Z_3-connectivity with independent number 2

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Abstract

Let G be a 3-edge-connected graph on n vertices. It is proved in this paper that if $\alpha(G) \le 2$, then either G can be Z_3-contracted to one of graphs $\{K_1, K_4\}$ or G is one of the graphs in Fig. 1.

1 Introduction

Graphs considered here are undirected, finite and may have multiple edges without loops[1]. Let G be a graph. Set $D = D(G)$ be an orientation of G. If an edge $e = uv \in E(G)$ is directed from a vertex u to a vertex v, then u is a tail of e, v is a head of e. For a vertex $v \in V(G)$, let $E^+(v)(E^-(v))$ denote the set of all edges with v as a tail(a head). Let A be an abelian group with the additive identity 0, and let $A^* = A - \{0\}$.

For every mapping $f : E(G) \to A$, the boundary of f is a function $\partial f : V(G) \to A$ defined by

$$\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. If $\partial f(v) = 0$ for each vertex $v \in V(G)$, then f is called an A-flow of G. Moreover, if $f(e) \ne 0$ for every $e \in E(G)$, then f is a nowhere-zero A-flow of G.

A graph G is A-*connected* if for any mapping $b : V(G) \to A$ with $\sum_{v \in V(G)} b(v) = 0$, there exists an orientation of G and a mapping $f : E(G) \to A^*$ such that $\partial f(v) = b(v) \pmod 3$ for each $v \in V(G)$. The concept of A-connectivity was firstly introduced by Jaeger et al in [7] as a generalization of nowhere-zero flows. Obviously, if G is A-connected, then G admits a nowhere-zero A-flow.

For $X \subseteq E(G)$, the contraction G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$. Let A be an abelian group with $|A| \ge 3$. Denote by G' the graph obtained by repeatedly contracting A-connected subgraphs

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of G until no such subgraph left. We say G can be A-contracted to G'. Clearly, if a graph G can be A-contracted to K_1, then G is A-connected.

In this paper, we focus on Z_3-connectivity. The following conjecture is due to Jaeger *et al.*

Conjecture 1.1 *[7] Every 5-edge-connected graph is Z_3-connected.*

It is still open. However, many authors are devoted to approach this conjecture. Chvátal and Erdős [3] proved a classical result: a graph G with at least 3 vertices is hamiltonian if its independence number is less than or equal to its connectivity (this condition is known as Chvátal-Erdős Condition). Therefore Chvátal-Erdős Condition guarantees the existence of nowhere-zero 4-flows. Recently, Luo, Miao, Xu [10] characterized the graphs satisfying Chvátal-Erdős Condition that admit a nowhere-zero 3-flow.

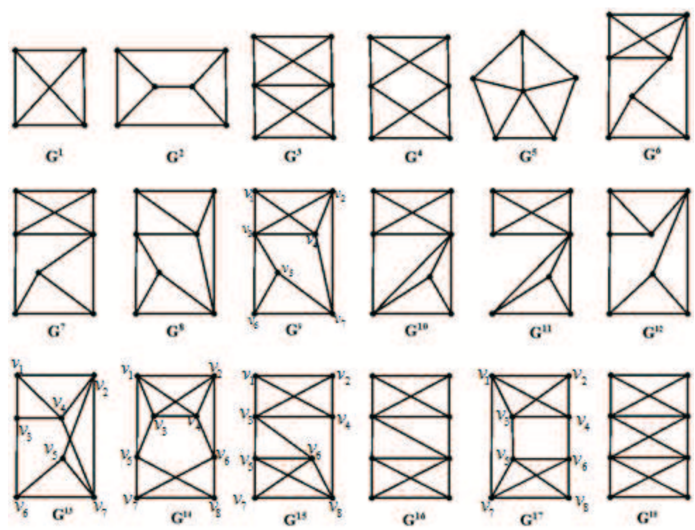

Fig. 1: 18 specified graphs which is Z_3-connected

Theorem 1.2 *(Luo et al. [10]) Let G be a bridgeless graph with independence number $\alpha(G) \leq 2$. Then G admits a nowhere-zero 3-flow if and only if G can not be contracted to a K_4 and G is not one of G^3, G^5, G^{18} in Fig. 1 or $G \notin G^{3'}$.*

Motivated by this, we consider the Z_3-connectivity of graphs satisfying the weaker Chvátal-Erdős Condition. In this paper, we extend Luo *et al.*'s result to group connectivity. The main theorem is as follows.

Theorem 1.3 *Let G be a 3-edge-connected simple graph and $\alpha(G) \leq 2$. G is not one of the 18 special graphs shown in Fig. 1 if and only if G can be Z_3-contracted to one of the graphs $\{K_1, K_4\}$.*

From Theorem 1.3, we obtain the following corollary immediately.

Corollary 1.4 *Let G be a 3-edge-connected graph and $\alpha(G) \leq 2$. Then one of the following holds:*
(i) G can be Z_3-contracted to one of the graphs $\{K_1, K_4\}$, or

(ii) G is one of the 18 special graphs shown in Fig. 1, or

(ii) G is one of the graphs $\{G^{3'}, G^{4'}, G^{10'}, G^{11'}\}$ shown in Fig. 2, where u, v are adjacent by m edges, $m \geq 2$ for $i = 3, 4, 10$ and $m \geq 3$ for $i = 11$.

