}\setminus\{0,2\})\setminus\mathcal{G}$ is included in the set of sequences of the form

$$\tau(w_0\overline{w_0})^{k_0}(w_0\overline{w_{i_1'}})^{k_0'}(w_{i_1}\overline{w_{i_1}})^{k_1}(w_{i_1}\overline{w_{i_2'}})^{k_1'}\dots(w_{i_n}\overline{w_{i_n}})^{k_n}(w_{i_n}\overline{w_{i_{n+1}'}})^{k_n'}\dots,$$

where $\tau\in\bigcup_{k=0}^\infty\Omega_{2N-1}^k,\ k_n\in\mathbb{N}\cup\{0\},\ k_n'\in\{0,1\}$ and $0<i_1'\leq i_1<i_2'\leq i_2<\dots\leq i_n<i_{n+1}'\leq i_{n+1}<\dots$, together with their reflections. Thus, since the length of the block w_n is growing exponentially, $\dim_H\mathcal{U}_{\beta_c,2N-1}=0$ (cf. [6, 7]). Note that $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta_c,2N-1})$ contains the set of sequences of the form

$$(w_0\overline{w_0})^{k_0}\dots(w_n\overline{w_n})^{k_n}\dots,\quad k_n\in\mathbb{N},$$

and the fact that $w_n\overline{w_n}$ can not be written as concatenation of two or more blocks of the form $w_\ell\overline{w_\ell}$ with $\ell<n$. Therefore, $|\mathcal{U}_{\beta_c,2N-1}|=2^{\aleph_0}$ which yields Part (2), and so finishes the proof of Theorem 4.6.

5. The critical point for $\mathcal{S}_{\beta,\pm N}$

In this section we show that there exist infinitely many $t\in\mathcal{S}_{\beta,\pm N}$, i.e., there exist infinitely many $t\in[-1,1]$ having a unique Ω_{2N-1}-code and making the intersection $\Gamma_{\beta,N}\cap(\Gamma_{\beta,N}+t)$ a self-similar set. Moreover, we find the critical point α_c for $\mathcal{S}_{\beta,\pm N}$, i.e., the set $\mathcal{S}_{\beta,\pm N}$ has positive Hausdorff dimension if $\beta\in(1/(2N-1),\alpha_c)$, and contains countably infinite many elements if $\beta \in [\alpha_c, 1/N)$. We are able to prove that α_c is strictly smaller than β_c, the critical point of $\mathcal{U}_{\beta,\pm N}$ which is the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code.

In order to using techniques from beta-expansions, we consider the set $\mathcal{S}_{\beta,2N-1} = \mathcal{S}_{\beta,\pm N} + 1$. Thus it follows from Theorem 3.2 that for $\beta \in (1/(2N-1), 1/N)$,

$$\mathcal{S}_{\beta,2N-1} = \left\{\pi_{2N-1}((\varepsilon_\ell)_{\ell=1}^\infty) \in \mathcal{U}_{\beta,2N-1} : (N-1-|\varepsilon_\ell - N + 1|)_{\ell=1}^\infty \text{ is strongly periodic}\right\}.$$

Let Ψ be a map from Ω_{2N-1} to Ω_N defined by

$$\Psi(\varepsilon) = N - 1 - |\varepsilon - N + 1|,$$

then Ψ induces a map on blocks (for $\xi = \xi_1 \dots \xi_k \in \Omega_{2N-1}^k$ we let $\Psi(\xi) = \Psi(\xi_1)\dots\Psi(\xi_k)$), and a map $\Psi_\infty : \Omega_{2N-1}^\infty \to \Omega_N^\infty$ given by $\Psi_\infty((\varepsilon_\ell)_{\ell=1}^\infty) = (\Psi(\varepsilon_\ell))_{\ell=1}^\infty$. Then $\mathcal{S}_{\beta,2N-1}$ can be rewritten as

$$\mathcal{S}_{\beta,2N-1} = \mathcal{U}_{\beta,2N-1} \cap \pi_{2N-1}\Big(\bigcup_{\mathbf{c}} \Psi_\infty^{-1}(\mathbf{c})\Big), \tag{14}$$

where the union is taken over all strongly periodic sequences $\mathbf{c} = (c_\ell)_{\ell=1}^\infty \in \Omega_N^\infty$.

