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Lie Algebroids and generalized projective structures on Riemann surfaces

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Abstract. The space of generalized projective structures on a Riemann surface Σ of genus g with n marked points is the affine space over the cotangent bundle to the space of SL(N)-opers. It is a phase space of W_N-gravity on $\Sigma \times \mathbb{R}$. This space is a generalization of the space of projective structures on the Riemann surface. We define the moduli space of W_N-gravity as a symplectic quotient with respect to the canonical action of a special class of Lie algebroids. They describe in particular the moduli space of deformations of complex structures on the Riemann surface by differential operators of finite order, or equivalently, by a quotient space of Volterra operators. We call these algebroids the Adler-Gelfand-Dikii (AGD) algebroids, because they are constructed by means of AGD bivector on the space of opers restricted on a circle. The AGD-algebroids are particular case of Lie algebroids related to a Poisson sigma-model. The moduli space of the generalized projective structure can be described by cohomology of a BRST-complex.

Contents

1. Introduction

The goal of this paper is to define a moduli space of generalized projective structures on a Riemann surface $\Sigma_{g,n}$ of genus g with n marked points. The standard projective structure is a pair of projective connection T and the Beltrami differential μ. The projective connection is defined locally by the second order differential operator $\partial_z^2 + T(z,\bar{z})$. It behaves as $(2,0)$-differential. The Beltrami differential defines a deformation of complex structure on $\Sigma_{g,n}$ as $\partial_{\bar{z}} \to \partial_{\bar{z}} + \mu\partial_z$.

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It is a $(-1,1)$-differential. The pair (T,μ) can be considered as coordinates in the affine space over the cotangent bundle to the space of projective connection. It is an infinite dimensional symplectic space with the canonical form $\int_{\Sigma_{g,n}} DT \wedge D\mu$. It has a field-theoretical interpretation as a phase space of W_N-gravity, describing a topological field theory on $\Sigma_{g,n} \times \mathbb{R}$ [**25, 11, 15**]. The chiral vector fields generate canonical transformations of this space. The symplectic quotient with respect to this action is a finite-dimensional space - the moduli space of projective structure on $\Sigma_{g,n}$. The moduli space of complex structures is a part of this space.

The projective connections have a higher order generalizations. It is a space of $\mathrm{SL}(N,\mathbb{C})$-opers on $\Sigma_{g,n}$ [**27, 4**]. The dual to them variables define generalized deformations of complex structures by $(-j,1)$-differentials ($j>1$). It can be equivalently described as a quotient space of Volterra operators on $\Sigma_{g,n}$. These differentials describe highest order integrals of motion in the Hitchin integrable systems [**19**]. It is a base of the Hitchin fibration of the Higgs bundles and thereby their moduli parameterize the base. In contrast with the space of projective structures (W_2-gravity) the canonical transformations of this space generate Lie algebroids [**23, 5**] rather than the Lie algebra of vector fields. The symplectic quotient with respect to this action is a moduli space of generalized projective structures (the moduli space of W_N-gravity). In particular, it describes the moduli space of generalized deformations of complex structures by means of higher order differentials.

This situation can be generalized in the following way. Remind that a Lie algebroid $\mathcal{A}$ is a vector bundle over a space M with Lie brackets defined on its sections $\Gamma(\mathcal{A})$ and a bundle map (the anchor) to the vector fields on the base $\delta : \Gamma(\mathcal{A}) \to TM$. Let $\mathcal{R}$ be an affine space over the cotangent bundle T^*M (the principle homogeneous space). It is a symplectic space with the canonical symplectic form. Define a representation of $\mathcal{A}$ in the space of sections of $\mathcal{R}$ in a such way that together with the anchor action they generate canonical transformations of $\mathcal{R}$. These transformations are classified by the first cohomology group $H^1(\mathcal{A})$ of the algebroid. We call Lie algebroid equipped with the canonical representation in the space of sections $\mathcal{R}$ *the Hamiltonian algebroid* $\mathcal{A}^H$. The Hamiltonian algebroids are analogs of the Lie algebra of symplectic vector fields.

The symplectic quotient of $\mathcal{R}$ with respect to the canonical transformations can be described by cohomology of the BRST operator of $\mathcal{A}^H$ [**3, 18**]. We prove that the BRST operator in this case has the same structure as for the transformations by Lie algebras.

The general example of this construction is based on the Poisson sigma-model [**21, 26**]. We consider a one-dimensional Poisson sigma-model $X(t) : S^1 \to M$, where M is a Poisson manifold. The Lie algebroid in this case is a vector bundle over $\mathbf{M} = \{X(t)\}$. The base $\mathbf{M}$ is Poisson with the Poisson brackets related to the Poisson brackets on M. The space of sections of the Lie algebroid is $X^*(T^*M)$. The Lie brackets on $X^*(T^*M)$ are defined by the Poisson brackets on $\mathbf{M}$ [**14, 12, 24**]. Though the Poisson structure exists only on maps $S^1 \to M$, in some important cases the Lie algebroid structure (the Lie brackets and the anchor) can be continued on the space of maps from a Riemann surface $\Sigma \supset S^1$ to M and, moreover, to the affine space $\mathbf{R}$ over $X^*(T^*M)$. The canonical transformations of the space of smooth maps $\mathbf{R}$ define a Hamiltonian algebroid. It turns out that the first class constraints, generating these transformations are consistency conditions for a linear system. Two of three linear equations define a deformation of operator $\bar{\partial}$ on Σ.

This deformation depends on the Poisson bivector on M. The symplectic quotient with respect to the canonical actions can be described by the BRST construction. In particular, it defines the moduli space of these deformations of the complex structure.

We apply this scheme to the space of opers on a Riemann surface $\Sigma_{g,n}$. We start with the space $M_N(D)$ of $\mathrm{GL}(N,\mathbb{C})$-opers on a disk $D\subset\Sigma$. The space $M_N(D)$ being restricted on the boundary $S^1=\partial D$ is a Poisson space with respect to the Adler-Gelfand-Dikii (AGD) brackets [**1, 13**]. It allows us to define a Lie algebroid (the AGD-algebroid) $\mathcal{A}_N$ over $M_N(D)$. The Lie brackets on the space of sections $\Gamma(\mathcal{A}_N)$ and the anchor are derived from the AGD bivector. The case $N=2$ corresponds to the projective structure on D and the sections $\Gamma(\mathcal{A}_N)$ is the Lie algebra of vector fields with coefficients depending on the projective connection. The case $N>2$ is more involved and we deal with a genuine Lie algebroids since differential operators with the principle symbol of order two or more do not form a Lie algebra with respect to the standard commutator. The AGD brackets define a new commutator on $\Gamma(\mathcal{A}_N)$ that depends on the projective connection and higher spin fields. According with the AGD construction the differential operators in $\Gamma(\mathcal{A}_N)$ can be replaced by a quotient space of the Volterra operators on D. This construction can be continued from D to $\Sigma_{g,n}$. We preserve the same notation for this algebroid and call it the AGD Lie algebroid on $\Sigma_{g,n}$. The space M_N of opers on $\Sigma_{g,n}$ plays the role of the configuration space of W_N-gravity [**25, 11, 15**]. The whole phase space $\mathbf{R}_N$ of W_N-gravity is an affine space over the cotangent bundle to the space of opers T^*M_N. The canonical transformations of $\mathbf{R}_N$ are sections of the Hamiltonian AGD-algebroid $\mathcal{A}^H_N$ over M_N. The symplectic quotient of the phase space is the moduli space $\mathcal{W}_N$ of the W_N-gravity on $\Sigma_{g,n}$. Roughly speaking, this space is a combination of the moduli of W_N-deformations of complex structures and the spin $2,\ldots,$ spin N fields as the dual variables. This moduli space can be described by the cohomology of the BRST complex for the Hamiltonian algebroid. As it follows from the general construction, the BRST operator has the same structure as in the Lie algebra case. We consider in detail the simplest nontrivial case $N=3$. In this case it is possible to describe explicitly the sections of the algebroid as the second order differential operators, instead of Volterra operators. It should be noted that the BRST operator for the W_3-algebras was constructed in [**28**]. Here we construct the BRST operator for the different object - the algebroid symmetries of W_3-gravity. Another BRST description of W-symmetries was proposed in Ref.[**2**]. We explain our formulae and the origin of the algebroid by a special gauge procedure in the $\mathrm{SL}(N,\mathbb{C})$ Chern-Simons theory using an approach developed in Ref.[**11**].

The paper is organized as follows. In next section we define Lie algebroids, their representations, cohomolgy, the Hamiltonian algebroids, and the BRST construction. In Section 3 we use a Poisson sigma model to construct Hamiltonian algebroids. In Section 4 we consider two examples of our construction. We analyze the moduli space of flat $\mathrm{SL}(N,\mathbb{C})$-bundles and the moduli of projective structures on $\Sigma_{g,n}$. A nontrivial example of this construction is W_3-gravity. It is considered in detail in Section 5. The general W_N case is analyzed in Section 6.

2. Lie algebroids and groupoids

2.1. Lie algebroids and groupoids. We remind a definition brief description of Lie algebroids and Lie groupoids. Details of this theory can be found in [**23, 5**].

Definition 2.1. *A* Lie algebroid *over a smooth manifold M is a vector bundle $\mathcal{A} \to M$ with a Lie algebra structure on the space of its sections $\Gamma(\mathcal{A})$ defined by the Lie brackets $\lfloor \varepsilon_1, \varepsilon_2 \rfloor$, $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A})$ and a bundle map (*the anchor*) $\delta : \mathcal{A} \to TM$, satisfying the following conditions: (i) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A})$*

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{\lfloor \varepsilon_1, \varepsilon_2 \rfloor}\,, \tag{2.1}$$

(ii) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$

$$\lfloor \varepsilon_1, f\varepsilon_2 \rfloor = f\lfloor \varepsilon_1, \varepsilon_2 \rfloor + (\delta_{\varepsilon_1} f)\varepsilon_2\,. \tag{2.2}$$

In other words, the anchor defines a representation of $\Gamma(\mathcal{A})$ in the Lie algebra of vector fields on M. The second condition is the Leibnitz rule with respect to the multiplication of sections by smooth functions.

Let $\{e^j(x)\}$ be a basis of sections. Then the brackets are defined by the structure functions $f_i^{jk}(x)$ of the algebroid

$$\lfloor e^j, e^k \rfloor = f_i^{jk}(x)e^i\,,\ x \in M\,. \tag{2.3}$$

Using the Jacobi identity for the brackets $\lfloor\,,\,\rfloor$, we find

$$f_i^{jk}(x)f_n^{im}(x) + \delta_{e^m} f_n^{jk}(x) + \text{c.p.}(j,k,m) = 0\,. \tag{2.4}$$ [1]

If the anchor is trivial, then $\mathcal{A}$ is just a bundle of Lie algebras.

There exists the global object - the *Lie groupoid* [**17**].

Definition 2.2. *A Lie groupoid G over a manifold M is a pair of smooth manifolds (G, M), two smooth mappings $l, r : G \to M$ and a partially defined smooth binary operation (the product) $(g, h) \mapsto g \cdot h$ satisfying the following conditions:*
(i) It is defined when $l(h) = r(g)$.
(ii) It is associative: $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ whenever the products are defined.
(iii) For any $g \in G$ there exist the left and right identity elements l_g and r_g in G such that $l_g \cdot g = g \cdot r_g = g$.
(iv) Each g has an inverse g^{-1} such that $g \cdot g^{-1} = l_g$ and $g^{-1} \cdot g = r_g$.

We denote an element of $g \in G$ by the triple $\langle\langle x|g|y\rangle\rangle$, where $x = l(g),\ y = r(g)$. Then the product $g \cdot h$ is

$$g \cdot h = \langle\langle x|g \cdot h|z\rangle\rangle = \langle\langle x|g|y\rangle\rangle\langle\langle y|h|z\rangle\rangle\,.$$

An orbit of the groupoid in the base M is defined as an equivalence $x \sim y$ if $x = l(g),\ y = r(g)$ for some $g \in G$. The isotropy subgroup G_x for $x \in M$ is defined as

$$G_x = \{g \in G \mid l(g) = x = r(g)\} = \{\langle\langle x|g|x\rangle\rangle\}\,.$$

The Lie algebroid is a infinitesimal version of the Lie groupoid. The anchor is determined in terms of the multiplication law. The problem of integration of a Lie algebroid to a Lie groupoid is treated in Ref. [**17**].

[1] The sums over repeated indices are understood throughout the paper, and c.p.(j,k,m) means the cycle permutation.

2.2. Representations and cohomology of Lie algebroids. The definition of algebroids representations is rather evident:

Definition 2.3. *A vector bundle representation (VBR) $(\rho,\mathcal{M})$ of a Lie algebroid $\mathcal{A}$ over M is a vector bundle $\mathcal{M}$ over M and a map ρ from $\mathcal{A}$ to the bundle of differential operators in $\mathcal{M}$ Diff $^{\leq 1}(\mathcal{M},\mathcal{M})$ of the order less or equal to 1, such that:*
(i) the principle symbol of $\rho(\varepsilon)$ is a scalar equal to the anchor of ε:

$$\mathrm{Symb}(\rho(\varepsilon)) = \mathrm{Id}_{\mathcal{M}}\delta_\varepsilon\,, \tag{2.5}$$

(ii) for any $\varepsilon_1,\varepsilon_2\in\Gamma(\mathcal{A})$

$$[\rho(\varepsilon_1),\rho(\varepsilon_2)] = \rho(\lfloor\varepsilon_1\varepsilon_2\rfloor)\,, \tag{2.6}$$

where the l.h.s. denotes the commutator of differential operators.

For example, the trivial bundle is a VBR representation (the map ρ is the anchor map δ).

Consider a small ball $U_\alpha\subset M$ with local coordinates $x=(x^1,\dots,x^n)$. Then the anchor can be written as

$$\delta_{e^j} = b^{ja}(x)\frac{\partial}{\partial x^a} = \langle b^j|\partial\rangle\,,\ (a=1,\dots,\dim M)\,.^2 \tag{2.7}$$

Let $\xi=\xi=\xi^a\partial_a$, $(\partial_a=\frac{\partial}{\partial x^a})$ be a section of the tangent bundle TM. Then the VBR on TM takes the form

$$\rho(e^j)\xi = \langle b^j|\partial\xi\rangle - \langle\partial b^j|\xi\rangle = [\delta_{e^j},\xi]\,. \tag{2.8}$$

Similarly, the VBR on a section $p=p_a dx^a=\langle p|dx\rangle$ of T^*M is

$$\rho(e^j)p = \langle b^j|\partial\rangle\langle p|dx\rangle + \langle dx|\langle\partial(b^j)p\rangle\rangle = \mathcal{L}_{\delta_{e^j}}p\,, \tag{2.9}$$

where $\mathcal{L}$ is the Lie derivative along the vector field δ^j_e. We omit a more general definition of the sheaf representation.

Now we define cohomology groups of algebroids. First, we consider the case of contractible base M. Let $\mathcal{A}^*$ be a bundle over M dual to $\mathcal{A}$. Consider the bundle of graded commutative algebras $\wedge^\bullet\mathcal{A}^*$. The space $\Gamma(M,\wedge^\bullet\mathcal{A}^*)$ is generated by the sections η_k: $\langle\eta_j|e^k\rangle=\delta^k_j$. It is a graded algebra

$$\Gamma(M,\wedge^\bullet\mathcal{A}^*) = \oplus\mathcal{A}^*_n\,,\ \mathcal{A}^*_n = \{c_n(x)=\frac{1}{n!}c_{j_1,\dots,j_n}(x)\eta_{j_1}\dots\eta_{j_n}\,,\ x\in U\}\,.$$
$$c_{j_1,\dots,j_n}(x)\in\mathcal{O}(M)\,.$$

Define the Cartan-Eilenberg operators “dual” to the brackets $\lfloor,\rfloor$

$$sc_n(x;e^1,\dots,e^n,e^{n+1}) = \sum_i(-1)^{i-1}\delta_{e^i}c_n(x;e^1,\dots,\hat{e}^i,\dots e^n) -$$

$$-\sum_{j<i}(-1)^{i+j}c_n(x;\lfloor e^i,e^j\rfloor,\dots,\hat{e}^j,\dots,\hat{e}^i,\dots,e^n)\,, \tag{2.10}$$

where

$$\delta_{e^i}c_n(x) = \langle\partial c_n(x)|b^i(x)\rangle\,.$$

It follows from (2.1) and (2.4) that $s^2=0$. Thus, s defines a complex of bundles $\mathcal{A}^*\to\wedge^2\mathcal{A}^*\to\cdots$.

[2]The brackets $\langle\ |\ \rangle$ mean summations over repeating indices.

The cohomology groups $H^k(\mathcal{A},\mathcal{O}(M))$ of this complex are called *the cohomology groups of algebroid with trivial coefficients*. This complex is a part of the BRST complex described below.

The action of the coboundary operator s takes the following form on the low cochains:

$$sc(x;\varepsilon)=\delta_\varepsilon c(x)\,, \tag{2.11}$$

$$sc(x;\varepsilon_1,\varepsilon_2)=\delta_{\varepsilon_1}c(x;\varepsilon_2)-\delta_{\varepsilon_2}c(x;\varepsilon_1)-c(x;\lfloor\varepsilon_1,\varepsilon_2\rfloor)\,, \tag{2.12}$$

$$sc(x;\varepsilon_1,\varepsilon_2,\varepsilon_3)=\delta_{\varepsilon_1}c(x;\varepsilon_2,\varepsilon_3)-\delta_{\varepsilon_2}c(x;\varepsilon_1,\varepsilon_3) \tag{2.13}$$

$$+\delta_{\varepsilon_3}c(x;\varepsilon_1,\varepsilon_2)-c(x;\lfloor\varepsilon_1,\varepsilon_2\rfloor,\varepsilon_3)+c(x;\lfloor\varepsilon_1,\varepsilon_3\rfloor,\varepsilon_2)-c(x;\lfloor\varepsilon_2,\varepsilon_3\rfloor,\varepsilon_1)\,.$$

It follows from (2.11) that $H^0(\mathcal{A},\mathcal{O}(M))$ is isomorphic to the invariants in the space $\mathcal{O}(M)$.

