THE HAUSDORFF DIMENSION OF THE BOUNDARY OF THE IMMEDIATE BASIN OF INFINITY OF McMULLEN MAPS

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Abstract. In this paper, we give a formula of the Hausdorff dimension of the boundary of the immediate basin of infinity of McMullen maps $f_p(z) = z^Q + p/z^Q$, where $Q \geq 3$ and p is small. This gives a lower bound of the Hausdorff dimension of the Julia sets of McMullen maps in the special cases.

1. Introduction

The dynamics of McMullen maps

$$f_p(z) = z^Q + p/z^Q$$

with $Q \geq 3$ have been studied a lot ([2–4, 11, 12]). These special rational maps can be viewed as a perturbation of the simple polynomial $f_0(z) = z^Q$.

It is known from [2, 7] that for small p, the Julia set $J(f_p)$ of f_p consists of uncountably many Jordan curves about the origin. This kind of Julia set is homeomorphic to $\mathcal{C} \times \mathbb{S}$, where $\mathcal{C}$ is the middle third Cantor set and $\mathbb{S}$ is the unit circle (See Figure 1). These Julia sets are called *Cantor circles*. In this case, all Fatou components are attracted by ∞. We denote by B_p the immediate attracting basin of ∞, then the boundary ∂B_p is a Jordan curve (actually quasicircle by Lemma 2.3). In fact, it is proven in [12] that ∂B_p is always a Jordan curve if $J(f_p)$ is not a Cantor set. In this paper, we obtain the following main theorem:

Theorem 1.1. *Let $Q \geq 3$, then for small p such that $J(f_p)$ is a Cantor circle, the Haudorff dimension of ∂B_p is*

$$\dim_H(\partial B_p) = 1 + \frac{|p|^2}{\log Q} + \mathcal{O}(|p|^3). \tag{1.1}$$

In particular, if $Q \neq 4$, then the higher order $\mathcal{O}(|p|^3)$ can be replaced by $\mathcal{O}(|p|^4)$.

As an immediate corollary, the main theorem gives a lower bound of the Hausdorff dimension of $J(f_p)$ with small p.

We would like to mention that for the polynomials $P_c(z) = z^d + c$ with $d \geq 2$ and small c such that P_c is hyperbolic, the Hausdorff dimension of the Julia set of P_c has been calculated in [10], [13] and [1], where the dimensional formula was expanded to the second order, third order and fourth order in c, respectively. In theory, terms of higher orders can be calculated successively. However, the calculation become more complicated as the rising of order.

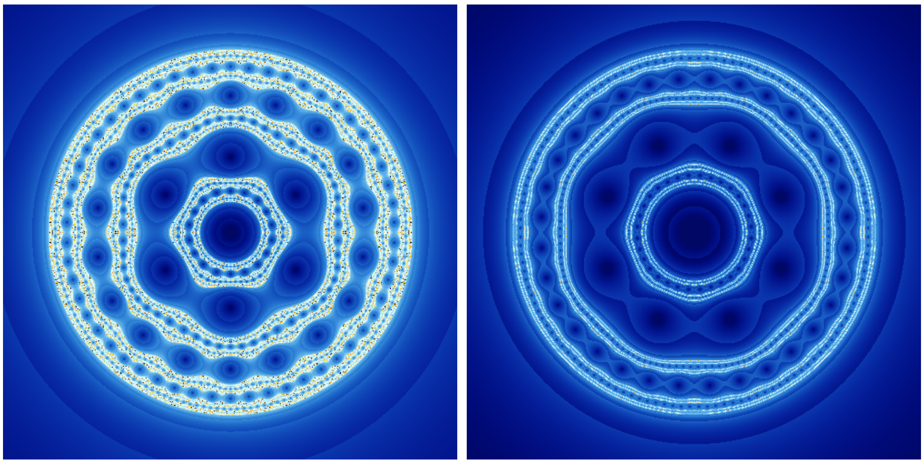

Figure 1: The Julia sets of $f_p(z) = z^Q + p/z^Q$, where $p = 0.005$ and $Q = 3, 4$ respectively. Both are Cantor circles. Figure range: $[-1.25, 1.25] \times [-1.25, 1.25]$.