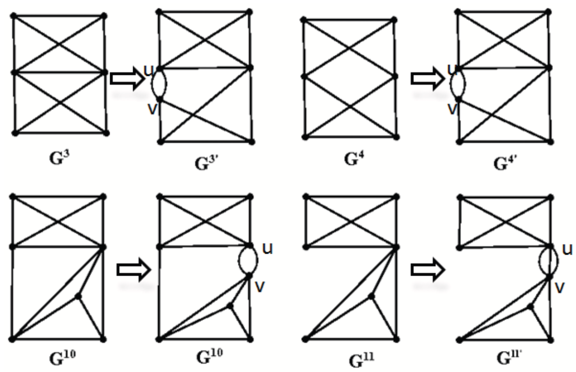

Fig. 2: Construction of graph of $G^{3'}$, $G^{4'}$, $G^{10'}$, $G^{11'}$

We end this section with some terminology and notation not define in [1]. For $V_1, V_2 \subseteq V(G)$ and $V_1 \cap V_2 = \emptyset$, denote by $e(V_1, V_2)$ the number of edges with one endpoint in V_1 and the other endpoint in V_2. For $S \subseteq V(G)$, $G[S]$ denotes an induced subgraph of G with vertex-set S. Let $N_G(v)$ denote the set of all vertices adjacent to vertex v; set $N_G[v] = N_G(v) \cup \{v\}$. We usually use $N(v)$ and $N[v]$ for $N_G(v)$ and $N_G[v]$ if there is no confusion. A k-vertex denotes a vertex of degree k. Let K_n denote a complete graph with n vertices, where $n \geq 3$. Moreover, K_3 denotes a 3-cycle. A *k-cycle* is a cycle of length k; a 3-cycle is also called a *triangle.* The wheel W_k is the graph obtained from a k-cycle by adding a new vertex and joining it to every vertex of the k-cycle. When k is odd (even), we say W_k is an *odd (even)* wheel. For convenience, we define W_1 as a triangle.

2 Preliminary

Here we state some lemmas which are essential to the proof of our result.

Lemma 2.1 *Let A be an abelian group with $|A| \geq 3$. The following results are known:*

(1) (Proposition 3.2 of [8]) K_1 is A-connected;

(2)(Corollary 3.5 of [8]) K_n and K_n^- are A-connected if $n \geq 5$;

(3) ([7] and Lemma 3.3 of [8])C_n is A-connected if and only if $|A| \geq n+1$;

(4) (Theorem 4.6 of [2]) $K_{m,n}$ is A-connected if $m \geq n \geq 4$; neither $K_{2,t}$ $(t \geq 2)$ nor $K_{3,s}$ $(s \geq 3)$ is Z_3-connected;

(5) (Lemma 2.8 of [2] and Proposition 2.4 of [4] and Lemma 2.6 of [5]) Each even wheel is Z_3-connected and each odd wheel is not;

(6) (Proposition 3.2 of [8]) Let $H \subseteq G$ and H be A-connected. G is A-connected if and only if G/H is A-connected;

(7) (Lemma 2.3 of [6]) Let v be not a vertex of G. If G is A-connected and $e(v,G) \geq 2$, then $G \cup \{v\}$ is A-connected.

Let G be a graph and u, v, w be three vertices of G with $uv, uw \in E(G)$, and $d_G(u) \geq 4$. Let $G_{[uv,uw]}$ be the graph $G \cup \{vw\} - \{uv, uw\}$.

Lemma 2.2 *(Theorem 3.1 of [2]) Let A be an abelian group with $|A| \geq 3$. If $G_{[uv,uw]}$ is A-connected, then so is G.*

Let H_1 and H_2 be two disjoint graphs. The 2-sum of H_1 and H_2, denoted by $H_1 \oplus H_2$, is the grpah obtained from $H_1 \cup H_2$ by identifying exactly one edge. A graph G is triangularly connected if for any two distinct edges e, e', there is a sequence of distinct cycles of length at most 3, $C_1, C_2, \ldots, C_m$ in G such that $e \in E(C_1)$, $e' \in E(C_m)$ and $|E(C_i) \cap E(C_{i+1})| = 1$ for $1 \leq i \leq m-1$.

Lemma 2.3 *(Fan et al.[5]) Let G be a triangularly connected graph. Then G is A-connected for all abelian group A with $|A| \geq 3$ if and only if $G \neq H_1 \oplus H_2 \oplus \ldots \oplus H_k$, where H_i is an odd wheel (including a triangle) for $1 \leq i \leq k$.*

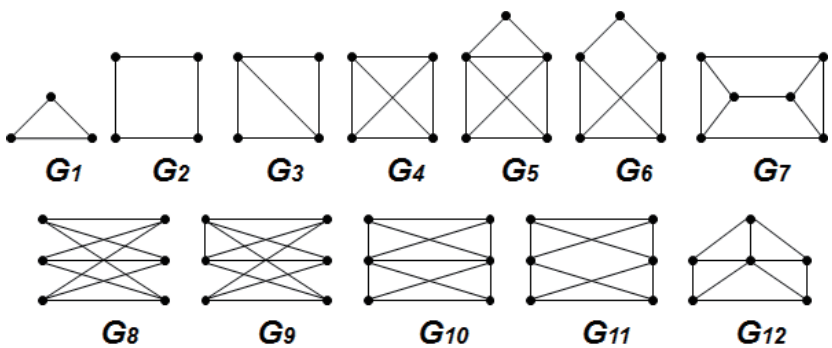

Fig. 3: 12 specified graphs

There are lots of results about Degree condition and Z_3-connectivity. We say G satisfies Ore-condition, if for each $uv \notin E(G)$, $d(u) + d(v) \geq |V(G)|$. We will discuss our result via the following Theorem.