The next cohomology group $H^1(\mathcal{A},\mathcal{O}(M))$ is responsible for the shift of the anchor action on $\mathcal{O}(M)$ $\delta_\varepsilon f(x)=\langle\delta_\varepsilon x|\partial f(x)\rangle)$

$$\hat{\delta}_\varepsilon f(x)=\delta_\varepsilon f(x)+c(x;\varepsilon)\,,\ \ sc(x;\varepsilon)=0\,. \tag{2.14}$$

Then due to (2.12), this action is consistent with the defining anchor property (2.1).

We modify the action (2.14). Let $\Psi=\exp f\in\mathcal{O}^*(M)$. Then the action on Ψ takes the form

$$\tilde{\delta}_\varepsilon\Psi(x)=(\delta_\varepsilon+c(x;\varepsilon))\,\Psi(x) \tag{2.15}$$

satisfies the the anchor property (2.1) $[\tilde{\delta}_{\varepsilon_1}\tilde{\delta}_{\varepsilon_2}]=\tilde{\delta}_{[\varepsilon_1,\varepsilon_2]}$.

If M is not contractible the definition of cohomology group is more complicated. We sketch the Ĉech version of it. Choose an acyclic covering U_α. Consider the Ĉech complex with coefficients in $\bigwedge^\bullet(\mathcal{A}^*)$ corresponding to this covering:

$$\bigoplus\Gamma(U_\alpha,\bigwedge{}^\bullet(\mathcal{A}^*))\stackrel{d^C}{\longrightarrow}\bigoplus\Gamma(U_{\alpha\beta},\bigwedge{}^\bullet(\mathcal{A}^*))\stackrel{d^C}{\longrightarrow}\cdots$$

The Ĉech differential d^C commutes with the Cartan-Eilenberg operator s, and cohomology of algebroid are cohomology of normalization of this bicomplex :

$$\bigoplus\Gamma(U_\alpha,\mathcal{A}_0^*)\stackrel{d^C,s}{\longrightarrow}\bigoplus\Gamma(U_{\alpha\beta},\mathcal{A}_0^*)\oplus\bigoplus\Gamma(U_\alpha,\mathcal{A}_1^*)\stackrel{\left(\begin{array}{ccc}d^C & s & \\ & -d^C & s\end{array}\right)}{\longrightarrow}$$

$$\bigoplus\Gamma(U_{\alpha\beta\gamma},\mathcal{A}_0^*)\oplus\bigoplus\Gamma(U_{\alpha\beta},\mathcal{A}_1^*)\oplus\bigoplus\Gamma(U_\alpha,\mathcal{A}_2^*)\longrightarrow\cdots\,.$$

The cochains $c^{i,j}\in\bigoplus_{\alpha_1\alpha_2\cdots\alpha_j}\Gamma(U_{\alpha_1\alpha_2\cdots\alpha_j},\mathcal{A}_i^*)$ are bigraded. The differential maps $c^{i,j}$ to $(-1)^jd^C\,c^{i,j}+sc^{i,j}$, has type $(i,j+1)$ for $(-1)^jd^C\,c^{i,j}$ and $(i+1,j)$ for $sc^{i,j}$.

Again, the group $H^0(\mathcal{A},\mathcal{O}(M))$ is isomorphic to the invariants in the whole space $\mathcal{O}(M)$.

Consider the next group $H^{(1)}(\mathcal{A},\mathcal{O}(M))$. It has two components $(c_\alpha(x,\varepsilon),c_{\alpha\beta}(x))$. They are characterized by the following conditions (see (2.12))

$$c_\alpha(x;\lfloor\varepsilon_1,\varepsilon_2\rfloor)=\delta_{\varepsilon_1}c_\alpha(x;\varepsilon_2)-\delta_{\varepsilon_2}c_\alpha(x;\varepsilon_1)\,,$$

$$\delta_\varepsilon c_{\alpha\beta}(x)=-c_\alpha(x;\varepsilon)+c_\beta(x;\varepsilon)\,, \tag{2.16}$$

$$c_{\alpha\gamma}(x)=c_{\alpha\beta}(x)+c_{\beta\gamma}(x)\,. \tag{2.17}$$

The group $H^{(2)}(\mathcal{A},\mathcal{O}(M))$ is responsible for the central extension of the the brackets on $\Gamma(\mathcal{A})$. Let $c(x;\varepsilon_1,\varepsilon_2)$ be a two-cocycles. Then

$$\lfloor(\varepsilon_1,0),(\varepsilon_2,0)\rfloor_{c.e.} = (\lfloor\varepsilon_1,\varepsilon_2\rfloor, c(x;\varepsilon_1,\varepsilon_2))\,. \tag{2.18}$$

The cocycle condition (2.13) means that the new brackets $\lfloor\ ,\ \rfloor_{c.e.}$ satisfies (2.4). The exact cocycles lead to the split extensions. There is an obstacle to this continuations in $H^{(3)}(\mathcal{A},\mathcal{O}(M))$. We do not dwell on this point.

EXAMPLE 2.1. *Consider a flag variety $Fl_N = G/B$, where $G = \mathrm{SL}(N,\mathbb{C})$ and B is the lower Borel subgroup. The flag variety is a base of a $\mathrm{sl}(N,\mathbb{C})$ algebroid. In particular, $sl(2,\mathbb{C})$ anchor acts on $Fl_2 \sim \mathbb{C}P^1$ in a neighborhood $\mathcal{U}_+$ of $z=0$ by the vector fields*

$$\delta_e = \partial_z\,,\ \delta_h = -2z\partial_z\,,\ \delta_f = -z^2\partial_z\,.$$

The one-cocycle representing $H^1(\mathrm{sl}(2,\mathbb{C}),\mathcal{O}(\mathbb{C}P^1))$

$$c(z,e) = 0\,,\ c(z,h) = \nu\,,\ c(z,f) = \nu z$$

defines the extension of the anchor action $\hat{\delta}_\varepsilon$ (2.14). Let w be a local coordinate in a neighborhood $\mathcal{U}_-$ of $z=\infty$. Then

$$\delta_e = -w^2\partial_w\,,\ \delta_h = 2w\partial_w\,,\ \delta_f = \partial_w\,.$$

and

$$c(w,e) = \nu w\,,\ c(w,h) = -\nu\,,\ c(w,f) = 0\,.$$

The another component of $H^1(\mathrm{sl}(2,\mathbb{C}),\mathcal{O}(\mathbb{C}P^1))$ satisfying (2.16) is the cocycle

$$c_{+-}(z) = \nu\log z\,. \tag{2.19}$$

EXAMPLE 2.2. *Consider the $\mathbb{C}^*$ bundle over $\mathbb{C}^2\setminus 0 = \{(z_1,z_2)\}$. Define the anchor map*

$$\delta_\varepsilon = \varepsilon(z_1\partial_{z_1} + z_2\partial_{z_2})\,. \tag{2.20}$$

According with (2.14) it can be extended as

$$\hat{\delta}_\varepsilon = \varepsilon(z_1\partial_{z_1} + z_2\partial_{z_2} - \nu)\,, \tag{2.21}$$

because $\varepsilon\nu \in \mathbb{C}$ represents a non-trivial one-cocycle.

2.3. Affine spaces over cotangent bundles. We shall consider Hamiltonian algebroids over cotangent bundles. The cotangent bundles are a special class of symplectic manifolds. There exist a generalization of cotangent bundles, that we include in our scheme. It is affine spaces over a cotangent bundles we are going to define.

Let V be a vector space and $\mathcal{R}$ is a manifold with an action of V on $\mathcal{R}$

$$\mathcal{R}\times V \to \mathcal{R} : (x,v) \to x+v \in \mathcal{R}\,.$$

DEFINITION 2.4. *The manifold $\mathcal{R}$ is an affine space over V (a* principle homogeneous space over *V) if the action of V on $\mathcal{R}$ is transitive and free. We denote it $\mathcal{R}/V$.*

In other words, for any pair $x_1,x_2\in\mathcal{R}$ there exists $v\in V$ such that $x_1+v = x_2$, and $x+v\neq x$ if $v\neq 0$.

This construction is generalized to vector bundles. Let E be a vector bundle over M and $\Gamma(E)$ is the linear space of its sections.

DEFINITION 2.5. *An affine space $\mathcal{R}/E$ over E is a bundle over M with the space of sections $\Gamma(\mathcal{R})$ defined as the affine space over $\Gamma(E)$.*

Consider a cotangent bundle T^*M and the corresponding affine space $\mathcal{R}/T^*M$. Let ξ_α be a section of $\mathcal{R}/T^*M$ over a contractible set $\mathcal{U}_\alpha \subset M$. It can be identified with a section of $T^*\mathcal{U}_\alpha$. To define $\mathcal{R}$ over M consider an intersection $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. We assume that local sections are related as

$$\xi_\alpha = \xi_\beta + \varsigma_{\alpha\beta}\,, \tag{2.22}$$

where $\varsigma_{\alpha\beta} \in \Gamma(\mathcal{Z}^{(1)}(\mathcal{U}_{\alpha\beta}))$ and

$$\mathcal{Z}^{(1)}(\mathcal{U}_{\alpha\beta}) = \{\varsigma_{\alpha\beta} \in \Omega^{(1)}(\mathcal{U}_{\alpha\beta}) \,|\, d\varsigma_{\alpha\beta} = 0\}\,.$$

We have $\varsigma_{\alpha\beta} = -\varsigma_{\beta\alpha}$ and on the intersection $\mathcal{U}_{\alpha\beta\gamma} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

$$\varsigma_{\alpha\beta} + \varsigma_{\beta\gamma} + \varsigma_{\gamma\alpha} = 0\,.$$

If $\varsigma_{\alpha\beta} = p_\alpha - p_\beta$ then $\mathcal{R}$ can be continued from $\mathcal{U}_\alpha$ to $\mathcal{U}_\beta$ as $T^*\mathcal{U}_\beta$. Therefore, the non-equivalent affine spaces over T^*M are classified by the elements $H^1(M, \mathcal{Z}^{(1)})$.

An affine space $\mathcal{R}/T^*M$ is a symplectic space with the canonical form ω. Locally on $\mathcal{U}_\alpha$, $\omega = \langle d\xi_\alpha \wedge dx_\alpha\rangle$. In fact, the symplectic form is well defined globally, because the transition forms $\varsigma_{\alpha\beta}$ are closed. But in contrast with T^*M, ω is not exact, since $\xi_\alpha dx_\alpha$ is defined only locally. The symplectic form on T^*M has vanishing cohomology class $[\omega_{T^*M}] = 0$, while the cohomology class of $[\omega_{\mathcal{R}}]$ is an nontrivial element in $H^2(M, \mathbb{C})$.

`Example` **2.3**
The cotangent bundle T^*Fl_N to the flag variety
$Fl_N = \mathrm{SL}(N,\mathbb{C})/B$, without the null section, can be identified with the coadjoint orbit passing through the Jordanian matrix $\sum_{j=1}^{N-1} E_{j,j+1}$. Consider in particular $Fl_2 \sim \mathbb{C}P^1$. The symplectic form on the cell $\mathcal{U}_+ = \{|z| < \infty\} \subset \mathbb{C}P^1$ is $\omega = dp_+ \wedge dz_+$. The cotangent bundle $T^*\mathcal{U}_+$ can be represented by the matrix

$$\begin{pmatrix} -z_+p_+ & p_+ \\ -z_+^2 p_+ & z_+p_+ \end{pmatrix}.$$

On $\mathcal{U}_- = \{|z| > 0\}$ the form ω is $dp_- \wedge dz_-$ $\;(z_- = \frac{1}{z_+})$. On the intersection the transition form vanishes $p_+dz_+ = p_-dz_-$.

The affine space $\mathcal{R}/T^*\mathbb{C}P^1$ is a generic coadjoint orbit $\mathcal{O}_\nu$ passing through $\mathrm{diag}(\nu, -\nu)$. Over $\mathcal{U}_+$ the coadjoint orbit has the parametrization (ξ_+, z_+)

$$\begin{pmatrix} -z_+\xi_+ + \frac{1}{2}\nu & \xi_+ \\ -z_+^2\xi_+ + \nu z_+ & z_+\xi_+ - \frac{1}{2}\nu \end{pmatrix}$$

with the form $d\xi_+ \wedge dz_+$. On the cell $\mathcal{U}_-$ the form is $d\xi_- \wedge dz_-$. The transition form is represented by the non-trivial cocycle from $H^1(\mathbb{C}P^1, T^*Fl_2)$

$$\xi_-dz_- - \xi_+dz_+ = \nu\, d(\log z_-) \tag{2.23}$$

(compare with (2.19)).

`Example` **2.4**
The basic example, though infinite-dimensional, is the affine space over the antiHiggs bundles. [3] The antiHiggs bundle $\mathcal{H}_N(\Sigma)$ is a cotangent bundle to the

[3] We use the antiHiggs bundles instead of the standard Higgs bundles for reasons, that will become clear in Sect. 4.

space of holomorphic connections $M = \{\nabla^{(1,0)} = \partial + A\}$ in a trivial vector bundle of rank N over a complex curve Σ. The cotangent vector (the antiHiggs field) is $\mathrm{sl}(N,\mathbb{C})$ valued $(0,1)$-form $\bar{\Phi}$. The symplectic form on $\mathcal{H}_N(\Sigma)$ is $-\int_\Sigma \mathrm{tr}(D\bar{\Phi}\wedge DA)$. An example of the affine space $\mathcal{R}/\mathcal{H}_N(\Sigma)$ is the space of the κ-connections $\{(\kappa\bar{\partial} + \bar{A}, \partial + A)\}$ $(\kappa \in \mathbb{C})$ with the symplectic form $\int_\Sigma \mathrm{tr}(DA \wedge D\bar{A})$.

2.4. Lie algebroids with representations in affine spaces. Let $\mathcal{A}$ be a Lie algebroid over M with the anchor δ. We shall define a representation ρ of $\mathcal{A}$ acting on sections of the affine space $\mathcal{R}/T^*M$ in such a way that prove that ρ combined with δ are hamiltonian vector fields on $\mathcal{R}$ with respect to $\omega = \langle d\xi|dx\rangle$. It means that

$$i_\varepsilon\omega = dh_\varepsilon\,,\ \delta_\varepsilon x = \{h_\varepsilon, x\}\,,\ \rho_\varepsilon\xi = \{h_\varepsilon, \xi\}\,.$$

LEMMA 2.1. *The anchor action (2.7) of a Lie algebroid $\mathcal{A}$ over M can be lifted to a hamiltonian action on $\mathcal{R}/T^*M$ with*

$$h_\varepsilon = \langle\delta_\varepsilon|\xi\rangle + c(x;\varepsilon)\,, \tag{2.24}$$

where $c(x;\varepsilon) \in H^1(\mathcal{A}, \mathcal{O}(M))$.

Proof.
Define first the VBR of $\mathcal{A}$ on sections of $\mathcal{R}$. Consider a contractible set $\mathcal{U}_\alpha \subset M$ with local coordinates $x_\alpha = (x^1_\alpha, \dots, x^d_\alpha)$ The anchor has the form

$$\delta_\varepsilon x^a_\alpha = V^a_\alpha\,,\ (\,\delta_{e^j} = \sum_a V^{j,a}\partial_a = \langle V^j|\partial\rangle\,) \tag{2.25}$$

Let $\xi_\alpha \in \Gamma(\mathcal{R}_\alpha/T^*\mathcal{U}_\alpha)$. Define an action ρ on the sections

$$\rho_\varepsilon\xi_\alpha = \mathcal{L}_\varepsilon\xi_\alpha + dc_\alpha(x;\varepsilon)\,, \tag{2.26}$$

where $\mathcal{L}_\varepsilon$ is the Lie derivative and $c_\alpha(x;\varepsilon)$ represents an element from $H^1(\mathcal{A}, \mathcal{O}(M))$. Since $c_\alpha(x;\varepsilon)$ is a cocycle, the action (2.26) satisfies (2.6). It also satisfies (2.5). Therefore, the action (2.26) is a VBR. The last term in (2.26) is responsible for the passage from $T^*\mathcal{U}_\alpha$ to the affine space $\mathcal{R}_\alpha$, otherwise ξ_α is transformed as a cotangent vector (see (2.9)).

To define the VBR globally we prove that on the intersection $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have

$$\rho_\varepsilon(\xi_\alpha - \xi_\beta) = \mathcal{L}_\varepsilon(\varsigma_{\alpha\beta} + dc_{\alpha\beta})\,, \tag{2.27}$$

where $\varsigma_{\alpha\beta}$ is a closed one-form representing an element from $H^1(M, \mathcal{Z}^{(1)})$ and $c_{\alpha\beta} \in H^1(M, \mathcal{O}(M))$ (see (2.17)). To prove (2.27) we apply (2.26) to its left hand side

$$\rho_\varepsilon(\xi_\alpha - \xi_\beta) = \mathcal{L}_\varepsilon(\xi_\alpha - \xi_\beta) + d(c_\alpha(x;\varepsilon) - c_\beta(x;\varepsilon))\,.$$

Since $d(\xi_\alpha - \xi_\beta) = d\varsigma_{\alpha\beta} = 0\,,\ \mathcal{L}_\varepsilon(\xi_\alpha - \xi_\beta) = d\imath_\varepsilon\varsigma_{\alpha\beta}$. Then using (2.16) we come to (2.27). The action (2.27) means that

$$\rho_\varepsilon\varsigma_{\alpha\beta} = \mathcal{L}_\varepsilon(\varsigma_{\alpha\beta} + dc_{\alpha\beta})$$

is the Lie derivative of a closed one-form. It allows us to define the VBR on sections of $\mathcal{R}/T^*M$.

