Geometric ergodicity for classes of homogeneous Markov chains *

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Abstract

The paper deals with non asymptotic computable bounds for the geometric convergence rate of homogeneous ergodic Markov processes. Some sufficient conditions are stated for simultaneous geometric ergodicity of Markov chain classes. This property is applied to nonparametric estimating in ergodic diffusion processes.

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1 Introduction

In this paper we study a class of homogeneous Markov chains $(\Phi)_{n\geq 0}$ with values in some measurable space $(\mathcal{X}\,,\,\mathcal{B}(\mathcal{X}))$ defined by a parametrized family of transition probabilities

$$(\mathbf{P}^{\vartheta}_{x})_{x\in\mathcal{X}\,,\,\vartheta\in\Theta}\,, \tag{1.1}$$

where Θ is a parametric set for this family. For each $\vartheta\in\Theta$ the sequence $\Phi=(\Phi_n)_{n\geq 0}$ is a homogeneous Markov chain defined on the measurable space $(\mathcal{X}\,,\,\mathcal{B}(\mathcal{X}))$ with a transition probability $\mathbf{P}^{\vartheta}$, i.e.

$$\mathbf{P}(\Phi_1\in\Gamma|\Phi_0=x)=\mathbf{P}^{\vartheta}_{x}(\Gamma)\,.$$

Our main goal is to state geometric ergodicity for this class simultaneously over all values of the parameter $\vartheta\in\Theta$.

Geometric ergodicity is studied in a number of papers (see, for example, [1], [16]-[19]). We remind that a chain $(\Phi_n)_{n\geq 0}$ on the space $(\mathcal{X},\mathcal{B}(\mathcal{X}))$ with a invariant measure π is called *geometrically ergodic* if there exist a $\mathcal{X}\to[1,\infty[$ function $V(x)$ and some constants $R>0$, $\kappa>0$ such that, for any $n\geq 1$,

$$\sup_{x\in\mathcal{X}}\sup_{0\leq g\leq V}\frac{1}{V(x)}|\mathbf{E}_x\,g(\Phi_n)-\pi(g)|\leq R\,e^{-\kappa n}\,. \tag{1.2}$$

As we shall see later (see, Definition 6.1 below) the function V, providing the drift condition, is given by the Lyapunov functions (see, e.g. [13] in the case of diffusion processes and [7], [11], [12] for Markov chains). For this reason, in the sequel, we shall call such functions by the *Lyapunov functions*.

The property (1.2) is useful in applied problems related to the identification of stochastic systems, described by stochastic processes with dependent values, in particular, governed by stochastic difference or stochastic differential equations. Necessity of simultaneous geometric ergodicity appears in statistics, when one studies a minimax risk with respect to some family of distributions related to a statistical experiment. In particular, in this paper we shall apply simultaneous geometric ergodicity to nonparametric estimating the drift coefficient in the stochastic differential equation (see [2]):

$$\mathrm{d}y_t=S(y_t)\,\mathrm{d}t+\sigma(y_t)\mathrm{d}W_t\,,\quad 0\leq t\leq T\,, \tag{1.3}$$

where S and σ are unknown functions and S has to be estimated from observations $(y_t)_{0\leq t\leq T}$. In studying minimax risks for kernel estimators we need to use geometric ergodicity for the process (1.3) simultaneously over all coefficients S and σ from some functional class (see (2.7) below). In this case, $\vartheta=(S,\sigma)$ is the class parameter.

It is clear that to apply the property (1.2) to some distribution family we have to find some explicit expressions for the parameters $R>0$ and $\kappa>$

0. It is a well-known problem in the Markov Chain Monte Carlo (MCMC) theory, when a stopping rule for simulations is based on the accuracy of n-step approximations. Therefore, we need to find computable bounds in (1.2). Note that some explicit expressions for R and κ were calculated in [1], [17], [21] and [22] for ψ-irreducible homogeneous Markov chains. These results are not applicable in our case because it is not clear what does it mean ψ-irreducibility for a class of parametrized Markov chains. Note that in [1] and [17] the parameters R and κ were obtained by making use of the Kendall theorem. In [22] these parameters are obtained through the direct coupling method for the Markov chain. To this end the authors impose in [21]–[22] some additional assumptions which are not satisfied in the case of the diffusion process (1.3) (see Remark 3.1 in Section 3). Moreover, it should be noted that the upper bound in (1.2) given in [21] is calculated under the assumption that the minorization condition holds on the whole state space. This assumption is never true for the model (1.3).

In the paper we apply the coupling method to the renewal process generated by entrance times of the process into the minorization set. Note, that Meyn and Tweedie use the same approach in order to obtain convergence results (see, [16], chapter 13). Their results imply the power convergence rate. To obtain the geometric rate we make use of the Lyapunov functions method for the related coupling process.

In order to explain the novelty of the method introduced in the paper, we give the scheme of proving the property (1.2). The first step consists in passing to a splitting chain, which yields a chain with an accessible atom. Then, one makes use of the *Regenerative Decomposition* for splitting chains in order to evaluate the convergence rate (see [10], [16]). Let us remind that the principal term in this decomposition gives the deviation in the renewal theorem, which may be evaluated thanks to the Kendall renewal theorem that provides a geometric convergence rate. In our case the same convergence rate is obtained thanks to making use of the Lyapunov functions method for the coupling renewal process (see Theorem 4.1 in Section 4). This upper bound enables us to find the explicit non asymptotic exponential upper bound in the ergodic theorem for which we can find the supremum over the transition probability family in (1.1).

In this paper we find some sufficient conditions which provide simultaneous geometric ergodicity of the family (1.1) over all values of the parameter ϑ. We check these conditions for the diffusion model (1.3). As corollary, we obtain explicit upper bounds for geometric convergence rate in the ergodic theorem for diffusion processes. These bounds may be used in the Monte Carlo technique to calculate some functionals of ergodic diffusion processes. In that case one can replace these functionals by the corresponding integrals with respect to the invariant density which has a simple explicit form. The accuracy of this approximation is given by the explicit non asymptotic bounds in the geometric convergence rate for diffusion processes.

The paper is organized as follows. In the next section the main results are stated. Section 3 provides the explicit formulas for parameters in the geometric convergence rate. Section 4 is devoted to related coupling renewal processes. In section 5 geometric ergodicity is proved for a parametrized class of homogeneous Markov chains. In section 6 we apply this property to stochastic differential equations. Some basic results on homogeneous Markov chains are given in the Appendix.

2 Main results

Assume now, that the family of transition probabilities $(\mathbf{P}^{\vartheta})_{\vartheta\in\Theta}$ satisfies the following conditions

$\mathbf{H}_1$) *There exist* $0<\delta<1$, *some set* $C\in\mathcal{B}_(\mathcal{X})$ *and some probability measure* ν *on* $\mathcal{B}(\mathcal{X})$ *with* $\nu(C)=1$ *such that, for any* $A\in\mathcal{B}(\mathcal{X})$ *with* $\nu(A)>0$,

$$\inf_{x\in C}\left(\inf_{\vartheta\in\Theta}\mathbf{P}^{\vartheta}(x,A)-\delta\nu(A)\right)\,>\,0\,. \tag{2.1}$$

For the sequel we denote

$$\eta\,=\,\inf_{x\in C}\left(\inf_{\vartheta\in\Theta}\mathbf{P}^{\vartheta}(x,C)-\delta\right)\,. \tag{2.2}$$

$\mathbf{H}_2$) *There exist* $\mathcal{X}\to[1,\infty)$ *function* V, *some constants* $0<\rho<1$, $D\ge 1$, *and a set* C *from* $\mathcal{B}(\mathcal{X})$ *such that*

$$V^{*}=\sup_{x\in C}V(x)\,<\infty$$

and, for any $x\in\mathcal{X}$,

$$\sup_{\vartheta\in\Theta}\mathbf{E}^{\vartheta}_{x}\left(V(\Phi_1)\right)\,\le\,(1-\rho)V(x)\,+\,D\mathbf{1}_C(x)\,. \tag{2.3}$$

Here $\mathbf{E}^{\vartheta}_{x}$ means the expectation with respect to the transition probability $\mathbf{P}^{\vartheta}(x,\cdot)$.

Remark 2.1. *Condition* $\mathbf{H}_2$) *is called the uniform drift condition and that of* $\mathbf{H}_1$) *is the uniform minorization condition.*

Theorem 2.1. *Assume the conditions* $\mathbf{H}_1$)–$\mathbf{H}_2$) *hold true with the same set* $C \in \mathcal{B}(\mathcal{X})$. *Then, for each* $\theta \in \Theta$, *the chain* Φ *admits an invariant distribution* π^ϑ *on* $\mathcal{B}(\mathcal{X})$. *Moreover, for any* $n \geq 2$,

$$\sup_{\vartheta \in \Theta} \sup_{x \in \mathcal{X}} \sup_{0<f\leq V} \frac{1}{V(x)} \left| \mathbf{E}_x^\vartheta f(\Phi_n) - \int_{\mathcal{X}} f(z)\, \pi^\vartheta(\mathrm{d}z) \right| \leq R^* e^{-\kappa^* n}, \tag{2.4}$$

where the parameters $R^* = R^*(\rho, \delta, D, \eta, V^*)$ *and* $\kappa^* = \kappa^*(\rho, \delta, D, \eta_1, V^*)$ *are given in* (3.5).

Apply now to the process (1.3). To this end we have to introduce some functional class of functions $\vartheta = (S, \sigma)$. First, for some $\mathbf{x}_* \geq 1$, $M > 0$ and $L > \beta > 0$, we denote by $\mathcal{V}_1$ the class of functions S from $\mathbf{C}^1(\mathbb{R})$ such that

$$\sup_{|x| \leq \mathbf{x}_*} \left(|S(x)| + |\dot{S}(x)| \right) \leq M$$

and

$$-L \leq \inf_{|x| \geq \mathbf{x}_*} \dot{S}(x) \leq \sup_{|x| \geq \mathbf{x}_*} \dot{S}(x) \leq -\beta\,.$$

Second, for some fixed reals $\sigma_{min} > 0$ and $\sigma_{max} > \sigma_{min}$, we denote by $\mathcal{V}_2$ the class of functions σ from $\mathbf{C}^2(\mathbb{R})$ such that, for all $x \in \mathbb{R}$,

$$\sigma_{min} \leq \min\left(|\sigma(x)|\,,\, |\dot{\sigma}(x)|\,,\, |\ddot{\sigma}(x)|\right) \leq \max\left(|\sigma(x)|\,,\, |\dot{\sigma}(x)|\,,\, |\ddot{\sigma}(x)|\right) \leq \sigma_{max}\,.$$

Finally, we set

$$\Theta = \mathcal{V}_1 \times \mathcal{V}_2\,. \tag{2.5}$$

Note that (see, for example, [6]), for any function ϑ from Θ, the equation (1.3) admits a unique strong solution, which is an ergodic process having an invariant measure π_ϑ with the invariant density q_ϑ defined as

$$q_\vartheta(x) = \frac{\sigma^{-2}(x) \exp\{\int_0^x S_1(v)\mathrm{d}v\}}{\int_{-\infty}^{+\infty} \sigma^{-2}(z) \exp\{\int_0^z S_1(v)\mathrm{d}v\}\mathrm{d}z}\,, \tag{2.6}$$

where $S_1(v) = 2S(v)/\sigma^2(v)$.

Theorem 2.2. *For any* $0 < \epsilon \leq 1/2$ *and* $t > 0$,

$$\sup_{\vartheta \in \Theta} \sup_{x \in \mathbb{R}} \sup_{0<g\leq 1} \frac{\left| \mathbf{E}_x^\vartheta\, g(y_t) - \int_{\mathbb{R}} g(x) q_\vartheta(x)\mathrm{d}x \right|}{(1+x^2)^\epsilon} \leq R_\epsilon\, e^{-\kappa_\epsilon t}\,, \tag{2.7}$$

where the parameters $R_\epsilon > 0$ *and* $\kappa_\epsilon > 0$ *are given in* (3.13).