Theorem 2.4 *(Luo et al. [9]) A simple graph G satisfying the Ore-condition with at least 3 vertices is not Z_3-connected if and only if G is one of the 12 graphs in Fig. 3.*

Lemma 2.5 *Let G be a graph. If for some mapping $b : V(G) \to Z_3$ with $\sum_{v \in V(G)} b(v) = 0$, there exists no orientation such that $|E^+(v)| - |E^-(v)| = b(v)$ (mod 3) for each $v \in V(G)$, then G is not Z_3-connected.*

Proof. By the definition of Z_3-connectivity, we know that G is Z_3-connected if and only if for any $b : V(G) \to Z_3$ with $\sum_{v \in V(G)} b(v) = 0$, there exists an orientation and function $f : E(G) \to Z_3^*$

such that $\partial f(v) = b(v) \pmod 3$ for each $v \in V(G)$. We know that Z_3-connectivity is independent on the orientation of graph. For above b and f, We only need to focus on edges of $f(e) = 2$. If $f(e) = 2$, then we can invert the orientation of e and let $f(e) = 1$, the others maintain. In this way we can get a new orientation of G and a new function f' on $E(G)$, such that $\partial f'(v) = b(v)$ for each $v \in V(G)$. Follow this way, finally we can get a new orientation and a new function $f'' : E(G) \to Z_3$ such that $f''(e) = 1$ and $\partial f''(v) = b(v)$ for each $e \in E(G)$ and each $v \in V(G)$. Thus we can deduce that G is Z_3-connected if and only if for any $b : V(G) \to Z_3$ with $\sum_{v\in V(G)} b(v) = 0$, there exists an orientation and function $f : E(G) \to Z_3^*$ such that $f(e) = 1$ and $\partial f(v) = b(v)$ for each $e \in E(G)$ and each $v \in V(G)$. In this case, $\partial f(v) = |E^+(v)| - |E^-(v)|$ since $f(e) = 1$. That is, G is Z_3-connected if and only if for any $b : V(G) \to Z_3$ with $\sum_{v\in V(G)} b(v) = 0$, there exists an orientation such that $|E^+(v)| - |E^-(v)| = b(v)$ for each $v \in V(G)$. We are done. ■

Lemma 2.6 *If G is one of graphs in Fig. 1, then G is not Z_3-connected.*

Proof. By Theorem 2.4, each graph in $\{G^1, G^2, G^3, G^4, G^5\}$ is not Z_3-connected.

If $G \cong G^9$, then let $b(v) = 0$ for each 3-vertex and 5-vertex in G and $b(v) = 1$ for each 4-vertex in G. By Lemma 2.5, we only need to proof that there exists no orientation such that $|E^+(v)| - |E^-(v)| = b(v) \pmod 3$ for each $v \in V(G)$. Since $b(v_5) = b(v_6) = 0$, we can orient edges such that $E^+(v_5) = 0$ and $E^+(v_6) = 3$ or $E^+(v_5) = 3$ and $E^+(v_6) = 0$. In the former case, we can orient edges v_3v_1, v_3v_2, v_3v_4 all with v_3 as a tail (or all with v_3 as a head) by $b(v_3) = 0$. WLOG, we assume edges v_3v_1, v_3v_2, v_3v_4 all with v_3 as a tail. Then $E^+(v_1) = 0$ by $b(v_1) = 0$. In this case, we must orient edges v_2v_4, v_4v_7 both with v_4 as a head. But we cannot orient v_2v_7, such that $|E^+(v_7)| - |E^-(v_7)| = 1 \pmod 3$. The proof of the latter case is similar as above. Thus G^9 is not Z_3-connected by Lemma 2.5. Since G^8 is a spanning subgraph of G^9, G^8 is not Z_3-connected by Lemma 2.1 (1) (6).

By Lemma 2.3, each graph of $\{G^{10}$, $G^{16}, G^{18}\}$ is not Z_3-connected.

Since $G \in \{G^7, G^{11}, G^{12}\}$ is a spanning subgraph of G^{10}, each graph of $\{G^7$, $G^{11}, G^{12}\}$ is not Z_3-connected by Lemma 2.1 (1) (6).

If $G \cong G^{13}$, then let $b(v) = 0$ for each 3-vertex in G and $b(v) = 1$ for each 4-vertex in G. That is $b(v_1) = b(v_3) = b(v_5) = b(v_6) = 0$, $b(v_2) = b(v_4) = b(v_7) = 1$. We first orient edges adjacent to vertices with $b(v) = 0$. Then in either case, we can orient edges $\{v_4v_2, v_4v_7\}$ both with v_4 as a head by $b(v_4) = 1$ and orient edges $\{v_4v_7, v_2v_7\}$ both with v_7 as a head by $b(v_7) = 1$. That is edge v_4v_7 has two orientations, a contradiction. By Lemma 2.5, G^{13} is not Z_3-connected.

If $G \cong G^{14}$, then let $b(v) = 0$ for each 3-vertex in G and $b(v) = 1$ for each 4-vertex in G. We first orient edges adjacent to vertices with $b(v) = 0$. WLOG, we assume $E^+(v_7) = 3$ and $E^+(v_8) = 0$. Then we can orient edges $\{v_1v_5, v_3v_5\}$ both with v_5 as a head by $b(v_5) = 1$ and orient edges $\{v_2v_6, v_4v_6\}$ both with v_6 as a head by $b(v_6) = 1$. Then we can orient edges $\{v_1v_3, v_2v_3, v_4v_3\}$ all with v_3 as a head (a tail) by $b(v_3) = 1$ and orient edges $\{v_1v_4, v_2v_4, v_4v_3\}$ all with v_4 as a tail (a head) by $b(v_4) = 1$. In either case, we cannot orient edge v_1v_2, such that $|E^+(v_1)| - |E^-(v_1)| = 0 \pmod 3$. By Lemma 2.5, G^{14} is not Z_3-connected.