The direct calculations show that δ_ε (2.25) and ρ_ε (2.26) are hamiltonian vector fields $\{h_\varepsilon\,,\ \}$ with h_ε (2.24). The Hamiltonians have the linear dependence on "momenta". The corresponding Hamiltonians (2.24) have the linear dependence on "momenta". The exact cocycle $c(x;\varepsilon) = \delta_\varepsilon f(x)$ shifts ξ in the Hamiltonian

$h_\varepsilon = \langle \xi + df(x)|\delta_\varepsilon)\rangle$. Thus, all nonequivalent lifts of anchors from M to $\mathcal{R}/T^*M$ are in one-to-one correspondence with $H^1(\mathcal{A}, \mathcal{O}(M))$. □

Remark 2.1. *There exists the map (see (2.24))*

$$\Gamma(\mathcal{A}) \to \mathcal{O}(\mathcal{R}), \ (\varepsilon \to h_\varepsilon = \langle \delta_\varepsilon|\xi\rangle + c(x;\varepsilon)). \tag{2.28}$$

Due to (2.1) and the cocycle property of $c(\varepsilon, x)$ it is the Lie algebras map

$$\{h_{\varepsilon_1}, h_{\varepsilon_2}\} = h_{[\varepsilon_1,\varepsilon_2]}. \tag{2.29}$$

The map to the hamiltonian vector fields is the bundle map

$$f\varepsilon \to f\{h_\varepsilon, \ \},$$

because the ring of functions is defined on the base M and thereby is Poisson commutative.

This remark suggests the following definition.

Definition 2.6. *We call the Lie algebroid $\mathcal{A}$ over M equipped with the VBR in the sections of $\mathcal{R}/T^*M$ the Hamiltonian algebroid $\mathcal{A}^H$. The anchor of $\mathcal{A}^H$ is the map (2.28).*

Then the Jacobi identity for the Hamiltonian algebroids assumes the form

$$f_i^{jk}(x)f_n^{im}(x) + \{h_{e^m}, f_n^{jk}(x)\} + c.p.(j,k,m) = 0. \tag{2.30}$$

`Example` **2.5**

Consider the action of $\mathbb{C}^*$: on $\mathbb{C}^2 \setminus \{0\}$, $(z_a \to \lambda z_a)$, $(\lambda \neq 0)$. Infinitesimally, we have (2.20) (see Example 2.2). The quotient $(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$ is isomorphic to $\mathbb{C}P^1 \sim Fl_2$. We lift the infinitesimal action (2.20) to the cotangent bundle $T^*(\mathbb{C}^2 \setminus \{0\})$. The cotangent bundle $T^*(\mathbb{C}^2 \setminus \{0\})$ is equipped with the canonical symplectic form

$$\omega = dp_1 \wedge dz_1 + dp_2 \wedge dz_2.$$

It is invariant under the action of $\mathbb{C}^*$: $z_a \to (\exp\varepsilon)z_a$, $p_a \to (\exp -\varepsilon)p_a$. The generating Hamiltonian h_ε of this transformation ($\imath_\varepsilon\omega = d\mu^*$) has the form

$$h_\varepsilon = -\varepsilon(p_1z_1 + p_2z_2).$$

It defines the Hamiltonian algebroid $\mathcal{A}^H$.

Consider the symplectic quotient of $T^*(\mathbb{C}^2 \setminus 0)$ with respect to the $\mathbb{C}^*$ action generating by the shifted Hamiltonian by the cocycle $\varepsilon\nu$

$$h_\varepsilon \to h_\varepsilon - \varepsilon\nu.$$

It defines the moment map constraint

$$p_1z_1 + p_2z_2 = \nu.$$

For $z_1 \neq 0$ one can fix the gauge $z_1 = 1$ and find from the moment constraint $p_1 = -p_2z_2 + \nu$. In this case the symplectic quotient is isomorphic to $T^*\mathbb{C}$ with coordinates (p_2, z_2) and the form $dp_2 \wedge dz_2$. Similarly, for $z_2 \neq 0$ one can take $z_2 = 1$, $p_2 = -p_1z_1 + \nu$ and the canonical coordinates (p_1, z_1) on the symplectic quotient $T^*\mathbb{C}$. On the intersection we come to the relation

$$p_1dz_1 = p_2dz_2 + d\log z_1^\nu.$$

Comparing with (2.23) we conclude that for a nonzero value of the moment map $\nu \neq 0$ the symplectic quotient is a generic coadjoint orbit, otherwise for $\nu = 0$ we come to $T^*\mathbb{C}P^1$.

2.5. Reduced phase space and its BRST description. Let e^j be a basis of sections in $\Gamma(\mathcal{A})$. Then the Hamiltonians (2.24) can be represented in the form $h^j = \langle e^j | F(x)\rangle$, where $F(x) \in \Gamma(\mathcal{A}^*)$ defines the *moment map*

$$F\, : \mathcal{R} \to \Gamma(\mathcal{A}^*)\,.$$

The coadjoint action $\mathrm{ad}^*_\varepsilon$ in $\Gamma((\mathcal{A}^H))$ is defined in the standard way

$$\langle \lfloor \varepsilon, e^j \rfloor | F(x)\rangle = \langle e^j | \mathrm{ad}^*_\varepsilon F(x)\rangle\,.$$

The zero-valued moment $F(x) = 0$ is preserved by the groupoid coadjoint action G generated by $\mathrm{ad}^*_\varepsilon$. The moment constraints $F(x) = 0$ generate canonical algebroid action on $\mathcal{R}$. The reduced phase space is defined as the quotient

$$\mathcal{R}^{red} = \mathcal{R}//G := \{x \in \mathcal{R} | (F(x) = 0)/G\}\,,$$

In other words, $\mathcal{R}^{red}$ is the set of orbits of G on the constraint surface $F(x) = 0$.

The BRST approach allows us to go around the reduction procedure by introducing additional fields (the ghosts). We shall construct the BRST complex for $\mathcal{A}^H$ in a similar way as the Cartan-Eilenberg complex for the Lie algebroid $\mathcal{A}$. The BRST complex is endowed with a Poisson structure and in this way it has the form of the quasi-classical limit of the Cartan-Eilenberg complex.

The sections $\eta \in \Gamma((\mathcal{A})^*)$ are the anti-commuting (odd) fields called *the ghosts*. We preserve the notation for the Hamiltonians in terms of the ghosts $h = \langle \eta | F(x)\rangle$, where $\{\eta_j\}$ is a basis in $\Gamma((\mathcal{A})^*)$. Introduce another type of odd variables (*the ghost momenta*) $\mathcal{P}^j \in \Gamma(\mathcal{A}^H)$ dual to the ghosts η_k. We attribute the ghost number one to the ghost fields $\mathrm{gh}(\eta) = 1$, minus one to the ghost momenta $\mathrm{gh}(\mathcal{P}) = -1$ and $\mathrm{gh}(x) = 0$ for $x \in \mathcal{R}$. Define the Poisson brackets in addition to the Poisson structure on $\mathcal{R}$

$$\{\eta_j, \mathcal{P}^k\} = \delta^k_j\,,\ \{\eta_j, x\} = \{\mathcal{P}^k, x\} = 0\,. \tag{2.31}$$

All fields are incorporated in the graded Poisson superalgebra

$$\mathcal{B}FV = (\Gamma(\wedge^\bullet((\mathcal{A})^* \oplus \mathcal{A})) \otimes \mathcal{O}(\mathcal{R}) = \Gamma(\wedge^\bullet(\mathcal{A})^*) \otimes \Gamma(\wedge^\bullet \mathcal{A}) \otimes \mathcal{O}(\mathcal{R})\,.$$

(*the Batalin-Fradkin-Volkovitsky ($\mathcal{B}FV$) algebra*).

There exists a nilpotent operator Q on the $\mathcal{B}FV$ algebra $Q^2 = 0,\ gh(Q) = 1$ (*the BRST operator*) transforming it into the BRST complex. The cohomology of the BRST complex give rise to the structure of the classical reduced phase space $\mathcal{R}^{red}$. Namely, $H^0(Q)$ is the space of invariants with respect to the symplectic action of G and in this way can be identified with the classical observables.

Suppose that the action of Q has the hamiltonian form:

$$Q\psi = \{\psi, \Omega\},\ \psi, \Omega \in \mathcal{B}FV\,.$$

Due to the Jacobi identity the nilpotency of Q is equivalent to $\{\Omega, \Omega\} = 0$. Since Ω is odd, the brackets are symmetric. For a generic Hamiltonian system with the first class constraints Ω can be represented as the expansion [**18**]

$$\Omega = h_\eta + \frac{1}{2}\langle \lfloor \eta, \eta' \rfloor | \mathcal{P}\rangle + \ldots\,,\ (h_\eta = \langle \eta | F\rangle)\,,$$

where the higher order terms in $\mathcal{P}$ are omitted. The highest order of $\mathcal{P}$ in Ω is called *the rank* of the BRST operator Q. If $\mathcal{A}$ is a Lie algebra defined together with its canonical action on $\mathcal{R}$ then Q has the rank one or less. In this case the BRST operator Q is the extension of the Cartan-Eilenberg operator giving rise to the cohomology of $\mathcal{A}$ with coefficients in $\mathcal{O}(\mathcal{R})$. Due to the Jacobi identity the first two terms in the previous expression provide the nilpotency of Q. It turns out that for the Hamiltonian algebroids $\mathcal{A}^H$ Ω has the same structure as for the Lie algebras, though the Jacobi identity (2.4) has additional terms.

THEOREM 2.1. *The BRST operator Q for the Hamiltonian algebroid $\mathcal{A}^H$ has the rank one:*

$$\Omega = \langle \eta | F \rangle + \frac{1}{2} \langle \lfloor \eta, \eta' \rfloor | \mathcal{P} \rangle \,. \tag{2.32}$$

Proof.
Straightforward calculations show that

$$\{\Omega, \Omega\} = \{h_{\eta_1}, h_{\eta_2}\} + \frac{1}{2}\langle \lfloor \eta_2, \eta_2' \rfloor | F \rangle - \frac{1}{2}\langle \lfloor \eta_1, \eta_1' \rfloor | F \rangle$$

$$+\frac{1}{2}\{h_{\eta_1}, \langle \lfloor \eta_2 \eta_2' \rfloor | \mathcal{P}_2 \rangle\} - \frac{1}{2}\{h_{\eta_2}, \langle \lfloor \eta_1, \eta_1' \rfloor | \mathcal{P}_1 \rangle\} + \frac{1}{4}\{\langle \lfloor \eta_1, \eta_1' \rfloor | \mathcal{P}_1 \rangle, \langle \lfloor \eta_2, \eta_2' \rfloor | \mathcal{P}_2 \rangle\} \,.$$

The sum of the first three terms vanishes due to (2.29). The sum of the rest terms is the left hand side of (2.4). The additional dangerous term may come from the Poisson brackets of the structure functions $\{\lfloor \eta_1, \eta_1' \rfloor, \lfloor \eta_2, \eta_2' \rfloor\}$. In fact, these brackets vanish because the structure functions do not depend on the ghost momenta. Thus, (2.4) leads to the desired identity $\{\Omega, \Omega\} = 0$. □

3. Lie algebroids and Poisson sigma-model

3.1. Cotangent bundles to Poisson manifolds as Lie algebroids. Let M be a Poisson manifold with the Poisson bivector $\pi = \pi(\varepsilon, \varepsilon')$, where $\varepsilon, \varepsilon'$ are sections of the bundle T^*M. It is a skewsymmetric tensor with vanishing Schouten brackets (the Jacobi identity) $[\pi, \pi]_S = 0$. In local coordinates $x = (x_1, \ldots, x_n)$ it means

$$\partial_i \pi^{jk}(x) \pi^{im}(x) + \text{c.p.}(j, k, m) = 0 \,. \tag{3.1}$$

The Poisson brackets are defined on the space $\mathcal{O}(M)$

$$\{f(x), g(x)\} := \langle dg | \pi | df \rangle, \; df, dg \in \Gamma(T^*M) \,.$$

The Poisson bivector gives rise to the map

$$V^\pi : T^*M \to TM \,, \; V^\pi_\varepsilon = \langle \varepsilon | \pi | \partial \rangle \,, \; \varepsilon \in \Gamma(T^*M) \,. \tag{3.2}$$

In particular,

$$\delta_\varepsilon x^k = \varepsilon_j \pi^{jk}(x) \,. \tag{3.3}$$

In this way we obtain a map from the space $\mathcal{O}(M)$ to the space of the Poisson vector fields

$$f \to V_f = \partial_i f \pi^{ik} \partial_k = \langle df | \pi | \partial \rangle \,, \; (\partial_i = \frac{\partial}{\partial x^i}) \,. \tag{3.4}$$

The Poisson brackets can be rewritten as $\{f(x), g(x)\} = -i_{V_f} dg$.

Define brackets on the sections $\varepsilon, \varepsilon' \in \Gamma(T^*M)$

$$\lfloor \varepsilon, \varepsilon' \rfloor = d\langle \varepsilon | \pi(x) | \varepsilon' \rangle + \langle d\varepsilon | \pi | \varepsilon' \rangle + \langle \varepsilon | \pi | d\varepsilon' \rangle \,, \tag{3.5}$$

$$\lfloor \varepsilon, \varepsilon' \rfloor_k = \varepsilon_j \partial_k (\pi^{ji}) \varepsilon'_i \,.$$

LEMMA 3.1. *The brackets (3.5) are the Lie brackets. T^*M is a Lie algebroid $\mathcal{A}$ over the Poisson manifold M with the Lie brackets (3.5) and the anchor (3.2).*

Proof.
It follows from the Jacobi identity (3.1), that the brackets (3.5) are the Lie brackets. The commutator of the vector fields satisfies (2.1)

$$[V_\varepsilon, V_{\varepsilon'}] = V_{\lfloor\varepsilon,\varepsilon'\rfloor}\,.$$

The property (2.2) follows from the definition of Lie brackets (3.5). □

The structure functions of T^*M are defined by the Poisson bivector

$$f_i^{jk}(x) = \partial_i \pi^{jk}(x)\,.$$

This type of the Lie brackets correspond to a particular choice of the Dirac structure in the Courant brackets on $TM \oplus T^*M$ [**8, 9**].

REMARK 3.1. *A linear space M with the linear Poisson brackets $\pi^{jk}(x) = f_i^{jk}x^i$ can be identified with a Lie coalgebra. Then the map V^π (3.2) is the coadjoint action*

$$V_\varepsilon^\pi \sim \mathrm{ad}_\varepsilon^*\,, \tag{3.6}$$

and the brackets (3.5) are the Lie brackets on M^*.

3.2. Poisson sigma-model and Lie algebroids. Consider a set $\mathbf{M}_{S^1}$ of smooth maps $\tilde{X} : S^1 \to M$

$$\mathbf{M}_{S^1} = \{\tilde{X}(t) : S^1 \to M\,,\ |t| = 1\,,\ \tilde{X} \in C[t, t^{-1}] \otimes \Omega^{(m)}(S^1)\,\}\,.$$

Assume that there exists a Poisson bivector π defined on $\mathbf{M}_{S^1}$, with a holomorphic dependence on $\tilde{X}$ and it is a $(2m-1)$-form on S^1

$$\pi = \pi(\tilde{X})\,,\ \pi \in \wedge^2(T\mathbf{M}_{S^1}) \otimes \Omega^{(2m-1)}(S^1)\,.$$

By means of π define a Lie algebroid $\mathcal{A}_{S^1}$ over $\mathbf{M}_{S^1}$, as it was described above. The sections of the algebroid $\Gamma(\mathcal{A}_{\mathbf{M_{S^1}}}) = \{\varepsilon_r\}$ are defined by the pairing

$$\oint_{S^1} \delta\tilde{\varepsilon}_j \cdot \delta\tilde{X}^j\,. \tag{3.7}$$

Therefore, $\tilde{\varepsilon}$ are $(1-m)$-forms on S^1 taking values in the pull-back by $\tilde{X}$ of the cotangent bundle T^*M

$$\Gamma(\mathcal{A}_{\mathbf{M_{S^1}}}) = \tilde{X}^*(T^*(M)) \otimes \Omega^{(1-m)}(S^1)\,.$$

According with (3.3)) and (3.5) the anchor and of the brackets take the form

$$\delta_\varepsilon \tilde{X} = \langle\varepsilon|\pi(\tilde{X})\,,\ (\delta_\varepsilon \tilde{X}^k = \varepsilon_j \pi^{jk}(\tilde{X})\frac{\delta}{\delta\tilde{X}^k})\,, \tag{3.8}$$

$$\lfloor\tilde{\varepsilon}, \tilde{\varepsilon}'\rfloor = d\langle\tilde{\varepsilon}|\pi(\tilde{X})|\tilde{\varepsilon}'\rangle + \langle d\tilde{\varepsilon}|\pi(\tilde{X})|\tilde{\varepsilon}'\rangle + \langle\tilde{\varepsilon}|\pi(\tilde{X})|d\tilde{\varepsilon}'\rangle\,. \tag{3.9}$$

Consider a complex curve Σ_g of genus g. Let $X(z,\bar{z})$ and $\varepsilon(z,\bar{z})$ be smooth continuations of $\tilde{X}(t)$ and $\tilde{\varepsilon}(t)$ from $S^1 \subset \Sigma_g$ on Σ_g and $\mathbf{M}$ be the set of the maps

$$X : \Sigma_g \to M\,,\ \tilde{X} = X|_{S^1}\,,$$

$$\mathbf{M} = \{X : C^\infty(\Sigma_g \to M\,)\,,\ \otimes\,\Omega^{(m,0)}(\Sigma_g)\,\}\,.$$

$$\varepsilon \in \mathcal{G} = X^*(T^*(M)) \otimes \Omega^{(1-m,0)}(\Sigma_g)\,,\ \tilde{\varepsilon} = \varepsilon|_{S^1}$$

The main assumption. *The anchor (3.8) and the Lie brackets (3.9) are defined on the maps $X \in \mathbf{M}$ and $\varepsilon \in \mathcal{G}$. Therefore there exists the Lie algebroid $\mathcal{A}$ over $\mathbf{M}$ with $\mathcal{G}$ as a space of sections.*

3.3. Global Hamiltonian algebroid. Let ξ be a $(1-m,1)$-form on Σ_g taking values in sections of the affine space over the pull-back by X of the cotangent space T^*M.

$$\xi \in \Gamma(\mathbf{R}X^*(T^*M)) \otimes \Omega^{(1-m,1)}(\Sigma_g)\,.$$

The affine space $= (\xi, X)$ plays the role of the phase space of the $2+1$ sigma-model

$$\Sigma_g \times \mathbb{R} \to M\,.$$

It is endowed with the canonical symplectic form

$$\omega = \int_{\Sigma_g} \langle DX \wedge D\xi\rangle\,. \tag{3.10}$$

Let $c(X,\varepsilon)$ be a cocycle

$$c(X,\varepsilon) = \int_{\Sigma_g} \langle \varepsilon|\bar{\partial}X\rangle\,. \tag{3.11}$$

Thereby, one can define the shifted anchor

$$\delta_\varepsilon X^k = \varepsilon_j \pi^{jk}(X) + c(X,\varepsilon)\,.$$

In principle the cocycle can be trivial and the shift can be removed (see (2.14) and (2.11)).