Remark 2.2. *Note that the property* (2.7) *is called simultaneous geometric ergodicity. As is shown in Section 6, the function* $(1+x^2)^\epsilon$ *is a Lyapunov function. It should be said that in [20], for the process* (1.3) *with a constant diffusion coefficient (i.e. for* $\sigma = 1$*), an exponential bound for deviation* (2.7) *was obtained. The proof of that result was based on the coupling method applied directly to the diffusion process* (1.3) *provided the existence of a Lyapunov function. In contrast with [20], in our paper an explicit family of Lyapunov functions is given that is of help in applications.*

Remark 2.3. *It should be noted that the inequality* (2.7) *may be applied to Monte Carlo calculation of the expectation* $\mathbf{E}_x^\vartheta\, g(y_t)$*. Indeed, the previous expectation can be replaced with the integral of* g *with respect to the invariant density* (2.6)*. The precision of such approximation is given in* (2.7)*.*

3 Computable bounds for geometric convergence rate

In this section we introduce the parameters R^* and κ^* which make explicit the upper bound for geometric ergodicity in (2.4). For any $0 < \gamma < 1$, we denote

$$B^* = \check{U}^* \left(1 + \frac{\check{U}^* V^*}{1-(1-\delta\eta_1)^\gamma}\right)\,, \tag{3.1}$$

where $\eta_1 = \eta/(1-\delta)$ and

$$\check{U}^* = \max\left(\frac{1-\rho+D}{(1-\delta)(1-\rho)^{1-\gamma}\,(1-(1-\rho)^\gamma)}\,,\,\frac{V^*}{(1-\rho)^{1-\gamma}}\right)\,.$$

We remind that the parameter η appears in the condition $\mathbf{H}_1$). Moreover, we put

$$\begin{cases} r_* &= \dfrac{(1-\gamma)^2|\ln(1-\rho)|\,|\ln(1-\delta\eta_1)|}{\ln(\check{U}^* V^*) + |\ln(1-\delta\eta_1)|}\,; \\ \widetilde{l} &= 2 + \left[\dfrac{\ln(V^* B^*)}{r_*} + \dfrac{\ln \widetilde{q}(1-e^{-r_*})^{-1}}{2r_*}\right]\,, \end{cases} \tag{3.2}$$

where

$$\widetilde{q} = \frac{1-\check{B}_1^*}{2} \quad \text{and} \quad \check{B}_1^* = \min\left(e^{-r_*}\,,\,\frac{\delta}{e^{r_*}-1+\delta\eta_1}\right)\,.$$

Here $[a]$ means the integer part of a reel number $a > 0$. Next,

$$\widetilde{A} = \frac{V^*B^*+1}{(1-e^{-r_*})(1-e^{-\gamma r_*})} \quad \text{and} \quad \widetilde{\varrho}_1 = \frac{(1-\gamma)^2\widetilde{r}\,|\ln(1-\widetilde{\epsilon})|}{\ln\widetilde{A} + r_*\widetilde{l} + |\ln(1-\widetilde{\epsilon})|}\,, \tag{3.3}$$

where $\widetilde{r} = |\ln(1-\widetilde{q})|$ and $\widetilde{\epsilon} = \delta\eta_1(1-\delta)^{\widetilde{l}-2}$. Now we introduce the following parameter

$$\widetilde{A}_3 = \frac{1-\gamma}{\gamma}\left(3 + 2\frac{e^{r_*}\,\widetilde{A}\left(1+\widetilde{A}e^{r_*\widetilde{l}}\right)}{(1-(1-\widetilde{\epsilon})^{\gamma})\,(e^{r_*}-1)}\right) V^* B^* + e^{\widetilde{\kappa}} \tag{3.4}$$

and

$$\widetilde{\kappa} = \frac{(1-\gamma)\widetilde{\varrho}_1 r_*}{\widetilde{\varrho}_1 + \ln V^* B^*}\,.$$

We define the parameters R^* and κ^* in the Theorem 2.1 as

$$\begin{cases} \kappa^* = \kappa^*(\rho,\delta,D,V^*) & = \dfrac{(1-\gamma)\widetilde{\varrho}_1 r_*}{2(\widetilde{\varrho}_1 + \ln V^* B^*)}\,; \\[2ex] R^* = R^*(\rho,\delta,D,V^*) & = 2\dfrac{(\widetilde{A}_2+1)e^{\kappa^*}+1}{e^{\kappa^*}-1}\, V^*\,(B^*)^2\,. \end{cases} \tag{3.5}$$

Further we define the upper bound in the ergodic Theorem 2.2. First of all, we take the set C in the minorization condition $\mathbf{H}_1$) as the interval $C = [-K, K]$ for some $K > 0$ and we chose the measure ν as the uniform distribution on C, i.e. for any measurable set A from $\mathcal{B}(\mathbb{R})$,

$$\nu(A) = \frac{1}{2K}\,\mathbf{mes}(A \cap [-K\,,\,K])\,, \tag{3.6}$$

where $\mathbf{mes}(\cdot)$ is the Lebesgue measure on $\mathbb{R}$.
In order to define the threshold $\delta > 0$ we need the quadratic function

$$\Omega_*(z) = \omega_1^*\,z^2 + \omega_2^* z + \omega_3^* \tag{3.7}$$

with $\omega_1^* = 4(L\sigma_*)^2 + L\sigma_* + 1$, $\omega_2^* = L\sigma_*\sigma_{max} + H_0^*$ and $\omega_3^* = H_1^* + 2(H_0^*)^2$, where

$$H_0^* = \frac{M+\sigma_*}{\sigma_{min}}\,,\quad H_1^* = M(1+\sigma_*) + L + 2\sigma_*^2 \quad \text{and} \quad \sigma_* = \frac{\sigma_{max}}{\sigma_{min}}\,.$$

We chose the parameter δ as

$$\delta_K = \frac{3Ke^{-\Omega_*(\widetilde{K})}}{2\sqrt{2\pi}\sigma_{max}}\,, \tag{3.8}$$

where $\widetilde{K} = K/\sigma_{min}$. Now we set

$$K_0 = \sqrt{\mathbf{M}_1} + 8\sigma_{min} + 8\check{\sigma}_* + 4\sqrt{\check{\sigma}_*(\mathbf{M}_1+\mathbf{M}_2) + 16\check{\sigma}_*^2}\,, \tag{3.9}$$

where $\check{\sigma}_* = \sigma^2_{max}/(\beta(1-e^{-\beta}))$,

$$\mathbf{M}_1 = \frac{(M+\beta\mathbf{x}_*)^2 + \sigma^2_{max}\beta}{\beta^2} \quad \text{and} \quad \mathbf{M}_2 = \mathbf{M}_1(1-e^{-\beta})\,.$$

We set

$$\eta_K = 1 - \delta_K - \frac{4\check{\sigma}_*\,(K^2+\mathbf{M}_2)}{(K^2-\mathbf{M}_1)^2}\,. \tag{3.10}$$

Note, that for any $K \ge K_0$

$$\delta_K \le \frac{3\widetilde{K}e^{-\widetilde{K}^2}}{2\sqrt{2\pi}} \le \frac{3}{4\sqrt{e\pi}} \quad \text{and} \quad \eta_K \ge \frac{3(\sqrt{e\pi}-1)}{4\sqrt{e\pi}}\,.$$

Therefore, Propositions A.7–A.8 imply the condition $\mathbf{H}_1$) with the parameters δ_K and η_K defined in (3.8) and (3.10) for any $K \ge K_0$. To check the condition $\mathbf{H}_2$) we set

$$b^*_0 = \beta(2+\mathbf{x}_*)(1+\mathbf{x}_*) + (M+3)(3+\mathbf{x}_*) \quad \text{and} \quad b^*_1 = \frac{b^*_0}{2\beta}\,. \tag{3.11}$$

Propositions 6.3–6.4 imply, that for any $0 < \epsilon \le 1/2$, the diffusion process (1.3) satisfies the drift condition $\mathbf{H}_2$) with $V(x) = (1+x^2)^\epsilon$, $C = [-K_\epsilon, K_\epsilon]$,

$$\rho_\epsilon = (1-\check{\epsilon})\,(1-e^{-2\epsilon\beta}) \quad \text{and} \quad D_\epsilon = V^*_\epsilon e^{-2\epsilon\beta} + b^*_1(1-e^{-2\epsilon\beta})\,, \tag{3.12}$$

where

$$K_\epsilon = K_0 + \sqrt{\left(\frac{b^*_1}{\check{\epsilon}}\right)^{1/\epsilon} - 1} \quad \text{and} \quad V^*_\epsilon = (1+K^2_\epsilon)^\epsilon\,.$$

Now we define

$$\kappa_\epsilon = \kappa^*(\rho_\epsilon, \delta_\epsilon, D_\epsilon, \eta_\epsilon, V^*_\epsilon) \quad \text{and} \quad R_\epsilon = R^*(\rho_\epsilon, \delta_\epsilon, D_\epsilon, \eta_\epsilon, V^*_\epsilon)\,, \tag{3.13}$$

where the functions κ^* and R^* are defined in (3.5), $\delta_\epsilon = \delta_{K_\epsilon}$, $\eta_\epsilon = \eta_{K_\epsilon}$ and $D_\epsilon = D_{K_\epsilon}$.

Remark 3.1. *Note that we can not apply the bounds for geometric convergence rate from the paper [22] (Theorem 12) and [21] (Theorem 8) since there the bounds were obtained under the condition*

$$(1+K^2)^\epsilon > \frac{2D_K}{\rho_\epsilon}\,. \tag{3.14}$$

This condition is not satisfied in our case for sufficiently small values of the parameter $\epsilon > 0$. In fact, we apply the bounds (2.7) in [5] to point-wise estimating the drift coefficient S under observations of the process (1.3) at discrete times $(t_k)_{k\ge1}$. The question of interest is the behavior of kernel estimators that are nonlinear functionals of observations. In order to study the non asymptotic estimating precision one needs of some concentration inequalities [4] based on the bounds (2.7) with the parameter $\epsilon > 0$ no matter how small.

In order to illustrate the behavior of the geometric rate, we suppose that the parameters satisfy the following conditions:

$$\begin{cases} \lim_{\delta\to 0}\eta = 1\,, \qquad \lim_{\delta\to 0}\dfrac{1-\eta}{\delta} = +\infty\,, \\ \lim_{\rho\to 1}\dfrac{D}{V^*} = 0 \qquad \text{and} \quad \lim_{\rho\to 1}\dfrac{\ln V^*}{|\ln(1-\rho)|} = 1\,. \end{cases} \tag{3.15}$$

It should remark that these conditions hold true for the process (1.3) with the parametric set (2.5) as $L > \beta \to \infty$ and $\check{\epsilon} \to 0$ for some fixed $\epsilon > 0$.
It should be noted, that the geometric rate in (1.2) obtained in [17] satisfies

$$\kappa^* = O(\delta^8) \quad \text{as} \quad \delta \to 0\,.$$

In [22] under the condition (3.14) this rate is "best", i.e.

$$\kappa^* = O(\delta) \quad \text{as} \quad \delta \to 0\,.$$

Under the conditions (3.15) the coefficient κ^* defined in (3.5)

$$\kappa^* = (c_1 + o(1))\frac{\delta^{13/2+\mu_0(\gamma)}}{|\ln\delta|^2} \quad \text{as} \quad \delta \to 0\,,\ \rho \to 1\,,$$

where $c_1 > 0$ and $\mu_0(\gamma) \to 0$ as $\gamma \to 0$.

Remark 3.2. *Note that the condition $L, \beta \to \infty$ on the drift function of the process* (1.3) *concerns the behavior of the function only outside of the interval* $[-\mathbf{x}_*\,,\,\mathbf{x}_*]$*, i.e. outside of the informative part of the function S. Remind (see, [3]), that the class* (2.5) *is used to bound the function S on the interval* $[-\mathbf{x}_*\,,\,\mathbf{x}_*]$ *and outside of the interval the conditions are imposed to preserve ergodicity.*

Remark 3.3. *It should remark that, unfortunately, the bounds from the papers [1], [17] are not applicable, in general case, to classes of Markov chains. Indeed, irreducibility is one of conditions providing the geometrical rate in [17]. Therefore, in the case of a parametric Markov chain class, irreducibility measure should depend on a class parameter. It is not clear, what will going on this measure when one takes the supremum over the class parameter and how one needs to change the irreducibility condition in order to obtain uniform bounds over the class parameter $\vartheta \in \Theta$ by using the proof in [17].*

4 Coupling Renewal Method

In this Section we shall obtain a non asymptotic upper bound with explicit constants in the renewal theorem by making use of the coupling method. The notions used here can be found in [8], [11], [14].

Let $(Y_j)_{j\geq 0}$ and $(Y'_j)_{j\geq 0}$ be two independent sequences of random variables taking values in $\mathbb{N}$. Assume that the initial random variables Y_0 and Y'_0 have distributions $\mathbf{a} = (\mathbf{a}(k))_{k\geq 0}$ and $\mathbf{b} = (\mathbf{b}(k))_{k\geq 0}$, respectively, i.e. for any $k \geq 0$,

$$\mathbf{P}(Y_0 = k) = \mathbf{a}(k) \quad \text{and} \quad \mathbf{P}(Y'_0 = k) = \mathbf{b}(k)\,.$$

The sequences $(Y_j)_{j\geq 1}$ and $(Y'_j)_{j\geq 1}$ are supposed to be the i.i.d. sequences with the same distribution $p = (p(k))_{k\geq 0}$, i.e. for any $k \geq 0$,

$$\mathbf{P}(Y_1 = k) = \mathbf{P}(Y'_1 = k) = p(k)\,.$$

We assume also that $p(0) = \mathbf{P}(Y_1 = 0) = 0$, i.e. the sequences $(Y_j)_{j\geq 1}$ and $(Y'_j)_{j\geq 1}$ take values in $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Moreover, we suppose that the distributions a, b and p satisfy the following condition

C) *There exists a real number* $r > 0$ *such that*

$$\ln\left(\max\left(\mathbf{E}\, e^{rY_0}\,,\, \mathbf{E}\, e^{rY_1}\right)\right) \leq \upsilon_*(r) \quad \text{and} \quad \ln\left(\mathbf{E}\, e^{rY'_0}\right) \leq \upsilon'_*(r)\,. \tag{4.1}$$

For any $n \geq 0$, we define the following stopping times

$$t_n = \inf\{k \geq 0\,:\, \sum_{i=0}^{k} Y_i > n\} \quad \text{and} \quad t'_n = \inf\{k \geq 0\,:\, \sum_{i=0}^{k} Y'_i > n\}\,.$$

Further, we set

$$W_n = \sum_{j=0}^{t_n} Y_j - n \quad \text{and} \quad W'_n = \sum_{j=0}^{t'_n} Y'_j - n\,. \tag{4.2}$$

It is easy to see that the sequences $(W_n)_{n\geq 1}$ and $(W'_n)_{n\geq 1}$ are homogeneous Markov chains taking values in $\mathbb{N}^*$ such that, for any n, k and l from $\mathbb{N}^*$,

$$\begin{aligned}\mathbf{P}\left(W_n = k | W_{n-1} = l\right) &= \mathbf{P}\left(W'_n = k | W'_{n-1} = l\right) \\ &= p(k)\mathbf{1}_{\{l=1\}} + \mathbf{1}_{\{k=l-1\}}\mathbf{1}_{\{l\geq 2\}}\,. \end{aligned} \tag{4.3}$$

Firstly we study the entrance times $(s_k)_{k\geq 0}$ of the chain $(W_n)_{n\geq 0}$ to the state $\{1\}$ which are defined as $s_{-1} = 0$ and for $k \geq 0$

$$s_k = \inf\{l \geq s_{k-1} + 1\,:\, W_l = 1\}\,. \tag{4.4}$$

One can check directly that the stopping times $s_k, k \geq 1$, can be represented as

$$s_k = s_0 + \sum_{j=1}^{k} \varsigma_j\,. \tag{4.5}$$

One can check directly that in this case $(\varsigma_j)_{j\ge 1}$ are i.i.d. random variables independent of s_0 and, for any $l\in\mathbb{N}^*$,

$$\mathbf{P}(\varsigma_1=l)=\mathbf{P}(s_0=l|W_0=1)=p(l)\,, \tag{4.6}$$

i.e. the random variables $(\varsigma_j)_{j\ge 1}$ have the same distribution as Y_1. Now we study the properties of the stopping time s_0.