Theorem 5.1. *Given $N \geq 2$ and $\beta \in (1/(2N-1), 1/N)$, let $\Gamma_{\beta,N}$ be the N-part homogeneous Cantor set, and $\mathcal{S}_{\beta,\pm N}$ be the set of $t \in [-1,1]$ having a unique $\Omega_{\pm N}$-code and making the intersection $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ a self-similar set. Denote $\alpha_c := [N+1-\sqrt{(N-1)(N+3)}\,]/2$. Then*

(1) If $\beta \in (1/(2N-1), \alpha_c)$, $\dim_H \mathcal{S}_{\beta,\pm N} > 0$;
(2) If $\beta \in [\alpha_c, 1/N)$, $|\mathcal{S}_{\beta,\pm N}| = \aleph_0$.

Since $\mathcal{S}_{\beta,2N-1} = \mathcal{S}_{\beta,\pm N} + 1$, we only need to consider the corresponding conclusions for $\mathcal{S}_{\beta,2N-1}$. A simple computation yields that α_c satisfies the equation

$$1 = N\alpha_c + \sum_{j=2}^{\infty} (N-1)\alpha_c^j.$$

Then it follows by Proposition 4.2 that $(\delta_\ell(\alpha_c))_{\ell=1}^\infty = N(N-1)^\infty = \lambda_1\lambda_2^\infty$ is the quasi-greedy α_c-expansion of 1. It follows from Proposition 4.1 and

$$(\delta_\ell(\alpha_c))_{\ell=1}^\infty = \lambda_1\lambda_2^\infty > (\lambda_\ell)_{\ell=1}^\infty = (\delta_\ell(\beta_c))_{\ell=1}^\infty$$

that $\alpha_c < \beta_c$. The proof of Theorem 5.1 will be divided into several lemmas.

Lemma 5.2. *Given $N \geq 2$ and $n \in \mathbb{N}$, let α_n be defined by $(\delta_\ell(\alpha_n))_{\ell=1}^\infty = (N(N-1)^{n-1})^\infty$. If $\beta < \alpha_n$, then $\dim_H \mathcal{S}_{\beta,2N-1} > 0$.*

Proof. Let $v_n = N(N-1)^{n-1}$ and $\overline{v_n} = (N-2)(N-1)^{n-1}$ be its reflection. It follows from $\beta < \alpha_n$ and Proposition 4.1 that $(\delta_\ell(\beta))_{\ell=1}^\infty > (\delta_\ell(\alpha_n))_{\ell=1}^\infty = (N(N-1)^{n-1})^\infty$, which implies that for any $k \in \{0, 1, \dots, n-1\}$

$$\sigma^k(v_n v_n) \leq \delta_1(\alpha_n)\dots\delta_{2n-k}(\alpha_n), \quad \sigma^k(\overline{v_n} v_n) < \delta_1(\alpha_n)\dots\delta_{2n-k}(\alpha_n),$$
$$\sigma^k(v_n \overline{v_n}) < \delta_1(\alpha_n)\dots\delta_{2n-k}(\alpha_n), \quad \sigma^k(\overline{v_n v_n}) < \delta_1(\alpha_n)\dots\delta_{2n-k}(\alpha_n).$$

Thus by Theorem 4.5 we obtain that

$$\prod_1^\infty \{v_n, \overline{v_n}\} \subseteq \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}).$$