2. Proof of the Main Theorem

The proof of the main theorem is similar to that in [9] and [13]. All details of complicated calculations will be included in the next section. In the following, we always assume that p is small ($p = 0$ is allowed).

Unlike the polynomials $P_c(z) = z^d + c$, the parameter space of McMullen family has a special point at $p = 0$. The whole Julia set $J(f_p)$ does not converge to $J(f_0)$ (the unit circle $\mathbb{S}$) in Hausdorff topology when p tends to 0, see [3]. However, the boundary of the immediately attracting basin of infinity ∂B_p does. In fact, we can show (Lemma 2.3) that ∂B_p is a holomorphic motion of the unit circle $\mathbb{S}$. For this, we first recall the definition of holomorphic motion.

Definition 2.1 (Holomorphic Motion, [6])**.** Let E be a subset of $\widehat{\mathbb{C}}$, a map $h : \mathbb{D} \times E \to \widehat{\mathbb{C}}$ is called a *holomorphic motion* of E parameterized by $\mathbb{D}$ and with base point 0 if

(1) For every $z \in E$, $\beta \mapsto h(\beta, z)$ is holomorphic for β in $\mathbb{D}$;
(2) For every $\beta \in \mathbb{D}$, $z \mapsto h(\beta, z)$ is injective on E; and
(3) $h(0, z) = z$ for all $z \in E$.

The unit disk $\mathbb{D}$ in Definition 2.1 can be replaced by any other topological disk.

Theorem 2.2 (The λ-Lemma, [6])**.** *A holomorphic motion $h : \mathbb{D} \times E \to \widehat{\mathbb{C}}$ of E has a unique extension to a holomorphic motion $h : \mathbb{D} \times \overline{E} \to \widehat{\mathbb{C}}$ of $\overline{E}$. The extension is a continuous map. For each $\beta \in \mathbb{D}$, the map $h(\beta, \cdot) : E \to \widehat{\mathbb{C}}$ extends to a quasiconformal map of the sphere to itself.*

It's known from [11] that in the parameter space of f_p, the McMullen domain $\mathcal{M} := \{p \in \mathbb{C} - \{0\}; J(f_p) \text{ is a Cantor circle}\}$ is a deleted neighborhood of the origin. It turns out that $\mathcal{V} = \mathcal{M} \cup \{0\}$ is a topological disk containing 0.

Lemma 2.3. *There is a holomorphic motion $H : \mathcal{V} \times \mathbb{S} \to \mathbb{C}$ parameterized by $\mathcal{V}$ and with base point 0 such that $H(p, \mathbb{S}) = \partial B_p$ for all $p \in \mathcal{V}$.*

Proof. We first prove that every repelling periodic point of $f_0(z) = z^Q$ moves holomorphically in $\mathcal{V}$. Let $z_0 \in \mathbb{S}$ be such a point with period k. For small p, the map f_p is a small perturbation of f_0. By implicit function theorem, there is a neighborhood U_0 of 0 such that z_0 becomes a repelling point z_p of f_p with the same period k, for all $p \in U_0$. On the other hand, for all $p \in \mathcal{M}$, since f_p has no non-repelling cycles, each repelling cycle of f_p moves holomorphically throughout $\mathcal{M}$ (See Theorem 4.2 in [8]).

Since $\mathcal{V}$ is simply connected, there is a holomorphic map $Z : \mathcal{V} \to \mathbb{C}$ such that $Z(p) = z_p$ for $p \in U_0$. Let $\mathrm{Fix}(f_0)$ be all repelling points of f_0. Then the map $h : \mathcal{V} \times \mathrm{Fix}(f_0) \to \mathbb{C}$ defined by $h(p, z_0) = Z(p)$ is a holomorphic motion. Notice that $\mathbb{S} = \overline{\mathrm{Fix}(f_0)}$, by Theorem 2.2, there is an extension of h, say $H : \mathcal{V} \times \mathbb{S} \to \mathbb{C}$. It's obvious that $H(p, \mathbb{S})$ is a connected component of $J(f_p)$.

To finish, we show $H(p, \mathbb{S}) = \partial B_p$ for all $p \in \mathcal{V}$. By the uniqueness of the holomorphic motion of hyperbolic Julia sets, it suffices to show $H(p, \mathbb{S}) = \partial B_p$ for small and real parameter $p \in (0, \epsilon)$, where $\epsilon > 0$.