If $G \cong G^{15}$, then let $b(v_1) = b(v_2) = b(v_3) = b(v_7) = 0$ and $b(v_5) = b(v_6) = 1$, $b(v_4) = b(v_8) = 2$. We first orient edges adjacent to vertices v_1 and v_2. Then in either case, we can orient edges $\{v_4v_3, v_4v_8\}$ both with v_4 as a tail by $b(v_4) = 2$. Then we orient edges v_3v_6, v_3v_5 both as v_3 as a head by $b(v_3) = 0$. Since $b(v_5) = b(v_6) = 1$, edges $\{v_5v_6, v_5v_8, v_5v_7\}$ with v_5 as a head (or a tail) and $\{v_5v_6, v_7v_6, v_8v_6\}$ all with v_6 as a tail (or a head). In either case, we cannot orient edges v_7v_8, such that $|E^+(v_7)| - |E^-(v_7)| = 0 \pmod 3$. By Lemma 2.5, G^{15} is not Z_3-connected.

If $G \cong G^{17}$, then let $b(v) = 0$ for each 3-vertex in G and $b(v) = 1$ for each 4-vertex in G. By the similar discussion, we can not find an orientation such that $|E^+(v)| - |E^-(v)| = 0 \pmod 3$ for each $v \in V(G)$. Thus G^{17} is not Z_3-connected by Lmmma 2.5 . G^6 is a subgraph of G^{17}. If G^6 is Z_3-connected, then G^{17} is Z_3-connected by Lemma 2.1 (6) (7), a contradiction. Thus G^6 is not Z_3-connected. ■

3 The case when $\delta(G) \geq 4$

Lemma 3.1 *Suppose G is a 3-edge-connected graph with $\delta(G) \geq 4$. If $\alpha(G) \leq 2$, then G is Z_3-connected.*

Proof. Clearly, we can assume G is simple; otherwise, we can contracted G into G' by contracting 2-cycles. By Lemma 2.1 (3) (6), G' is Z_3-connected if and only if G is Z_3-connected. Since $\delta(G) \geq 4$, $n \geq 5$. When $n = 5$, $G \cong K_5$, by Lemma 2.1 (2), G is Z_3-connected. Then we assume $n \geq 6$. By Lemma 2.1 (2), We only need to discuss the case $\alpha(G) = 2$.

If $d(u) + d(v) \geq n$ for each $uv \notin E(G)$, then G satisfies the Ore-condition. By Theorem 2.4 and since $\delta(G) \geq 4$, G is Z_3-connected.

Thus there exists two non-adjacent vertices u, v such that $d(u) + d(v) \leq n - 1$.

Set x, y be such vertices of G, that is $d(x) + d(y) \leq n - 1$ and $xy \notin E(G)$. Since $\alpha(G) = 2$, $e(v, \{x, y\}) \geq 1$ for each $v \in V(G) - \{x, y\}$. Then $|N(x) \cap N(y)| \leq 1$ by $d(x) + d(y) \leq n - 1$.

Case 1. $|N(x) \cap N(y)| = 0$.

In this case, $G[N[x]]$ and $G[N[y]]$ is a complete graph K_{m_1}, K_{m_2} ($m_1, m_2 \geq 5$) since $\alpha(G) = 2$ and $\delta(G) \geq 4$. By Lemma 2.1 (2), $G[N[x]]$ and $G[N[y]]$ is Z_3-connected. Since G is 3-edge connected, G is Z_3-connected by Lemma 2.1 (2) (3) (6).

Case 2. $|N(x) \cap N(y)| = 1$.

Suppose $u \in N(x) \cap N(y)$. Similarly, we know that $G[N[x] - \{u\}]$ and $G[N[y] - \{u\}]$ is a complete graph. Suppose $G[N[x] - \{u\}] \cong K_{m_1}$, $G[N[y] - \{u\}] \cong K_{m_2}$. Clearly, $m_i \geq 4$ for each $i \in \{1, 2\}$.

If $m_i = 4$ for each $i \in \{1, 2\}$, then $G[N[x] - \{u\}] \cong K_4$, $G[N[y] - \{u\}] \cong K_4$. Suppose $N(x) = \{x_1, x_2, x_3, u\}$ and $N(y) = \{y_1, y_2, y_3, u\}$. If $e(u, N[x]) \geq 3$, then $N[x]$ contains a K_5^- as a subgraph, by Lemma 2.1 (2), $G[N[x]]$ is Z_3-connected. Since G is 3-edge-connected, $e(N[x], N[y] - \{u\}) \geq 3$, by Lemma 2.1 (2) (6), G is Z_3-connected. WLOG, we assume $1 \leq e(u, N[x]) \leq 2$ and $1 \leq e(u, N[y]) \leq 2$. Then there are at least two vertices in $\{x_1, x_2, x_3\}$ which is not adjacent to u and at least two vertices in $\{y_1, y_2, y_3\}$ which is not adjacent to u. WLOG, we assume

$x_1, x_2, y_1, y_2 \notin N(u)$. In this case, $x_iy_j \in E(G)$ for each $i, j \in \{1, 2\}$ since $\alpha(G) = 2$. Then we can get a trivial graph K_1 by contracting 2-cycles from $G_{[yy_1,yy_2]}$. By Lemma 2.1 (3) (6), $G_{[yy_1,yy_2]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected.