Since $\Psi(v_n) = (N-2)(N-1)^{n-1} = \Psi(\overline{v_n})$, it is easy to see that

$$\prod_1^\infty \{v_n, \overline{v_n}\} \subseteq \Psi_\infty^{-1}\big(((N-2)(N-1)^{n-1})^\infty\big).$$

Thus noting that $((N-2)(N-1)^{n-1})^\infty$ is obviously a strongly periodic sequence in Ω_N^∞, it follows from (14) that

$$\prod_1^\infty \{v_n, \overline{v_n}\} \subseteq \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}\big(((N-2)(N-1)^{n-1})^\infty\big) \subseteq \pi_{2N-1}^{-1}(\mathcal{S}_{\beta,2N-1})$$

which implies $\dim_H \mathcal{S}_{\beta,2N-1} \geq \dim_H \pi_{2N-1}(\prod_1^\infty \{v_n, \overline{v_n}\}) > 0$. □

Since $(\delta_\ell(\alpha_n))_{\ell=1}^\infty = (N(N-1)^{n-1})^\infty$ decreases to $N(N-1)^\infty = (\delta_\ell(\alpha_c))_{\ell=1}^\infty$ in the sense of lexicographical order as $n \to \infty$, we obtain from Proposition 4.1 that α_n increases to α_c. Thus for each $\beta < \alpha_c$, there exists some $n \in \mathbb{N}$ such that $\beta < \alpha_n$ and then $\dim_H \mathcal{S}_{\beta,2N-1} > 0$ by Lemma 5.2. This finishes the proof of Part (1) of Theorem 5.1.

In the following we will show Part (2). For $\beta \in [\alpha_c, 1/N)$, it follows by Proposition 4.1 that $(\delta_\ell(\beta))_{\ell=1}^\infty \leq (\delta_\ell(\alpha_c))_{\ell=1}^\infty = N(N-1)^\infty$, which together with Theorem 4.5 imply the following property:

(P9) For $N \geq 2$ and $\beta \in [\alpha_c, 1/N)$, any block in $\mathcal{F}$ is forbidden in $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$ where

$$\mathcal{F} = \bigcup_{k=0}^\infty \bigcup_{\tau=N-2}^{N-1} \{\tau N(N-1)^k N,\ \overline{\tau}(N-2)(N-1)^k(N-2)\}.$$

For a positive integer n, let $\mathcal{B}_n$ be the set of blocks of length n occurring in elements of $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$, i.e.,

$$\mathcal{B}_n := \big\{\varepsilon_{i+1}\varepsilon_{i+2}\dots\varepsilon_{i+n} :\ i \geq 0,\ (\varepsilon_\ell)_{\ell=1}^\infty \in \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})\big\}.$$

Lemma 5.3. *Given $N \geq 2$ and $\beta \in [\alpha_c, 1/N)$, let $\mathbf{b} = b_1 \dots b_p \in \{N-2, N-1\}^p$ with $b_1 = N-1$ for some $p \in \mathbb{N}$. Then $\Psi^{-1}(\mathbf{b}) \cap \mathcal{B}_p = \{(N-1)^p\}$ or $\{\xi, \overline{\xi}\}$ for some $\xi \in \{N-2, N-1, N\}^p$.*

Proof. Let $\xi = \xi_1 \dots \xi_p \in \Psi^{-1}(\mathbf{b}) \cap \mathcal{B}_p$. Then it follows from $\mathbf{b} \in \{N-2, N-1\}^p$ and the definition of Ψ that $\xi \in \{N-2, N-1, N\}^p$. Note that $\Psi^{-1}(N-1) = \{N-1\}$ and $\Psi^{-1}(N-2) = \{N-2, N\}$.

(I) $\mathbf{b} = (N-1)^p$. Then $\Psi^{-1}(\mathbf{b}) \cap \mathcal{B}_p = \{(N-1)^p\}$.