Under the small perturbation f_p with $p \in (0, \epsilon)$, the fixed point $z_0 = 1$ of f_0 becomes the repelling fixed points z_p of f_p, which is real and close to 1. The map f_p has exactly two real and positive fixed points. One is z_p and the other is z_p^*, which is near 0. It's obvious that z_p is the landing point of the zero external ray of f_p. So $H(p, 1) = z_p \in \partial B_p$. This implies $H(p, \mathbb{S}) = \partial B_p$ for all $p \in (0, \epsilon)$. □

The boundary ∂B_p is a 'repeller' of the map f_p in the sense of Ruelle [10].

Theorem 2.4 (Ruelle, [10])**.** *If the repeller J_λ of a family of real analytic conformal maps f_λ depends analytically on λ, then the Hausdorff dimension of J_λ depends real analytically on λ.*

We define a function $H : \mathcal{V} \to \mathbb{R}^+$ by $H(p) = \dim_H(\partial B_p)$. We first derive some basic properties of H. The fact $\partial B_0 = \mathbb{S}$ implies $H(0) = 1$. By Ruelle's theorem, we know that H is a real analytic function. Thus when p is near 0, we have

$$H(p) = \sum_{s,t \geq 0} a_{st} p^s \bar{p}^t, \ a_{00} = 1. \tag{2.1}$$

It follows from $\overline{f_p(\bar{z})} = f_{\bar{p}}(z)$ and $f^{\circ 2}_{e^{2\pi i/(Q-1)}p}(e^{\pi i/(Q-1)} z) = e^{\pi i/(Q-1)} f_p^{\circ 2}(z)$ that

$$\overline{H(p)} = H(p) = H(\bar{p}), \quad H(e^{2\pi i/(Q-1)} p) = H(p).$$

So the coefficients satisfy

$$a_{st} = \overline{a_{st}} = a_{ts}, \ a_{st} = a_{st} e^{2\pi i(s-t)/(Q-1)}.$$

In particular, if $s - t \neq 0 \mathrm{mod}(Q-1)$, then $a_{st} = 0$. Thus we have

$$H(p) = \begin{cases} 1 + a_{20}(p^2 + \bar{p}^2) + a_{11}|p|^2 + \mathcal{O}(|p|^4), & \text{if } Q = 3, \\ 1 + a_{11}|p|^2 + \mathcal{O}(|p|^3), & \text{if } Q = 4, \\ 1 + a_{11}|p|^2 + \mathcal{O}(|p|^4), & \text{if } Q \geq 5. \end{cases}$$

To compute the Hausdorff dimension of ∂B_p, we need the following result (See [5], Theorem 9.1, Propositions 9.6 and 9.7)

Theorem 2.5 (Falconer, [5])**.** *Let $S_1, \cdots, S_m$ be contractive maps on a closed subset D of $\mathbb{R}^n$ such that $|S_i(x) - S_i(y)| \leq c_i|x - y|$ with $c_i < 1$. Then*

(1) *There there exists a unique non-empty compact set J such that $J = \bigcup_{i=1}^m S_i(J)$.*

(2) *The Hausdorff dimension $H(J)$ of J satisfies $H(J) \leq s$, where $\sum_{i=1}^m c_i^s = 1$.*

(3) *If we require further $|S_i(x) - S_i(y)| \geq b_i|x - y|$ for $i = 1, \cdots, m$, then $H(J) \geq \widetilde{s}$, where $\sum_{i=1}^m b_i^{\widetilde{s}} = 1$.*

Now, we have

Lemma 2.6. *For any $p \in \mathcal{V}$, the Hausdorff dimension $D = H(p)$ of ∂B_p is determined by the following equation*

$$\sum_{z \in \mathrm{Fix}(f_p^{\circ n}) \cap \partial B_p} |(f_p^{\circ n})'(z)|^{-D} = \mathcal{O}(1). \tag{2.2}$$