Proposition 4.1. *Assume that the condition* $\mathbf{C}$) *holds. Then*

$$\mathbf{E}\,e^{rs_0}\le 3\,e^{\upsilon_*(r)}\,.$$

Proof. First of all, note that, for $k\ge 2$ and $l\ge 0$,

$$\mathbf{P}(s_0=l|W_0=k)=\mathbf{1}_{\{l=k-1\}}\,.$$

Moreover, taking into account that, for any $k\ge 1$,

$$\mathbf{P}(W_0=k)=\mathbf{a}(0)p(k)+\mathbf{a}(k)\,, \tag{4.7}$$

we obtain that

$$\mathbf{E}\,e^{rs_0}=\mathbf{P}(W_0=1)\,\mathbf{E}\,e^{rY_1}+\sum_{k=2}^{\infty}e^{r(k-1)}\,\mathbf{P}(W_0=k)$$

$$\le\mathbf{E}\,e^{rY_1}+\mathbf{a}(0)e^{-r}\,\mathbf{E}\,e^{rY_1}+e^{-r}\,\mathbf{E}\,e^{rY_0}\le 3\,e^{\upsilon_*(r)}\,.$$

Hence Proposition 4.1. $\square$

Now, we introduce the embedded Markov chain $(Z_k)_{k\ge 0}$ by

$$Z_k=W'_{s_k} \tag{4.8}$$

and the corresponding entrance time to the state $\{1\}$:

$$\varpi=\inf\{k\ge 1\,:\,Z_k=1\}\,. \tag{4.9}$$

In order to study the property of this stopping time, we need of the following notations

$$l_*=l_*(r)=2+\left[\frac{\ln\left(e^{2\upsilon_*(r)}(1-e^{-r})^{-1}\mathbf{q}(r)\right)}{2r}\right]\,, \tag{4.10}$$

where $\mathbf{q}(r)=(1-e^{\upsilon_1(r)})/2$ and the parameter $\upsilon_1(r)<0$ will be specified below. Moreover, for any $0<\gamma\,,\,\epsilon_*<1$, we set

$$A^*_1(r)=A^*(r)\,\frac{1+A^*(r)e^{rl_*}}{1-(1-\epsilon_*)^{\gamma}}\,, \tag{4.11}$$

where

$$A^*(r)=\frac{1+Q^*(r)e^{rl_*+\upsilon_*(r)}}{(1-\mathbf{q}(r))^{1-\gamma}\,(1-(1-\mathbf{q}(r))^{\gamma})}\quad\text{and}\quad Q^*(r)=\frac{e^{\upsilon_*(r)}}{1-e^{-r}}\,.$$

Proposition 4.2. *Assume the condition* **C**). *Then, for any* $0 < \gamma < 1$, *for any* $\epsilon_* > 0$ *and* $\upsilon_1(r) \le -r$ *for which*

$$0 < \epsilon_* \le \min_{1 \le j \le l_* - 1} p(j) \quad and \quad \ln \mathbf{E} e^{-rY_1} \le \upsilon_1(r)\,, \tag{4.12}$$

one has

$$\mathbf{E}\, e^{\varrho_* \varpi} \le\ Q_1^*(r)\, A_1^*(r)\,,$$

where $Q_1^*(r) = e^{\upsilon_*(r)} + e^{\upsilon'_*(r)} + Q^*(r)$ *and*

$$\varrho_* = \varrho_*(r) = \frac{(1-\gamma)^2 |\ln(1-\mathbf{q}(r))|\, |\ln(1-\epsilon_*)|}{\ln A^*(r) + rl_* + |\ln(1-\epsilon_*)|}\,. \tag{4.13}$$

Proof. Firstly we note that the sequence (4.8) is a homogeneous Markov chain with values in $\mathbb{N}^*$ such that, for any m and l from $\mathbb{N}^*$ and for any $k \ge 1$,

$$\mathbf{P}(Z_k = m | Z_{k-1} = l) = \mathbf{P}(W'_{\varsigma_1 - l} = m | Y'_0 = 0)\,. \tag{4.14}$$

Note, that $W'_k = -k$ for $k < 0$ under the condition $Y'_0 = 0$. Therefore, for any positive function V,

$$\mathbf{E}\,[V(Z_1)|Z_0 = l] = \mathbf{E}\, V(l - \varsigma_1)\, \mathbf{1}_{\{\varsigma_1 < l\}} + \mathbf{E}\, Q(\varsigma_1 - l) \mathbf{1}_{\{\varsigma_1 \ge l\}}\,, \tag{4.15}$$

where $Q(n) = \mathbf{E}\,(V(W_n)|Y_0 = 0)$. Using the distribution (4.3) yields

$$Q(n) \le \mathbf{E}\left(V(W_{n-1} - 1)|Y_0 = 0\right) + \mathbf{E} V(Y_1) \mathbf{P}\left(W_{n-1} = 1 | Y_0 = 0\right)\,.$$

Choosing now $V(x) = e^{rx}$ one has

$$Q(n) \le e^{-r} Q(n-1) + \mathbf{E} V(Y_1)\,.$$

From the last inequality, taking into account that $Q(0) = \mathbf{E} V(Y_1)$, it follows that, for any $n \ge 1$,

$$Q(n) \le e^{-nr} Q(0) + \mathbf{E} V(Y_1) \sum_{j=1}^{n} e^{-(n-j)r} \le Q^*(r)\,, \tag{4.16}$$

where the upper bound $Q^*(r)$ is defined in (4.10). This implies that the last term in (4.15) can be estimated as

$$\mathbf{E}\, Q(\varsigma_1 - l) \mathbf{1}_{\{\varsigma_1 \ge l\}} \le Q^*(r) e^{\upsilon_*(r) - rl}\,.$$

Therefore,

$$\frac{\mathbf{E}\ (V(Z_1)|Z_0 = l)}{V(l)} \le e^{\upsilon_1(r)} + Q^*(r) e^{\upsilon_*(r) - 2rl}\,.$$

By making use of the definition of l_* in (4.10), we obtain

$$\sup_{l\ge l_*} \frac{\mathbf{E}\,(V(Z_1)|Z_0=l)}{V(l)} \le \frac{1+e^{\upsilon_1(r)}}{2} = 1-\mathbf{q}(r) < 1\,. \tag{4.17}$$

Moreover, for any $1 \le l \le l_*$,

$$\mathbf{E}\,(V(Z_1)|Z_0=l) \le (1-\rho)V(l) + e^{\upsilon_1(r)}V(l) + Q^*(r)e^{\upsilon_*(r)-rl}\,,$$

i.e. the chain $(Z_k)_{k\ge1}$ satisfies the condition (A.2) in the Appendix with

$$C=\{1,\ldots,l_*-1\} \quad \text{and} \quad D=e^{rl_*+\upsilon_*(r)}Q^*(r)\,.$$

Therefore, by Proposition A.2 for $a_* = -(1-\gamma)\ln(1-\mathbf{q}(r))$, one gets

$$\sup_{l\ge1} \frac{U_C(l,a_*,V)}{V(l)} \le \, A^*(r)\,,$$

where the upper bound $A^*(r)$ is given in (4.11). Moreover, from (4.12) and (4.14) we get that, for $2 \le l \le l_*-1$,

$$\mathbf{P}(Z_1=1|Z_0=l) \ge p(l-1) \ge \epsilon_*\,.$$

Therefore, putting in Proposition A.3 $\mathbf{k}_*=\epsilon_*$, $a=\varrho_*$ defined in (4.13) and the set $B=\{1\}$ we obtain that, for any $l\ge1$,

$$\mathbf{E}\,(e^{\varrho_*\varpi}|Z_0=l) \le V(l)A_1^*(r)\,,$$

where the parameters ϱ_* and $A_1^*(r)$ are defined in (4.11) and (4.13). This upper bound implies

$$\mathbf{E}\,(e^{\varrho_*\varpi}) \le A_1^*(r)\,\mathbf{E}\,V(Z_0)\,.$$

Moreover, note now that

$$\mathbf{E}\,V(Z_0) = \sum_{j=1}^{\infty} \mathbf{E}\,\left(V(Z_0)|W_0^{'}=j\right)\,\mathbf{P}(W_0^{'}=j)\,.$$

Similarly to (4.15) we obtain

$$\mathbf{E}\,\left(V(Z_0)|W_0^{'}=j\right) = \mathbf{E}\,V(j-s_0)\,\mathbf{1}_{\{s_0<j\}} + \mathbf{E}\,Q(s_0-j)\mathbf{1}_{\{s_0\ge j\}}\,.$$

Using here the inequality (4.16) yields

$$\mathbf{E}\,\left(V(Z_0)|W_0^{'}=j\right) \le V(j)+Q^*(r)\,.$$

Moreover, similarly to (4.7) we obtain that, for any $j\ge1$,

$$\mathbf{P}(W_0^{\prime}=j) = \mathbf{b}(0)p(j)+\mathbf{b}(j)\,.$$

Thus,

$$\mathbf{E}V(Z_0) \le \mathbf{b}(0)\mathbf{E}V(Y_1) + \mathbf{E}V(Y_0^{'}) + Q^*(r)$$

and we come to the inequality (4.12). Hence Proposition 4.2. $\square$

Proposition 4.3. *Assume the condition* **C**). *Then, for any* $\epsilon_* > 0$ *satisfying the condition* (4.2) *and for any* $0 < \gamma < 1$, *there exists* $\kappa > 0$ *such that*

$$\mathbf{E}e^{\kappa s_\varpi} \le A_2^*(r)\,, \tag{4.18}$$

where

$$A_2^*(r) = \frac{(1-\gamma)\left(3e^{2\upsilon_*(r)} + Q_1^*(r)A_1^*(r)\right)}{\gamma} \quad \textit{and} \quad \kappa = \kappa(r) = \frac{(1-\gamma)\varrho_* r}{\varrho_* + \upsilon_*(r)}\,,$$

the coefficients $A_1^*(r)$ *and* ϱ_* *are defined in* (4.11) *and* (4.13).

Proof. Indeed, we have

$$\mathbf{E}\,e^{\kappa s_\varpi} = \kappa \int_0^\infty e^{\kappa t}\,\mathbf{P}(s_\varpi > t)\,\mathrm{d}t$$

$$\le \kappa \int_0^\infty e^{\kappa t}\,\left(\mathbf{P}(s_N > t)\,\mathrm{d}t + \mathbf{P}(\varpi > N)\right)\,\mathrm{d}t\,,$$

where $N = N(t) = 1 + [\vartheta t]$, and ϑ is some positive parameter which will be chosen later. Note now that, for $0 < \vartheta < r/\upsilon_*(r)$,

$$\mathbf{P}(s_N > t) \le 3e^{\upsilon_*(r)(N+1)-rt} \le 3e^{2\upsilon_*(r)}\,e^{-(r-\upsilon_*(r)\vartheta)t}\,.$$

Moreover, due to Proposition 4.2

$$\mathbf{P}(\varpi > N) \le Q_1^*(r)\,A_1^*(r)\,e^{-N\varrho_*} \le Q_1^*(r)\,A_1^*(r)\,e^{-\vartheta\varrho_* t}\,.$$

Therefore, denoting

$$\iota_*(\vartheta) = \min\left((r - \upsilon_*(r)\vartheta)\,,\,\vartheta\varrho_*\right)\,,$$

one gets

$$\mathbf{E}\,e^{\kappa s_\varpi} \le \kappa\,\left(3e^{2\upsilon_*(r)} + Q_1^*(r)\,A_1^*(r)\right)\int_0^\infty e^{-(\iota_*(\vartheta)-\kappa)t}\,\mathrm{d}t\,.$$

Maximizing now $\iota_*(\vartheta)$ yields

$$\max_{0<\vartheta<\upsilon_*(r)/r} \iota_*(\vartheta) = \iota_*(\vartheta_{max}) = \frac{r\,\varrho_*}{\varrho_* + \upsilon_*(r)}\,, \quad \vartheta_{max} = \frac{r}{\varrho_* + \upsilon_*(r)}\,.$$

Therefore, choosing now $\vartheta = \vartheta_{max}$ and $\kappa = (1-\gamma)\iota_*(\vartheta_{max})$, we come to the inequality (4.18). Hence Proposition 4.3. □

Let us define the renewal sequence $(u(n))_{n\ge 0}$ as follows

$$u(n) = \sum_{j=0}^\infty p^{*j}(n)\,, \tag{4.19}$$

where p^{*j} denotes the jth convolution power. For $j = 0$ we set $p^0(n) = 1$ for $n = 0$ and $p^0(n) = 0$ for $n \geq 1$. We remind that, for two sequences $(\mathbf{a}(j))_{j\geq 0}$ and $(u(j))_{j\geq 0}$, the convolution sequence $(\mathbf{a} * u(j))_{j\geq 0}$ is defined for any $j \geq 0$ as

$$\mathbf{a} * u(j) = \sum_{i=0}^{j} \mathbf{a}(i) u(j-i) \,.$$

Proposition 4.4. *Assume that the condition* $\mathbf{C}$) *holds and there exists* $\epsilon_* > 0$ *satisfying the inequality* (4.12). *Then, for any* $0 < \gamma < 1$ *and* $n \geq 2$,

$$|\Delta(n)| \,\leq A_2^* \, e^{-\kappa n} \,,$$

where $\Delta(n) = \mathbf{a} * u(n) - \mathbf{b} * u(n)$, *the coefficients* κ *and* A_2^* *are given in* (4.18).