(II) $\mathbf{b} \neq (N-1)^p$. Let $b_{k_1} = b_{k_2} = \cdots = b_{k_s} = N-2$ for $1 < k_1 < k_2 < \cdots < k_s \leq p$, and $b_k = N-1$ for $k \neq k_i$. Then also $\xi_k = N-1$ for $k \neq k_i$. Moreover, if $\xi_{k_1} = N$, then it follows from (P9) that $\xi_{k_2} = N-2,\ \xi_{k_3} = N,\ \xi_{k_4} = N-2$ and so on. Similarly, if $\xi_{k_1} = N-2$ we will obtain by (P9) that $\xi_{k_2} = N,\ \xi_{k_3} = N-2,\ \xi_{k_4} = N$ and so on. Thus, $\Psi^{-1}(\mathbf{b}) \cap \mathcal{B}_p = \{\xi, \overline{\xi}\}$. □

Lemma 5.4. *Given $N \geq 2$ and $\beta \in [\alpha_c, 1/N)$, let $\boldsymbol{c} = (c_\ell)_{\ell=1}^\infty \in \Omega_N^\infty$ be a strongly periodic sequence. Then $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\boldsymbol{c})$ is at most countable.*

Proof. Note by $\beta \geq \alpha_c$ that $(\delta_\ell(\beta))_{\ell=1}^\infty \leq (\delta_\ell(\alpha_c))_{\ell=1}^\infty = N(N-1)^\infty$. Thus for any sequence $(\varepsilon_\ell)_{\ell=1}^\infty \in \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$, we obtain by the same argument as in Lemma 4.12 that

$$\varepsilon_k \varepsilon_{k+1} \cdots \in \prod_1^\infty \{N-2, N-1, N\}$$

for some large $k \in \mathbb{N}$, which implies that $\Psi_\infty(\varepsilon_k\varepsilon_{k+1}\ldots) \in \{N-2, N-1\}^\infty$. Let $\mathbf{c} = a_1 \ldots a_q (b_1 \ldots b_q)^\infty$ with $a_\ell \leq b_\ell,\ 1 \leq \ell \leq q$ be a strongly periodic sequence in Ω_N^∞ such that $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\mathbf{c}) \neq \emptyset$. Then

$$b_1 \ldots b_q \in \{N-2, N-1\}^q.$$

Case I. $b_1 \ldots b_q = (N-2)^q$. It follows from $\Psi^{-1}(N-2) = \{N-2, N\}$ that $\Psi_\infty^{-1}(\mathbf{c}) \subseteq \Psi^{-1}(a_1 \ldots a_q)\{N-2, N\}^\infty$. Note by (P9) (with $\tau = N-2, k = 0$) that blocks $N(N-2)^2$ and $(N-2)N^2$ are forbidden in $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$. Thus

$$\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\mathbf{c}) \subseteq \Psi^{-1}(a_1 \ldots a_q)\{N^\infty, (N(N-2))^\infty, ((N-2)N)^\infty, (N-2)^\infty\}$$

which is at most countable.

Case II. $b_1 \ldots b_q \neq (N-2)^q$. Then there exists $b_k = N-1$ for some $k \in \{1, \ldots, q\}$. Note that

$$\mathbf{c} = a_1 \ldots a_q (b_1 \ldots b_q)^\infty = a_1 \ldots a_q b_1 \ldots b_{k-1} (b_k \ldots b_q b_1 \ldots b_{k-1})^\infty.$$

It follows from Lemma 5.3 that there exists a q-block $\xi = \xi_1 \ldots \xi_q \in \{N-2, N-1, N\}^q$ such that $\Psi^{-1}(b_k \ldots b_q b_1 \ldots b_{k-1}) \cap \mathcal{B}_q = \{\xi, \overline{\xi}\}$. Thus

$$\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\mathbf{c}) \subseteq \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Big(\Psi^{-1}(a_1 \ldots a_q b_1 \ldots b_{k-1}) \prod_1^\infty \{\xi, \overline{\xi}\}\Big).$$

Note that since $\Psi^{-1}(\mathbf{c})$ and $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1})$ are all reflection invariant, $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\mathbf{c})$ is also reflection invariant. Thus we only need to consider the following three cases.