If $m_1 = 4$ and $m_2 \geq 5$, then $G[N[x] - \{u\}] \cong K_4$ and $G[N[y] - \{u\}]$ is Z_3-connected by Lemma 2.1 (2). If $e(u, N[y] - \{u\}) \geq 2$, then $G[N[y]]$ is Z_3-connected by Lemma 2.1 (7). Since G is 3-edge-connected, $e(N[x] - \{u\}, N[y]) \geq 3$, by Lemma 2.1 (2) (6), G is Z_3-connected. Suppose $e(u, N[y] - \{u\}) = 1$. If there exist $v \in N(x)$ such that $vu \notin E(G)$, then $e(v, N(y)) = m_2 - 1 \geq 4$. Thus $G[N[y] \cup \{v\} - \{u\}]$ is Z_3-connected by Lemma 2.1 (7). Since G is 3-edge connected, G is Z_3-connected by Lemma 2.1 (2) (6). Thus $e(u, N[x] - \{u\}) = 4$, this means $G[N[x]]$ is K_5, by Lemma 2.1 (2), $G[N[x]]$ is Z_3-connected. Since G is 3-edge connected, G is Z_3-connected by Lemma 2.1 (6) (7).

If $m_i \geq 5$ for each $i \in \{1, 2\}$, then $G[N[x]-\{u\}]$ and $G[N[y]-\{u\}]$ is Z_3-connected by Lemma 2.1 (2). Since G is 3-edge connected, G is Z_3-connected by Lemma 2.1 (6) (7). ■

4 Proof of Theorem 1.3

In this section, we define $\mathcal{F}$ be a family of 3-edge connected simple graphs G which satisfies $\alpha(G) = 2$ and $\delta(G) = 3$.

Lemma 4.1 *Suppose $G \in \mathcal{F}$. If there exists two non-adjacent vertices u, v such that $d(u) + d(v) = n - 2$, then either G is one of the graphs $\{G^{15}, G^{16}, G^{17}, G^{18}\}$ shown in Fig. 1 or G can be Z_3-contracted in to $\{K_1, K_4\}$.*

Proof. Set x, y be such vertices of G, that is $d(x) + d(y) = n - 2$ and $xy \notin E(G)$. Since $\alpha(G) = 2$, $e(v, \{x, y\}) \geq 1$ for each $v \in V(G) - \{x, y\}$. Since $d(x) + d(y) = n - 2$, $|N(x) \cap N(y)| = 0$.

In this case, $G[N[x]]$ and $G[N[y]]$ is a complete graph K_{m_1}, K_{m_2} $(m_1, m_2 \geq 4)$ by $\alpha(G) = 2$ and $\delta(G) = 3$. If $m_i \geq 5$ for each $i \in \{1, 2\}$, then $\delta(G) \geq 4$, contrary to $\delta(G) = 3$. Thus we need to discuss cases of $m_i = 4$ for some $i \in \{1, 2\}$. Assume $m_1 = 4$. Let $N(x) = \{x_1, x_2, x_3\}$.

Case 1. $m_2 = 4$.

Let $N(y) = \{y_1, y_2, y_3\}$. Since G is 3-edge connected, $e(N(x), N(y)) \geq 3$. If there exists one vertex, say x_1, such that $e(x_1, N(y)) \geq 3$, then $G[N[y] \cup x_1]$ is Z_3-connected by Lemma 2.1 (2). In this case, if $e(v, N[y] \cup \{x_1\}) = 1$ for each $v \in N[x] - \{x_1\}$, then G can be contracted into K_4; otherwise, G is Z_3-connected. Thus we assume $e(u, N[y]) \leq 2$ and $e(v, N[x]) \leq 2$ for each $u \in N(x)$ and $v \in N(y)$.

When $e(N(x), N(y)) = 3$. Then G is one of graphs $\{G^{15}, G^{16}, G^{17}\}$ in Fig. 1.

When $e(N(x), N(y)) = 4$. Then either $G \cong G^{18}$ in Fig. 1 or G contains a 4-vertex. In the latter case, we assume $d(x_1) = 4$ and $N(x_1) = \{x, x_2, x_3, y_1\}$. Then $e(\{x, x_2, x_3\}, N(y)) = 3$. Considering graph $G_{[x_1x,x_1x_2]}$. Clearly, $\{x, x_2, x_3\}$ can be contracted into one vertex v^* by contracting two 2-cycles and we called this new graph G^*. Since $e(\{x, x_2, x_3\}, N(y)) = 3$, $d_{G^*}(v^*) = 3 + 1 = 4$. That is G^* contains a K_5^- or C_2 as a subgraph. In either case, by Lemma 2.1 (2) (3) (6) (7), G^*

is Z_3-connected. By Lemma 2.1 (3) (6), $G_{[x_1x,x_1x_2]}$ is Z_3-connected. Thus G is Z_3-connected by Lemma 2.2.

When $e(N(x),N(y)) \geq 5$. We only need to prove the case $e(N(x),N(y)) = 5$. In this case, we may assume $e(x_i,N(y)) = 2$ for each $i = 1,2$ and $e(x_3,N(y)) = 1$. WLOG, we assume $x_1y_1, x_1y_2 \in E(G)$. We can get a trivial graph K_1 by contracting 2-cycles from graph $G_{[x_1y_1,x_1y_2]}$. By Lemma 2.1 (1) (3) (6), $G_{[x_1y_1,x_1y_2]}$ is Z_3-connected. Thus G is Z_3-connected by Lemma 2.2.

Case 2. $m_2 \geq 5$.