Proof. Obviously, that for $n \geq 1$,

$$\mathbf{a} * u(n) = \mathbf{P}\left(\cup_{j=0}\left\{\sum_{i=0}^{j} Y_i = n\right\}\right) = \mathbf{P}(W_{n-1} = 1)$$

and

$$\mathbf{b} * u(n) = \mathbf{P}\left(\cup_{j=0}\left\{\sum_{i=0}^{j} Y'_i = n\right\}\right) = \mathbf{P}(W'_{n-1} = 1) \,.$$

Therefore,

$$\Delta(n) = \mathbf{P}(W_{n-1} = 1 \,,\, W'_{n-1} \geq 2) - \mathbf{P}(W'_{n-1} = 1 \,,\, W_{n-1} \geq 2) \,.$$

Now, we introduce the "coupling" stopping time τ as

$$\tau = \inf\{k \geq 1 \,:\, (W_k, W'_k) = (1,1)\} \,.$$

Note that, for any $n \geq 2$, by the Markov property for the chain $(W_k, W'_k)_{k\geq 1}$, one has

$$\mathbf{P}(W_n = 1 \,,\, W'_n \geq 2, \tau \leq n-1) = \sum_{k=1}^{n-1} \mathbf{P}(W_n = 1 \,,\, W'_n \geq 2, \tau = k)$$

$$= \sum_{k=1}^{n-1} \mathbf{P}(\tau = k) \, v_{n-k} \,.$$

where $v_k = \mathbf{P}(W_k = 1 \,|\, W_0 = 1)\mathbf{P}(W_k \geq 2 \,|\, W_0 = 1)$. Similarly, one gets

$$\mathbf{P}(W'_n = 1 \,,\, W_n \geq 2, \tau \leq n-1) = \sum_{k=1}^{n-1} \mathbf{P}(\tau = k) v_{n-k} \,.$$

This implies that

$$\Delta(n) = \alpha_1(n-1) - \alpha_2(n-1)\,,$$

where $\alpha_1(n) = \mathbf{P}(W_n = 1\,,\, W'_n \ge 2\,, \tau > n)$ and

$$\alpha_2(n) = \mathbf{P}(W'_n = 1\,,\, W_n \ge 2\,, \tau > n)\,.$$

Therefore, for any $n \ge 2$,

$$|\Delta(n)| \le \max\left(\alpha_1(n-1)\,,\, \alpha_2(n-1)\right) \le \mathbf{P}(\tau > n)\,.$$

Taking into account that $\tau \le s_\varpi$ a.s., we obtain

$$|\Delta(n)| \le \mathbf{P}(s_\varpi > n) \le e^{-\kappa n}\, \mathbf{E} e^{\kappa s_\varpi}\,.$$

Proposition 4.3 implies the upper bound (4.18). Hence Proposition 4.4 □

Theorem 4.1. *Assume that there exists $r > 0$ such that*

$$\ln \mathbf{E}\, e^{rY_1} \le \upsilon_*(r)\,.$$

Then, for any $0 < \gamma < 1$, $n \ge 2$ and $\epsilon_* > 0$ satisfying the inequality (4.12),

$$\left| u(n) - \frac{1}{\mathbf{E}\, Y_1} \right| \le A^*_3(r)\, e^{-\kappa n}\,,$$

where

$$A^*_3(r) = \frac{1-\gamma}{\gamma} \left(3 + 2 \frac{e^r\, A^*(r)\, \left(1 + A^*(r) e^{r l_*}\right)}{\left(1 - (1-\epsilon_*)^\gamma\right) (e^r - 1)} \right) e^{\upsilon_*(r)} \tag{4.20}$$

and the parameter $\kappa > 0$ is defined in (4.18).

Proof. We obtain the inequality (4.20) through Proposition 4.4 in which we choose $\mathbf{a}(0) = 1$ with $\mathbf{a}(j) = 0$ for $j \ge 1$. Moreover, we choose the distribution $(\mathbf{b}(j))_{j \ge 0}$ as

$$\mathbf{b}(j) = \frac{1}{\mathbf{E} Y_1} \mathbf{P}(Y_1 > j) = \frac{1}{\mathbf{E} Y_1} \sum_{i=j+1}^{\infty} p(i)\,.$$

It is easy to see directly that, for any $j \ge 1$,

$$\mathbf{b} * u(j) = \frac{1}{\mathbf{E} Y_1}\,.$$

Note now, that through the condition of this theorem we obtain

$$e^{rn}\, \mathbf{b}(n) = \frac{e^{rn} \mathbf{P}(Y_1 > n)}{\mathbf{E} Y_1} \le \frac{\mathbf{E} e^{rY_1} \mathbf{1}(Y_1 > n)}{\mathbf{E} Y_1} \to 0\,, \quad \text{as} \quad n \to \infty\,.$$

Therefore, the summing by parts yields

$$\sum_{j \ge 0} e^{rj} \mathbf{b}(j) = \frac{\mathbf{E}\, e^{rY_1} - 1}{(e^r - 1) \mathbf{E} Y_1} \le \frac{e^{\upsilon_*(r)} - 1}{e^r - 1} := e^{\upsilon'_*(r)}\,.$$

and Proposition 4.4 implies the inequality (4.20). Hence Theorem 4.1. □

5 Proof of Theorem 2.1

First we fix some $0 < \gamma < 1$ and we set $a_1 = -(1-\gamma)\ln(1-\rho)$. We start with studying the properties of the function

$$U_C^\vartheta(x, a_1, V) = \mathbf{E}_x^\vartheta \sum_{j=1}^{\tau_C} e^{a_1 j}\, V(\Phi_j)$$

where $\tau_C = \inf\{n \geq 1 \,:\, \Phi_n \in C\}$. The condition $\mathbf{H}_2$) and Proposition A.2 imply immediately that, for any $0 < \gamma < 1$,

$$\sup_{\vartheta\in\Theta} \sup_{x\in\mathcal{X}} \frac{U_C^\vartheta(x, a_1, V)}{V(x)} \leq \frac{1-\rho+D}{(1-\rho)^{1-\gamma}(1-(1-\rho)^\gamma)} := U^* \,. \tag{5.1}$$

Now, we introduce a splitting chain family as in [16], p. 108 (see also [18]). We set $\check{\mathcal{X}} = \mathcal{X} \times \{0,1\}$, $\mathcal{X}_0 = \mathcal{X} \times \{0\}$ and $\mathcal{X}_1 = \mathcal{X} \times \{1\}$. Let $\mathcal{B}(\mathcal{X}_i)$ be the $\sigma-$fields generated by the set $A_i = A \times \{i\}$ with $A \in \mathcal{B}(\mathcal{X})$, $i = 0, 1$. In the sequel we will denote by $< \check{x} >_i$ the ith component of $\check{x} \in \check{\mathcal{X}}$. It is clear, that $< \check{x} >_0 \in \mathcal{X}$ and $< \check{x} >_1 \in \{0,1\}$. Furthermore, we define the σ - field $\mathcal{B}(\check{\mathcal{X}})$ as a $\sigma-$field generated by $\mathcal{B}(\mathcal{X}_0) \cup \mathcal{B}(\mathcal{X}_1)$ and for any measure λ on $\mathcal{B}(\mathcal{X})$ we relate the measure λ^* on $\mathcal{B}(\check{\mathcal{X}})$ as

$$\lambda^*(A_0) = (1-\delta)\lambda(A\cap C) + \lambda(A \cap C^c) \quad \text{and} \quad \lambda^*(A_1) = \delta\lambda(A\cap C)\,.$$

Now, for each $\vartheta \in \Theta$, we introduce a homogeneous Markov chain $(\check{\Phi}_n)_{n\geq 0}$ by the following transition probabilities

$$\check{\mathbf{P}}^\vartheta(\check{x}, \cdot) = \begin{cases} \mathbf{P}^\vartheta(x,\cdot)^*\,, & \text{if} \quad \check{x} \in \mathcal{X}_0 \setminus C_0\,; \\[2mm] \dfrac{\mathbf{P}^\vartheta(x,\cdot)^* - \delta\nu^*(\cdot)}{1-\delta}\,, & \text{if} \quad \check{x} \in C_0\,; \\[2mm] \nu^*(\cdot)\,, & \text{if} \quad \check{x} \in \mathcal{X}_1\,. \end{cases} \tag{5.2}$$

Note, that for any $\check{x} \in \mathcal{X}_1$,

$$\check{\mathbf{P}}^\vartheta(\check{x}, C_0 \cup C_1) = \nu^*(C_0 \cup C_1) = \nu(C) = 1\,. \tag{5.3}$$

Obviously, that the set $\alpha = C_1$ is an accessible atom for the chain $(\check{\Phi}_n)_{n\geq 1}$, i.e. for any positive $\check{\mathcal{X}} \to \mathbb{R}$ function g

$$\check{\mathbf{E}}_{\check{x}}^\vartheta\, g(\check{\Phi}_1) = \check{\mathbf{E}}_{\check{y}}^\vartheta\, g(\check{\Phi}_1)\,, \quad \text{for any} \quad \check{x}\,,\, \check{y} \in \alpha\,.$$

This implies directly that, for any nonnegative random variable ξ measurable with respect to the $\sigma-$field generated by the chain $(\check{\Phi}_n)_{n\geq 1}$, one has

$$\check{\mathbf{E}}_{\check{x}}^\vartheta\, \xi = \check{\mathbf{E}}_{\check{y}}^\vartheta\, \xi \quad \text{for any} \quad \check{x}\,,\, \check{y} \in \alpha\,.$$

In the sequel we denote by $\check{\mathbf{E}}^{\vartheta}_{\alpha}(\cdot)$ the such expectations. Moreover, one can check directly that the chain $(\check{\Phi}_n)_{n\ge 1}$ is ν^*-irreducible. Next, for any set $\check{C}$ from $\mathcal{B}(\check{\mathcal{X}})$ we introduce the corresponding entrance time

$$\check{\tau}_{\check{C}} = \inf\left\{n \ge 1 \,:\, \check{\Phi}_n \in \check{C}\right\} \tag{5.4}$$

and the corresponding entrance function

$$\check{U}^{\vartheta}_{\check{C}}(\check{x}, a_1, \check{V}) = \check{\mathbf{E}}^{\vartheta}_{\check{x}} \sum_{j=1}^{\check{\tau}_{\check{C}}} e^{a_1 j}\, \check{V}(\check{\Phi}_j)\,, \tag{5.5}$$

where $\check{V}(\check{x}) = V(<\check{x}>_1)$. By Proposition A.5 we obtain that, for any $x \in \mathcal{X}$,

$$U^{\vartheta}_{C}(x, a_1, V) = (1-\delta)\check{U}^{\vartheta}_{C_0\cup C_1}(x_0, a_1, \check{V})\,\mathbf{1}_{\{x\in C\}} + \check{U}^{\vartheta}_{C_0\cup C_1}(x_0, a_1, \check{V})\,\mathbf{1}_{\{x\in C^c\}} + \delta\check{U}^{\vartheta}_{C_0\cup C_1}(x_1, a_1, \check{V})\,\mathbf{1}_{\{x\in C\}}\,,$$

where $x_i = (x, i)$ for $i = 0, 1$. Note now, that due to the property (5.3),

$$\check{U}^{\vartheta}_{C_0\cup C_1}(x_1, a_1, \check{V}) = \check{\mathbf{E}}^{\vartheta}_{x_1}\, e^{a_1}\, \check{V}(\check{\Phi}_1) \le e^{a_1}\, V^* = \frac{V^*}{(1-\rho)^{1-\gamma}}\,.$$

Therefore, using the upper bound (5.1) and the coefficient $\check{U}^*$ given in (3.1), we obtain

$$\sup_{\vartheta\in\Theta}\sup_{\check{x}\in\check{\mathcal{X}}} \frac{\check{U}^{\vartheta}_{C_0\cup C_1}(\check{x}, a_1, \check{V})}{\check{V}(\check{x})} \le \check{U}^*\,.$$

Note now that, for $\check{x} \in C_0$ by the definition (5.2)

$$\check{\mathbf{P}}^{\vartheta}(\check{x}, \alpha) = \check{\mathbf{P}}^{\vartheta}(\check{x}, C_1) = \delta\frac{\mathbf{P}^{\vartheta}(x, C) - \delta}{1-\delta} \ge \delta\eta_1\,,$$

where the parameter $0 < \eta_1 \le 1$ is defined in (3.1). By making use of Proposition A.3 with $a_* = a_1 = -(1-\gamma)\ln(1-\rho)$ and $\mathbf{k}_* = \delta\eta_1$, one gets

$$\sup_{\vartheta\in\Theta}\sup_{\check{x}\in\check{\mathcal{X}}} \frac{\check{U}^{\vartheta}_{\alpha}(\check{x}, r_*, \check{V})}{\check{V}(\check{x})} \le B^*\,, \tag{5.6}$$

where B^* and r_* are given in (3.1) and (3.2). Therefore, by Proposition A.1, the chain $(\check{\Phi}_n)_{n\ge 0}$ is ergodic for each $\vartheta \in \Theta$ with the invariant measure given as

$$\check{\pi}^{\vartheta}(\check{\Gamma}) = \frac{1}{\check{\mathbf{E}}^{\vartheta}_{\alpha}\check{\tau}_{\alpha}}\, \check{\mathbf{E}}^{\vartheta}_{\alpha} \sum_{j=1}^{\check{\tau}_{\alpha}} \mathbf{1}_{\{\check{\Phi}_j\in\check{\Gamma}\}}\,. \tag{5.7}$$

Now, for any $n \geq 2$, we define $\breve{\iota} = \max\{1 \leq j \leq n-1 \, : \, \check{\Phi}_j \in \alpha\}$ and we put $\breve{\iota} = 0$ if $\check{\tau}_\alpha \geq n$. Moreover, note that, for any $\check{\mathcal{X}} \to \mathbb{R}$ function f and any $n \geq 2$,

$$\check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}_n) \, \mathbf{1}_{\{\check{\tau}_\alpha < n\}} = \sum_{j=1}^{n-1} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}_n) \, \mathbf{1}_{\{\check{\tau}_\alpha \leq j\}} \mathbf{1}_{\{\breve{\iota} = j\}}$$

$$= \sum_{j=1}^{n-1} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, \mathbf{1}_{\{\check{\tau}_\alpha \leq j\}} \, \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, \left(f(\check{\Phi}_n) \, \mathbf{1}_{\{\breve{\iota} = j\}} | \check{\Phi}_1, \ldots, \check{\Phi}_j \right) \, .$$