(IIa) $\xi = (N-1)^q$. Then $\prod_1^\infty \{\xi, \overline{\xi}\}$ collapses to a single point $(N-1)^\infty$.

(IIb) $\xi = (N-1)^\ell N \xi_{\ell+2} \ldots \xi_{q-r-1} N (N-1)^r$ with $\ell \geq 1, r \geq 0$ and $\ell + r \leq q-1$ (note that $\xi = (N-1)^\ell N (N-1)^r$ if $\ell + r = q-1$). It follows by (P9) that blocks $\xi\xi$ and $\overline{\xi\xi}$ are forbidden in $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\mathbf{c})$. Thus $\prod_1^\infty \{\xi, \overline{\xi}\}$ collapses to two points $(\xi\overline{\xi})^\infty$ and $(\overline{\xi}\xi)^\infty$.

(IIc) $\xi = (N-1)^\ell N \xi_{\ell+2} \ldots \xi_{q-r-1} (N-2)(N-1)^r$ with $\ell \geq 1, r \geq 0$ and $\ell + r \leq q-2$. By the same argument as in (IIb) we also obtain that $\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi^{-1}(\mathbf{c})$ is at most countable. □

It follows from Lemma 5.4 and (14) that for $\beta \in [\alpha_c, 1/N)$, the set

$$\pi_{2N-1}^{-1}(\mathcal{S}_{\beta,2N-1}) = \pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \bigcup_{\mathbf{c}} \Psi_\infty^{-1}(\mathbf{c}) = \bigcup_{\mathbf{c}} \big(\pi_{2N-1}^{-1}(\mathcal{U}_{\beta,2N-1}) \cap \Psi_\infty^{-1}(\mathbf{c})\big)$$

is at most countable since the union on the right is countable. Note that for $\beta \in [\alpha_c, 1/N)$, $\{0^q(N-1)^\infty : q \in \mathbb{N}\} \subseteq \pi_{2N-1}^{-1}(\mathcal{S}_{\beta,2N-1})$. This gives Part (2), finishing the proof of Theorem 5.1.

6. Final remarks

In this paper we determined the size of two types of sets $\mathcal{U}_{\beta,\pm N}$, and $\mathcal{S}_{\beta,\pm N}$, where $\mathcal{U}_{\beta,\pm N}$ is the set of $t \in \Gamma_{\beta,N} - \Gamma_{\beta,N}$ having a unique $\Omega_{\pm N}$-code and $\mathcal{S}_{\beta,\pm N}$ is the set of t not only having a unique code but also making the intersection $\Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t)$ a self-similar set. It follows from [19] that for $\beta \in (1/(2N-1), 1/N)$ there also exist a lot of $t \in \Gamma_{\beta,N} - \Gamma_{\beta,N} = [-1,1]$ having exactly p different $\Omega_{\pm N}$-codes for any integer $p \geq 2$. Let

$$\mathcal{F}_{\beta,\pm N}^{(p)} := \{t \in \Gamma_{\beta,N} - \Gamma_{\beta,N} : t \text{ has exactly } p \text{ different } \Omega_{\pm N}\text{-codes}\},$$

and

$$\mathcal{S}_{\beta,\pm N}^{(p)} := \{t \in \mathcal{F}_{\beta,\pm N}^{(p)} : \Gamma_{\beta,N} \cap (\Gamma_{\beta,N} + t) \ \text{ is a self-similar set}\}.$$

Problem. How large is the set $\mathcal{F}_{\beta,\pm N}^{(p)}$ for a given positive integer $p \geq 2$? How to characterize this set? This is also an open problem for beta-expansions. Moreover, how large is the set $\mathcal{S}_{\beta,\pm N}^{(p)}$?

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