By Lemma 2.1 (2), $G[N[y]]$ is Z_3-connected. Since G is 3-edge connected, $e(N(x),N(y)) \geq 3$. We can deduce that G can be contracted to K_4 or G is Z_3-connected by Lemma 2.1 (2) (6). ■

Lemma 4.2 *Suppose $G \in \mathcal{F}$ and $d(u)+d(v) \geq n-1$ for each $uv \notin E(G)$. If there exists two non-adjacent vertices u,v such that $d(u)+d(v) = n-1$, then either G is one of the graphs $\{G^6, G^7, \ldots, G^{14}\}$ shown in Fig. 1 or G can be Z_3-contracted in to $\{K_1, K_4\}$.*

Proof. Set x,y be such vertices of G, that is $d(x)+d(y) = n-1$ and $xy \notin E(G)$. Since $\alpha(G) = 2$, $e(v,\{x,y\}) \geq 1$ for each $v \in V(G) - \{x,y\}$. Since $d(x)+d(y) = n-1$, $|N(x) \cap N(y)| = 1$.

Suppose $u \in N(x) \cap N(y)$. Similarly, we know that $G[N[x]-\{u\}]$ and $G[N[y]-\{u\}]$ is a complete graph. Suppose $G[N[x]-\{u\}] \cong K_{m_1}$, $G[N[y]-\{u\}] \cong K_{m_2}$. Clearly, $m_i \geq 3$ for each $i \in \{1,2\}$.

Case 1. Suppose $m_i = 3$ for each $i = 1,2$.

Let $N(x) = \{x_1,x_2,u\}$, $N(y) = \{y_1,y_2,u\}$. Since $\delta(G) = 3$, $d(u) \geq 3$.

If $d(u) = 3$, then WLOG, we assume $N(u) = \{x,x_1,y\}$. Since $\alpha(G) = 2$, $x_2y_i \in E(G)$ for each $i \in \{1,2\}$. Then either G is one of graphs $\{G^{12}, G^{13}\}$ in Fig. 1 or $x_1y_i \in E(G)$ for each $i \in \{1,2\}$. In the latter case, we can get a trivial graph K_1 by contracting 2-cycles till no 2-cycles exist from graph $G_{[x_2y_1,x_2y_2]}$. By Lemma 2.1 (1) (3) (6), $G_{[x_2y_1,x_2y_2]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected.

If $d(u) = 4$, then $e(u,N[x]) = 2$ or $e(u,N[x]) = 3$ by symmetry. Suppose $e(u,N[x]) = 2$. WLOG, we assume $N(u) = \{x,x_1,y,y_1\}$. Since $\alpha(G) = 2$, $x_2y_2 \in E(G)$. If $e(\{x_1,x_2\},\{y_1,y_2\}) = 1$, then $G \cong G^{12}$. If $e(\{x_1,x_2\},\{y_1,y_2\}) = 2$, then $G \in \{G^8, G^{13}\}$. If $e(\{x_1,x_2\},\{y_1,y_2\}) \geq 3$, then we can get a trivial graph K_1 by contracting 2-cycles till no such subgraph exists from $G_{[ux,ux_1]}$. By Lemma 2.1 (1) (3) (6), $G_{[ux,ux_1]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected. Suppose $e(u,N[x]) = 3$. Then $N(u) = \{x,x_1,x_2,y\}$. Since G is 3-edge connected, $e(\{x_1,x_2\},\{y_1,y_2\}) \geq 2$. If $e(\{x_1,x_2\},\{y_1,y_2\}) = 2$, then G is one of graphs $\{G^6, G^7\}$. If $e(\{x_1,x_2\},\{y_1,y_2\}) \geq 3$, then we can get a trivial graph K_1 by contracting 2-cycles till no such subgraph exists from $G_{[ux,ux_1]}$. By Lemma 2.1 (1) (3) (6), $G_{[ux,ux_1]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected.

If $d(u) = 5$, then WLOG, we assume $N(u) = \{x,x_1,x_2,y,y_1\}$. If $e(x_i,\{y_1,y_2\}) = 2$, then G contains a 4-wheel with y_1 as a hub. Then we can deduce that G is Z_3-connected by Lemma 2.1 (5) (6) (7). Thus we assume $e(x_i,\{y_1,y_2\}) \leq 1$ for each $i \in \{1,2\}$. If $e(\{x_1,x_2\},\{y_1,y_2\}) = 1$, then $G \cong G^7$. If $e(\{x_1,x_2\},\{y_1,y_2\}) = 2$, then either $G \cong G^9$ or we can get a trivial graph K_1 by contracting 2-cycles till no such subgraph exists from $G_{[x_ix,x_ix_j]}$, where $x_iy_1 \notin E(G)$ and $i \neq j$. By Lemma 2.1 (1) (3) (6), $G_{[x_ix,x_ix_j]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected..

If $d(u) = 6$, then $N(u) = \{x, x_1, x_2, y, y_1, y_2\}$. If $e(\{x_1, x_2\}, \{y_1, y_2\}) = 0$, then $G \cong G^{11}$. If $e(\{x_1, x_2\}, \{y_1, y_2\}) = 1$, then $G \cong G^{10}$. Thus we assume $e(\{x_1, x_2\}, \{y_1, y_2\}) \geq 2$. If there exists i, say x_1, such that $e(\{x_1\}, \{y_1, y_2\}) = 2$, then $G[N[y] \cup x_1]$ is a K_5^-. By Lemma 2.1 (2) (6), we can deduce that G is Z_3-connected. Thus WLOG, we assume $x_1y_1, x_2y_2 \in E(G)$. Considering graph $G_{[x_1y_1,x_1x_2]}$. $G_{[x_1y_1,x_1x_2]}$ contains a 4-wheel with y_1 as a hub. We can get a new graph with 3 vertices and 4 edges by contracting this 4-wheel from $G_{[x_1y_1,x_1x_2]}$, which is Z_3-connected by Lemma 2.1 (5) (6). By Lemma 2.2, G is Z_3-connected.