The canonical transformations of ω (3.10) according with (2.25) and (2.26) are represented by

$$\delta_\varepsilon X = \langle \varepsilon|\pi(X)\,, \tag{3.12}$$

(see (3.8)), and by

$$\rho_\varepsilon \xi = dc(X,\varepsilon) + \mathcal{L}_\varepsilon \xi\,, \tag{3.13}$$

$$(\rho_\varepsilon \xi_k = \bar{\partial}\varepsilon_k + \partial_k(\varepsilon_n \pi^{nj})\xi_j + \varepsilon_n \pi^{nj})\partial_k \xi_j)\,,$$

where $\mathcal{L}_\varepsilon$ is the Lie derivative with the vector fields $\langle \varepsilon|\pi(X)$.

The transformations (3.12), (3.13) and the brackets (3.9) are consistent with the orders of forms on Σ_g:

(3.14)

ε	X	ξ	π
$(1-m,0)$	$(m,0)$	$(1-m,1)$	$(2m-1,0)$

Assume that M is flat with global coordinates (x^j), $(j=1,\dots,d)$. Then another consistent assignment depending on coordinates will be used in Section 5.

(3.15)

ε_j	X^j	ξ_j	π^{jk}
$(1-m_j,0)$	$(m_j,0)$	$(1-m_j,1)$	$(m_j+m_k-1,0)$

In Section 6 we will use another pairing between the sections of algebroids (the covectors) and the tangent to the base vectors (6.4) It leads to another correspondence between the forms and the fields:

(3.16)

ε	X	ξ	π
$(-m,0)$	$(m,0)$	$(-m,1)$	$(2m,0)$

The transformations (3.12), (3.13) are generated by the first class constraints

$$F:=\bar{\partial}X+\pi(X)|\xi\rangle=0\,. \quad (3.17)$$

If we use (3.14) then F is a $(m,1)$-form on Σ_g. In the case (3.15) we come $d=\dim M$ constraints

$$F^j=\bar{\partial}X^j+\pi^{jk}(X)\xi_k=0\,,\ (\dim(F^j)=(m_j,1)\,. \quad (3.18)$$

Due to Lemma 2.1 the action (3.13) means the lift of the anchor action from $\mathbf{M}$ to $\mathbf{R}/T^*\mathbf{M}$ by the cocycle (3.11). The canonical transformations are the Hamiltonian transformation

$$\delta_{h_\varepsilon}f(X,\xi)=\{h_\varepsilon,f(X,\xi)\}\,. \quad (3.19)$$

Here the Poisson brackets are inverse to the symplectic form ω (3.10) and

$$h_\varepsilon=\int\langle\varepsilon|F\rangle=\langle\delta_\varepsilon X|\xi\rangle+c(X,\varepsilon)\,. \quad (3.20)$$

(see (2.24)).

Summarizing, we have defined the symplectic manifold $\mathcal{R}=\{(\xi,X)\}$ and the hamiltonian action of the algebroid $\mathcal{A}$ defined by the Hamiltonian (3.20).

3.4. Deformation of complex structure on complex curves. Following our approach we interpret the constraints (3.17) as consistency conditions for a linear system. In this and next subsections we take the order of forms from (3.14). The passage to (3.15) and (3.16) is straightforward.

Let $\psi\,,\varphi$ be sections of $V_m=X^*(T^*M)\otimes\Omega^{(-m+1,0)}(\Sigma_g)$ and $V'_m=X^*(TM)\otimes\Omega^{(m,0)}(\Sigma_g)$, and B is a linear map $V_m\to V'_m$

$$B(X)\psi=\varphi\,,\ B(X)=\pi(X)\,. \quad (3.21)$$

Define, in addition, two maps

$$A\,:\,V_m\to V_m\otimes\Omega^{(0,1)}(\Sigma_g)\,,\ A^*\,:\,V'_m\to V'_m\otimes\Omega^{(0,1)}(\Sigma_g)\,,$$

$$A=-\bar{\partial}+d\pi(X)|\xi\rangle\,,\ A^*=-\bar{\partial}-d\pi(X)|\xi\rangle\,. \quad (3.22)$$

Locally, the operators are defined as

$$A^l_k\varphi^k=\left(-\bar{\partial}\delta^l_k-\frac{\delta}{\delta X^k}\pi^{ln}\xi_n\right)\varphi^k\,,$$

$$(A^*)^l_k\psi_l=\left(-\bar{\partial}\delta^l_k+\frac{\delta}{\delta X^k}\pi^{ln}\xi_n\right)\psi_l\,.$$

Consider the linear system

$$\pi(X)\psi=0\,, \quad (3.23)$$

$$\left(-\bar{\partial} - d\pi(X)|\xi\rangle\right)\varphi = 0\,, \tag{3.24}$$

$$\left(-\bar{\partial} + d\pi(X)|\xi\rangle\right)\psi = 0\,, \tag{3.25}$$

LEMMA 3.2. *Let the Poisson bivector satisfies the non-degeneracy condition:* $\det(d\pi|\psi\rangle) \neq 0$, $(\det \frac{\delta}{\delta X^i}\pi^{jm}(\psi)_m \neq 0)$ *for any* $\psi \in X^*(T^*\mathbf{M})$. *Then constraints (3.17) are the consistency conditions for (3.23),(3.24) and (3.25).*

Proof. The consistency condition of these equations is the operator equation $BA - A^*B = 0$ for B (3.8), A, A^*, (3.22). After substitution the expressions for A, A^*, B and applying the Jacobi identity (3.1) one comes to the equality

$$(\bar{\partial}X^i + \pi^{is}\xi_s)\frac{\delta}{\delta X^i}\pi^{jm}(\psi)_m = 0.$$

The later is equivalent to the constraint equation (3.17) if π is non-degenerate in the above sense. □

REMARK 3.2. *The spaces* V_m *and* V'_m *are analogs of coadjoint and adjoint spaces. In the first example in next Section they coincide with coadjoint and and adjoint representations of* $\mathrm{sl}(N,\mathbb{C})$. *In what follows we shall consider "vector representations".*

The equations (3.24) and (3.25) define the generalized deformations of the operator $\bar{\partial}$ on Σ_g acting in the space of sections of V_m and V'_m. This deformation is provided by the Poisson bivector π and by sections ξ of the affine bundle. We shall apply this scheme for the concrete Poisson structures below.

3.5. BRST construction. Let G be the Lie groupoid corresponding to the Lie algebroid $\mathcal{A}_{\mathbf{M}}$, and
$\mathbf{R}^{red} = \mathbf{R}//G$ is the corresponding symplectic quotient.

The symplectic quotient $\mathbf{R}^{red}$ can be described by the BRST technique. Define the BRST anticommuting ghosts

$$\eta \in \Gamma(X^*(T^*M) \otimes \Omega^{(1-m,0)}(\Sigma_g)\,,$$

and their momenta

$$\mathcal{P} \in \Gamma(X^*(TM) \otimes \Omega^{(m,1)}(\Sigma_g)\,.$$

The classical BRST complex is the set of fields

$$\bigwedge\nolimits^{\bullet}\Big(\Gamma(X^*(TM) \otimes \Omega^{(1-m,0)}(\Sigma_g)) \oplus \Gamma(X^*(T^*M) \otimes \Omega^{(m,1)}(\Sigma_g))\Big) \otimes \mathcal{O}(\mathcal{R})\,. \tag{3.26}$$

Theorem 2.1 states that the BRST operator has the rank one

$$\Omega = \int_{\Sigma_g}\langle\eta|F\rangle + \int_{\Sigma_g}\langle\lfloor\eta,\eta\rfloor|\mathcal{P}\rangle\,. \tag{3.27}$$

Remind that the classical observables on $\mathbf{R}^{red}$ are elements from $H^0(Q)$. The moduli space of deformations of complex structures is a part of $\mathcal{R}^{red}$.

Let us briefly repeat the steps that lead to the moduli space of deformations of complex structure on the disk Σ_g.

- We start with a Poisson manifold M and define the Lie algebroid $\mathcal{A}$ over M (Lemma 3.1).
- This algebroid has infinite-dimensional version $\mathcal{A}_{\mathbf{M}}$ if one consider the maps X from Σ_g to M.

- We define the set of fields (X, ξ), where $\xi \in X^*(T^*M)$. It is the affine space $\mathbf{R}/T^*\mathbf{M}$.
- The anchor action and the representation of $\mathcal{A}$ in $\mathbf{R}$ is generated by the first class constraints (3.17). They are the compatibility conditions for the linear system (3.23), (3.24) and (3.25). Two last equations define the generalized deformation of complex structures on in the space of sections V_m, V'_m.
- The reduced phase space $\mathbf{R}^{red}$ can be described in terms of the BRST complex (3.26) with Ω (3.27).

We repeat these steps in concrete cases considered in the rest part of paper.

4. Two examples of Hamiltonian algebroids with Lie algebra symmetries

In this section we consider two examples, where the spaces of sections of the gauge algebras are replaced by sections of Lie algebroids, though the results can be obtained within standards Lie algebras symmetries implying a trivial anchor action. We construct Lie algebroids to illustrate our approach.

Let $\Sigma_{g,n}$ be a complex curve of genus g with n marked points. The first example is the moduli space of flat bundles over $\Sigma_{g,n}$. It will become clear later, that it is an universal system containing hidden algebroid symmetries. The second example is the moduli space of the projective structures ($\mathcal{W}_2$-structures) on $\Sigma_{g,n}$.

The generalization of the latter example is the $\mathcal{W}_N$-structures, where the symmetries are defined by a nontrivial Lie algebroid, will be considered in last Sections.

4.1. Flat bundles with regular singularities. We consider a rank N trivial vector bundle E over $\Sigma_{g,n}$. Define the derivatives $d' : E \to E \otimes \Omega^{(1,0)}(\Sigma_{g,n})$, $d'' : E \to E \otimes \Omega^{(0,1)}(\Sigma_{g,n})$.

4.1.1. *Local Lie algebroid.* On a disk $D \subset \Sigma_{g,n}$ one can choose the derivatives in the form

$$d' = (\kappa\partial + A) \otimes dz\,,\ d'' = \bar{\partial} + \bar{A}\,,\ \kappa \in \mathbb{C} \tag{4.1}$$

where $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$, z is a local coordinate and $A(z,\bar{z})$, $\bar{A}(z,\bar{z})$ are $\mathrm{sl}(N,\mathbb{C})$ valued $C^\infty(D)$ functions. Let $M_{SL_N}(S^1)$ be the set $\{d'\}$ restricted on the boundary S^1 of D. It has a structure of the affine Lie coalgebra $\hat{L}^*(\mathrm{sl}(N,\mathbb{C}))$ with the Lie-Poisson brackets on the space of functionals $\mathcal{O}(M_{SL_N}(S^1))$

$$\{f(A), g(A)\} = \oint_{S^1} \mathrm{tr}\left([df(A), dg(A)]A) + df(A)\partial(dg(A))\right)\,,\ \{f(A), \kappa\} = 0\,,$$

where $df(A)$ is a variation of $f(A)$. Thereby, $M_{SL_N}(S^1)$ can be considered as the base of the Lie algebroid $\mathcal{A}_{SL_N}(S^1)$ (see Remark 3.1). It corresponds to $m = 1$ in (3.14).

Following (3.5) we define the space of sections of the algebroid. It is the Lie algebra of smooth functionals $\mathcal{G}_{SL_N}(S^1) = \hat{L}(\mathrm{sl}(N,\mathbb{C}))$ with coefficients from $\mathcal{O}(M_{SL_N}(S^1))$. The variable dual to κ corresponds to the central charge of $\hat{L}(\mathrm{sl}(N,\mathbb{C}))$. In this way we come to the Lie brackets

$$[(\varepsilon(t), 0), (\varepsilon'(t')), 0)]_{jk} = (\delta(t,t')\sum_{l=1}^{N}\varepsilon_{jl}(t)\varepsilon'_{lk}(t') - \varepsilon'_{jl}(t')\varepsilon_{lk}(t), \oint_{S^1}\mathrm{tr}(\varepsilon(t)\partial\varepsilon'(t)))\,, \tag{4.2}$$

where $\delta(t,t') = \sum_{i\in\mathbb{Z}} t^i (t')^{-i-1} dt$. The integral defining the central extension represents a nontrivial 2-cocycle (see (2.12)).

The anchor is defined by the gauge transformations

$$\delta_\varepsilon A = \kappa\partial\varepsilon + [A,\varepsilon]\,. \tag{4.3}$$

Note, that the multiplication of sections on functionals from $\mathcal{O}(M_{SL_N}(S^1))$ modifies the brackets (4.2) according with (2.2) Similarly, we have for the gauge transformations

$$\delta_{f(A)\varepsilon} A = f(A)(\kappa\partial\varepsilon + [A,\varepsilon])\,.$$

4.1.2. *Global Lie algebroid.* Though the Poisson structure is defined only on S^1, the gauge algebra (4.2) and the gauge transformations (4.3) are well defined on the curve. Thereby it is possible to define the global Lie algebroid. We specify the behavior of the fields in neighborhoods of the marked points. Assume that A has first order holomorphic poles at the marked points

$$A|_{z\to x_a} = \frac{A_a}{z-x_a}\,. \tag{4.4}$$

We denote by $\mathcal{G}_{SL_N} = \{\varepsilon\}$ the Lie algebra of the smooth gauge transformations on $\Sigma_{g,n}$. Assume that at the marked points the gauge transformations are nontrivial

$$\varepsilon|_{z\to x_a} = r_a + O(z-x_a)\,,\ \ r_a \in \mathrm{sl}(N,\mathbb{C}) \neq 0\,. \tag{4.5}$$

At the marked points we add a collection P of n elements from the Lie coalgebra $\mathrm{sl}^*(N,\mathbb{C})$

$$P = \{\mathbf{p} = (p_1,\ldots,p_n)\}\ ,$$

endowed with the Lie-Poisson structure $\{f(p_a), g(p_b)\} = \delta_{ab}\langle p_a|[df,dg]\rangle$.

The gauge algebra $\mathcal{G}_{SL_N}$ acts on P by the evaluation maps as

$$\delta_\varepsilon p_a = [p_a, r_a]\,,\ \ \varepsilon\in\mathcal{G}_{SL_N}\,. \tag{4.6}$$

Thereby, we have defined a Lie algebroid $\mathcal{A}_{SL_N} = \mathcal{G}_{SL_N}\times M_{SL_N}$ over $M_{SL_N} = \{d', P\}$ with the anchor map (4.3), (4.6).

The cohomology $H^i(\mathcal{A}_{SL_N}) = H^i(\mathcal{G}_{SL_N}, \mathcal{O}(M_{SL_N}))$ are the standard cohomology of the gauge algebra $\mathcal{G}_{SL_N}$ with the cochains taking values in holomorphic functionals on M_{SL_N}. There is a nontrivial one-cocycle

$$\begin{aligned} c(A,\mathbf{p};\varepsilon) &= \int_{\Sigma_{g,n}} \mathrm{tr}\left(\varepsilon(\bar\partial A - 2\pi i\sum_{a=1}^n \delta(x_a)p_a)\right) \\ &= \langle\varepsilon|\bar\partial A\rangle - 2\pi i\sum_{a=1}^n \mathrm{tr}(r_a\cdot p_a) \end{aligned} \tag{4.7}$$

representing an element of $H^1(\mathcal{A}_{SL_N})$. This cocycle provides a nontrivial extension of the anchor action (see (2.14))

$$\hat\delta_\varepsilon f(A,\mathbf{p}) = \langle\varepsilon|\bar\partial A - \partial(df(A)) + [df(A),A]\rangle - 2\pi i\sum_{a=1}^n \mathrm{tr}(r_a p_a)\,.$$

Next consider $2g$ contours γ_α, $(\alpha = 1,\ldots,2g)$ generating $\pi_1(\Sigma_g)$. The contours determine the 2-cocycles

$$c_\alpha(\varepsilon_1,\varepsilon_2) = \int_{\gamma_\alpha} \mathrm{tr}(\varepsilon_1\partial\varepsilon_2) \tag{4.8}$$

(see (4.2). The cocycles (4.8) lead to $2g$ central extensions $\hat{\mathcal{G}}_{SL_N}$ of $\mathcal{G}_{SL_N}$

$$\hat{\mathcal{G}}_{SL_N} = \mathcal{G}_{SL_N} \oplus_{\alpha=1}^{2g} \mathbf{C}\Lambda_\alpha \,,$$

$$[(\varepsilon_1,0),(\varepsilon_2,0)]_{c.e.} = \left([\varepsilon_1,\varepsilon_2], \sum_\alpha c_\alpha(\varepsilon_1,\varepsilon_2)\right) .$$

4.2. Global Hamiltonian algebroid. To define the corresponding Hamiltonian algebroid we consider the cotangent bundle $T^*M_{SL_N}(\Sigma_{g,n})$ with the sections $\bar{\Phi} \in \Omega^{(0,1)}(\Sigma_{g,n}, \mathrm{sl}(N,\mathbb{C}))$.