Now, taking into account that $(\check{\Phi}_n)_{n \geq 1}$ is a homogeneous Markov chain, we can calculate the last conditional expectation as follows

$$\check{\mathbf{E}}^{\vartheta}_{\check{x}} \, \left(f(\check{\Phi}_n) \, \mathbf{1}_{\{\breve{\iota} = j\}} | \check{\Phi}_1, \ldots, \check{\Phi}_j \right) = \mathbf{1}_{\{\check{\Phi}_j \in \alpha\}} \check{\mathbf{E}}^{\vartheta}_{\alpha} \, \left(f(\check{\Phi}_{n-j}) \, \mathbf{1}_{\{\check{\Phi}_1 \notin \alpha, \ldots, \check{\Phi}_{n-j-1} \notin \alpha\}} \right)$$

$$= \mathbf{1}_{\{\check{\Phi}_j \in \alpha\}} g_{f,\alpha}(n-j) \, , \tag{5.8}$$

where $g_{f,\alpha}(k) = \check{\mathbf{E}}^{\vartheta}_{\alpha} \, f(\check{\Phi}_k) \, \mathbf{1}_{\{\check{\tau}_\alpha \geq k\}}$. By convention, we set $g_{f,\alpha}(0) = 0$. Therefore,

$$\check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}_n) \, \mathbf{1}_{\{\check{\tau}_\alpha < n\}} = \sum_{j=1}^{n} \check{\mathbf{P}}^{\vartheta}_{\check{x}} \, (\check{\tau}_\alpha \leq j) \; g_{f,\alpha}(n-j) = h_{\check{x}} * g_{f,\alpha}(n) \, ,$$

where $h_{\check{x}}(0) = 0$ and, for $j \geq 1$,

$$h_{\check{x}}(j) = \check{\mathbf{P}}^{\vartheta}_{\check{x}} \, (\check{\tau}_\alpha \leq j) = \check{\mathbf{P}}^{\vartheta}_{\check{x}} \, \left(\check{\Phi}_j \in \alpha \right) \, .$$

Moreover, for $j \geq 1$,

$$h_{\check{x}}(j) = \sum_{l=1}^{j} \check{\mathbf{P}}^{\vartheta}_{\check{x}} \, \left(\check{\tau}_\alpha = l \, , \, \check{\Phi}_j \in \alpha \right) = \sum_{l=1}^{j} \mathbf{a}_{\check{x}}(l) \, u(l-j) = \mathbf{a}_{\check{x}} * u(j) \, ,$$

where

$$\mathbf{a}_{\check{x}}(l) = \check{\mathbf{P}}^{\vartheta}_{\check{x}} \, (\check{\tau}_\alpha = l) \quad \text{and} \quad u(l) = \check{\mathbf{P}}^{\vartheta}_{\alpha} \, \left(\check{\Phi}_l \in \alpha \right) \, . \tag{5.9}$$

It is clear that $\mathbf{a}_{\check{x}}(0) = 0$ and $u(0) = 1$, i.e. $\mathbf{a}_{\check{x}} * u(0) = 0$. This implies that

$$h_{\check{x}}(j) = \mathbf{a}_{\check{x}} * u(j) \, ,$$

for all $j \geq 0$. Finally, taking into account that $\mathbf{a}_{\check{x}} * u * g_{f,\alpha}(1) = 0$, we obtain that for any $n \geq 1$,

$$\check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}_n) \, \mathbf{1}_{\{\check{\tau}_\alpha < n\}} = \mathbf{a}_{\check{x}} * u * g_{f,\alpha}(n) \, . \tag{5.10}$$

Note that the sequence $(u(n))_{n\geq 0}$ is a renewal sequence, i.e.

$$u(n) = \sum_{j=0}^{\infty} p^{*j}(n) \quad \text{with} \quad p(k) = \check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_{\alpha} = k)\,.$$

Now, we set $Y_0 = \check{\tau}_{\alpha}$ and $Y_j = \inf\{j \geq Y_{j-1} + 1 \,:\, \check{\Phi}_j \in \alpha\}$. One can check directly that $(Y_j)_{j\geq 1}$ is i.i.d. sequence with the distribution $(p(k))_{k\geq 1}$, i.e. $u(n)_{n\geq 0}$ is the renewal function for the sequence $(Y_j)_{j\geq 1}$. We denote

$$\omega(n) = \left| u(n) - \frac{1}{\check{\mathbf{E}}^{\vartheta}_{\alpha} Y_1} \right|\,.$$

We estimate this term by Theorem 4.1. First we have to check the condition $\mathbf{C}_1$) uniformly over the parameter $\vartheta \in \Theta$, i.e. to show that, for any $j \geq 1$,

$$\inf_{\vartheta\in\Theta} \check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_{\alpha} = j) \geq \delta\,\eta_1 (1-\delta)^{j-1}\,. \tag{5.11}$$

Let us check this property for $j = 1$. We remind that, by the condition $\mathbf{H}_1$), one has $\nu(C) = 1$. Thus, the definition (5.2) implies

$$\check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_{\alpha} = 1) = \nu^{*}(C_1) = \delta \geq \delta\,\eta_1\,.$$

To show the property (5.11) for $j \geq 2$ note, that $\check{\mathbf{P}}^{\vartheta}_{\alpha}(\mathcal{X}_0) = \nu^{*}(\mathcal{X}_0) = 1 - \delta$. Moreover, taking into account that $\mathbf{P}^{\vartheta}(z, \mathcal{X}_0)^{*} \geq 1 - \delta$ we obtain the same lower bound for the splitting distribution (5.2), i.e. for any $\check{z} \in \mathcal{X}_0$

$$\check{\mathbf{P}}^{\vartheta}(\check{z}, \mathcal{X}_0) = \mathbf{1}_{\{\check{z}\in C_0\}} \left(\frac{\mathbf{P}^{\vartheta}(z, \mathcal{X}_0)^{*} - \delta\nu^{*}(\mathcal{X}_0)}{1-\delta} \right) + \mathbf{1}_{\{\check{z}\in \mathcal{X}_0\setminus C_0\}} \mathbf{P}^{\vartheta}(z, \mathcal{X}_0)^{*} \geq (1-\delta)\,.$$

Similarly, for any $\check{z} \in \mathcal{X}_0$ we obtain

$$\check{\mathbf{P}}^{\vartheta}(\check{z}, C_1) = \mathbf{1}_{\{\check{z}\in C_0\}} \left(\frac{\mathbf{P}^{\vartheta}(z, C_1)^{*} - \delta\nu^{*}(C_1)}{1-\delta} \right) + \mathbf{1}_{\{\check{z}\in \mathcal{X}_0\setminus C_0\}} \mathbf{P}^{\vartheta}(z, C_1)^{*} \geq \delta \frac{\mathbf{P}^{\vartheta}(z, C) - \delta}{1-\delta} \geq \delta\eta_1\,.$$

Now through the induction we can show, that for any $j \geq 1$

$$\check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \in \mathcal{X}_0\,, \ldots, \check{\Phi}_j \in \mathcal{X}_0\right) \geq (1-\delta)^{j}\,.$$

Therefore, taking into account that for any $\check{x} \in \check{X}$ we have $\check{\mathbf{P}}^{\vartheta}(\check{x}, \mathcal{X}_1 \setminus C_1) = 0$, we obtain for $j \geq 2$

$$\begin{aligned}
\check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_{\alpha} = j) &= \check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \notin \alpha \,, \ldots, \check{\Phi}_{j-1} \notin \alpha \,, \check{\Phi}_j \in \alpha\right) \\
&= \check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \in \mathcal{X}_0 \,, \ldots, \check{\Phi}_{j-1} \in \mathcal{X}_0 \,, \check{\Phi}_j \in C_1\right) \\
&\geq \delta\, \eta_1\, \check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \in \mathcal{X}_0 \,, \ldots, \check{\Phi}_{j-1} \in \mathcal{X}_0\right)\,.
\end{aligned}$$

This yields the lower bound (5.11). Similarly we can show

$$\sup_{j\geq 1} \frac{\sup_{\vartheta\in\Theta} \check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_{\alpha} = j)}{(1-\delta\eta_1)^{j-1}} \leq \delta$$

and

$$\sup_{\vartheta\in\Theta} \check{\mathbf{E}}^{\vartheta}_{\alpha} e^{-r_* Y_1} \leq \frac{\delta}{e^{r_*} - 1 + \delta\eta_1}\,.$$

Moreover, taking into account that $\check{\mathbf{E}}^{\vartheta}_{\alpha} e^{-r_* Y_1} \leq e^{-r_*}$ we obtain that

$$\sup_{\vartheta\in\Theta} \check{\mathbf{E}}^{\vartheta}_{\alpha} e^{-r_* Y_1} \leq \check{B}^*_1 < 1\,,$$

where $\check{B}^*_1$ is given in (3.2). Therefore, the sequence $(Y_j)_{j\geq 0}$ satisfies the condition of Theorem 4.1 with $r = r_*$, $\upsilon_*(r) = \ln(V^* B^*)$, $l_* = \widetilde{l}$,

$$\epsilon_* = \delta\, \eta_1\, (1-\delta)^{\widetilde{l}-2} \quad \text{and} \quad \upsilon_1(r) = \ln \check{B}^*_1\,,$$

where $\widetilde{l}$ is defined in (3.2). Therefore, for $n \geq 2$,

$$\omega(n) \leq \widetilde{A}_3\, e^{-\widetilde{\kappa} n}\,,$$

where the parameters $\widetilde{A}_3$ and $\widetilde{\kappa}$ are defined in (3.4). Taking into account that $\widetilde{A}_3 \geq e^{\widetilde{\kappa}}$ we obtain that, for any $n \geq 0$,

$$\omega(n) \leq \widetilde{A}_3\, e^{-\widetilde{\kappa} n}\,.$$

Therefore, for any $0 < \varkappa < \widetilde{\kappa}$

$$\widehat{\omega}(\varkappa) = \sum_{n\geq 0} e^{\varkappa n} \omega(n) \leq \widetilde{A}_3 \frac{e^{\widetilde{\kappa}-\varkappa}}{e^{\widetilde{\kappa}-\varkappa} - 1}\,. \tag{5.12}$$

Now, taking into account that

$$\check{\pi}^{\vartheta}(f) = \frac{1}{\check{\mathbf{E}}_{\alpha} Y_1} \sum_{j=0}^{+\infty} g_{f,\alpha}(j)$$

and that $\check{\mathbf{E}}_\alpha Y_1 \ge 1$, one obtains, for any $n \ge 1$,

$$|\check{\mathbf{E}}^\vartheta_{\check{x}}\, f(\check{\Phi}_n)\, \mathbf{1}_{\{\check{\tau}_\alpha < n\}} - \check{\pi}^\vartheta(f)| \,\le\, \mathbf{a}_{\check{x}} * \omega * g_{f,\alpha}(n) + Q_{\check{x}} * g_{f,\alpha}(n) + G_{f,\alpha}(n)\,,$$

where

$$Q_{\check{x}}(n) = \check{\mathbf{P}}^\vartheta_{\check{x}}\,(\tau_\alpha > n) \quad \text{and} \quad G_{f,\alpha}(n) = \sum_{j=n+1} g_{f,\alpha}(j)\,.$$

Therefore, for any $n \ge 0$,

$$\Delta_{\check{x}}(n) = |\check{\mathbf{E}}^\vartheta_{\check{x}}\, f(\check{\Phi}_n)\; - \check{\pi}^\vartheta(f)| \le \mathbf{a}_{\check{x}} * \omega * g_{f,\alpha}(n) \\ + Q_{\check{x}} * g_{f,\alpha}(n) + g_{f,\check{x}}(n) + G_{f,\alpha}(n)\,, \qquad (5.13)$$

where $g_{f,\check{x}}(n) = \check{\mathbf{E}}^\vartheta_{\check{x}}\, f(\check{\Phi}_n)\, \mathbf{1}_{\{\check{\tau}_\alpha \ge n\}}$. Now for any sequence $(b(n))_{n\ge 0}$ we denote by $\widehat{b}(\varkappa)$ the Laplace transformation, i.e.

$$\widehat{b}(\varkappa) = \sum_{n\ge 0} e^{\varkappa n}\, b(n)\,.$$

Therefore, from (5.13) we obtain for any $0 < \varkappa < r_*$,

$$\widehat{\Delta}_{\check{x}}(\varkappa) \le \widehat{a}_{\check{x}}(\varkappa)\widehat{\omega}(\varkappa)\widehat{g}_{f,\alpha}(\varkappa) + \widehat{Q}_{\check{x}}(\varkappa)\widehat{g}_{f,\alpha}(\varkappa) + \widehat{g}_{f,\check{x}}(\varkappa) + \widehat{G}_{f,\alpha}(\varkappa)\,. \qquad (5.14)$$

It is clear

$$\widehat{g}_{f,\check{x}}(\varkappa) = \check{U}^\vartheta_\alpha(\check{x},\varkappa,\check{f}) \quad \text{with} \quad \check{f}(\check{x}) = f(<x>_1)\,.$$