Case 2. Suppose $m_i = 4$ for each $i = 1, 2$.

Suppose $N(x) = \{x_1, x_2, x_3, u\}$, $N(y) = \{y_1, y_2, y_3, u\}$. If $e(u, N[x]) \geq 3$, then $G[N[x]]$ contains a K_5^- as a subgraph. By Lemma 2.1 (2), $G[N[x]]$ is Z_3-connected. Since G is 3-edge connected, $e(N[x], N[y] - \{u\}) \geq 3$. Then $G/G[N[x]]$ contains a K_5^- as a subgraph. By Lemma 2.1 (2), $G/G[N[x]]$ is Z_3-connected. By Lemma 2.1 (6), G is Z_3-connected. Thus we may assume $1 \leq e(u, N[x]) \leq 2$ and $1 \leq e(u, N[y]) \leq 2$. Since G is 3-edge connected, $d(u) \geq 3$.

If $e(u, N[x]) = 2$ and $e(u, N[y]) = 2$, then we assume, WLOG, we assume $N(u) = \{x, x_1, y, y_1\}$. Since $\alpha(G) = 2$, $x_2y_i, x_3y_i \in E(G)$ for each $i = 2, 3$. In this case, $\delta(G) \geq 4$, contrary to $\delta(G) = 3$.

If $e(u, N[x]) = 2$ and $e(u, N[y]) = 1$, then we assume, WLOG, we assume $N(u) = \{x, x_1, y\}$. Since $\alpha(G) = 2$, $x_2y_i, x_3y_i \in E(G)$ for each $i = 1, 2, 3$. In this case, $G[N[y] \cup \{x_2, x_3\} - \{u\}]$ is Z_3-connected by Lemma 2.1 (2) (7). By Lemma 2.1 (7), G is Z_3-connected.

Case 3. Suppose $m_i \geq 5$ for each $i = 1, 2$. Clearly, $G[N[x] - \{u\}]$ and $G[N[y] - \{u\}]$ is Z_3-connected by Lemma 2.1 (2). Since $\delta(G) = 3$, WLOG, we assume $e(u, N[x] - \{u\}) = 2$. Thus $G[N[x]]$ is Z_3-connected by Lemma 2.1 (7). Since G is 3-edge connected, $e(N[x], N[y] - \{u\}) \geq 3$, G is Z_3-connected by Lemma 2.1 (2) (6).

Case 4. Suppose $m_1 = 3$ and $m_2 = 4$.

Suppose $N(x) = \{x_1, x_2, u\}$, $N(y) = \{y_1, y_2, y_3, u\}$. If $e(u, N[y] - \{u\}) \geq 3$, then $G[N[y]]$ contains a K_5^- as a subgraph. By Lemma 2.1 (2), $G[N[y]]$ is Z_3-connected. Since G is 3-edge connected, $e(N[x] - \{u\}, N[y]) \geq 3$. Then either G can be contracted to K_4 or G is Z_3-connected. Thus we assume $1 \leq e(u, N[y] - \{u\}) \leq 2$ and $d(u) \geq 3$.

If $d(u) = 3$, then $e(u, N[x] - \{u\}) = 1$ and $e(u, N[y] - \{u\}) = 2$ or $e(u, N[x] - \{u\}) = 2$ and $e(u, N[y] - \{u\}) = 1$. In the former case, we assume, WLOG, $N(u) = \{x, y, y_1\}$. Since $\alpha(G) = 2$, $x_1y_i, x_2y_i \in E(G)$ for each $i = 2, 3$. In this case, we can get a trivial graph K_1 by contracting 2-cycles from $G_{[x_1y_2,x_1y_3]}$. By Lemma 2.1 (1) (3) (6), $G_{[x_1y_2,x_1y_3]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected. In the latter case, we assume, WLOG, $N(u) = \{x_1, x, y\}$. Since $\alpha(G) = 2$, $x_2y_i \in E(G)$ for each $i = 1, 2, 3$. In this case, $G[N[y] \cup \{x_2\} - \{u\}]$ is Z_3-connected by Lemma 2.1 (2). Then either G is Z_3-connected or G can contracted into K_4.

If $d(u) = 4$, then $e(u, N[x] - \{u\}) = 2$ and $e(u, N[y] - \{u\}) = 2$ or $e(u, N[x] - \{u\}) = 3$ and $e(u, N[y] - \{u\}) = 1$. Suppose $e(u, N[x] - \{u\}) = 2$ and $e(u, N[y] - \{u\}) = 2$. We assume, WLOG, $N(u) = \{x, x_1, y, y_1\}$. Since $\alpha(G) = 2$, $x_2y_i \in E(G)$ for each $i = 2, 3$. If no other edge, then $G \cong G^{14}$; otherwise, either we can get a trivial graph K_1 by contracting 2-cycles from $G_{[uy,uy_1]}$ or $G[N[y]] \cup \{x_2\} - \{u\}$ contains a K_5^- as a subgraph. In the former case, by Lemma 2.1 (1) (3) (6), $G_{[uy,uy_1]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected; in the latter case, $G[N[y]] \cup \{x_2\} - \{u\}$ is Z_3-connected by Lemma 2.1 (2). By Lemma 2.1 (7), G is also Z_3-connected. Then suppose $e(u, N[x]-\{u\}) = 3$ and $e(u, N[y]-\{u\}) = 1$. Then $N(u) = \{x_1, x_2, x, y\}$. Since G is 3-edge connected, $e(\{x_1, x_2\}, \{y_1, y_2, y_3\}) \geq 2$. Since $d(u) + d(v) \geq n - 1 = 7$ for each $uv \notin E(G)$, $d(y_i) \geq 4$ for each $i = 1, 2, 3$. In this case, $e(\{x_1, x_2\}, \{y_1, y_2, y_3\}) \geq 3$. Then either we can get a trivial graph K_1 by contracting 2-cycles from $G_{[yy_1,yy_2]}$ or $G[N[y] \cup \{x_i\} - \{u\}]$ contains K_5^- as spanning subgraph for some $i \in \{1, 2\}$. In the former case, $G_{[yy_1,yy_2]}$ is Z_3-connected by Lemma 2.1 (1) (3) (6). By Lemma 2.2, G is Z_3-connected; in the latter case, we can deduce G is Z_3-connected by Lemma 2.1 (2) (7).