REMARK 4.1. *The form $\bar{\Phi}$ is dual to d'. Here and in what follows we do not consider the dual to κ variables.*

Define the affine space $\mathcal{R}^0_{SL_N}$ over $T^*M_{SL_N}(\Sigma_{g,n})$ as the space of sections $\{d'' = \bar\partial + \bar{A}\}$.

The symplectic form on $\mathcal{R}_{SL_N}$ is

$$\omega^0 = \int_{\Sigma_{g,n}} \mathrm{tr}(DA \wedge D\bar{A}) = \langle DA \wedge D\bar{A}\rangle \,. \tag{4.9}$$

Consider the contributions of the marked points in the symplectic structure. We define there the symplectic manifold

$$(T^*G_1 \times \ldots, \times T^*G_n) \,.$$

with the form

$$\sum_{a=1}^n \omega_a = \sum_{a=1}^n \langle (D(p_a g_a^{-1}) \wedge Dg_a\rangle \,. \tag{4.10}$$

Here ω_a is the canonical symplectic form on $T^*G_a \sim T^*\mathrm{SL}(N,\mathbb{C})$. We pass from T^*G_a to the coadjoint orbits

$$\mathcal{O}_a = \{p_a = g_a^{-1} p_a^{(0)} g_a \mid p_a^{(0)} = \mathrm{diag}(\lambda_{a,1},\ldots,\lambda_{a,N}) \,,\ g_a \in \mathrm{SL}(N,\mathbf{C})\} \,. \tag{4.11}$$

and assume that the orbits are generic $\lambda_{a_j} \neq \lambda_{a_k}$, for $j \neq k$. The orbits are the symplectic quotient $\mathcal{O}_a \sim \mathrm{SL}(N,\mathbb{C})\backslash\backslash T^*G_a$ with respect to the action $g_a \to f_a g_a$, $f_a \in \mathrm{SL}(N,\mathbb{C})$. The form ω_a coincides on $\mathcal{O}_a$ with the Kirillov-Kostant form $\omega_a = \langle D(g_a^{-1}p_a^{(0)}) \wedge Dg_a\rangle$. It was mentioned above (Example 1, in **2.4**) that the orbits $\mathcal{O}_a$ are the affine spaces $Aff(T^*Fl_a(N))$ over the cotangent bundles $T^*Fl_a(N)$ to the flag varieties $Fl_a(N)$.

Eventually we come to the symplectic manifold

$$\mathcal{R}_{SL_N} = (\mathcal{R}^0_{SL_N}; \mathcal{O}_1 \times \ldots \mathcal{O}_n) \sim (Aff(T^*E); Aff(T^*Fl_1) \times \ldots Aff(T^*Fl_n)) \,,$$

$$\omega = \omega^0 + \sum_{a=1}^n \omega_a = \int_{\Sigma_g} \mathrm{tr}(DA \wedge D\bar{A}) + \sum_{a=1}^n \mathrm{tr}(D(g_a^{-1}p_a^0) \wedge Dg_a) \,. \tag{4.12}$$

According with (3.11) the pass from $T^*M_{SL_N}(D^\times)$ to $\mathcal{R}_{SL_N}$ is provided by the cocycle (4.7)

Consider the Hamiltonian

$$h_\varepsilon = \int_{\Sigma_g} \mathrm{tr}\,\varepsilon \left(F(A,\bar{A}) - 2\pi i \sum_{a=1}^n \delta(x_a) p_a \right) ,\ F(A,\bar{A}) = \bar\partial A - \kappa\partial\bar{A} + [\bar{A}, A] \,.$$

The Hamiltonian generates the canonical vector fields (4.3) and

$$\rho_\varepsilon \bar{A} = \bar{\partial}\varepsilon + [\bar{A},\varepsilon]\,,\ \rho_\varepsilon g_a = g_a r_a\,. \tag{4.13}$$

(see (2.26)). The global version of this transformations is the gauge groupoid G_{SL_N} acting on $\mathcal{R}_{SL_N}$. The flatness condition

$$F(A,\bar{A}) - 2\pi i\sum_{a=1}^{n}\delta(x_a)p_a = 0 \tag{4.14}$$

is the moment constraint with respect to this action.

The flatness is the compatibility condition for the linear system

$$\begin{cases} (\kappa\partial + A)\psi = 0\,, \\ (\bar{\partial} + \bar{A})\psi = 0\,, \end{cases} \tag{4.15}$$

where $\psi \in \Omega^0(\Sigma_{g,n}, \mathrm{Aut}E)$ as in **Lemma 3.2**. We can consider the same system for $\psi \in \Omega^0(\Sigma_{g,n}, E)$ (see **Remark 3.1**).

The second equation describes the deformation of the holomorphic structure on E.

4.2.1. *The moduli space of flat bundles.* The moduli space $\mathcal{M}_N^{flat}$ of flat $\mathrm{SL}(N,\mathbb{C})$-bundles is the symplectic quotient $\mathcal{R}_{SL_N}//G_{SL_N}$. It has dimension

$$\dim \mathcal{M}_N^{flat} = 2(N^2-1)(g-1) + N(N-1)n\,, \tag{4.16}$$

where the last term is the contribution of the coadjoint orbits $\mathcal{O}_a$.

As in the general case $\mathcal{O}(\mathcal{M}_N^{flat})$ can be identified with cohomology group $H^0(Q)$ of the BRST operator Q which we are going to define. Let $\eta \in \Omega^{(0)}(\Sigma_{g,n}, \mathrm{End}E)$ be the ghost field and $\mathcal{P}$ is its momentum $\mathcal{P} \in \Omega^{(1,1)}(\Sigma_{g,n}, \mathrm{End}E)$. Consider the algebra

$$\mathcal{O}(\mathcal{R}_N) \otimes \wedge^\bullet \left(\mathcal{G}_{SL_N}) \oplus \mathcal{G}^*_{SL_N}\right)\,.$$

Then the BRST operator Q acts on functionals on this algebra as

$$Q\Psi(A,\bar{A},p_a,\eta,\mathcal{P}) = \{\Omega, \Psi(A,\bar{A},p_a\eta,\mathcal{P})\}\,,$$

where

$$\Omega = \int_{\Sigma_g} \mathrm{tr}\,\eta\left(F(A,\bar{A}) - 2\pi\imath\sum_{a=1}^{n} p_a\delta(x_a)\right)\rangle + \frac{1}{2}\int_{\Sigma_g}\mathrm{tr}([\eta,\eta']|\mathcal{P})\,.$$

4.3. Projective structures on $\Sigma_{g,n}$.

4.3.1. *Local Lie algebroid.* The set of projective connections $M_2(D)$ on a disk $D \subset \Sigma_{g,n}$ is represented by the second order differential operators $\kappa^2\partial^2 - T$, where $T = T(z,\bar{z}) \in C^\infty(D)$ $T(z,\bar{z}) \in \Omega^{(2,0)}(D)\,,\ \kappa \in \mathbb{C}$.

The set $M_2(S^1)$ ($S^1 \sim \partial D$) is a Poisson manifold with the brackets

$$\{T(t),T(s)\} = \left(-\frac{1}{2}\kappa^3\partial^3 + 2T\kappa\partial + \kappa\partial T\right)\delta(t,s)\,,\ \{T,\kappa\} = 0\,, \tag{4.17}$$

where $\partial = \partial_t$ and $\delta(t,s) = \sum_{k\in\mathbb{Z}} t^k s^{-k-1}dt$. This case corresponds to (3.14) $m = 2$

The dual space with respect to the pairing

$$\langle (T,\kappa)|(\varepsilon,c)\rangle = \oint_{S^1}\delta\varepsilon\cdot\delta T + \kappa c$$

is the central extended Lie algebra of vector fields $\mathcal{G}_2(S^1) = \{(\varepsilon, c)\}$ on S^1

$$(\varepsilon, c)\,,\ \varepsilon = \varepsilon(z)\frac{\partial}{\partial z} \in \Omega^{(-1,0)}(S^1)\,,\ \varepsilon(z) \in C^\infty(S^1)\,.$$

The commutation relations can be read off from the Poisson brackets (see (3.5))

$$[(\varepsilon_1, 0), (\varepsilon_2, 0)] = (\varepsilon_1\kappa\partial\varepsilon_2 - \varepsilon_2\kappa\partial\varepsilon_1, \oint_{S^1} \varepsilon_1(-\frac{3}{2}\kappa^2\partial^3 + 2T\partial + \partial T)\varepsilon_2)\,. \tag{4.18}$$

One can omit the last two terms under the integral since they form an exact cocycle (see (2.12) and we deal with the standard cocycle.

The coadjoint action of $\mathcal{G}_2(S^1)$ on $M_2(S^1)$

$$\delta_\varepsilon T(z, \bar{z}) = -\varepsilon\kappa\partial T - 2T\kappa\partial\varepsilon + \frac{1}{2}\kappa^3\partial^3\varepsilon \tag{4.19}$$

defines the anchor in the vector bundle $\mathcal{A}_2(S^1)$ over $M_2(S^1)$.

4.3.2. *Global Lie algebroid.* The algebroid (4.18), (4.19) can be defined globally over the space $M_2 = M_2(\Sigma_{g,n})$ of projective connections T on $\Sigma_{g,n}$. We assume that $T(z, \bar{z})$ is smooth on $\Sigma_{g,n}$ and has poles at the marked points $x_a, (a = 1, \dots, n)$ up to the second order:

$$T|_{z\to x_a} \sim \frac{T^a_{-2}}{(z - x_a)^2} + \frac{T^a_{-1}}{(z - x_a)} + \dots\,, \tag{4.20}$$

The section of the algebroid are smooth chiral vector fields

$$\mathcal{G}_2(\Sigma_{g,n}) = \Gamma(\Omega^{(-1,0)}(\Sigma_{g,n})) = \{\varepsilon(z, \bar{z})\frac{\partial}{\partial z}\}\,.$$

Assume that the vector fields have the first order holomorphic nulls at the marked points

$$\varepsilon|_{z\to x_a} = r_a(z - x_a) + o(z - x_a),\ r_a \neq 0\,. \tag{4.21}$$

We denote this global algebroid $\mathcal{A}_2 = \mathcal{G}_2(\Sigma_{g,n}) \oplus M_2(\Sigma_{g,n})$.

Consider the cohomology $H^\bullet(\mathcal{A}_2) \sim H^\bullet(\mathcal{G}_2, M_2)$. Due to (4.19) and (4.21) $\delta_\varepsilon T^a_{-2} = 0$ and thereby T^a_{-2} in (4.20) represents an element from $H^0(\mathcal{A}_2)$.

The anchor action (4.19) can be extended by the one-cocycle $c(T;\varepsilon)$ representing a nontrivial element of $H^1(\mathcal{A}_2)$

$$c(T;\varepsilon) = \int_{\Sigma_{g,n}} \varepsilon\bar{\partial}T = \langle\varepsilon|\bar{\partial}T\rangle\,, \tag{4.22}$$

$$\hat{\delta}_\varepsilon f(T) = \langle\delta_\varepsilon T|df(T)\rangle + c(T;\varepsilon)\,.$$

The contribution of the marked points in (4.22) is $2\pi i r_a T^a_{-2}$.

There exist $2g$ nontrivial two-cocycles defined by the integrals over non contractible contours γ_α:

$$c_\alpha(\varepsilon_1, \varepsilon_2) = \kappa^3 \oint_{\gamma_\alpha} \varepsilon_1\partial^3\varepsilon_2\,.$$

The cocycles give rise to the central extension $\hat{\mathcal{G}}_2$ of the Lie algebra of the vector fields on $\Sigma_{g,n}$ (see (4.18)).

4.3.3. *Global Hamiltonian algebroid.* The affine space $\mathcal{R}_2(\Sigma_{g,n})/T^*M_2(\Sigma_{g,n})$ has the Darboux coordinates T and μ, where $\mu \in \Omega^{(-1,1)}(\Sigma_{g,n})$ is the Beltrami differential. The anchor (4.19) is lifted to $\mathcal{R}_2(\Sigma_{g,n})$ as

$$\delta_\varepsilon \mu = -\varepsilon\kappa\partial\mu + \mu\kappa\partial\varepsilon + \bar{\partial}\varepsilon \,, \tag{4.23}$$

where the last term occurs due to the cocycle (4.22). The symplectic form on $\mathcal{R}_2(\Sigma_{g,n})$ is

$$\omega = \int_{\Sigma_{g,n}} DT \wedge D\mu \,. \tag{4.24}$$

Remark 4.2. *The space $\mathcal{R}_2$ is the classical phase space of the $2+1$-gravity on $\Sigma_{g,n} \times I$ [**6**]. The Beltrami differential μ is related to the conformal class of metrics on $\Sigma_{g,n}$ and plays the role of a coordinate, while T is a momentum. In our construction the role of μ and T is interchanged.*

We specify the dependence of μ on the positions of the marked points in the following way. Let $\mathcal{U}'_a$ be neighborhoods of the marked points x_a , $(a = 1,\dots,n)$ such that $\mathcal{U}'_a \cap \mathcal{U}'_b = \emptyset$ for $a \neq b$. Define a smooth function $\chi_a(z,\bar{z})$

$$\chi_a(z,\bar{z}) = \begin{cases} 1, & z \in \mathcal{U}_a \,,\ \mathcal{U}'_a \supset \mathcal{U}_a \\ 0, & z \in \Sigma_g \setminus \mathcal{U}'_a \,. \end{cases} \tag{4.25}$$

Due to (4.23) at the neighborhoods of the marked points μ is defined up to the term $\bar{\partial}(z-x_a)\chi(z,\bar{z})$. Then μ can be represented as

$$\mu = \sum_{a=1}^{n} [t_{0,a} + t_{1,a}(z-x_a) + \dots]\mu^0_a \,,\ \mu^0_a = \bar{\partial}\chi_a(z,\bar{z}) \,,\ (t_{0,a} = x_a - x^0_a) \,, \tag{4.26}$$

where only $t_{0,a}$ can not be removed by the gauge transformations (4.21), (4.23).

Contribution of the marked points to the symplectic form (4.24) takes the form

$$\sum_{a=1}^{n} DT^a_{-2} \wedge Dt_{1,a} + DT^a_{-1} \wedge Dt_{0,a} \,. \tag{4.27}$$

The canonical transformations are generated by the Hamiltonian

$$h_\varepsilon = \int_{\Sigma_{g,n}} \varepsilon F(T,\mu) = \int_{\Sigma_{g,n}} \mu\delta_\varepsilon T + c(T,\varepsilon) \,, \tag{4.28}$$

where

$$F(T,\mu) = (\bar{\partial} + \mu\kappa\partial + 2\kappa\partial\mu)T - \frac{1}{2}\kappa^3\partial^3\mu \,. \tag{4.29}$$

We put $F(T,\mu) = 0$

$$(\bar{\partial} + \mu\kappa\partial + 2\kappa\partial\mu)T - \frac{1}{2}\kappa^3\partial^3\mu = 0 \,. \tag{4.30}$$

Let ψ be a $(-\frac{1}{2},0)$ differential. Then (4.30) is the compatibility condition for the linear system

$$\begin{cases} (\kappa^2\partial^2 - T)\psi = 0 \,, \\ (\bar{\partial} + \mu\kappa\partial - \frac{1}{2}\kappa\partial\mu)\psi = 0 \,. \end{cases} \tag{4.31}$$

It is analog of the vector representation mentioned in **Remark 3.1**. The system (3.23) - (3.25) in this case has the form

$$\begin{cases} (-\frac{1}{2}\kappa^3\partial^3 + 2T\kappa\partial + \kappa\partial T))\phi = 0\,, \\ (\bar{\partial} + \mu\kappa\partial - \kappa\partial\mu)\phi = 0\,. \end{cases}$$

Here ϕ is a section of the adjoint representation $\Omega^{(-1,0)}(\Sigma_{g,n})$.

It follows from the second equations in both systems that the Beltrami differential μ provides the deformation of complex structure on $\Sigma_{g,n}$.

4.3.4. *The moduli space $\mathcal{W}_2$.* Let G_2 be the group corresponding to the Lie algebra $\mathcal{G}_2$.

Definition 4.1. *The moduli space $\mathcal{W}_2$ of W_2-gravity on $\Sigma_{g,n}$ is the symplectic quotient of $\mathcal{R}_2$ with respect to the action of G_2,*

$$\mathcal{W}_2 = \mathcal{R}_2//G_2 = \{F(T,\mu) = 0\}/G_2\,.$$

It has dimension $6(g-1)+2n$. The space of observables is isomorphic to the cohomology H^0 of the BRST complex. It is generated by the fields $T, \mu \in \mathcal{R}_2$, the ghosts fields $\eta \in \Omega^{(-1,0)}(\Sigma_{g,n})$ and the ghosts momenta $\mathcal{P} \in \Omega^{(2,1)}(\Sigma_{g,n})$. The BRST operator Q is defined by Ω

$$\Omega = \int_{\Sigma_{g,n}} \eta F(T,\mu) + \frac{1}{2}\int_{\Sigma_{g,n}} [\eta,\eta']\mathcal{P}.$$

The first term is just the Hamiltonian (4.28), where the vector fields are replaced by the ghosts.

5. Hamiltonian algebroid structure in $\mathcal{W}_3$-gravity

Now consider the concrete example of the general construction with a nontrivial algebroid structure. It is the W_N structures on $\Sigma_{g,n}$ [**25, 11, 15**]. They generalize the W_2 structure described in previous Section. In this Section we consider in details the W_3 case.