Proof. Let $w_p \in \partial B_p$ be the landing point of the zero external ray of f_p. We can split w_p into two point w_p^+ and w_p^- and view ∂B_p as a closed segment with extreme points w_p^+ and w_p^-. The map $f_p^{\circ n} : \partial B_p \to \partial B_p$ has Q^n inverse branches, say $S_1, \cdots, S_{Q^n}$, each maps ∂B_p to a closed segment such that their images are in anticlockwise order. Moreover, $\partial B_p = \bigcup S_i(\partial B_p)$. In particular, both $S_1(\partial B_p)$ and $S_{Q^n}(\partial B_p)$ contain w_p as an end point, for $1 < j < Q^n$, $S_j(\partial B_p)$ contains exactly one fixed point of $f_p^{\circ n}$. By Koebe distortion theorem and the fact that ∂B_p is a quasicircle, there exist two constants C_1, C_2 both independent of n, such that

$$\frac{C_1}{|(f_p^{\circ n})'(\zeta)|} \leq \frac{|S_i(x) - S_i(y)|}{|x - y|} \leq \frac{C_2}{|(f_p^{\circ n})'(\zeta)|}, \ \forall 1 \leq i \leq Q^n, \ x, y \in S_i(\partial B_p),$$

where ζ is the unique fixed point of $f_p^{\circ n}$ in $S_i(\partial B_p)$.

By Theorem 2.5, we have $s_1 \leq D \leq s_2$, where $\sum_\zeta C_j^{s_j} |(f_p^{\circ n})'(\zeta)|^{-s_j} = 1, \ j = 1, 2$. It turns out that when n is large, the sum $\sum_\zeta |(f_p^{\circ n})'(\zeta)|^{-D}$ is a number between C_1^{-D} and C_2^{-D} and

$$\sum_{z \in \mathrm{Fix}(f_p^{\circ n}) \cap \partial B_p} |(f_p^{\circ n})'(z)|^{-D} = \sum_\zeta |(f_p^{\circ n})'(\zeta)|^{-D} - |(f_p^{\circ n})'(w_p)|^{-D} = \mathcal{O}(1).$$

The proof is completed. □

Proof of Theorem 1.1. Note that when $p = 0$, the Julia set $J(f_p)$ is the unit circle which can be parameterized by $z(t) = e^{2\pi i t}$ such that

$$f_p(z(t)) = z(Qt). \tag{2.3}$$

For small $p \neq 0$, the restriction $f_p : \partial B_p \to \partial B_p$ is a covering map with degree Q. Then ∂B_p can parameterized such that (2.3) holds since ∂B_p is homeomorphic to the unit circle. By Lemma 2.3, we know that the point $z(t)$ on ∂B_p moves holomorphically on p. This means that, in a neighborhood of 0, we can expand $z(t)$ by

$$z(t) = e^{2\pi i t}(1 + p\, U_1(t) + p^2\, U_2(t) + \mathcal{O}(p^3)), \tag{2.4}$$

where $U_m(t)$ satisfies $U_m(t+1) = U_m(t)$ for $m \geq 1$. Substituting (2.4) into (2.3), then comparing the same order in p, we have the following equations

$$U_1(Qt) - QU_1(t) = e^{-2\pi i(2Q)t}, \tag{2.5}$$

$$U_2(Qt) - QU_2(t) = \frac{Q(Q-1)}{2} U_1^2(t) - e^{-2\pi i(2Q)t} Q U_1(t). \tag{2.6}$$

It is easy to verify the linear functional equation $\phi(Qt) - Q\phi(t) = e^{-2\pi i t}$ has the solution

$$\phi(t) = -\frac{1}{Q} \sum_{l=0}^{\infty} Q^{-l} e^{-2\pi i Q^l t}. \tag{2.7}$$

Hence we can solve the equations (2.5) and (2.6) by

$$U_1(t) = \phi(2Qt), \tag{2.8}$$

$$U_2(t) = \frac{Q(Q-1)}{2}\sum_{l_1,l_2=1}^{\infty} Q^{-(l_1+l_2)}\phi(2(Q^{l_1}+Q^{l_2})t) + Q\sum_{l=1}^{\infty} Q^{-l}\phi(2(Q^l+Q)t). \tag{2.9}$$

Actually, the higher order terms $U_m(t)$ with $m \geq 3$ can also be calculated by induction. But it will be extremely complicated.