If $d(u) = 5$, then $e(u, N[x] - \{u\}) = 3$ and $e(u, N[y] - \{u\}) = 2$. WLOG, we assume $N(u) = \{x_1, x_2, x, y, y_1\}$. Since $d(u) + d(v) \geq n - 1 = 7$ for each $uv \notin E(G)$, $d(y_i) \geq 4$ for each $i = 1, 2, 3$. Then $e(\{x_1, x_2\}, \{y_1, y_2, y_3\}) \geq 2$. In this case, we can get a trivial graph K_1 by contracting 2-cycles and K_5^- from graph $G_{[yy_1,yy_2]}$. By Lemma 2.1 (1) (6), $G_{[yy_1,yy_2]}$ is Z_3-connected. By Lemma 2.2, G is Z_3-connected.

Case 5. Suppose $m_1 = 3$ and $m_2 \geq 5$.

Clearly, $G[N[y] - \{u\}]$ is Z_3-connected by Lemma 2.1 (2). Since $\delta(G) \geq 3$, $e(u, N[x] - \{u\}) \geq 2$ or $e(u, N[y] - \{u\}) \geq 2$. If $e(u, N[y] - \{u\}) \geq 2$, then $G[N[y]]$ is Z_3-connected by Lemma 2.1 (7). Since G is 3-edge connected, $e(N[x] - \{u\}, N[y]) \geq 3$. Thus either G is Z_3-connected or G can be contracted into K_4. Thus we assume $e(u, N[y] - \{u\}) = 1$ and $e(u, N[x] - \{u\}) \geq 2$. Set $N(x) = \{x_1, x_2, u\}$. WLOG, we assume $x_1u \in E(G)$. If $x_2u \in E(G)$, then $G[N[x]]$ is K_4. Since G is 3-edge connected, $e(N[x], N[y] - \{u\}) \geq 3$. Thus $G/G[N[y] - \{u\}]$ contains K_5^- or C_2. In either case, G is Z_3-connected by Lemma 2.1 (2) (3) (6). If $x_2u \notin E(G)$, then $x_2v \in E(G)$ for each $v \in N(y) - \{u\}$ by $\alpha(G) = 2$. Thus $G[N[y] \cup \{x_2\} - \{u\}]$ is Z_3-connected by Lemma 2.1 (7). Thus either G is Z_3-connected or G can be contracted into K_4.

Case 6. Suppose $m_1 = 4$ and $m_2 \geq 5$.

Clearly, $G[N[y] - \{u\}]$ is Z_3-connected by Lemma 2.1 (2). Since $\delta(G) \geq 3$, $e(u, N[x] - \{u\}) \geq 2$ or $e(u, N[y] - \{u\}) \geq 2$. If $e(u, N[y] - \{u\}) \geq 2$, then $G[N[y]]$ is Z_3-connected by Lemma 2.1 (7). Since G is 3-edge connected, $e(N[x] - \{u\}, N[y]) \geq 3$. Thus $G/G[N[y]]$ contains K_5^- or C_2. In either case, G is Z_3-connected by Lemma 2.1 (1) (2) (3) (6). Thus $e(u, N[y] - \{u\}) = 1$ and $e(u, N[x] - \{u\}) \geq 2$. Set $N(x) = \{x_1, x_2, x_3, u\}$. WLOG, we assume $x_1u \in E(G)$. If $x_iu \in E(G)$ for some $i \in \{2, 3\}$, then $G[N[x]]$ contains K_5^- as a subgraph, is Z_3-connected. Since G is 3-edge connected, $e(N[x], N[y] - \{u\}) \geq 3$. Thus $G/G[N[x]]$ contains K_5^- or C_2. By Lemma 2.1 (2) (3) (6), G is Z_3-connected. If $x_iu \notin E(G)$ for each $i \in \{2, 3\}$, then $x_iv \in E(G)$ for each $v \in N(y) - \{u\}$ and $i \in \{2, 3\}$ by $\alpha(G) = 2$. Thus $G[N[y] \cup \{x_2, x_3\} - \{u\}]$ is Z_3-connected by Lemma 2.1 (7). By Lemma 2.1 (7), G is Z_3-connected. ■

Proof of Theorem 1.3 Since G a is 3-edge connected simple graph, $n \geq 4$. When $n = 4$, $G \cong K_4$, that is G^1. When $n = 5$, G contains W_4 as a subgraph, by Lemma 2.1 (1) (5) (6), G is Z_3-connected.

By Lemma 4.1 and 4.2, we only need prove the case of G which satisfies the Ore-condition. By Theorem 2.4 and since G is 3-edge-connected, either G is Z_3-connected or G is one of graphs $\{G^2, G^3, G^4, G^5\}$ shown in Fig. 1. Thus we prove that G is not one of the 18 special graphs shown in Fig. 1 if and only if G can be Z_3-contracted to one of the graphs $\{K_1, K_4\}$. ■

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