5.1. SL$(N,\mathbb{C})$**-opers.** Opers are G-bundles over complex curves with additional structures [**27, 4**]. Let E_N be a SL$(N,\mathbb{C})$-bundle over $\Sigma_{g,n}$. It is a SL$(N,\mathbb{C})$-*oper* if there exists a flag filtration $E_N \supset \ldots \supset E_1 \supset E_0 = 0$ and a covariant derivative, that acts as $\nabla :\ E_j \subset E_{j+1}\otimes\Omega^{(1,0)}(\Sigma_{g,n})$. Moreover, ∇ induces an isomorphism $E_j/E_{j-1} \to E_{j+1}/E_j \otimes \Omega^{(1,0)}(\Sigma_{g,n})$. It means that locally

$$\nabla = \kappa\partial - \begin{pmatrix} 0 & 1 & 0 & \ldots & & 0 \\ 0 & 0 & 1 & \ldots & & \\ & & & \cdot & & \\ 0 & 0 & & & 0 & 1 \\ W_N & W_{N-1} & \ldots & & W_2 & 0 \end{pmatrix}, \tag{5.1}$$

where the matrix elements $W_k = W_k(z,\bar{z})$ are smooth. In other words, we define the N-order differential operator on $\Sigma_{g,n}$

$$L_N = \kappa^N\partial^N - W_2\kappa^{N-2}\partial^{N-2}\ldots - W_N\ :\ \Omega^{(-\frac{N-1}{2},0)}(\Sigma_{g,n}) \to \Omega^{(\frac{N+1}{2},0)}(\Sigma_{g,n}) \tag{5.2}$$

with vanishing subprinciple symbol. The GL$(N,\mathbb{C})$-opers come from the GL$(N,\mathbb{C})$-bundles and have the additional term $-W_1\partial^{N-1}$ in (5.2).

In this section we consider $\mathrm{SL}(3,\mathbb{C})$-opers and postpone the general case to next Section. It is possible to choose $E_1=\Omega^{-1,0}(\Sigma_{g,n})$. For $N=3$ we have

$$\nabla=\kappa\partial-\left(\begin{array}{ccc}0&1&0\\0&0&1\\W&T&0\end{array}\right), \tag{5.3}$$

and the third order differential operator

$$L_3=\kappa^3\partial^3-T\kappa\partial-W:\ \Omega^{(-1,0)}(\Sigma_{g,n})\to\Omega^{(2,0)}(\Sigma_{g,n})\,. \tag{5.4}$$

5.2. Local Lie algebroid over SL$(3,\mathbb{C})$-opers. Consider the set $M_3(D)=\{L_3\}$ of $\mathrm{SL}(3,\mathbb{C})$-opers on a disk $D\subset\Sigma_{g,n}$. On $S^1=\partial D$ this set becomes a Poisson manifold with respect to the AGD brackets [**1, 13**]

$$\{T(t),T(t')\}=\bigl(-2\kappa^3\partial^3+2T(t)\kappa\partial+\kappa\partial T(t)\bigr)\,\delta(t-t')\,, \tag{5.5}$$

$$\{T(t),W(t')\}=\bigl(\kappa^4\partial^4-T(t)\kappa^2\partial^2+3W(t)\kappa\partial-\kappa\partial W(t)\bigr)\,\delta(t-t')\,, \tag{5.6}$$

$$\{W(t),W(t')\}= \tag{5.7}$$

$$+\left(\frac{2}{3}\kappa^5\partial^5-\frac{4}{3}T(t)\kappa^3\partial^3-2\partial T(t)\kappa^2\partial^2+\left(\frac{2}{3}T(t)^2-2\kappa^2\partial^2T(t)+2\kappa\partial W(t)\right)\kappa\partial\right.$$
$$\left.+\left(\kappa^2\partial^2W(t)-\frac{2}{3}\kappa^3\partial^3T(t)+\frac{2}{3}T(t)\kappa\partial T(t)\right)\right)\delta(t-t')\,.$$

Remark 5.1. *These brackets can be obtained in two ways. It was pointed in In Ref.* [**10**] *they derived via the Poisson reduction from the Lie-Poisson brackets on the Lie coalgebra of the Borel subalgebra $\hat{L}(sl(3,mC))$ by the with respect the action of the unipotent subgroup. Another scheme was proposed in* [**11**]*, where the original coalgebra is $\hat{L}^*(sl(3,mC))$ and the action is generated by a maximal parabolic subgroup.*

In this way $M_3(S^1)$ can be considered as a base of a Lie algebroid $\mathcal{A}_3(S^1)\sim T^*M_3(S^1)$. This situation corresponds to (3.15) with $M\sim\mathbb{C}^2$, $m_1=2$ and $m_2=3$. To define the space of its sections we consider the dual space $M_3^*(S^1)$ of second order differential operators on S^1 with a central extension

$$M_3^*(S^1)=\{(\varepsilon^{(1)}\frac{d}{dt}+\varepsilon^{(2)}\frac{d^2}{dt^2},c)\}\,. \tag{5.8}$$

This space is defined by the pairing (see (3.7))

$$\langle(T,W,\kappa)|(\varepsilon^{(1)},\varepsilon^{(2)},c)\rangle=\oint_{S^1}(\varepsilon^{(1)}T+\varepsilon^{(2)}W)+\kappa c\,. \tag{5.9}$$

Following (3.5) we define the Lie brackets on $T^*M_3(S^1)$ by means of the AGD Poisson structure

$$\lfloor(\varepsilon_1^{(1)}0),(\varepsilon_2^{(1)},0)\rfloor=(\kappa(\varepsilon_1^{(1)}\partial\varepsilon_2^{(1)}-\varepsilon_2^{(1)}\partial\varepsilon_1^{(1)})\frac{d}{dt}\,,\,c(\varepsilon_1^{(1)},\varepsilon_2^{(1)}))\,, \tag{5.10}$$

(5.11)
$$\lfloor(\varepsilon^{(1)},0),(\varepsilon^{(2),0})\rfloor=\left(-\varepsilon^{(2)}\kappa^2\partial^2\varepsilon^{(1)})\frac{d}{dt}+(-2\varepsilon^{(2)}\kappa\partial\varepsilon^{(1)}+\varepsilon^{(1)}\kappa\partial\varepsilon^{(2)})\frac{d^2}{dt^2}\,,\,c(\varepsilon_1^{(1)},\varepsilon_2^{(2)})\right)\,,$$

$$\lfloor\varepsilon_1^{(2)},\varepsilon_2^{(2)}\rfloor=\left((\frac{2}{3}[\kappa\partial(\kappa^2\partial^2-T)\varepsilon_1^{(2)}]\varepsilon_2^{(2)}-\frac{2}{3}[\kappa\partial(\kappa^2\partial^2-T)\varepsilon_2^{(2)}]\varepsilon_1^{(2)})\frac{d}{dt}+\right. \tag{5.12}$$

$$(\varepsilon_2^{(2)}\kappa^2\partial^2\varepsilon_1^{(2)} - \varepsilon_1^{(2)}\kappa^2\partial^2\varepsilon_2^{(2)})\frac{d^2}{dt^2}\,,\ c(\varepsilon_1^{(2)},\varepsilon_2^{(2)}))\Big)\,.$$

Here $c(\varepsilon_1^{(j)},\varepsilon_2^{(k)})$ are the cocycles

$$c(\varepsilon_1^{(j)},\varepsilon_2^{(k)}) = \oint_{S^1}\lambda(\varepsilon_1^{(j)},\varepsilon_2^{(k)}),\ (j,k=1,2)\,, \tag{5.13}$$

$$\lambda(\varepsilon_1^{(1)},\varepsilon_2^{(1)}) = -2\varepsilon_1^{(1)}\kappa^2\partial^3\varepsilon_2^{(1)} + \dots\,,\ \lambda(\varepsilon_1^{(1)},\varepsilon_2^{(2)}) = \varepsilon_1^{(1)}\kappa^3\partial^4\varepsilon_2^{(2)} + \dots\,,$$

$$\lambda(\varepsilon_1^{(2)},\varepsilon_2^{(2)}) = \frac{2}{3}\varepsilon_1^{(2)}\kappa^4\partial^5\varepsilon_2^{(2)} + \dots\,,$$

and ellipses means the terms depending on lesser degrees of κ. It can be proved that $sc = 0$ (2.13) and that c is not exact. Note, that the brackets (5.10) define the algebra of the vector fields and the commutation relations are their generalization to the second order differential operators.

According with (3.3) the anchor action in $\mathcal{A}_3(S^1)$ has the form

$$\delta_{\varepsilon^{(1)}}T = -2\kappa^3\partial^3\varepsilon^{(1)} + 2T\kappa\partial\varepsilon^{(1)} + \kappa\partial T\varepsilon^{(1)}\,, \tag{5.14}$$

$$\delta_{\varepsilon^{(1)}}W = -\kappa^4\partial^4\varepsilon^{(1)} + 3W\kappa\partial\varepsilon^{(1)} + \kappa\partial W\varepsilon^{(1)} + T\kappa^2\partial^2\varepsilon^{(1)}\,, \tag{5.15}$$

$$\delta_{\varepsilon^{(2)}}T = \kappa^4\partial^4\varepsilon^{(2)} - T\kappa^2\partial^2\varepsilon^{(2)} + (3W - 2\kappa\partial T)\kappa\partial\varepsilon^{(2)} + (2\kappa\partial W - \kappa^2\partial^2 T)\varepsilon^{(2)}\,, \tag{5.16}$$

$$\delta_{\varepsilon^{(2)}}W = \frac{2}{3}\kappa^5\partial^5\varepsilon^{(2)} - \frac{4}{3}T\kappa^3\partial^3\varepsilon^{(2)} - 2\kappa^3\partial T\partial^2\varepsilon^{(2)} + \tag{5.17}$$

$$\kappa(\frac{2}{3}T^2 - 2\kappa^2\partial^2 T + 2\kappa\partial W)\partial\varepsilon^{(2)} + (\kappa^2\partial^2 W - \frac{2}{3}\kappa^3\partial^3 T + \frac{2}{3}\kappa T\partial T)\varepsilon^{(2)}\,.$$

Thereby, we obtain the Lie algebroid $\mathcal{A}_3(S^1)$ over $M_3(S^1)$. Note that in $\mathcal{A}_3(S^1)$ in contrast with the previous cases we encounter with the structure functions - the r.h.s of (5.12) depends on the projective connection T.

The Jacobi identuty (2.4) in $\mathcal{A}_3(S^1)$ takes the form

$$\lfloor\lfloor\varepsilon_1^{(2)},\varepsilon_2^{(2)}\rfloor,\varepsilon_3^{(2)}\rfloor^{(1)} - (\varepsilon_1^{(2)}\kappa\partial\varepsilon_2^{(2)} - \varepsilon_2^{(2)}\kappa\partial\varepsilon_1^{(2)})\delta_{\varepsilon_3^{(2)}}T + \text{c.p.}(1,2,3) = 0, \tag{5.18}$$

$$\lfloor\lfloor\varepsilon_1^{(2)},\varepsilon_2^{(2)}\rfloor,\varepsilon_3^{(1)}\rfloor^{(1)} - (\varepsilon_1^{(2)}\kappa\partial\varepsilon_2^{(2)} - \varepsilon_2^{(2)}\kappa\partial\varepsilon_1^{(2)})\delta_{\varepsilon_3^{(1)}}T = 0. \tag{5.19}$$

The brackets here correspond to the product of structure functions in the left hand side of (2.4) and the superscript (1) corresponds to the first order differential operators. For the rest brackets the Jacobi identity has the standard form.

5.3. Global Lie algebroid over SL$(3,\mathbb{C})$-opers. The base $M_3(\Sigma_{g,n})$ of the global Lie algebroid $\mathcal{A}_3(\Sigma_{g,n})$ are SL$(3,\mathbb{C})$-opers. They are well defined globally on $\Sigma_{g,n}$. The space of its sections $\mathcal{G}(\mathcal{A}_3)$ are the second order differential operators without free terms. To define this space properly we use the formalism of Volterra operators on $\Sigma_{g,n}$ in Section 6. They are well defined on $\Sigma_{g,n}$. There is map from a quotient space of the Volterra operators to $\mathcal{G}(\mathcal{A}_3)$. It will be defined in subsection 6.2.

5.3.1. *Derivation of the brackets.* To derive the Lie brackets (5.10) - (5.12) and the anchor action (5.14) – (5.17) globally we use the matrix description of $\mathrm{SL}(3,\mathbb{C})$-opers (5.3).

Consider the set $G_3(\Sigma_{g,n})$ of automorphisms of the bundle E over $\Sigma_{g,n}$

$$A \to f^{-1}\kappa\partial f - f^{-1}Af\,, \tag{5.20}$$

that preserve the $\mathrm{SL}(3,\mathbb{C})$-oper structure

$$f^{-1}\kappa\partial f - f^{-1}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix} f = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W' & T' & 0 \end{pmatrix}. \tag{5.21}$$

It is clear that $G_3(\Sigma_{g,n})$ is the Lie groupoid over $M_3(\Sigma_{g,n}) = \{(W,T)\}$ with $l(f) = (W,T)$, $r(f) = (W',T')$, $f \to \langle\langle W,T|f|W',T'\rangle\rangle$. The left identity map is

$$P\exp(\int_{z_0}^{z} A(W,T)) \cdot C \cdot P\exp(\int_{z_0}^{z} A(W,T))\,,$$

where C is an arbitrary matrix from $\mathrm{SL}(3,\mathbb{C})$ and $A(W,T))$ has the oper structure (5.3). The right identity map has the same form with (W,T) replaced by (W',T').

The infinitesimal version of (5.21) takes the form

$$\kappa\partial X - \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix}, X\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta W & \delta T & 0 \end{pmatrix}. \tag{5.22}$$

It is a linear differential system for the matrix elements of the traceless matrix X. The matrix elements $x_{j,k} \in \Omega^{(j-k,0)}(\Sigma_{g,n})$ depend on two arbitrary fields $x_{23} = \varepsilon^{(1)}$, $x_{13} = \varepsilon^{(2)}$. The solution takes the form

$$X = \begin{pmatrix} x_{11} & x_{12} & \varepsilon^{(2)} \\ x_{21} & x_{22} & \varepsilon^{(1)} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \tag{5.23}$$

$$x_{11} = \frac{2}{3}(\kappa^2\partial^2 - T)\varepsilon^{(2)} - \kappa\partial\varepsilon^{(1)}\,,\; x_{12} = \varepsilon^{(1)} - \kappa\partial\varepsilon^{(2)}\,,$$

$$x_{21} = \frac{2}{3}\kappa\partial(\kappa^2\partial^2 - T)\varepsilon^{(2)} - \kappa^2\partial^2\varepsilon^{(1)} + W\varepsilon^{(2)}\,,\; x_{22} = -\frac{1}{3}(\kappa^2\partial^2 - T)\varepsilon^{(2)}\,,$$

$$x_{31} = \frac{2}{3}\kappa^2\partial^2(\partial^2 - T)\varepsilon^{(2)} - \kappa^3\partial^3\varepsilon^{(1)} + \kappa\partial(W\varepsilon^{(2)}) + W\varepsilon^{(1)}\,,$$

$$x_{32} = \frac{1}{3}\kappa\partial(\kappa^2\partial^2 - T)\varepsilon^{(2)} - \kappa^2\partial^2\varepsilon^{(1)} + W\varepsilon^{(2)} + T\varepsilon^{(1)}\,,$$

$$x_{33} = -\frac{1}{3}(\kappa^2\partial^2 - T)\varepsilon^{(2)} + \kappa\partial\varepsilon^{(1)}\,.$$

The matrix elements of the commutator $[X_1, X_2]_{13}$, $[X_1, X_2]_{23}$ give rise to the brackets (5.10), (5.11), (5.12). Simultaneously, from (5.22) one obtain the anchor action (5.14)–(5.17).

5.3.2. *Contribution of the marked points.* Assume that the coefficients of opers $M_3 = M_3(\Sigma_{g,n})$ have holomorphic poles at the marked points

$$T|_{z\to x_a} \sim \frac{T^a_{-2}}{(z-x_a)^2} + \frac{T^a_{-1}}{(z-x_a)} + \dots\,, \tag{5.24}$$

$$W|_{z\to x_a} \sim \frac{W^a_{-3}}{(z-x_a)^3} + \frac{W^a_{-2}}{(z-x_a)^2} + \frac{W^a_{-1}}{(z-x_a)} + \dots\,. \tag{5.25}$$

The space of sections $\mathcal{G}_3 \sim \Gamma(\mathcal{A}_3) = \{\varepsilon^{(1)},\ \varepsilon^{(2)}\}$ of the global algebroid $\mathcal{A}_3$ was defined above. In addition, assume the coefficients of the first and second order differential operators vanish holomorphically at the marked points

$$\varepsilon^{(1)} \sim r^{(1)}_a (z-x_a) + o(z-x_a)\,,\ \varepsilon^{(2)} \sim r^{(2)}_a (z-x_a)^2 + o(z-x_a)^2\,,\ r^{(j)} \neq 0\,. \tag{5.26}$$

Note that these asymptotics are consistent with the Lie brackets, the anchor action and with the asymptotics (5.24) and (5.25).