Notice that the fixed point of $f_p^{\circ n}$ forms the following set

$$\mathrm{Fix}(f_p^{\circ n}) \cap \partial B_p = \{z(t_j) : t_j = j/(Q^n-1), j = 0, 1, \cdots, Q^n-2\}. \tag{2.10}$$

Following [13], it is convenient to introduce the *average notation*

$$\langle G(t)\rangle_n = \frac{1}{Q^n-1}\sum_{j=0}^{Q^n-2} G(t_j). \tag{2.11}$$

A very useful property of this average is

$$\langle e^{2\pi i m t}\rangle_n = \begin{cases} 1 & \text{if } m \equiv 0 \bmod Q^n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

By the fact that

$$(f_p^{\circ n})'(z(t_j)) = \prod_{m=0}^{n-1} f_p'(z(Q^m t_j)) = Q^n \prod_{m=0}^{n-1}\left(z^{Q-1}(Q^m t_j) - \frac{p}{z^{Q+1}(Q^m t_j)}\right), \tag{2.13}$$

we can write (2.2) as

$$\mathcal{O}(1) = Q^{-nD}(Q^n-1)\left\langle \prod_{m=0}^{n-1}\left| z^{Q-1}(Q^m t) - \frac{p}{z^{Q+1}(Q^m t)}\right|^{-D}\right\rangle_n. \tag{2.14}$$

The calculation in Appendix shows that for all sufficiently large n, we have

$$\mathcal{O}(1) = Q^{-nD}(Q^n-1)(1 + D^2 n|p|^2 + \mathcal{O}(np^3)). \tag{2.15}$$

Fix some large n, when p is small enough, we have

$$\mathcal{O}(1) = \exp\Big(n(-(D-1)\log Q + D^2|p|^2)\Big). \tag{2.16}$$

This means that

$$D = 1 + \frac{|p|^2}{\log Q} + \mathcal{O}(|p|^3), \tag{2.17}$$

which is the required formula in the main theorem. □

3. Appendix

This section will devote to prove (2.15). Firstly, we do some simplifications on notations. We use z_m, $U_{1,m}$ and $U_{2,m}$ to denote $z(Q^m t)$, $U_1(Q^m t)$ and $U_2(Q^m t)$ respectively. Let $\sigma = e^{2\pi i t}$, by (2.4), we have

$$\begin{aligned} \left|z_m^{Q-1} - p/z_m^{Q+1}\right| &= \left|(1+V_m)^{Q-1} - \sigma^{-2Q^{m+1}} p/(1+V_m)^{Q+1}\right| \\ &= \left|1 + (Q-1)V_m + (Q-1)(Q-2)V_m^2/2 - \sigma^{-2Q^{m+1}} p\,(1-(Q+1)V_m) + \mathcal{O}(p^3)\right|, \end{aligned} \tag{3.1}$$

where $V_m = U_{1,m}\,p + U_{2,m}\,p^2 + \mathcal{O}(p^3)$. So

$$\begin{aligned} \left|z_m^{Q-1} - p/z_m^{Q+1}\right|^{-\frac{D}{2}} &= \Big|1 + \Big[(Q-1)U_{1,m} - \sigma^{-2Q^{m+1}}\Big]\,p \\ &+ \Big[(Q-1)(Q-2)U_{1,m}^2/2 + (Q+1)\sigma^{-2Q^{m+1}}U_{1,m} + (Q-1)U_{2,m}\Big]\,p^2 + \mathcal{O}(p^3)\Big|^{-\frac{D}{2}} \\ &= \left|1 - \frac{D}{2}A_m p + \frac{D}{8}B_m p^2 + \mathcal{O}(p^3)\right|, \end{aligned} \tag{3.2}$$

where

$$A_m = (Q-1)\,U_{1,m} - \sigma^{-2Q^{m+1}}, \tag{3.3}$$

$$\begin{aligned} B_m = {} & (Q-1)\,(D(Q-1)+2)U_{1,m}^2 - 2(D(Q-1)+4Q)\,\sigma^{-2Q^{m+1}}U_{1,m} \\ & -4(Q-1)\,U_{2,m} + (D+2)\,\sigma^{-4Q^{m+1}}. \end{aligned} \tag{3.4}$$