Moreover, note that

$$\widehat{a}_{\check{x}}(\varkappa) = \check{\mathbf{E}}_{\check{x}}\, e^{\varkappa\check{\tau}_\alpha} \le \check{U}^\vartheta_\alpha(\check{x},\varkappa,\check{f})\,, \quad \widehat{Q}_{\check{x}}(\varkappa) = \frac{\check{\mathbf{E}}_{\check{x}}\, e^{\varkappa\check{\tau}_\alpha} - 1}{e^\varkappa - 1} \le \frac{\check{U}^\vartheta_\alpha(\check{x},\varkappa,\check{f})}{e^\varkappa - 1}$$

and

$$\widehat{G}_{f,\alpha}(\varkappa) = \frac{\widehat{g}_{f,\check{x}}(\varkappa) - \widehat{g}_{f,\check{x}}(0)}{e^\varkappa - 1} \le \frac{\check{U}^\vartheta_\alpha(\alpha,\varkappa,\check{f})}{e^\varkappa - 1}\,.$$

Therefore, taking into account that $\widetilde{\kappa} \le r_*$, we obtain, that for any function $1 \le f \le V$ and for any $0 < \varkappa < \widetilde{\kappa}$,

$$\widehat{\Delta}_{\check{x}}(\varkappa) \le \left(\widehat{\omega}(\varkappa) + \frac{e^\varkappa + 1}{e^\varkappa - 1}\right) \check{U}^\vartheta_\alpha(\check{x},\varkappa,\check{V})\check{U}^\vartheta_\alpha(\alpha,\varkappa,\check{V}) \\ \le \left(\widetilde{A}_3 \frac{e^{\widetilde{\kappa}-\varkappa}}{e^{\widetilde{\kappa}-\varkappa} - 1} + \frac{e^\varkappa + 1}{e^\varkappa - 1}\right) \check{U}^\vartheta_\alpha(\check{x},\varkappa,\check{V})\check{U}^\vartheta_\alpha(\alpha,\varkappa,\check{V})\,.$$

Now, putting here $\varkappa = \kappa^* = \widetilde{\kappa}/2$ and taking into account the inequality (5.6) we get, for any $\check{x} \in \check{\mathcal{X}}$, that

$$\widehat{\Delta}_{\check{x}}(\kappa^*) \le \left(\frac{(\widetilde{A}_3 + 1)e^{\kappa^*} + 1}{e^{\kappa^*} - 1}\right) V(<x>_1)\, V^*\; (B^*)^2\,.$$

Moreover, note that the chain $(\Phi_n)_{n\ge 1}$ is ergodic with the invariant measure π^ϑ defined in (5.7) and (A.12). By applying Proposition A.5 with λ equals to the Dirac measure at x, we obtain that, for any function $0 < f \le V$,

$$\mathbf{E}_x^\vartheta f(\Phi_n) - \pi^\vartheta(f) = (1-\delta)\left(\check{\mathbf{E}}_{x_0}^\vartheta\, \check{f}(\check{\Phi}_n) - \check{\pi}^\vartheta(\check{f})\right) \mathbf{1}_{\{x\in C\}}$$
$$+ \delta\left(\check{\mathbf{E}}_{x_1}^\vartheta\, \check{f}(\check{\Phi}_n) - \check{\pi}^\vartheta(\check{f})\right) \mathbf{1}_{\{x\in C\}}$$
$$+ \left(\check{\mathbf{E}}_{x_0}^\vartheta\, \check{f}(\check{\Phi}_n) - \check{\pi}^\vartheta(\check{f})\right) \mathbf{1}_{\{x\in C^c\}}\,.$$

Therefore, for any $x \in \mathcal{X}$, one gets

$$|\mathbf{E}_x^\vartheta f(\Phi_n) - \pi^\vartheta(f)| \le |\check{\mathbf{E}}_{x_0}^\vartheta\, \check{f}(\check{\Phi}_n) - \check{\pi}^\vartheta(\check{f})| + |\check{\mathbf{E}}_{x_1}^\vartheta\, \check{f}(\check{\Phi}_n) - \check{\pi}^\vartheta(\check{f})|\,,$$

i.e.

$$\sum_{n\ge 0} e^{\kappa^* n}\, |\mathbf{E}_x^\vartheta f(\Phi_n) - \pi^\vartheta(f)| \le \widehat{\Delta}_{x_0}(\kappa^*) + \widehat{\Delta}_{x_1}(\kappa^*)\,.$$

From here it follows the inequality (2.4). Hence Theorem 2.1. □

6 Application to diffusion processes

In order to study geometric ergodicity for the process (1.3) we start with the chain $(\Phi_n^y)_{n\ge 0}$, where $\Phi_n^y = y_n$.

Proposition 6.1. *For any $\vartheta \in \Theta$, the sequence $(\Phi_n^y)_{n\ge 0}$ is a homogeneous Markov chain aperiodic and ψ-irreducible, where ψ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.*

Proof. Taking into account (see, for example, [6]) that the solution of the equation (1.3) is a homogeneous Markov process, we obtain immediately that $(\Phi_n^y)_{n\ge 0}$ is a homogeneous Markov chain. In this case (see [6]), for any ϑ, the process $(y_t)_{t\ge 0}$ admits the transition density $\upsilon_\vartheta(t,x,y)$ as follows :

$$\upsilon_\vartheta(t,x,y) = \frac{\Upsilon(t,x,y)}{\sqrt{2\pi t}\,\sigma(y)} \exp\left\{\int_{\varsigma(x)}^{\varsigma(y)} H_\vartheta(u)\mathrm{d}u - \frac{(\varsigma(y)-\varsigma(x))^2}{2t}\right\}\,, \qquad (6.1)$$

Here,

$$H_\vartheta(z) = \frac{S(\check{\varsigma}(z))}{\sigma(\check{\varsigma}(z))} - \frac{\dot{\sigma}(\check{\varsigma}(z))}{2\sigma^2(\check{\varsigma}(z)}\,, \quad \varsigma(x) = \int_0^x \sigma^{-1}(u)\,\mathrm{d}u$$

and $\check{\varsigma}(\cdot)$ is the inverse function of ς, i.e. it is the unique solution of the equation $z = \varsigma(\check{\varsigma})$. Moreover,

$$\Upsilon(t,x,y) = \mathbf{E}\,\exp\left\{-\frac{1}{2}\int_0^t \widetilde{H}_\vartheta(w_{u,t}^*(x,y))\mathrm{d}u\right\}\,,$$

$\widetilde{H}_\vartheta(x) = \dot{H}_\vartheta(x) + H^2_\vartheta(x)$ and

$$w^*_{u,t}(x,y) = \varsigma(x) + \frac{u}{t}(\varsigma(y) - \varsigma(x)) + w_u - \frac{u}{t}\, w_t\,.$$

It means that, for any $n \ge 1$, for any $A \in \mathcal{B}(\mathbb{R})$ and for any $x \in \mathbb{R}$,

$$\mathbf{P}^\vartheta(\Phi^y_n \in A | \Phi^y_0 = x) = \int_A \upsilon_\vartheta(n,x,z)\mathrm{d}z\,. \tag{6.2}$$

Thus, the chain $(\Phi^y_n)_{n\ge 0}$ is ψ-irreducible, where ψ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Moreover, in this case the chain is aperiodic. □

Now, we check the minorization condition $\mathbf{H}_1$) for the chain $(\Phi^y_n)_{n\ge 0}$.

Proposition 6.2. *For any $K \ge 8\sigma_{min}$, the chain $(\Phi^y_n)_{n\ge 0}$ satisfies the minorization condition $\mathbf{H}_1$) with $C = [-K, K]$, $\delta = \delta_K$ and $\eta = \eta_K$ defined in* (3.10)*, and the probability measure ν_K defined in* (3.6)*.*

Proof. We start with studying the properties of the function H_ϑ defined in (6.1). From definition of the class Θ we find immediately that, for any $z \in \mathbb{R}$,

$$|\check{\varsigma}(z)| \le |z|\,\sigma_{max}\,.$$

Moreover, note that, for any ϑ from Θ and for any y from $\mathbb{R}$,

$$|\dot{S}(y)| \le M + L \quad \text{and} \quad |S(y)| \le M + L|y|\,. \tag{6.3}$$

Therefore,

$$\sup_{|z|\le z_*}\sup_{\vartheta\in\Theta} |H_\vartheta(z)| \le H^*_0 + L\sigma_*\, z_*\,,$$

where H^*_0 is given in (3.6). Note now, that the derivative of H_ϑ can be represented as $\dot{H}_\vartheta(z) = F_\vartheta(\check{\varsigma}(z))$ with

$$F_\vartheta(y) = \dot{S}(y) - \frac{S(y)\dot{\sigma}(y)}{\sigma(y)} - \frac{\ddot{\sigma}(y)}{2\sigma(y)} + \frac{(\dot{\sigma}(y))^2}{\sigma^2(y)}\,.$$

By making use of the upper bounds (6.3), we obtain

$$\sup_{\vartheta\in\Theta}|F_\vartheta(y)| \le H^*_1 + L\sigma_*|y|$$

and, therefore, for any $z_* > 0$,

$$\sup_{|z|\le z_*}\sup_{\vartheta\in\Theta}|\widetilde{H}_\vartheta(z)| \le \widetilde{H}_*(z_*)\,,$$

where $\widetilde{H}_*(z) = H^*_1 + 2(H^*_0)^2 + L\sigma_*\sigma_{max}\, z + 2(L\sigma_*)^2\, z^2$ and the coefficient H^*_1 is given in (3.6). Moreover, we note that on the set

$$\Gamma_K = \{ \sup_{0\le u\le 1} |w_u| \le \widetilde{K}/2\} \quad \text{with} \quad \widetilde{K} = K/\sigma_{min}\,,$$

the process $(w^*_{v,t}(x,y))_{0\le v\le t\le 1}$ is bounded :

$$\sup_{x,y\in C}\ \sup_{0\le u\le t\le 1} |w^*_{u,t}(x,y)| \le 2\widetilde{K}\,.$$

Therefore, for any x, y from C,

$$\Upsilon(1,x,y) \ge \mathbf{P}(\Gamma_K)\, e^{-\frac{1}{2}\widetilde{H}_*(2\widetilde{K})}\,.$$

By making use of the Doob inequality we obtain

$$\mathbf{P}(\Gamma_K) \ge 1 - \frac{4\mathbf{E}\sup_{0\le t\le 1} w^2_t}{K^2_1} \ge 1 - \frac{16\sigma^2_{min}}{K^2}\,,$$

i.e. for $K \ge 8\sigma_{min}$,

$$\mathbf{P}(\Gamma_K) \ge 3/4 \quad \text{and} \quad \Upsilon(1,x,y) \ge \frac{3}{4}\, e^{-\frac{1}{2}\widetilde{H}_*(2\widetilde{K})}\,.$$

Moreover, for any $x, y \in C$,

$$\sup_{\vartheta\in\Theta} \left| \int_{\varsigma(x)}^{\varsigma(y)} H_\vartheta(u)\mathrm{d}u \right| \le \left(H^*_0 + L\sigma_*\widetilde{K} \right)\, \widetilde{K}\,.$$

This implies that

$$\inf_{x,z\in C}\, \upsilon_\vartheta(1,x,z) \ge 3(4\sqrt{2\pi}\sigma_{max})^{-1}\, e^{-\Omega_*(\widetilde{K})}\,,$$

where $\Omega_*(z)$ is introduced in (3.10). Therefore, taking into account that, for any A from $\mathcal{B}(\mathcal{X})$,

$$\mathbf{P}^\vartheta(\Phi^y_1 \in A|\Phi^y_0 = x) = \int_A \upsilon_\vartheta(1,x,z)\mathrm{d}z\,,$$

yields the inequality (2.1) with δ_K and $\nu_K(\cdot)$ defined in (3.10) and (3.6) respectively. Moreover, the inequality (2.2) follows directly from Proposition A.8. Hence Proposition 6.2. $\square$

Now, for any $\mathbf{C}^2(\mathbb{R}) \to \mathbb{R}$ function V, we introduce the generator

$$\mathbf{A}_\vartheta(V)(x) = \dot{V}(x)S(x) + \frac{1}{2}\,\sigma^2(x)\ddot{V}(x)\,. \tag{6.4}$$

Definition 6.1. *Any $\mathbb{R} \to [1,\infty)$ twice continuously differentiable function V is called uniform over $\vartheta \in \Theta$ Lyapunov function for the equation* (1.3) *if the following conditions fulfill:*

- *for some constants $\gamma > 0$, $b^* > 0$ and for any $x \in \mathbb{R}$,*

$$\sup_{\vartheta\in\Theta} \mathbf{A}_\vartheta(V)(x) \le -\gamma V(x) + b^* \,; \tag{6.5}$$

- $\lim_{x\to\infty} V(x) = \infty$ *and there exists $m > 0$ such that*

$$\sup_{x\in\mathbb{R}} \frac{V(x) + |\dot{V}(x)|}{1 + |x|^m} < \infty \,. \tag{6.6}$$

Proposition 6.3. *For any $0 < \epsilon \le 1/2$, the function $V(x) = (1+x^2)^\epsilon$ satisfies the inequality* (6.5) *with $\gamma = 2\epsilon\beta$ and $b^* = \epsilon b^*_0$, where b^*_0 is given in* (3.11).

Proof. The definition of the space Θ implies directly that, for $|x| \ge \mathbf{x}_*$,

$$xS(x) \le |x|(M + \beta\mathbf{x}_*) - \beta x^2 \,.$$

Therefore, we get, for any $\vartheta \in \Theta$,

$$\begin{aligned}
\mathbf{A}_\vartheta(V)(x) &\le \frac{2\epsilon V(x) x S(x)}{1+x^2} + \epsilon\sigma^2_{max} \\
&\le \frac{2\epsilon\, V(x) x S(x)}{1+x^2} \mathbf{1}_{\{|x|\ge \mathbf{x}_*\}} + \epsilon\left((1+\mathbf{x}^2_*)^\epsilon M + \sigma^2_{max}\right) \\
&\le -2\epsilon V(x)\beta + b^* \,.
\end{aligned}$$

Hence Proposition 6.3. ☐

Proposition 6.4. *Let $V(x)$ be a uniform over $\vartheta \in \Theta$ Lyapunov function for equation* (1.3) *from definition 6.1 with constants γ and b^*. Then, for any $K > 0$ and $0 < \check{\varepsilon} < 1$ for which*

$$\inf_{|x|\ge K} V(x) \ge \frac{b^*}{\check{\varepsilon}\gamma} \,,$$

the chain $(\Phi^y_n)_{n\ge 0}$ satisfies the following inequality

$$\sup_{\vartheta\in\Theta} \mathbf{E}^\vartheta_x\, V(\Phi^y_1) \le (1-\rho)V(x) + D_K\, \mathbf{1}_{\{|x|\le K\}} \,, \tag{6.7}$$

where $\rho = (1-\check{\varepsilon})(1-e^{-\gamma})$, $D_K = V^*_K e^{-\gamma} + b^*(1-e^{-\gamma})/\gamma$ and $V^*_K = \sup_{|x|\le K} V(x)$.