5.3.3. *Cohomology of the Lie algebroid.* It follows from (5.14) – (5.17), (5.24), (5.25), and (5.26) that $\delta_{\varepsilon_j} T^a_{-2} = 0$, $\delta_{\varepsilon_j} W^a_{-3} = 0$, $(j = 1, 2,\ a = 1, \dots n)$. Therefore,

$$T^a_{-2}\,,\ W^a_{-3} \in H^0(\mathcal{A}_3, M_3) \tag{5.27}$$

Define the cocycles

$$c^{(1)} = \int_{\Sigma_{g,n}} \varepsilon^{(1)} \bar\partial T\,,\ c^{(2)} = \int_{\Sigma_{g,n}} \varepsilon^{(2)} \bar\partial W \tag{5.28}$$

from $H^1(\mathcal{A}_3, M_3)$. The contribution of the marked points to the cocycles is equal to

$$c^{(1)} \to \sum_{a=1}^n r^{(1)}_a T^a_{-2}\,,\ c^{(2)} \to \sum_{a=1}^n r^{(2)}_a W^a_{-3}\,.$$

The cocycles lead to the shift of the anchor action

$$\hat\delta_{\varepsilon^{(j)}} f(W,T) = \langle \delta_{\varepsilon^{(j)}} W | \frac{\delta f}{\delta W}\rangle + \langle \delta_{\varepsilon^{(j)}} T | \frac{\delta f}{\delta T}\rangle + c^{(j)}\,.$$

There exists $2g$ central extensions c_α of $\mathcal{G}_3$, provided by the nontrivial cocycles from $H^2(\mathcal{A}_3, M_3)$. They are the contour integrals γ_α

$$c_\alpha(\varepsilon^{(j)}_1, \varepsilon^{(k)}_2) = \oint_{\gamma_\alpha} \lambda(\varepsilon^{(j)}_1, \varepsilon^{(k)}_2),\ (j,k = 1,2)\,, \tag{5.29}$$

where γ_α are the fundamental cycles of $\Sigma_{g,n}$ and $\lambda(\varepsilon^{(j)}_1, \varepsilon^{(k)}_2)$ are defined by (5.13). These cocycles allow us to construct the extended brackets:

$$\lfloor (\varepsilon^{(j)}_1, 0), (\varepsilon^{(m)}_2, 0) \rfloor_{c.e.} = (\lfloor \varepsilon^{(j)}_1, \varepsilon^{(m)}_2 \rfloor, \sum_\alpha c_\alpha(\varepsilon^{(j)}_1, \varepsilon^{(m)}_2))\,,\ (j,m = 1,2)\,.$$

5.4. Global Hamiltonian algebroid. The affine space $\mathcal{R}_3$ over T^*M_3 is the classical phase space for the W_3-gravity on $\Sigma_g \times \mathbb{R}$ [**25, 11, 15**]. Its sections are the Beltrami differentials $\mu \in \Omega^{(-1,1)}(\Sigma_{g,n})$ and the differentials $\rho \in \Omega^{(-2,1)}(\Sigma_{g,n})$. They are smooth and near the marked points behave as (4.26) and

$$\rho|_{z\to x_a} \sim (t^{(2)}_{a,0} + t^{(2)}_{a,1}(z - x^0_a))\bar\partial \chi_a(z,\bar z)\,. \tag{5.30}$$

According with the general theory the anchor (5.14)–(5.17) can be lifted from M_3 to $\mathcal{R}_3$. This lift is nontrivial owing to the cocycle (5.28). It follows from (2.26) that the anchor action on μ and ρ takes the form

$$\delta_{\varepsilon^{(1)}}\mu = -\bar{\partial}\varepsilon^{(1)} - \mu\kappa\partial\varepsilon^{(1)} + \kappa\partial\mu\varepsilon^{(1)} - \rho\kappa^2\partial^2\varepsilon^{(1)}\,, \tag{5.31}$$

$$\delta_{\varepsilon^{(1)}}\rho = -2\rho\kappa\partial\varepsilon^{(1)} + \kappa\partial\rho\varepsilon^{(1)}\,, \tag{5.32}$$

$$\delta_{\varepsilon^{(2)}}\mu = \kappa^2\partial^2\mu\varepsilon^{(2)} - \frac{2}{3}\left[(\kappa\partial(\kappa^2\partial^2 - T)\rho)\varepsilon^{(2)} - (\kappa\partial(\kappa^2\partial^2 - T)\varepsilon^{(2)})\rho\right]\,, \tag{5.33}$$

$$\delta_{\varepsilon^{(2)}}\rho = -\bar{\partial}\varepsilon^{(2)} + (\rho\kappa^2\partial^2\varepsilon^{(2)} - \kappa^2\partial^2\rho\varepsilon^{(2)}) + 2\kappa\partial\mu\varepsilon^{(2)} - \mu\kappa\partial\varepsilon^{(2)}\,. \tag{5.34}$$

The transformations (5.14) – (5.17) and (5.31) – (5.34) are canonical with respect to the symplectic form

$$\omega = \int_{\Sigma_{g,n}} DT \wedge D\mu + DW \wedge D\rho\,.$$

They are defined by the Hamiltonians

$$h^{(1)} = \int_{\Sigma_{g,n}} (\mu\delta_{\varepsilon^{(1)}}T + \rho\delta_{\varepsilon^{(1)}}W) + c^{(1)}\,, \tag{5.35}$$

$$h^{(2)} = \int_{\Sigma_{g,n}} (\mu\delta_{\varepsilon^{(2)}}T + \rho\delta_{\varepsilon^{(2)}}W) + c^{(2)}\,. \tag{5.36}$$

After the integration by parts they take the form

$$h^{(1)} = \int_{\Sigma_{g,n}} \varepsilon^{(1)}F^{(1)}\,,\; h^{(2)} = \int_{\Sigma_{g,n}} \varepsilon^{(2)}F^{(2)}\,,$$

where $F^{(1)} \in \Omega^{(2,1)}(\Sigma_{g,n}),\; F^2 \in \Omega^{(3,1)}(\Sigma_{g,n})$

$$\begin{aligned} F^{(1)} = &-\bar{\partial}T - \kappa^4\partial^4\rho + T\kappa^2\partial^2\rho - \kappa(3W - 2\kappa\partial T)\partial\rho - \\ &-(2\kappa\partial W - \kappa^2\partial^2 T)\rho + 2\kappa^3\partial^3\mu - 2\kappa\partial T\mu - \kappa\partial T\mu\,, \end{aligned} \tag{5.37}$$

(5.38)

$$\begin{aligned} F^{(2)} = &-\bar{\partial}W - \frac{2}{5}\kappa^5\partial^5\rho + \frac{4}{3}T\kappa^3\partial^3\rho + 2\kappa\partial T\kappa^2\partial^2\rho + \kappa(-\frac{2}{3}T^2 + 2\kappa^2\partial^2 T - 2\partial W)\partial\rho \\ &-(\kappa^2\partial^2 W - \frac{2}{3}\kappa^3\partial^3 T + \frac{2}{3}\kappa T\partial T)\rho + \kappa^4\partial^4\mu - 3W\kappa\partial\mu - \kappa\partial W\mu - \kappa^2 T\partial^2\mu\,. \end{aligned}$$

5.5. The moduli space $\mathcal{W}_3$ of the W_3 gravity. Let G_3 be the groupoid corresponding to the algebroid $\mathcal{G}_3$.

DEFINITION 5.1. *The moduli space $\mathcal{W}_3$ of the W_3-gravity is the symplectic quotient*

$$\mathcal{W}_3 = \mathcal{R}_3//G_3 = \{F^1 = 0, F^2 = 0\}/G_3.$$

It has dimension $\dim\mathcal{W}_3 = 16(g-1) + 6n$. The term $6n$ comes from the coefficients $T^a_{-1}, W^a_{-1}, W^a_{-2}$, and the dual to them $t^{(1)}_{a,0}, t^{(2)}_{a,0}, t^{(2)}_{a,1}$, $(a = 1, \ldots, n)$ in (4.26) and (5.30).

The moment equations $F^{(1)} = 0,\; F^{(2)} = 0$ are the consistency conditions for the linear system

$$\begin{cases} (\kappa^3\partial^3 - T\kappa\partial - W)\psi(z,\bar{z}) = 0\,, \\ \left(\bar{\partial} + (\mu - \kappa\partial\rho)\partial + \kappa^2\rho\partial^2 + \frac{2}{3}(\kappa^2\partial^2 - T)\rho - \kappa\partial\mu\right)\psi(z,\bar{z}) = 0\,, \end{cases} \tag{5.39}$$

where $\psi(z,\bar{z}) \in \Omega^{(-1,0)}(\Sigma_{g,n})$. It is an analog of the vector representation. We will prove this statement in next subsection. This system defines W_3-*projective structure* on $\Sigma_{g,n}$. In the first equation the Schrodinger operator is replaced by the third order differential operator depending on two fields T and W. The second equation represents the deformation of the operator $\bar{\partial}$ (or more general $\bar{\partial}+\mu\partial$ as in (4.31)) by the second order differential operator. The left hand side is the explicit form of the deformed operator when it acts on the space $\Omega^{-1,0}(\Sigma_{g,n})$. This deformation cannot be supported by the structure of a Lie algebra and one leaves with the algebroid symmetries.

Now we construct the BRST complex. We introduce the ghosts fields $\eta^{(1)},\eta^{(2)}$ and their momenta $\mathcal{P}^{(1)},\mathcal{P}^{(2)}$. It follows from Theorem 2.1 that for

$$\Omega=\sum_{j=1,2}h^{(j)}(\eta^{(j)})+\frac{1}{2}\sum_{j,k,l=1,2}\int_{\Sigma_{g,n}}(\lfloor\eta^{(j)},\eta^{(k)}\rfloor\mathcal{P}^{(l)})$$

the operator $QF=\{F,\Omega\}$ is nilpotent and define the BRST cohomology in the complex

$$\bigwedge{}^{\bullet}(\mathcal{G}_3\oplus\mathcal{G}_3^*)\otimes C^\infty(\mathcal{R}_3)\,.$$

5.6. Chern-Simons derivation. We follow here the derivation of W-gravity proposed in Ref. [**11**]. We add to this construction a contribution of the Wilson lines due to the presence of the marked points on $\Sigma_{g,n}$.

Consider the Chern-Simons functional on $\Sigma_{g,n}\oplus\mathbb{R}^+$

$$S=\int_{\Sigma_{g,n}\oplus\mathbb{R}^+}\mathrm{tr}(\mathbf{A}d\mathbf{A}+\frac{2}{3}\mathbf{A}^3)+\sum_{a=1}^{n}\int_{\mathbb{R}^+}\mathrm{tr}(p_a^0\partial_t g_a g_a^{-1})\,,\ (\mathbf{A}=(A,\bar{A},A_t))\,,$$

where the last sum is the geometric action coming from the Kirillov-Kostant forms on the coadjoint orbits $\mathcal{O}_a$ (4.11). Introduce n Wilson lines $W_a(A_t)$ along the time directions and located at the marked points

$$W_a(A_t)=P\exp\mathrm{tr}(p_a\int A_t),\ a=1,\dots,n\,.$$

In the hamiltonian picture the phase space, corresponding to the Chern-Simons functional is

$$\mathcal{R}_{SL_3}=\{A\,,\bar{A}\,,\mathcal{O}_1,\dots,\mathcal{O}_n\}\,, \tag{5.40}$$

endowed with the symplectic form (4.12). The field A_t is the Lagrange multiplier for the first class constraints (4.14).

The phase space of W_3-gravity $\mathcal{R}_3$ can be derived from $\mathcal{R}_{SL_3}$. The flatness condition (4.14) generates the gauge transformations

$$A\to f^{-1}\kappa\partial f-f^{-1}Af\,,\ \bar{A}\to f^{-1}\bar{\partial}f-f^{-1}\bar{A}f\,,\ g_a\to g_af_a\,. \tag{5.41}$$

The symplectic quotient with respect to the gauge group $G_{\mathrm{SL}(3,\mathbb{C})}$ is the moduli space $\mathcal{M}_3^{flat}$ of the flat $\mathrm{SL}(3,\mathbb{C})$ bundles over $\Sigma_{g,n}$.

Let P be the maximal parabolic subgroup of $\mathrm{SL}(3,\mathbb{C})$ of the form

$$P=\begin{pmatrix}*&*&0\\ *&*&0\\ *&*&*\end{pmatrix},$$

and G_P be the corresponding gauge group. We partly fix first the gauge with respect to G_P. A generic connection ∇ can be transformed by $f\in G_P$ to the form

(5.3). Taking into account (4.14) we assume that A has simple poles at the marked points. To come to M_3 one should respect the behavior of the matrix elements at the marked points (5.24), (5.25). For this purpose we use an additional singular gauge transform by the diagonal matrix

$$h = \prod_{a=1}^{n} \chi_a(z,\bar{z}) \mathrm{diag}((z-x_a)^{-1}, 1, (z-x_a))\,.$$

The resulting gauge group we denote $G_{(P\cdot h)}$, where χ_a is defined by (4.25).

The form of $\bar{A}$ can be read off from (4.14)

$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & -\rho \\ a_{21} & a_{22} & -\mu \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{5.42}$$

$$a_{11} = -\frac{2}{3}(\kappa^2\partial^2 - T)\rho + \kappa\partial\mu\,,\ a_{12} = -\mu + \kappa\partial\rho\,,$$

$$a_{21} = -\frac{2}{3}\kappa\partial(\kappa^2\partial^2 - T)\rho + \kappa^2\partial^2\mu - W\rho\,,\ a_{22} = \frac{1}{3}(\kappa^2\partial^2 - T)\rho\,,$$

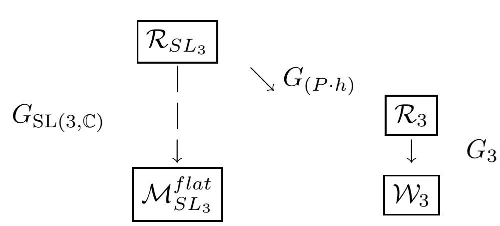
$$a_{31} = -\frac{2}{3}\kappa^2\partial^2(\kappa^2\partial^2 - T)\rho + \kappa^3\partial^3\mu - \kappa\partial(W\rho) - W\mu\,,$$

$$a_{32} = -\frac{1}{3}\kappa\partial(\kappa^2\partial^2 - T)\rho + \kappa^2\partial^2\mu - W\rho - T\mu\,,\ a_{33} = \frac{1}{3}(\kappa^2\partial^2 - T)\rho - \kappa\partial\mu\,.$$

The condition (4.14) for the special choice A (5.3) and $\bar{A}$ (5.42) gives rise to the relations $F(A,\bar{A})|_{(3,1)} = F^{(2)}$ (5.37), $F(A,\bar{A})|_{(2,1)} = F^{(1)}$ (5.38), while the other matrix elements of $F(A,\bar{A})$ vanish identically. At the same time, the matrix linear system (4.15) coincides with (5.39). In this way, we come to the matrix description of the moduli space $\mathcal{W}_3$.

The cocycles $c_\alpha(\varepsilon_1^{(j)}, \varepsilon_2^{(k)})$ (5.29) can be derived from the two-cocycle (4.8) of $\mathcal{A}_{SL_3}$. Substituting in (4.8) the matrix realization of $\Gamma(\mathcal{A}_3)$ (5.23), one comes to (5.29).

The action of groupoid G_3 on $A, \bar{A}$ plays the role of the rest gauge transformations that complete the $G_{(P\cdot h)}$ action to the G_{SL_3} action. The algebroid symmetry with non-trivial structure functions arises in this theory as a result of the partial gauge fixing by $G_{(P\cdot h)}$. Thus we come to the following diagram



The tangent space to $\mathcal{M}^{flat}_{SL_3}$ at the point $A = 0$, $\bar{A} = 0$, $p_a = 0$, $g_a = id$ coincides with the tangent space to $\mathcal{W}_3$ at the point $W = 0$, $T = 0$, $\mu = 0$, $\rho = 0$. Their dimension is $16(g-1)+6n$. But their global structure is different and the diagram cannot be closed by the horizontal isomorphisms. The interrelations between $\mathcal{M}^{flat}_{SL_N}$ and $\mathcal{W}_N$ were analyzed in [**20, 16**].

6. AGD algebroids and generalized projective structures

In this section we define generalized projective structures on $\Sigma_{g,n}$ related the $\mathrm{GL}(N,\mathbb{C})$ and $\mathrm{SL}(N,\mathbb{C})$-opers. It is a phase space of W_N-gravity. In particular, we define deformations of complex structures (the W_N-deformations) by the Volterra operators and by the opers. To construct a Lie algebroid over the space of $\mathrm{GL}(N,\mathbb{C})$-opers we use the pairing (6.4) corresponding to the case (3.16) with $m = N$. As a result the space of sections of the Lie algebroid is the space the Volterra operators instead of the space the differential operators, considered in previous Section. We start with the description of the local AGD algebroid following Ref. [**13**] and then give its global version. The passage from the Lie algebroid to the Hamiltonian algebroid allows us to describe the generalized projective structures and their moduli.

6.1. Local AGD algebroid. Consider a set $B = \Psi DO(D)$ of pseudo-differential operators on a disk $D \subset \Sigma_{g,n}$ and their restriction on the boundary $S^1 \sim \partial D$. It is a ring of formal Laurent series

$$B = B((\partial^{-1})) = \{B_{r,N}(S^1)\,,\ r,N \in \mathbb{Z}\} = \{X(t,\partial)\}\,,\ t \in S^1\,,\ \partial = \partial_t$$

$$X(t,\partial) = \sum_{k=-\infty}^{r-1} a_k(t)\partial^k,\ \ (a_k(z) \in \Omega^{-N-k+1}(S^1)\,. \tag{6.1}$$

The multiplication on B is defined as the non-commutative multiplication of their symbols

$$X(t,\lambda)\circ Y(t,\lambda) = \sum_{k\geq 0}\frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}X(t,\lambda)\frac{\partial^k}{\partial t^k}Y(t,\lambda)\,. \tag{6.2}$$

In what follows we omit the multiplication symbol $\circ$.

Note that $B_{r,N}(S^1) \in \Psi DO(S^1)$ can be considered as the formal map of the sheaves

$$B_{r,N}(S^1)\ :\ \Omega^{\frac{N-1}{2}}(S^1) \to \Omega^{-\frac{N-1}{2}}(S^1)\,. \tag{6.3}$$

Let $M_N^G(S^1)$ be a space of differential operators $L_N = \kappa^N\partial^N + W_1\kappa^{N-1}\partial^{N-1} + \ldots + W_N$ on S^1 $(\partial = \partial_t)$ with smooth coefficients, corresponding to the $\mathrm{GL}(N,\mathbb{C})$-oper on D. For brevity we call them the $\mathrm{GL}(N,\mathbb{C})$-opers. Then we have

$$XL_N\ :\ \Omega^{-\frac{N-1}{2}}(S^1) \to \Omega^{-\frac{N-1}{2}}(S^1)\,,\ L_NX\ :\ \Omega^{\frac{N+1}{2}}(S^1) \to \Omega^{\frac{N+1}{2}}(S^1)\,,$$

where the product is defined by (6.2).

Define a pairing between $M_N^G(S^1)$ and $B_{r,N}(S^1)$. For $L_NX = \sum_k c_k\partial^k$ let $Res\, L_NX = c_{-1}$. It is a one-form on S^1 and one can define a pairing

$$\langle L_NX\rangle = \frac{1}{2\pi}\oint_{S^1} Res(L_NX)dt\,. \tag{6.4}$$

Because $Res\,[L_N,X]$ is a derivative, $\langle L_NX\rangle = \langle XL_N\rangle$. The pairing corresponds to the case (3.16).