Then we have

$$\begin{aligned} \left|z_m^{Q-1} - p/z_m^{Q+1}\right|^{-D} &= \left(1 - \frac{D}{2}A_m p + \frac{D}{8}B_m p^2\right)\left(1 - \frac{D}{2}\overline{A}_m \overline{p} + \frac{D}{8}\overline{B}_m \overline{p}^2\right) + \mathcal{O}(p^3) \\ &= 1 - \frac{D}{2}(A_m p + \overline{A}_m \overline{p}) + \frac{D^2}{4}|p|^2 A_m \overline{A}_m + \frac{D}{8}(B_m p^2 + \overline{B}_m \overline{p}^2) + \mathcal{O}(p^3). \end{aligned} \tag{3.5}$$

Lemma 3.1. *Let $u, v \in \mathbb{N}$, for any large n, then* (1) $2Q^v/(Q^n-1) \not\equiv 0 \bmod 1$; (2) $2Q^v(Q^u+1)/(Q^n-1) \not\equiv 0 \bmod 1$.

Proof. Since $(Q, Q^n-1) = 1$, it follows that $(Q^v, Q^n-1) = 1$. This means that $2Q^v/(Q^n-1)$ can not be an integer since $0 < 2 < Q^n - 1$ for large n.

For the second assertion, suppose that $u = ns + r$ for $0 \le r < n$, then

$$\frac{2Q^v(Q^u+1)}{Q^n-1} \equiv 2Q^v\,\frac{Q^{ns}(Q^r-1)+2}{Q^n-1} \equiv \frac{2Q^v(Q^r+1)}{Q^n-1} \bmod 1.$$

Since $(Q^v, Q^n-1) = 1$, this means that $2Q^v(Q^u+1)/(Q^n-1)$ can not be an integer because $0 < 2(Q^r+1) < Q^n - 1$ for large n. □

By Lemma 3.1, combine the average property of (2.12), it is easy to verify the following

Corollary 3.2. $\langle A_m\rangle_n = 0$, $\langle A_m A_k\rangle_n = 0$ *and* $\langle B_m\rangle_n = 0$ *for* $0 \le m, k \le n-1$.

Now, by (3.5), we have

$$\left\langle \prod_{m=0}^{n-1} \left|z_m^{Q-1} - p/z_m^{Q+1}\right|^{-D} \right\rangle_n = 1 + \frac{D^2}{4}|p|^2 \sum_{m,\,k=0}^{n-1} \langle A_m \overline{A}_k \rangle_n + \mathcal{O}(|p|^3). \tag{3.6}$$

Substituting (3.3) to $A_m\overline{A}_k$, we have

$$\begin{aligned} \langle A_m \overline{A}_k \rangle_n = {} & (Q-1)^2 \langle U_{1,m} \overline{U}_{1,k} \rangle_n + \langle \sigma^{2Q(Q^k - Q^m)} \rangle_n \\ & - (Q-1)\left(\langle \sigma^{-2Q^{m+1}} \overline{U}_{1,k} \rangle_n + \langle \sigma^{2Q^{k+1}} U_{1,m} \rangle_n \right). \end{aligned} \tag{3.7}$$

Lemma 3.3. *Let $u \in \mathbb{N}$, $(Q^u - 1)/(Q^n - 1)$ is an integer if and only if $u = ns$ for some $s \in \mathbb{N}$.*

Proof. The "if" part is trivial, we only prove the "only if" part. Suppose that $u = ns + r$ for $0 \le r < n$, according to the assumption, we have

$$\frac{Q^u - 1}{Q^n - 1} \equiv \frac{Q^{ns}(Q^r - 1)}{Q^n - 1} \mod 1.$$

Since $(Q^{ns}, Q^n - 1) = 1$, we conclude that $(Q^u - 1)/(Q^n - 1)$ is an integer if and only if $(Q^r - 1)/(Q^n - 1)$ is an integer, namely $r = 0$. ☐