Proof. By the Ito formula, one gets

$$V(y_t) = V(y_0) + \int_0^t \mathbf{A}_\vartheta(V)(y_s)\mathrm{d}s + \int_0^t \dot{V}(y_s)\sigma(y_s)\mathrm{d}w_s\,.$$

In Proposition A.7, we have proved that the moments of the solution of equation (1.3) are bounded. This implies that the stochastic integral is a martingale in the above Ito formula. Therefore, by setting $Z(t) = \mathbf{E}_x^\vartheta\, V(y_t)$, one has

$$\dot{Z}(t) = \mathbf{E}_x^\vartheta\, \mathbf{A}_\vartheta(V)(y_t) = -\gamma Z(t) + \psi_t\,,$$

where $\psi_t = \mathbf{E}_x^\vartheta\,(\mathbf{A}_\vartheta(V)(y_t) + \gamma y_t)$. The inequality (6.5) gives $\psi_t \le b^*$. Resolving this differential equation, we obtain, that for $0 \le t \le 1$

$$\begin{aligned} Z(t) &= Z(0)e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\psi_s\mathrm{d}s \\ &\le Z(0)e^{-\gamma t} + b^*\,\frac{1-e^{-\gamma}}{\gamma} = V(x)e^{-\gamma t} + b^*\frac{1-e^{-\gamma}}{\gamma}\,. \end{aligned}$$

Therefore,

$$\sup_{\vartheta\in\Theta} \mathbf{E}_x^\vartheta\, V(\Phi_1^y)\, \le V(x)e^{-\gamma} + b^*\frac{1-e^{-\gamma}}{\gamma}\,.$$

From here we obtain the inequality (6.7). Hence Proposition 6.4. □

6.1 Proof of Theorem 2.2

First note, that thanks to Propositions 6.1 the diffusion process (1.3) with the parameter $\vartheta = (S,\sigma)$ from Θ introduced in (2.5), satisfies the condition $\mathbf{H}_1$) with the set $C = [-K, K]$ for $K \ge K_0$ given (3.7) and the measure ν defined in (3.6). Moreover, Propositions 6.3 – 6.4 imply that for any $0 < \epsilon \le 1/2$ this process satisfies the condition $\mathbf{H}_2$) with the parameters (3.12). Moreover, for any $t \ge 1$ and any $\mathbb{R} \to]0,1]$ function g, we set

$$\widetilde{g}(x) = \mathbf{E}_x^\vartheta\, g(y_t)\, = \mathbf{E}_x^\vartheta\, g(y_{\{t\}})\,.$$

Moreover, taking into account that $\pi(g) = \pi(\tilde{g})$, one has

$$\mathbf{E}_x^\vartheta\, g(y_t) - \pi(g) = \mathbf{E}_x^\vartheta\, \widetilde{g}(\Phi_{[t]}^y) - \pi(\tilde{g})\,.$$

Therefore by applying Theorem 2.1 to the chain $(\Phi_n^y)_{n\ge 0}$, we come to the upper bound (2.7). Hence Theorem 2.2. □

A Appendix

A.1 Homogeneous Markov chains with atoms

We follow the Meyn-Tweedie approach (see [16]). We remind some definitions from [16] for a homogeneous Markov chains $(\Phi_n)_{n\geq 0}$ defined on a measurable state space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Denote by $P(x, \cdot)\,, x \in \mathcal{X}$, the transition probability of this chain, i.e. for any $A \in \mathcal{B}(\mathcal{X}), x \in \mathcal{X}$,

$$P(x, A) \,=\, \mathbf{P}_x(\Phi_1 \in A) = \mathbf{P}(\Phi_1 \in A | \Phi_0 \,=\, x)\,.$$

The $n-$step transition probability is

$$P^n(x, A) \,=\, \mathbf{P}_x(\Phi_n \in A)\,.$$

We remind that a measure π on $\mathcal{B}(\mathcal{X}))$ is called *invariant* for this chain if, for any $A \in \mathcal{B}(\mathcal{X})$,

$$\pi(A) \,=\, \int_{\mathcal{X}} P(x, A)\pi(\mathrm{d}x)\,.$$

If there exists an invariant positive measure π with $\pi(\mathcal{X}) \,=\, 1$ then the chain is called *positive.*

Definition A.1. *The chain $(\Phi_n)_{n\geq 0}$ is φ-irreducible if there exists a nontrivial measure φ on $\mathcal{B}(\mathcal{X})$ such that, whenever $\varphi(A) \,>\, 0$, one has*

$$L(x, A) = \mathbf{P}_x(\cup_{n=1}^{\infty}\{\Phi_n \,\in\, A\}) \,>\, 0 \quad \text{for any} \quad x \,\in\, \mathcal{X}\,.$$

One can show that, for any φ-irreducible chain, there exists a "maximal" irreducible measure which is noted as ψ and the chain is called ψ-irreducible. A irreducible measure ψ is maximal if and only if $\psi(A) = 0$ implies

$$\psi(x \in \mathbb{R} \,:\, L(x, A) > 0) = 0\,.$$

In the sequel, we denote

$$\mathcal{B}_+(\mathcal{X}) \,=\, \{A \,\in\, \mathcal{B}(\mathcal{X}) : \psi(A) \,>\, 0\}\,.$$

Definition A.2. *The chain $(\Phi_n)_{n\geq 0}$ is Harris recurrent if it is ψ-irreducible and, for any $A \,\in\, \mathcal{B}_+(\mathcal{X})$, one has*

$$\mathbf{P}_x\left(\sum_{n=1}^{\infty} \mathbf{1}_{\{\Phi_n \in A\}}\right) = 1\,, \quad \text{for any} \quad x \,\in\, A\,.$$

Definition A.3. *The Markov ψ-irreducible chain $(\Phi_n)_{n\geq 0}$ is called periodic of period d if there exist disjoint sets $\Gamma_1, \ldots, \Gamma_d$ in $\mathcal{B}(\mathcal{X})$ with*

$$\psi\left(\cap_{j=1}^{d}\Gamma_j^c\right) = 0$$

such that, for $1 \leq i \leq d-1$ and for any $x \in \Gamma_i$, one has

$$\mathbf{P}_x(\Phi_1 \in \Gamma_{i+1}) = 1$$

and for $x \in \Gamma_d$ one has $\mathbf{P}_x(\Phi_1 \in \Gamma_1) = 1$. The chain is aperiodic if $d = 1$.

Definition A.4. *We will say that the chain $(\Phi_n)_{n\geq 0}$ satisfies the minorization condition if, for some $\delta > 0$, some set $C \in \mathcal{B}(\mathcal{X})$ and some probability measure ν with $\nu(C) = 1$, one has*

$$\inf_{A\in\mathcal{B}(\mathcal{X})}\left(\inf_{x\in C} P(x,A) - \delta\,\nu(A)\right) \geq 0\,. \tag{A.1}$$

Definition A.5. *A set $\alpha \in \mathcal{B}_+(\mathcal{X})$ is called accessible atom if, for any x and y from α,*

$$\mathbf{P}(x,\Gamma) = \mathbf{P}(y,\Gamma)\,, \quad \forall\,\Gamma \in \mathcal{B}(\mathcal{X})\,.$$

In order to study the ergodicity property, we associate to any set $C \in \mathcal{B}(\mathcal{X})$ the stopping time

$$\tau_C = \inf\{k \geq 1\,:\,\Phi_k \in C\}\,.$$

Proposition A.1. *Suppose that the Markov chain Φ is ψ-irreducible and contains an accessible atom α such that*

$$\mathbf{E}_\alpha \tau_\alpha < \infty\,.$$

Then the chain is ergodic with the invariant probability measure π defined as

$$\pi(\Gamma) = \frac{1}{\mathbf{E}_\alpha\,\tau_\alpha}\,\mathbf{E}_\alpha \sum_{j=1}^{\tau_\alpha} \mathbf{1}_{\{\Phi_j\in\Gamma\}}\,.$$

Proof. Indeed, by the definition of π, for any set $\Gamma \in \mathcal{B}(\mathcal{X})$, one has

$$\int_{\mathcal{X}} \pi(\mathrm{d}z)\,\mathbf{P}(z,\Gamma) = \frac{1}{\mathbf{E}_\alpha\,\tau_\alpha}\,\mathbf{E}_\alpha \sum_{j=1}^{\infty} \mathbf{1}_{\{j\leq\tau_\alpha\}}\,\mathbf{E}_\alpha\left(\mathbf{1}_{\{\Phi_{j+1}\in\Gamma\}}|\Phi_1,\ldots,\Phi_j\right)$$

$$= \frac{1}{\mathbf{E}_\alpha\,\tau_\alpha}\left(\mathbf{E}_\alpha \sum_{j=2}^{\tau_\alpha} \mathbf{1}_{\{\Phi_j\in\Gamma\}} + \mathbf{P}_\alpha\left(\Phi_{\tau_\alpha+1} \in \Gamma\right)\right)\,.$$

Moreover, it is easy to see that

$$\mathbf{P}_\alpha\left(\Phi_{\tau_\alpha+1}\in\Gamma\right)=\mathbf{P}_\alpha\left(\Phi_1\in\Gamma\right)\,.$$

This implies the relationship

$$\int_{\mathcal{X}}\pi(\mathrm{d}z)\,\mathbf{P}(z,\Gamma)=\pi(\Gamma)\,,$$

i.e. the measure π is invariant. Obviously, that $\pi(\mathcal{X})=1$, i.e. π is a probability measure. □

A.2 Lyapunov functions method for Markov chains

We start with the definition of a "Lyapunov function". For this we impose the following drift condition to the chain $(\Phi_n)_{n\ge0}$, i.e.

D) *There exist a* $\mathcal{X}\to[1,\infty)$ *function* V, *constants* $0<\rho<1$, $D\ge1$ *and a set* C *from* $\mathcal{B}(\mathcal{X})$ *such that for all* $x\in\mathcal{X}$

$$\mathbf{E}_x\left(V(\Phi_1)\right)\,\le\,(1-\rho)V(x)\,+\,D\mathbf{1}_C(x)\,. \tag{A.2}$$

In this case we call V the *Lyapunov function.*

Now, for any $\mathcal{X}\to[1,+\infty)$ function f and any set $A\in\mathcal{B}(\mathcal{X})$, we set

$$U_A(x,r,f)=\mathbf{E}_x\sum_{j=1}^{\tau_A}e^{rj}\,f(\Phi_j)\,. \tag{A.3}$$

Proposition A.2. *Assume, that the condition* **D**) *holds. Then, for any* $0<a<-\ln(1-\rho)$, *one has*

$$\sup_{x\in\mathcal{X}}\frac{U_C(x,a,V)}{V(x)}\le U^*(a)\,, \tag{A.4}$$

where

$$U^*(a)=\frac{(1-\rho)e^a+D\,e^a}{1-(1-\rho)e^a}\,.$$

Proof. The condition (A.2) implies immediately

$$U_C(x,a,V)\le\,(1-\rho)e^aV(x)+\,(1-\rho)e^aU_C(x,a,V)+D\,e^a\,.$$

Taking into account that $V(x)\ge1$, we obtain the inequality (A.4). □

Proposition A.3. *Assume that for some $a = a_* > 0$ the Markov chain $(\Phi_n)_{n\ge 1}$ satisfies the property* (A.4) *with the C - bounded function V, i.e.*

$$V^* = \sup_{x\in C} V(x) < \infty\,. \tag{A.5}$$

Let B be a set from $\mathcal{B}(\mathcal{X})$ such that, for some $\mathbf{k}_* > 0$,

$$\inf_{x\in C\setminus B} \mathbf{P}(x,B) \ge \mathbf{k}_*\,. \tag{A.6}$$

Then, for any $0<\gamma<1$ and

$$0 < a \le \frac{(1-\gamma)|\ln(1-\mathbf{k}_*)|}{\ln V^* U^*(a_*) + |\ln(1-\mathbf{k}_*)|}\, a_*\,, \tag{A.7}$$

one has

$$\sup_{x\in\mathcal{X}} \frac{1}{V(x)}\, U_B(x,a,V) \le U_1^*(a)\,, \tag{A.8}$$

where

$$U_1^*(a) = U^*(a)\left(1 + \frac{U^*(a)V^*}{1-(1-\mathbf{k}_*)^\gamma}\right)\,.$$

Proof. First, we introduce the sequence of stopping times $(\tau_C(n))_{n\ge 0}$ as follows : $\tau_C(0) = 0$ and, for $n\ge 1$,

$$\tau_C(n) = \inf\{k \ge \tau_C(n-1)+1\,:\, \Phi_k\in C\}\,.$$

Obviously, that $\tau_C(1) = \tau_C$. Moreover, the condition (A.4) implies that $\mathbf{E}_x\tau_C(n) < \infty$ for any $n\ge 1$ and $x\in\mathcal{X}$. Using this sequence we obtain that for $0<a\le a_*$

$$U_B(x,a,V) = \sum_{n=0}^{\infty} \mathbf{E}_x \sum_{j=\tau_C(n)+1}^{\tau_C(n+1)} e^{aj}\, V(\Phi_j)\, \mathbf{1}_{\{\tau_B\ge j\}}$$

$$\le U_C(x,a,V) + \sum_{n=1}^{\infty} \mathbf{E}_x\, \mathbf{1}_{\{\tau_B>\tau_C(n)\}}\, e^{a\tau_C(n)}\, U_C(z_n,a,V)\,,$$

where $z_n = \Phi_{\tau_C(n)}$. Moreover, taking into account the inequalities (A.4) and (A.5), we get that, for $a\le a_*$,

$$U_B(x,a,V) \le U^*(a)V(x) + U^*(a)V^* \sum_{n=1}^{\infty} \Upsilon_n(x,a)\,,$$

where

$$\Upsilon_n(x,a) = \mathbf{E}_x\, \mathbf{1}_{\{\tau_B>\tau_C(n)\}}\, e^{a\tau_C(n)}\,.$$