6.1.1. *Local AGD algebroid over* $\mathrm{GL}(N,\mathbb{C})$*-opers.* The local AGD algebroid over $\mathrm{GL}(N,\mathbb{C})$-opers was constructed implicitly in [**13**]. The AGD brackets on the space $M_N^G(S^1)$ are defined as follows. The space of sections of the cotangent bundle $T^*M_N^G(S^1)$ can be identified with the quotient space of the Volterra operators

$$\Gamma(T^*M_N^G(S^1)) = B_{0,N}(S^1)/B_{-N-2,N}(S^1)\,. \tag{6.5}$$

As before consider κ as an independent variable of an oper L_N and extend the pairing (6.4) by the additional term $\kappa\cdot c$, where c corresponds to the central extension. In this case instead of (ref9.1a) we come to the extended space of sections

$$\hat\Gamma(T^*M_N^G(S^1)) = B_{0,N}(S^1)/B_{-N-2,N}(S^1)\oplus\mathbb{C}\,.$$

For L_N and $X\in\Gamma(T^*M_N^G(S^1))$ define the functional $l_X=\langle L_NX\rangle$. In particular, for $l_X=W_1(z)\,,\ X=\delta(z/t)\partial_t^{-N}$. The AGD brackets have the form

$$\{l_X,l_Y\} = \langle L_NX(L_NY)_+\rangle - \langle XL_N(YL_N)_+\rangle\,, \tag{6.6}$$

where $X_+=\sum_{k=0}^N a_k\partial^k$ is the differential part of X. Equivalently, (6.6) can be rewritten as

$$\{l_X,l_Y\} = \langle XL_N(YL_N)_-\rangle - \langle L_NX(L_NY)_-\rangle\,, \tag{6.7}$$

where A_- is the integral part of $A\in\Psi DO\,,\ (A_+=A-A_-)$.

Using the general prescription (3.5) we find from (6.7) the Lie brackets in the space of sections $\hat\Gamma(T^*M_N^G(S^1))$

$$\lfloor(X,0),(Y,0)\rfloor = (((YL_N)_-X - X(L_NY)_- + X(L_NY)_+ - (Y(L_N)_+X),c)\,, \tag{6.8}$$

where

$$c=\frac{\partial}{\partial\kappa}\{l_X,l_Y\}\,. \tag{6.9}$$

Due to the Jacobi identity for the brackets (6.6) this term leads to the central extension of the Lie algebra $\Gamma(T^*M_N^G(S^1))$.

The anchor map assumes the form (see (3.12))

$$\delta_YL_N = (L_NY)_+L_N - L_N(YL_N)_+\,. \tag{6.10}$$

Definition 6.1. *The AGD algebroid* $\mathcal{A}_N^G(S^1)$ *over* $M_N^G(S^1)$ *is a bundle* $T^*M_N^G(S^1)$ *with the brackets (6.8) and the anchor (6.10).*

It follows from (6.6) and (6.10) that the Poisson brackets can be rewritten as

$$\{l_X,l_Y\} = \langle X\delta_YL_N\rangle\,, \tag{6.11}$$

or in the form of the "Poisson-Lie brackets"

$$\{l_X,l_Y\} = \frac{1}{2}\langle\lfloor X,Y\rfloor L_N\rangle = l_{\frac{1}{2}\lfloor X,Y\rfloor}\,. \tag{6.12}$$

The coefficient 1/2 arises from the quadratic form of the Poisson bivector. These two representations implies that the anchor plays the role of the coadjoint action.

Remark 6.1. *It is assumed in (6.8) and (6.10) that the sections* X *and* Y *are independent on a point in the base* $M_N^G(S^1)$*. To pass to a generic section one should use the defining properties of Lie algebroids. In fact, the Lie brackets (6.8) and the anchor (6.10) were appeared already in* [**13**].

6.1.2. *Local AGD Lie algebroid over* SL$(N,\mathbb{C})$*-opers.* Now we consider the space $M_N(S^1)$ of SL$(N,\mathbb{C})$-opers and construct the corresponding $\mathcal{A}_N(S^1)$ algebroid. Remind that an SL$(N,\mathbb{C})$-oper L_N^S on D is defined by the condition $W_1=0$, $(L_N^S=\partial^N+W_2\partial^{N-2}+\ldots)$.

The description of the AGD SL$(N,\mathbb{C})$-algebroids is based on the following statement

PROPOSITION 6.1.
- *The anchor action (6.10) preserves the coefficient W_1 of a* GL$(N,\mathbb{C})$*-oper iff*

$$Res\,[Y,L_N]=0\,. \tag{6.13}$$

  In particular it preserves an SL$(N,\mathbb{C})$*-oper L_N^S.*
- *Sections satisfying (6.13) generate a Lie algebra $\mathcal{G}$.*

This statement leads to the following definition.

DEFINITION 6.2. *The AGD Lie algebroid $\mathcal{A}_N(S^1)$ over the* SL$(N,\mathbb{C})$*-opers on S^1 is defined by the brackets (6.8) and the anchor map (6.10) with condition (6.13).*

Proof of Proposition. If $\delta_Y L_N^S$ does not change W_1 then $\langle a\partial_t^{-N}\delta_Y L_N\rangle=0$ for any continues functional a on the space $C^\infty(S^1)$. Due to (6.10) it implies

$$\langle a\partial_t^{-N}(L_NY)_+L_N-a\partial_t^{-N}L_N(YL_N)_+\rangle=0\,.$$

We rewrite the l.h.s. as

$$\langle a\partial_t^{-N}L_N(YL_N^S)_- - a\partial_t^{-N}(L_NY)_-L_N\rangle$$

$$=\langle(a\partial_t^{-N}L_N)_+YL_N-(L_Na\partial_t^{-N})_+L_NY=\langle a[Y,L_N]\rangle\,.$$

Vanishing of this expression for any a is equivalent to (6.13).

Let δ_X and δ_Y preserve the structure of the SL$(N,\mathbb{C})$-oper. Since $[\delta_X,\delta_Y]=\delta_{\lfloor X,Y\rfloor}$ the sections, satisfying (6.13) generate a Lie algebra $\mathcal{G}$. Moreover, SL$(N,\mathbb{C})$-opers generate a Poisson subalgebra. In fact, we have from (6.11) and (6.12)

$$\delta_{\lfloor X,Y\rfloor}W_1=\{l_{\lfloor X,Y\rfloor},W_1\}=2\{\{l_X,l_Y\},W_1\}=0\,.$$

□

Now prove that $\mathcal{G}$ is isomorphic to $\hat{\Gamma}(T^*M_N(S^1))\subset\hat{\Gamma}(T^*M_N^G(S^1))$. Consider the cotangent bundle $T^*M_N(S^1)$ to the space of SL$(N,\mathbb{C})$-opers $M_N(S^1)$. The space of its sections is quotient space of $T^*M_N(S^1)=T^*M_N^G(S^1)/\{a(t)\partial_t^{-N}\}$. It is possible to choose a section X of the SL$(N,\mathbb{C})$ Lie algebroid $\mathcal{A}_N(S^1)$ such that $X\in\Gamma(T^*M_N(S^1)$.

LEMMA 6.1. *For any $X\in\Gamma(T^*M_N^G(S^1))$ one can find $a(t)\partial_t^{-N}$ such that $Y=X+a(t)\partial_t^{-N}$ obeys (6.13).*

Proof. It easy to find that for SL$(N,\mathbb{C})$-oper L_N^S $Res\,[a\partial_t^{-N},L_N^S]=-N\partial_ta$. For any $X\in\Gamma(T^*M_N^G(S^1)$ $Res\,[X,L_N^S]=\partial_tF(t)$. Then $Res\,[(X+a\partial_t^{-N},L_N^S]=\partial_t(F(t)-Na(t))$. Choosing $a(t)=\frac{1}{N}F(t)$ we obtain a section of the SL$(N,\mathbb{C})$ algebroid. □

6.2. Global AGD Lie algebroid. As it was mentioned above the opers are well defined globally on the curves. We assume that in neighborhoods of the marked points the coefficients W_j behave as

$$W_j|_{z\to x_a} \sim W^a_{-j}(j)(z-x_a)^{-j} + W^a_{-j}(j-1)(z-x_a)^{-j+1} + \dots \,. \tag{6.14}$$

The base of the global AGD $\mathrm{GL}(N,\mathbb{C})$ algebroid $\mathcal{A}^G_N(\Sigma_{g,n})$ $(\mathcal{A}_N(\Sigma_{g,n}))$ is the space $M^G_N(\Sigma_{g,n})$ of global $\mathrm{GL}(N,\mathbb{C})$-opers on $\Sigma_{g,n}$. Similarly, the base of the global AGD $\mathrm{SL}(N,\mathbb{C})$ algebroid $\mathcal{A}_N(\Sigma_{g,n})$ is the space $M_N(\Sigma_{g,n})$ of global $\mathrm{SL}(N,\mathbb{C})$-opers. They have the prescribed behavior near the marked points. The spaces of their sections $\mathcal{G}^G_N$ $(\mathcal{G}_N)$ are the quotient spaces of the Volterra operators

$$\mathcal{G}^G_N = \Gamma(\mathcal{A}^G_N(\Sigma_{g,n})) = B_{0,N}(\Sigma_{g,n})/B_{-N-2,N}(\Sigma_{g,n})\,,$$

where

$$B_{r,N}(S^1) \;:\; \Omega^{\frac{N+1}{2}}(\Sigma_{g,n}) \to \Omega^{-\frac{N-1}{2}}(\Sigma_{g,n})\,.$$

Near a marked point x_a with a local coordinate z a section $X \in \mathcal{G}_N$ has the expansion

$$X = \sum_{j=1}^{N+1} \epsilon^{(j)}\partial_z^{-j}\,, \tag{6.15}$$

where $\epsilon^{(j)} \sim r^{(j)}_a(z-x_a)^j + o(z-x_a)^j\,,\ r^{(j)}_a \neq 0\,.$

Similarly, the base of the global AGD $\mathrm{SL}(N,\mathbb{C})$ algebroid $\mathcal{A}_N(\Sigma_{g,n})$ is the space $M_N(\Sigma_{g,n})$ of global $\mathrm{SL}(N,\mathbb{C})$-opers. The space of sections $\mathcal{G}^G_N$ of $\mathcal{A}_N(\Sigma_{g,n})$ satisfy (6.13).

Consider in detail the case $N=3$. Locally the sections of $\mathcal{G}^G_3$ can be represented by the operators

$$X = \epsilon^{(1)}\partial_z^{-1} + \epsilon^{(2)}\partial_z^{-2} + \epsilon^{(3)}\partial_z^{-3} + \partial_z^{-4}\,.$$

The space of sections described in Sections 5.2 and 5.3 (5.8) are the second order differential operators $\{(\varepsilon^{(1)}\frac{d}{dz} + \varepsilon^{(2)}\frac{d^2}{dz^2})\}$. We express $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ in terms of $\epsilon^{(j)}$. Note that for $\mathrm{SL}(3,\mathbb{C})$-opers (6.13) takes the form

$$\partial_z((\partial_z^2 - T)\epsilon^{(1)} + 3\epsilon^{(2)} + 3\epsilon^{(3)}) = 0\,.$$

Then the anchor (6.10) and the brackets (6.8) coincide with the anchor (5.14) – (5.17) and the brackets (5.10) – (5.12), if one puts $\epsilon^{(1)} = \varepsilon^{(2)}$ and $\epsilon^{(2)} = \varepsilon^{(1)}$.

The one-cocycle representing $H^1(\mathcal{A}_N)$ comes from the integration over $\Sigma_{g,n}$

$$c(L_N, X) = \int_{\Sigma_{g,n}} Res(X\bar{\partial}L_N)\,.$$

The contribution of the marked points is

$$c(L_N, X)|_{z=x_a} = \sum_{j=1}^{N} r^{(j)}_a W^a_{-j}(j)\,.$$

Let γ_α be a set of $2g$ fundamental cycles of $\Sigma_{g,n}$. One can define local AGD brackets $\{l_X, l_Y\}_\alpha$ (6.6) by the pairing (6.4) using γ_α. In this way we obtain $2g$ generators of $H^2(\mathcal{A}_N)$ of type (6.9).

6.3. Global Hamiltonian AGD algebroid. Let $B_{(r,N,1)}(\Sigma_{g,n}) = B_{r,N}(\Sigma_{g,n})\otimes \bar{K}(\Sigma_{g,n})$,, where $\bar{K}$ is the anti-canonical class,

$$B_{(r,N,1)}(\Sigma_{g,n})\,:\,\Gamma(\Omega^{(\frac{N+1}{2},0)}(\Sigma_{g,n}))\to\Gamma(\Omega^{(\frac{-N-1}{2},1)}(\Sigma_{g,n}))\,.$$

Consider the set of the SL$(N,\mathbb{C})$-opers $M_N = M_N(\Sigma_{g,n})$. The affine space $\mathcal{R}_N = Aff\,T^*M_N(\Sigma_{g,n})$ over $T^*M_N(\Sigma_{g,n})$ is the set of fields $\xi\in B_{(0,N,1)}(\Sigma_{g,n})/B_{(-N-2,N,1)}(\Sigma_{g,n})$. Near the marked points ξ behaves as

$$\xi=\sum_{j=1}^{N+1}\nu_j\partial^{-j}\,,\ (\nu_{N+1}=1)\,,$$

$$\nu_j\sim(t^{(j)}_{a,0}+\ldots+t^{(j)}_{a,j-1}(z-x_a)^{j-1})\bar{\partial}\chi_a(z,\bar{z})\,,\ (\nu_j\in\Omega^{(j-N,1)}(\Sigma_{g,n}))\,.$$

The symplectic form on $\mathcal{R}_N$ is

$$\omega=\int_{\Sigma_{g,n}}Res\,(DL_N\wedge D\xi)\,.$$

The anchor action on M_N (6.10) can be lifted from M_N to $\mathcal{R}_N$ as the canonical transformations of ω

$$\delta_Y\xi=-\bar{\partial}Y+Y(L_N\xi)_+-(\xi L_N)_+Y+(YL_N)_+\xi-\xi(L_NY)_+\,. \tag{6.16}$$

The transformations are generated by the Hamiltonians

$$h_Y=\int_{\Sigma_{g,n}}Res\,(\xi\delta_YL_N)+c(L_N,Y)\,.$$

The anchor action

$$\delta_YL_N=\{h_Y,L_N\}\,,\ \delta_Y\xi=\{h_Y,\xi\}\,.$$

defines the global Hamiltonian AGD-algebroid $\mathcal{A}^H_N(\Sigma_{g,n})$. The Hamiltonian can be represented in the form

$$h_Y=\int_{\Sigma_{g,n}}Res\,(YF(L_N,\xi))\,,$$

where

$$F:=\bar{\partial}L_N-(L_N\xi)_+L_N+L_N(\xi L_N)_+\,. \tag{6.17}$$

The space $\mathcal{R}_N$ is a phase space of W_N-gravity in the space $\Sigma_g\times\mathbb{R}$. The canonical transformations (6.10), (6.16) are the gauge transformations of the theory.

6.4. Generalized projective structures. Define the operator

$$\bar{\partial}+A\,:\,\Omega^{(\frac{N+1}{2},0)}(\Sigma_{g,n})\to\Omega^{(\frac{N+1}{2},1)}(\Sigma_{g,n})\,,\ A=-(L_N\xi)_+ \tag{6.18}$$

and the dual operator

$$\bar{\partial}+A^*\,:\,\Omega^{(-\frac{N-1}{2},0)}(\Sigma_{g,n})\to\Omega^{(-\frac{N-1}{2},1)}(\Sigma_{g,n})\,,\ A^*=(\xi L_N)_+\,.$$

Let $\psi=(\psi^-,\psi^+)$, $\psi^-\in\Omega^{(-\frac{N-1}{2},0)}(\Sigma_{g,n})$, $\psi^+\in\Omega^{(\frac{N+1}{2},0)}(\Sigma_{g,n})$. Similar to Lemma 3.2 we find that the constraints $F=0$ (see (6.17)) are equivalent to the linear problem

$$L_N\psi^-(z,\bar{z})=0\,, \tag{6.19}$$

$$(\bar{\partial}-(L_N\xi)_+)\psi^-(z,\bar{z})=0\,, \tag{6.20}$$

$$(\bar{\partial}+(\xi L_N)_+)\psi^+(z,\bar{z})=0\,. \tag{6.21}$$

Here we use the "vector representation". It means that in the matrix forms of opers (5.1) $\psi^{\pm}$ are vectors. The linear system defines *the generalized projective structures* on $\Sigma_{g,n}$.

In this way an oper L_N together with the dual element ξ defines W_N*-deformation of complex structures* on $\Sigma_{g,n}$. The equations (6.20) and (6.21) are equivalent to the deformed holomorphity condition for the sections
$\Omega^{(-\frac{N-1}{2},0)}(\Sigma_{g,n})$ and $\Omega^{(\frac{N+1}{2},0)}(\Sigma_{g,n})$.

Let G_N be the groupoid corresponding to the AGD-algebroid $\mathcal{A}_N(\Sigma_{g,n})$.

DEFINITION 6.3. *The moduli space $\mathcal{W}_N$ of the W_N-gravity is the symplectic quotient*

$$\mathcal{R}_N//G_N = \{\bar{\partial}L_N - (L_N\xi)_+L_N + L_N(\xi L_N)_+ = 0\}/G_N\,.$$

The moduli space of the W_N-deformations of complex structures on $\Sigma_{g,n}$ is a part of the symplectic quotient $\mathcal{W}_N \sim \mathcal{R}_N//G_N$. The cohomology of the classical BRST operator are defined by

$$\Omega = h_\eta + \frac{1}{2}\int_{\Sigma_{g,n}} Res\,((\lfloor\eta,\eta'\rfloor\mathcal{P})\ ,$$

where η is the ghost field corresponding to the gauge field Y and $\mathcal{P}$ is its momenta.

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