From Lemma 3.3 and the property (2.12) of average notation, it follows that

$$\begin{aligned} \sum_{m,\,k=0}^{n-1} \langle U_{1,m} \overline{U}_{1,k} \rangle_n &= \frac{1}{Q^2} \sum_{m,\,k=0}^{n-1} \sum_{l_1,\,l_2=0}^{\infty} \frac{1}{Q^{\,l_1+l_2}} \left\langle \sigma^{-2Q(Q^{l_1+m} - Q^{l_2+k})} \right\rangle_n \\ &= \frac{1}{Q^2} \sum_{m,\,k=0}^{n-1} \left(\sum_{l_1+m=l_2+k} \frac{1}{Q^{\,l_1+l_2}} + \sum_{v\neq 0} \sum_{l_1+m=l_2+k+nv} \frac{1}{Q^{\,l_1+l_2}} \right) \\ &= \frac{1}{Q^2} \left(\sum_{m=0}^{n-1} \sum_{k=0}^{m} \sum_{l_1=0}^{\infty} \frac{1}{Q^{\,2l_1+m-k}} + \sum_{k=0}^{n-1} \sum_{m=0}^{k-1} \sum_{l_2=0}^{\infty} \frac{1}{Q^{\,2l_2+k-m}} \right) \\ &\quad + \frac{1}{Q^2} \left(\sum_{v=1}^{+\infty} \sum_{m,\,k=0}^{n-1} \sum_{l_2=0}^{\infty} \frac{1}{Q^{\,2l_2+k-m+nv}} + \sum_{v=-1}^{-\infty} \sum_{m,\,k=0}^{n-1} \sum_{l_1=0}^{\infty} \frac{1}{Q^{\,2l_1+m-k-nv}} \right) \\ &= \frac{1}{Q^2 - 1} \left(\frac{Q+1}{Q-1} n + \mathcal{O}(1) \right) + \mathcal{O}(1) = \frac{n}{(Q-1)^2} + \mathcal{O}(1). \end{aligned} \tag{3.8}$$

Here we have used the following formulas

$$\sum_{m=0}^{n-1} \sum_{k=0}^{m} \frac{1}{Q^{m-k}} = \frac{nQ}{Q-1} - \frac{Q - Q^{-(n-1)}}{(Q-1)^2} = \frac{nQ}{Q-1} + \mathcal{O}(1), \tag{3.9}$$

$$\sum_{m=0}^{n-1} \sum_{k=0}^{m-1} \frac{1}{Q^{m-k}} = \frac{n}{Q-1} - \frac{Q - Q^{-(n-1)}}{(Q-1)^2} = \frac{n}{Q-1} + \mathcal{O}(1). \tag{3.10}$$

The calculation in (3.8) shows that the sum of the case for $l_1 + m \neq l_2 + k$ is bounded above by a constant depending only on Q when n tends to ∞, which we marked by $\mathcal{O}(1)$. This observation is important in the following similar calculations. Namely, the main ingredients of the result is derived from the case for $l_1 + m = l_2 + k$.

Similar to the calculation in (3.8), we have

$$\sum_{m,\,k=0}^{n-1} \left\langle \sigma^{2Q(Q^k - Q^m)} \right\rangle_n = n + \mathcal{O}(1) \tag{3.11}$$

and

$$\begin{aligned} &\sum_{m,\,k=0}^{n-1} \left(\left\langle \sigma^{-2Q^{m+1}} \overline{U}_{1,k} \right\rangle_n + \left\langle \sigma^{2Q^{k+1}} U_{1,m} \right\rangle_n \right) = -\frac{2}{Q} \sum_{m,\,k=0}^{n-1} \sum_{l=0}^{\infty} \frac{1}{Q^l} \left\langle \sigma^{-2Q(Q^m - Q^{k+l})} \right\rangle_n \\ = &- \frac{2}{Q} \sum_{m=0}^{n-1} \sum_{k=0}^{m} \frac{1}{Q^{m-k}} + \mathcal{O}(1) = -\frac{2n}{Q-1} + \mathcal{O}(1). \end{aligned} \tag{3.12}$$

Combine (3.7), (3.8), (3.11) and (3.12), we have

$$\sum_{m,\,k=0}^{n-1} \langle A_m \overline{A}_k \rangle_n = n + n + 2n + \mathcal{O}(1) = 4n + \mathcal{O}(1). \tag{3.13}$$

From (3.6), this means that

$$\left\langle \prod_{m=0}^{n-1} \left| z_m^{Q-1} - p/z_m^{Q+1} \right|^{-D} \right\rangle_n = 1 + D^2 n |p|^2 + \mathcal{O}(np^3). \tag{3.14}$$

The proof of (2.15) is completed.

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