Note now that, for $n = 1$,

$$\Upsilon_1(x,a) \le V(x)U^*(a)\,.$$

Let now $n \ge 2$. We have

$$\Upsilon_n(x,a) = \mathbf{E}_x\,\mathbf{1}_{\{\tau_B > \tau_C(n)\}}\,e^{a\tau_C(n)}$$
$$= \mathbf{E}_x\,\mathbf{1}_{\{\tau_B > \tau_C(n-1)\}}\,e^{a\tau_C(n-1)}\mathbf{E}_{z_{n-1}}\,e^{a\tau_C}\,\mathbf{1}_{\{\tau_B > \tau_C\}}\,.$$

Therefore, for $n \ge 2$,

$$\Upsilon_n(x,a) \le \Upsilon_{n-1}(x,a)\,\sup_{z\in C\setminus B}\mathbf{E}_z\,e^{a\tau_C}\,\mathbf{1}_{\{\tau_B>1\}}\,.$$

By the Hölder inequality and the condition (A.5), we get

$$\sup_{z\in C\setminus B}\mathbf{E}_z\,e^{a\tau_C}\,\mathbf{1}_{\{\tau_B>1\}}\;\le\;(V^*U^*(a_*))^{a/a_*}\;(1-\mathbf{k}_*)^{1-a/a_*}:=\mathbf{g}(a)\,.$$

Therefore, for any a satisfying the condition (A.7) we obtain

$$\mathbf{g}(a) \le (1-\mathbf{k}_*)^{\gamma}$$

and, for any $n \ge 2$,

$$\Upsilon_n(x,a) \le \Upsilon_1(x,a)\,\mathbf{g}(a)^{n-1} \le V(x)U^*(a)\,(1-\mathbf{k}_*)^{\gamma(n-1)}\,.$$

This implies directly the bound (A.8). Hence Proposition A.3. $\square$

A.3 Properties of splitting chains

Now, we study some property of the splitting chain $(\check{\Phi}_n)_{n\ge1}$ constructed in Section 4 , which we represent as

$$\check{\Phi}_n = (\check{\phi}_n, \check{\iota}_n)\,, \tag{A.9}$$

where $\check{\phi}_n \in \mathcal{X}$ and $\check{\iota}_n \in \{0,1\}$.

Proposition A.4. *For any measure λ on $\mathcal{B}(\mathcal{X})$ and any set $\check{\Gamma} \in \mathcal{B}(\check{\mathcal{X}})$,*

$$\int_{\check{\mathcal{X}}}\check{\mathbf{P}}^{\vartheta}(\check{x},\check{\Gamma})\,\lambda^*(\mathrm{d}\check{x}) = \lambda_1^*(\check{\Gamma})\,, \tag{A.10}$$

where

$$\lambda_1(\cdot) = \int_{\mathcal{X}}\mathbf{P}^{\vartheta}(x,\cdot)\,\lambda(\mathrm{d}x)\,.$$

Proof. Indeed, by the definition of the $*$ operation and of the transition probability $\check{\mathbf{P}}^{\vartheta}(\cdot,\cdot)$ we obtain

$$\int_{\check{\mathcal{X}}} \check{\mathbf{P}}^{\vartheta}(\check{x},\check{\Gamma})\,\lambda^*(\mathrm{d}\check{x}) = \int_{\mathcal{X}} \mathbf{P}^{\vartheta}(x,\check{\Gamma})^*\,\lambda(\mathrm{d}x) = \lambda_1^*(\check{\Gamma})\,.$$

□

Proposition A.5. *For any $n \geq 1$, any measurable positive $\mathcal{X}^n \to \mathbb{R}$ function G and for any measure λ on $\mathcal{B}(\mathcal{X})$, one has*

$$\int_{\mathcal{X}} \mathbf{E}_x^{\vartheta}\, G_n(\Phi_1,\ldots,\Phi_n)\,\lambda(\mathrm{d}x) = \int_{\check{\mathcal{X}}} \check{\mathbf{E}}_{\check{x}}^{\vartheta}\, G_n(\check{\phi}_1,\ldots,\check{\phi}_n)\,\lambda^*(\mathrm{d}\check{x})\,. \tag{A.11}$$

Proof. It is clear, that it suffices to check this equality for positive functions of the form

$$G_n(x_1,\ldots,x_n) = \prod_{j=1}^{n} g_j(x_j)\,.$$

First, we check this equality for $n = 1$. Note that, for any $\mathcal{X} \to \mathbb{R}$ function g and for any $x \in \mathcal{X}$, one has

$$\int_{\check{\mathcal{X}}} g(<\check{y}>_1)\,\mathbf{P}^*(x,\mathrm{d}\check{y}) = \int_{\mathcal{X}} g(y)\,\mathbf{P}(x,\mathrm{d}y)\,,$$

where $<\check{y}>_1$ denotes the first component of the $\check{y} \in \check{\mathcal{X}} = \mathcal{X} \times \{0,1\}$. Making use of this equality implies easy (A.11) for $n = 1$. Assume now that the equality (A.11) is true until $n-1$. We check it for n. Indeed, we have

$$\check{\mathbf{E}}_{\check{x}}^{\vartheta} G_n(\check{\phi}_1,\ldots,\check{\phi}_n) = \check{\mathbf{E}}_{\check{x}}^{\vartheta} \prod_{j=1}^{n} g_j(x_j) = \check{\mathbf{E}}_{\check{x}}^{\vartheta}\, g_1(\check{\phi}_1)\,T(\check{\Phi}_1)\,,$$

where

$$T(\check{y}) = \check{\mathbf{E}}_{\check{y}}^{\vartheta} \prod_{j=1}^{n-1} g_{j+1}(\check{\phi}_j)\,.$$

Now, we set

$$\mu(\Gamma) = \int_{\Gamma} g_1(y)\,\lambda_1(\mathrm{d}y)\,,$$

where the measure $\lambda_1(\cdot)$ is defined in (A.10). Therefore, taking into account Proposition A.4 , we can represent the integral on the right hand side of the equality (A.11) as

$$\int_{\check{\mathcal{X}}} \check{\mathbf{E}}_{\check{x}}^{\vartheta}\, G_n(\check{\phi}_1,\ldots,\check{\phi}_n)\,\lambda^*(\mathrm{d}\check{x}) = \int_{\check{\mathcal{X}}} T(\check{y})\mu^*(\mathrm{d}\check{y})\,.$$

By the induction assumption, one has

$$\int_{\check{\mathcal{X}}} T(\check{y})\mu^*(\mathrm{d}\check{y}) = \int_{\mathcal{X}} \mathbf{E}_y \prod_{j=1}^{n-1} g_{j+1}(\Phi_j)\ \mu(\mathrm{d}y) = \int_{\mathcal{X}} \mathbf{E}_x\, G_n(\Phi_1,\ldots,\Phi_n)\,\lambda(\mathrm{d}x)\,.$$

Hence, the Proposition A.5. $\square$

Proposition A.6. *Assume that the splitting chain $(\check{\Phi}_n)_{n\geq 1}$ has an invariant probability measure $\check{\pi}$. Then, the chain $(\Phi_n)_{n\geq 1}$ has the invariant probability measure π on $\mathcal{B}(\mathcal{X})$ which is given as*

$$\pi(\Gamma) = \check{\pi}(\Gamma_0) + \check{\pi}(\Gamma_1)\,. \tag{A.12}$$

Moreover, $\check{\pi} = \pi^*$.

Proof. First we check directly that $\check{\pi} = \pi^*$. Moreover, for any $\Gamma \in \mathcal{B}(\mathcal{X})$

$$\pi(\Gamma) = \check{\pi}(\Gamma_0 \cup \Gamma_1) = \int_{\check{\mathcal{X}}} \check{\pi}(\mathrm{d}\check{z})\,\check{\mathbf{P}}^{\vartheta}(\check{z},\Gamma_0\cup\Gamma_1)$$

$$= \int_{\check{\mathcal{X}}} \pi^*(\mathrm{d}\check{z})\,\check{\mathbf{P}}^{\vartheta}(\check{z},\Gamma_0\cup\Gamma_1)\,.$$

Therefore, applying here Proposition A.5 we obtain that

$$\pi(\Gamma) = \int_{\mathcal{X}} \mathbf{P}(z,\Gamma)\,\pi(\mathrm{d}z)\,,$$

i.e. π is the invariant measure for the chain $(\Phi_n)_{n\geq 1}$. Hence Proposition A.6. $\square$

A.4 Moment inequality for the process (1.3)

Proposition A.7. *Let $(y_t)_{t\geq 0}$ be a solution of the equation* (1.3)*. Then, for any $m \geq 1$ and for any $x \in \mathbb{R}$,*

$$\sup_{t\geq 0}\sup_{\vartheta\in\Theta} \mathbf{E}_x^{\vartheta}\,(y_t)^{2m} \leq (2m-1)!!\,(x^2 + \mathbf{M}_*/\beta)^m\,, \tag{A.13}$$

where $\mathbf{M}_* = (M + \beta\mathbf{x}_*)^2/\beta + 2\sigma^2_{max}$.

Proof. To obtain this inequality we make use of the method proposed in ([9], p.20) for linear stochastic equation. First of all note that thanks to Theorem 4.7 from [15], for any $T > 0$, there exists some $\epsilon > 0$ such that for each $\vartheta \in \Theta$ and $x \in \mathbb{R}$

$$\sup_{0\leq t\leq T} \mathbf{E}_x^{\vartheta}\, e^{\epsilon y_t^2} < \infty\,. \tag{A.14}$$

Applying the Ito formula to y_t^{2m} and denoting

$$\mathbf{B}_\vartheta(y) = 2yS(y) + \sigma^2(y) + \beta y^2 \,,$$

yield

$$\begin{aligned}\mathrm{d}y_t^{2m} &= -m\beta y_t^{2m}\mathrm{d}t + m y_t^{2(m-1)} \left(\mathbf{B}_\vartheta(y_t) + 2(m-1)\sigma^2(y_t)\right) \mathrm{d}t \\ &\quad + 2m y_t^{2m-1}\sigma(y_t)\mathrm{d}W_t \,.\end{aligned}$$

Therefore, taking into account that $y_0 = x$ we can represent the last equation in the following integral form

$$\begin{aligned}y_t^{2m} &= e^{-m\beta t}\, x^{2m} + m\int_0^t e^{-m\beta(t-s)}\, y_s^{2(m-1)} \left(\mathbf{B}_\vartheta(y_s) + 2(m-1)\sigma^2(y_s)\right) \mathrm{d}s \\ &\quad + 2m\int_0^t e^{-m\beta(t-s)}\, y_s^{2m-1}\sigma(y_s)\mathrm{d}W_s \,. \qquad \text{(A.15)}\end{aligned}$$

One can check directly that

$$\sup_{\vartheta\in\Theta}\sup_{y\in\mathbb{R}} |\mathbf{B}_\vartheta(y)| \le \frac{(M+\beta\mathbf{x}_*)^2}{\beta} + \sigma^2_{max} \,.$$

Moreover, the property (A.14) yields that, for any $m \ge 1$,

$$\mathbf{E}^\vartheta \int_0^t e^{-m\beta(t-s)}\, y_s^{2m-1}\sigma(y_s)\mathrm{d}W_s = 0 \,.$$

Therefore, setting $z_t(m) = \mathbf{E}_x^\vartheta y_t^{2m}$, we obtain

$$z_t(m) \le x^{2m} + m(2m-1)\mathbf{M}_* \int_0^t e^{-m\beta(t-s)}\, z_s(m-1)\, \mathrm{d}s \,.$$

The induction implies directly the bound (A.13). Hence Proposition A.7. □

Proposition A.8. *Let $(y_t)_{t\ge0}$ be a solution of the equation* (1.3). *Then, for any $K > \sqrt{\mathbf{M}_1}$,*

$$\sup_{|x|\le K}\sup_{\vartheta\in\Theta} \mathbf{P}_x^\vartheta\left(|y_1| \ge K\right) \le \frac{4\sigma^2_{max}\,(K^2+\mathbf{M}_2)}{\beta(1-e^{-\beta})\,(K^2-\mathbf{M}_1)^2} \,, \qquad \text{(A.16)}$$

where $\mathbf{M}_2$ and $\mathbf{M}_1$ are given in (3.9).

Proof. First, putting in (A.15) $m = 1$, we obtain

$$\sup_{t\geq 0} \mathbf{E}_x^\vartheta \, y_t^2 \,\leq\, x^2 + \mathbf{M}_2$$

and

$$\mathbf{P}_x^\vartheta \left(y_1^2 \geq K^2\right) \leq \mathbf{P}\left(2\zeta \geq \left(K^2 - \mathbf{M}_1\right)(1 - e^{-\beta})\right) \,,$$

where $\zeta = \int_0^1 e^{-\beta(1-s)} \, y_s \sigma(y_s) \mathrm{d}W_s$. Now, taking into account that for $|x| \leq K$

$$\mathbf{E}_x^\vartheta \, \zeta^2 = \int_0^1 e^{-2\beta(1-s)} \, \mathbf{E}_x^\vartheta \, y_s^2 \sigma^2(y_s) \mathrm{d}s \leq \sigma_{max}^2 \left(K^2 + \mathbf{M}_2\right) \frac{1 - e^{-\beta}}{\beta} \,.$$

The Chebyshev inequality implies now the bound (A.16). Hence Proposition A.8. □

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